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# Harmonic Sobolev–Besov spaces, layer potentials and regularity for the Neumann problems in Lipschitz domains

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**Abstract**

Let $\Omega$ be a bounded Lipschitz domain in $R^n (n \geqslant 3)$. After giving several equivalent characterization of harmonic Sobolev–Besov spaces and studying the mapping properties of the single layer potential operator, we study the Neumann problem for the Laplacian with data in quasi-Banach boundary Besov spaces and obtain the regularity results.


*MSC:* 35J25; 42B20

*Keywords:* Neumann problem; Besov space; Hardy space; Lipschitz domain

## 1. Introduction

We study the Neumann Problem (NP) for the Laplacian

$$\Delta u = 0 \text{ in } \Omega, \quad \partial u / \partial v = f \text{ on } \partial\Omega \tag{1.1}$$

with data in quasi-Banach boundary Besov spaces of domain $\Omega$ in $R^n (n \geqslant 3)$ with Lipschitz boundary, where $v$ denotes outward unit normal to $\partial\Omega$.

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The Laplace's equation in Lipschitz domains have been studied by many mathematicians, see [Br,JK,DK,FMM,Z] and references therein. In [FMM], the authors studied the Neumann boundary problem conditions

$$\Delta u = f \text{ in } \Omega, \quad \partial u/\partial v = g \text{ on } \partial\Omega \tag{1.2}$$

by establishing the invertibility of the classical layer potential operators on scales of Sobolev–Besov spaces on Lipschitz boundaries. And they obtained the following theorem.

**Theorem A.** *Let $\Omega \subset R^n (n \geqslant 3)$ be a bounded Lipschitz domain, then there exists $\varepsilon = \varepsilon(\Omega, n) \in (0, 1]$ such that the following relevance holds. If $1 < p < \infty$ and $0 < s < 1$ are such that $(s, 1/p)$ belongs to the region $R_\varepsilon = APQA'P'Q'$ in Fig. 1 then, for every $f \in L^p_{s-2}(\Omega)$ and $g \in B^p_{s-1-1/p}(\partial\Omega)$ satisfying the compatibility condition $\langle f, 1\rangle = \langle g, 1\rangle$, (1.2) has a unique (modulo additive constants) solution $u$ with the estimate*

$$\|u\|_{L^p_s(\Omega)} \leqslant c(\|f\|_{L^p_{s-2}(\Omega)} + \|g\|_{B^p_{s-1-1/p}(\partial\Omega)}).$$

*Finally, if $\partial\Omega \in C^1$ then we may take $\varepsilon = 1$.*

In Theorem A, we recall that $(s, 1/p)$ belongs to the region $R_\varepsilon = APQA'P'Q'$ iff there exists $\varepsilon = \varepsilon(M, n) \in (0, 1]$, such that one of the following statement holds:

(a) $p_0 < p < p'_0$ and $1/p < s < 1 + 1/p$,
(b) $1 < p \leqslant p_0$ and $3/p - 1 - \varepsilon < s < 1 + 1/p$,
(c) $p'_0 \leqslant p < \infty$ and $1/p < s < 3/p + \varepsilon$.

Wherein $1/p_0 = 1/2 + \varepsilon/2$ and $1/p'_0 = 1/2 - \varepsilon/2$.

In [Z], the author studied the inhomogeneous Neumann problem for $f \in L^p_s(\Omega)$ and $g = 0$ with $(s, 1/p)$ as in Theorem A via the estimates for the inverse Calderon operator.

In [JK], the authors studied the inhomogeneous Dirichlet problem

$$\Delta u = f \text{ in } \Omega, \quad Tr\, u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

via harmonic measure techniques, here $Tr\, u$ stands for the trace of function $u$.

In this paper, we mainly studied (NP) in Lipschitz domains with data in quasi-Banach Besov spaces of order less than one for appropriate range $p$ and $s$, where $p \leqslant 1$.

Before stating our main theorem, we first recall the B- or F-spaces on domains. For $s \in R, 0 < p, q \leqslant \infty$, the definitions of B-spaces $B^{p,q}_s = B^{p,q}_s(R^n)$ (or $\dot{B}^{p,q}_s = \dot{B}^{p,q}_s(R^n)$) and F-spaces $F^{p,q}_s = \dot{F}^{p,q}_s(R^n)$ (or $\dot{F}^{p,q}_s = \dot{F}^{p,q}_s(R^n)$) $(p \neq \infty)$ can be found in [FJ,Tr1,ET], here we omit the details. In particular, we have the potential spaces $L^p_s(R^n) = F^{p,2}_s(R^n)(p > 1)$, the Hardy–Sobolev spaces $h^p_s(R^n) = F^{p,2}_s(R^n)(0 < p \leqslant 1)$ and the Besov spaces $B^p_s(R^n) = B^{p,p}_s(R^n)(0 < p \leqslant \infty)$. Let $\Omega$ be a domain in $R^n$, we define

$$B_s^{p,q}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists F \in B_s^{p,q}(R^n), s.t.\ F|_\Omega = f\},$$

$$F_s^{p,q}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists F \in F_s^{p,q}(R^n), s.t.\ F|_\Omega = f\}$$

with the usual norm.

For Lipschitz domain $\Omega$, the scales of B- and F-spaces can be well-defined on the boundary, and we use the notations $B_s^{p,q}(\partial\Omega)$ (or $\dot{B}_s^{p,q}(\partial\Omega)$) and $F_s^{p,q}(\partial\Omega)$ (or $\dot{F}_s^{p,q}(\partial\Omega)$), respectively, for appropriate $s, p, q$, more details can be found in [FMM,Z].

Throughout this paper, we use the notations $h_s^p(\Omega) = F_s^{p,2}(\Omega)(p \leqslant 1)$, $B_s^p(\Omega) = B_s^{p,p}(\Omega)(0 < p \leqslant \infty)$ and $\dot{B}_s^p(\partial\Omega) = \dot{B}_s^{p,p}(\partial\Omega)$ (See [ET,CKS,Se,FMM,Z]). If no other claim, the domain $\Omega \subset R^n (n \geqslant 3)$ always means a bounded Lipschitz domain and let $M$ denote the Lipschitz constant of Lipschitz domain $\Omega$, see [St] for more details.

Now we state our main theorem as follows.

**Theorem 1.1.** *Let $\Omega$ be a bounded Lipschitz domain. There exists $\varepsilon \in (0, 1]$, which may be dependent in the Lipschitz constant of $\Omega$. Suppose*

$$n/(n+\varepsilon) < p \leqslant 1, \quad n(1/p - 1) - \varepsilon + 1 + 1/p < s < 1 + 1/p, \tag{1.4}$$

*then for all $f \in \dot{B}^p_{s-1-1/p}(\partial\Omega)$, there is a unique solution $u \in B_s^p(\Omega)$ to $(NP)$*

$$\Delta u = 0 \text{ in } B^p_{s-2}(\Omega),\ \partial u/\partial v = f \text{ on } \dot{B}^p_{s-1-1/p}(\partial\Omega) \tag{1.5}$$

*with the estimate*

$$||u||_{B_s^p(\Omega)} \leqslant c||f||_{\dot{B}^p_{s-1-1/p}(\partial\Omega)}.$$

*When the domain is $C^1$, $\varepsilon$ may be taken to be 1.*

**Remarks.** (i) We know that the collection of $s$ and $1/p$ for which Theorem 1.1 holds is best understood as the open triangle $QA'F$ in Fig. 1. When $\partial\Omega \in C^1$, then the collection of $s$ and $1/p$ is understood as the open triangle $CA'E$ in Fig. 1. Also note that $QF$ is parallel to $CE$ in Fig. 1.

(ii) When we were making up and revising our paper, Mitrea obtained the similar results as in Theorem 1.1 by using a Besov-based non-tangential maximal function (see [MM1]) or by establishing suitable square-function estimates for singular integral of potential type (see [MM2]).

In Theorem 1.1, the important ingredients in this regard are establishing the equivalent characterization to harmonic Hardy–Sobolev spaces $h_s^p(\Omega)$ for $p \leqslant 1$, and the techniques of the single potential estimates on the boundary Besov spaces $\dot{B}^p_{s-1/p-1}(\partial\Omega)$ for $p \leqslant 1$ as the following theorems.

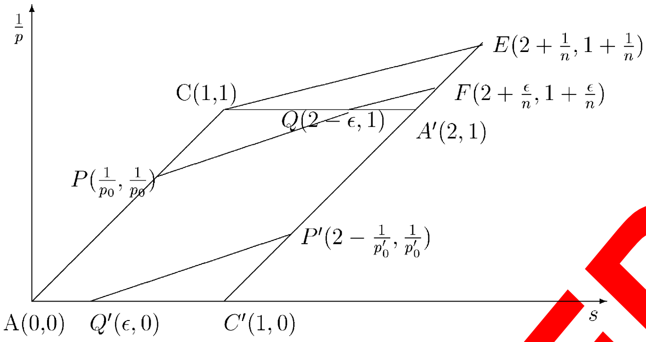


Fig. 1.

**Theorem 1.2.** *Let $\Omega$ be a bounded Lipschitz domain. Suppose that $u$ is a harmonic function in $\Omega$. Let $0<s<1, 0<p\leqslant 1$ and $k \in \mathbb{N} \cup \{0\}$. Consider the following statements*:

(a) *$u$ belongs to $h^p_{s+k}(\Omega)$*,
(b) *$u$ belongs to $B^p_{s+k}(\Omega)$*,
(c) *$\delta^{1-s}\sum_{|\alpha|=k+1} D^\alpha u$, $\sum_{|\alpha|\leqslant k} D^\alpha u$ belong to $h^p(\Omega)$*,
(d) *$\delta^{-s}\sum_{|\alpha|=k} D^\alpha u$, $\sum_{|\alpha|\leqslant k-1} D^\alpha u$ belong to $h^p(\Omega)$.*

*Then*, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftarrow$ (d).

When $1<p<\infty$, Theorem 1.2 was studied in [JK,FMM] by using the abstract interpolation theorems. Here, we have not utilized the abstract theorems.

Let $\omega_n = |B(0,1)|/n$, recall that the Newtonian potential is defined by

$$N(X) := (2-n)^{-1}\omega_n^{-1}|X|^{2-n}, \quad X \in R^n - \{0\}. \tag{1.6}$$

For $f$ on $\partial\Omega$, the single layer potential operator is defined by

$$(\mathcal{S}f)(X) := \langle f, N(X-\cdot)\rangle, \quad X \in \Omega. \tag{1.7}$$

$\mathcal{S}f$ is harmonic functions in $R^n - \partial\Omega$.

**Theorem 1.3.** *For any $0<s<1$ and $(n-1)/n<p\leqslant 1$ with $s+n(1/p-1)<1/p$, the single layer potential $\mathcal{S}$ is a bounded linear map from $B^p_{-s}(\partial\Omega)$ into $B^p_{1+1/p-s}(\Omega)$, that is, there exists a constant $C = C(M,n)$, such that*

$$\|\mathcal{S}f\|_{B^p_{1+1/p-s}(\Omega)} \leqslant c\|f\|_{B^p_{-s}(\partial\Omega)}.$$

*In particular, $S = Tr\mathcal{S}$ maps $B^p_{-s}(\partial\Omega)$ into $B^p_{1-s}(\partial\Omega)$.*

The paper is arranged as follows. In Section 2, we will recall some fundamental definitions and notations, including the atomic characterization of the boundary Besov spaces. Section 3 contains the proof of several equivalent characterization to harmonic Hardy–Besov spaces, i.e., Theorem 1.2. Section 4 contains general estimates and their invertibility results of the single potential operators on the boundary Besov spaces, i.e., Theorem 1.3. We also discuss the trace theorems in there. In Section 5, we will prove our main Theorem, Theorem 1.1.

## 2. Some fundamental definitions and notations

In this section, we first recall some fundamental definitions and notations.

For $s \in R$, denote $s_+ = \max(s, 0)$, and $[s]$ stands for the integer functions. If Q is a cube whose sides are parallel to the axes, the $x_Q, l_Q$ denote the center, sidelength of Q, respectively. Let $c$ be a positive constant, $cQ$ represents the cube $Q(x_Q, cl_Q)$. If no other claim, $Q$ always stands for a dyadic cube.

The dilates or dyadic dilates of $g$ are defined by $g_t(x) = t^{-n}g(x/t)$ or $g_j(x) = 2^{jn}g(2^j x)$ for $j \in \mathbb{N} \cup \{0\}$.

The definitions of the scales of Besov and Triebel–Lizorkin spaces on $R^n$ and their related matters, such as the embedding theorems, the dual theorems, the atomic decompositions and so on, can be found in [FJ,Tr1,ET], here we omit the details.

The scales of Besov and Triebel–Lizorkin spaces can be naturally transported from $R^{n-1}$ to the boundaries of (bounded) Lipschitz domain $\Omega$ via pull-back and a partition of unity. And we denote them by $B_s^{p,q}(\partial\Omega)$ and $F_s^{p,q}(\partial\Omega)$, respectively, for appropriate $s, p, q$. These boundary spaces were deliberately studied by [FMM,Z].

Let $d\sigma$ denote the surface measure on $\Omega$ so that $v$, the outward unit normal to $\partial\Omega$, is well-defined $d\sigma$-a.e..

We say that $\Omega$ is a Lipschitz graph, if there exists a Lipschitz function $\varphi : R^{n-1} \to R$, such that $\Omega = \{(x,t) \in R^n : t > \varphi(x)\}$. The definitions of the bounded Lipschitz domain can be found in [St], here we omit the details.

When $(n-1)/n < p, q < \infty, (n-1)(1/p-1)_+ < s < 1$, we say that $f$ belongs to $B_s^{p,q}(\partial\Omega)$ if $f(x, \varphi(x))$ belongs to $B_s^{p,q}(R^{n-1})$. When $(n-1)/n < p, q < \infty, (n-1)(1/p-1)_+ < 1 - s < 1$, we say that $f$ belongs to $B_{-s}^{p,q}(\partial\Omega)$ if $f(x, \varphi(x))\sqrt{1 + |\nabla\varphi(x)|}$ belongs to $B_{-s}^{p,q}(R^{n-1})$.

The definitions of the boundary function spaces can be extended to the case of the bounded Lipschitz domain via a simple partition of unity argument. The related matters of the boundary function spaces, such as the embedding theorems, the dual theorems, remain valid. For example, for $-1 < s < 0$ and $1 < p, q < \infty$, we have $B_s^{p,q}(\partial\Omega) = (B_{-s}^{p',q'}(\partial\Omega))^*$, where $1/p + 1/p' = 1, 1/q + 1/q' = 1$, and the duality pairing between $f \in B_s^{p,q}(\partial\Omega)$ and $g \in B_{-s}^{p',q'}(\partial\Omega)$ is $\int_{\partial\Omega} fg\, d\sigma$. See [FMM,Z] for more details.

For a bounded Lipschitz domain, a set $Q \subset \partial\Omega$ is said to be a "dyadic" cube if there exists a constant $\delta_0$ which is dependent in the Lipschitz constant, and a dyadic cube $Q'$ in $R^{n-1}$, such that $Q = \{(x, \varphi(x)) : x \in Q'\}$ with the length $l_{Q'} \leqslant \delta_0$. If no other claim, we always set $\delta_0 = 1$.

**Definition 2.1.** Let $\Omega$ be a bounded Lipschitz domain and $(n-1)/n < p \leqslant 1$, $(n-1)(1/p-1) < 1+s < 1$. A function $a_Q$ associated to a dyadic cube $Q \subset \partial\Omega$ with $l_Q < 1$, is said to be a boundary atom of the Besov spaces $\dot{B}_s^{p,q}(\partial\Omega)$ iff

$$supp\ a_Q \subset 2Q,\ |a_Q| \leqslant |Q|^{s/(n-1)-1/p} \text{ and } \int_{\partial\Omega} a_Q\, d\sigma = 0.$$

Now the atomic characterization [FJ] can be lifted to $\partial\Omega$ as below. The function $f$ belongs to the Besov space $\dot{B}_s^{p,q}(\partial\Omega)$ if only if $f$ has the atomic decomposition $f = \sum s_Q a_Q$ with $||s||_{\dot{b}_s^{p,q}(\partial\Omega)} < \infty$, where $a_Q$ is a boundary atom of $\dot{B}_s^{p,q}(\partial\Omega)$ as in Definition 2.1, and $||s||_{\dot{b}_s^{p,q}(\partial\Omega)} = \left\{\sum_{j=0}^{\infty}\left(\sum_Q |s_Q|^p\right)^{q/p}\right\}^{1/q}$. Moreover, we have

$$||f||_{\dot{B}_s^{p,q}(\partial\Omega)} \approx \inf\left\{||s||_{\dot{b}_s^{p,q}(\partial\Omega)} : f = \sum_Q s_Q a_Q\right\}.$$

## 3. Harmonic Sobolev and Besov spaces of order less than one

For any domain $\Omega$, let $dist(x, \partial\Omega)$ be the distance from $x$ to $\partial\Omega$. We know that there exists a function $\delta(x)$ defined in $\Omega$, such that (i) $c_1 dist(x, \partial\Omega) \leqslant \delta(x) \leqslant c_2 dist(x, \partial\Omega)$ for any $x \in \Omega$, (ii) $\delta(x) \in C^\infty(\Omega)$ and $|\partial_x^\alpha \delta(x)| \leqslant c_\alpha dist(x, \partial\Omega)^{1-|\alpha|}$ for all $\alpha$. Where $c_1, c_2, c_\alpha$ are independent of $\Omega$ (see [St]).

To prove Theorem 1.2, we first introduce a kind of a partition of unit for a bounded Lipschitz domain $\Omega$. That is, there exist $\{\chi_k : k \geqslant 0\} \subset C_0^\infty(\Omega)$, such that

(i) $0 \leqslant \chi_k \leqslant 1$, and $\sum_k \chi_k = 1$ in $\Omega$;
(ii) $supp\ \chi_k \subset \{x \in \Omega : (1-\varepsilon)2^{-k-1} \leqslant \delta(x) \leqslant (1+\varepsilon)2^{-k}\}$;
(iii) $\chi_k = 1$ for $\{(1-\varepsilon/2)2^{-k-1} \leqslant \delta(x) \leqslant (1+\varepsilon/2)2^{-k}\}$ ($\varepsilon$ small enough).

When $\Omega$ is a bounded Lipschitz domain, for $s, p$ with $sp < 1$, there exists an equivalent characterization to $F_s^{p,q}(\Omega)$ which can be found in [W1] as follows.

**Lemma 3.1.** *Let $\Omega$ be a bounded Lipschitz domain, let $s \in R$, $0 < p, q < \infty$ with $sp < 1$, then*

(i) $C_0^\infty(\Omega)$ *is dense in* $F_s^{p,q}(\Omega)$.

(ii) $f \in F_s^{p,q}(\Omega)$ *iff* $\left\{\sum_{k=0}^\infty (2^{k(s+n(1-1/p))}||(f\chi_k)_{2^k}||_{F_s^{p,q}})^p\right\}^{1/p} < \infty$. *Moreover, we have*

$$||f||_{F_s^{p,q}(\Omega)} \approx \left\{\sum_{k=0}^{\infty} (2^{k(s+n(1-1/p))}||(f\chi_k)_{2^k}||_{F_s^{p,q}})^p\right\}^{1/p}. \tag{3.1}$$

In particular, when $p = q$, by $B_s^{p,p} = F_s^{p,p}$, we have

$$||f||_{B_s^{p,p}(\Omega)} \approx \left\{\sum_{k=0}^{\infty} (2^{k(s+n(1-1/p))}||(f\chi_k)_{2^k}||_{B_s^{p,p}})^p\right\}^{1/p}. \tag{3.2}$$

**Proof of Theorem 1.2.** To prove Theorem 1.2, it is enough to show the case $k = 0$. In fact, let $\Omega$ be a bounded Lipschitz domain, for $s \in R, 0 < p, q < \infty$, we know $u \in F_s^{p,q}(\Omega)$ iff $u, \nabla u \in F_p^{s-1,q}(\Omega)$. For example, when $k \in N$, then we can found $u \in W^{-k,q}(\Omega)$ iff $u$, $\nabla u \in W^{-k-1,q}(\Omega)$ in [Nec], and $u \in W^{k,q}(\Omega)$ iff $u$, $\nabla u \in W^{k-1,q}(\Omega)$ in [St]. For general $s, p, q$, these facts can be deduced by the atomic characterization in Lipschitz domains, see [TW] or [W2].

In particular, when $s \in \mathbf{R}$ and $p = q$, then

$$u \in B_s^p(\Omega) \text{ iff } u,\ \nabla u \in B_{s-1}^p(\Omega). \tag{3.3}$$

*(b) implies (c)*: By $0 < p \leq 1$, $0 < s < 1$, we get $sp < 1$ and so we can use Lemma 3.1.

Let $u \in B_s^p(\Omega)$, we should prove $\delta^{1-s}\partial_i u$, $u \in h^p(\Omega)$, where $\partial_i = \partial/\partial x_i$. Choose a radial function $\phi$ with $\phi(x) \in C_0^\infty(B(0,1))$, $0 \leq \phi(x) \leq 2$ with $\int \phi(x)dx = 1$. For $i = 1, 2, \dots, n$, by $ps < 1$, for every harmonic function $u \in B_s^p(\Omega)$, we consider

$$||\delta^{1-s}\partial_i u||_{h^p(\Omega)} = \left\{\sum_{k=0}^{\infty}(2^{kn(1-1/p)}||(\delta^{1-s}\partial_i u\chi_k)_{2^k}||_{F_0^{p,2}})^p\right\}^{1/p} \quad \text{(i)}$$

$$= \left\{\sum_{k=0}^{\infty}(2^{kn(1-1/p)}||(\delta^{1-s}\phi_{t_k} * (\partial_i u)\chi_k)_{2^k}||_{F_0^{p,2}})^p\right\}^{1/p} \quad (t_k = 2^{-k}/100) \quad \text{(ii)}$$

$$= \left\{ \sum_{k=0}^{\infty} \left(2^{kn(1-1/p)} \| (\delta^{1-s} t_k^{-1} (\partial_i \phi)_{t_k} * u\chi_k)_{2^k} \|_{F_0^{p,2}}\right)^p \right\}^{1/p}$$

$$= \Bigg\{ \sum_{k=0}^{\infty} \Bigg(2^{kn(1-1/p)} \Bigg\| (\delta^{1-s} t_k^{-1} \chi_k)(2^k \cdot) \times \left( (\partial_i \phi)_{t_k} * \left( \sum_{i=k-1}^{k+1} u\chi_i \right) \right)_{2^k} \Bigg\|_{F_0^{p,2}} \Bigg)^p \Bigg\}^{1/p} \quad \text{(iii)}$$

$$\leqslant c \left\{ \sum_{k=0}^{\infty} \left( 2^{k(s+n(1-1/p))} \left\| \left( (\partial_i \phi)_{t_k} * \left( \sum_{i=k-1}^{k+1} u\chi_i \right) \right)_{2^k} \right\|_{F_0^{p,2}} \right)^p \right\}^{1/p} \quad \text{(iv)}$$

$$\leqslant c \left\{ \sum_{k=0}^{\infty} \left( 2^{k(s+n(1-1/p))} \left\| \left( \sum_{i=k-1}^{k+1} u\chi_i \right)_{2^k} \right\|_{F_0^{p,2}} \right)^p \right\}^{1/p} \quad \text{(v)}$$

$$\leqslant c \left\{ \sum_{k=0}^{\infty} \left(2^{k(s+n(1-1/p))} \| (u\chi_k)_{2^k} \|_{F_0^{p,2}}\right)^p \right\}^{1/p}$$

$$\leqslant c \left\{ \sum_{k=0}^{\infty} \left(2^{k(s+n(1-1/p))} \| (u\chi_k)_{2^k} \|_{B_0^{p,p}}\right)^p \right\}^{1/p} \quad \text{(vi)}$$

$$\leqslant c \left\{ \sum_{k=0}^{\infty} \left(2^{k(s+n(1-1/p))} \| (u\chi_k)_{2^k} \|_{B_s^{p}}\right)^p \right\}^{1/p} \leqslant c \|u\|_{B_s^p(\Omega)}. \quad \text{(vii)}$$

Here, the formula (i) is obtained by (3.1),

(ii) is obtained by $u * \varphi_{t_k}(x) = u(x)$ since $u$ is harmonic in $\Omega$ and $\phi$ is a radial function with $\int \phi = 1$,

(iii) is obtained by the support of the function $\chi_k$ and $\phi_{t_k}$,

The inequality (iv) is obtained by the pointwise multiplier theorem. In fact, $(\delta^{1-s} t_k^{-1} \chi_k)(2^k \cdot) \in C_0^\infty$ and $|\partial_i^m (\delta^{1-s} t_k^{-1} \chi_k)(2^k \cdot)| \leqslant c_m 2^{ks}$ for all $m$, see [Tr1] for more details.

(v) is obtained by the convolution properties. More details can be found in Remark 3 of [Tr1, p. 127].

(vi) is obtained by the imbedding theorem $B_0^{p,p} \subset F_0^{p,2}$ for $p \leqslant 1 < 2$, see [Tr1, p. 47] for more details.

(vii) is obtained by $B_s^p \subset B_0^{p,p}$ for $s > 0$ and (3.2).

So we obtain the proof of the case $(b) \Rightarrow (c)$.

*(c) implies (b)*: Given any harmonic function $u$ with $\delta^{1-s} \nabla u$, $u \in h^p(\Omega)$, we should prove $u \in B_s^p(\Omega)$, i.e., $u$, $\nabla u \in B_{s-1}^p(\Omega)$ by (3.3). In fact, by $s < 1$, if $u \in h^p(\Omega)$, then $u \in B_{s-1}^p(\Omega)$ by $h^p(\Omega) = F_0^{p,2}(\Omega) \subset B_0^{p,\infty}(\Omega) \subset B_{s-1}^{p,p}(\Omega) = B_{s-1}^p(\Omega)$, see [Tr1, p. 47] for

more details. Now by Lemma 3.1 and using the imbedding theorem $h^p(\Omega) \subset B^p_{s-1}(\Omega)$ again, we can obtain the case of (c) implies (b) by the following formulas as similar to the proof in the case of that (b) implies (c) (for $i = 1, \dots, n$).

$$\begin{aligned}
\|\partial_i u\|_{B^p_{s-1}(\Omega)} &= \left\{ \sum_{k=0}^{\infty} (2^{k(s-1+n(1-1/p))} \|(\partial_i u \chi_k)_{2^k}\|_{B^p_{s-1}})^p \right\}^{1/p} \\
&\leqslant c \left\{ \sum_{k=0}^{\infty} (2^{kn(1-1/p)}) \|(\delta^{1-s} \partial_i u \chi_k)_{2^k}\|_{B^p_{s-1}})^p \right\}^{1/p} \\
&\leqslant c \left\{ \sum_{k=0}^{\infty} (2^{kn(1-1/p)}) \|(\delta^{1-s} \partial_i u \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} = c\|\delta^{1-s} \partial_i u\|_{h^p(\Omega)}.
\end{aligned}$$

*(d) implies (c)*: The proof of this case is similar to the proof of the assertion $(c) \Leftrightarrow (b)$. In fact, by the facts that $0 < s < 1$ and u is harmonic in $\Omega$, we can complete the proof by the following:

$$\begin{aligned}
\|\delta^{1-s} \partial_i u\|_{h^p(\Omega)} &= \left\{ \sum_{k=0}^{\infty} (2^{kn(1-1/p)} \|(\delta^{1-s} \partial_i u \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} \\
&= \left\{ \sum_{k=0}^{\infty} (2^{kn(1-1/p)} \|(\delta^{1-s} \phi_{t_k} * (\partial_i u) \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} \qquad (t_k = 2^{-k}/100) \\
&= \left\{ \sum_{k=0}^{\infty} (2^{kn(1-1/p)} \|(\delta^{1-s} t_k^{-1} (\partial_i \phi)_{t_k} * u \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} \\
&\leqslant c \left\{ \sum_{k=0}^{\infty} (2^{k(s+n(1-1/p))} \|((\partial_i \phi)_{t_k} * u \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} \\
&\leqslant c \left\{ \sum_{k=0}^{\infty} (2^{k(s+n(1-1/p))} \|(\delta^s \delta^{-s} u \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} \\
&\leqslant c \left\{ \sum_{k=0}^{\infty} (2^{kn(1-1/p)} \|(\delta^{-s} u \chi_k)_{2^k}\|_{F_0^{p,2}})^p \right\}^{1/p} = \|\delta^{-s} u\|_{h^p(\Omega)}
\end{aligned}$$

(a) $\Leftrightarrow$ (c): Using the similar methods as the previous proof, we can also prove this assertion and so we omit the details.

## 4. The single layer potential

In this section, we study the bounds and the invertibility of the single layer potential on the boundary Besov spaces of order less than one.

Recall that the Newtonian potential is defined by

$$N(X) := (2-n)^{-1}\omega_n^{-1}|X|^{2-n}, \quad X \in R^n - \{0\}.$$

For $f$ on $\partial\Omega$, the single potential operator is defined by

$$(\mathcal{S}f)(X) := \langle f, N(X-\cdot)\rangle, \quad X \in \Omega.$$

The boundary layer potential and its adjoint are defined by

$$\begin{aligned} Kf(P) &= p.v.\frac{1}{\omega_n}\int_{\partial\Omega} \langle Q-P, n(Q)\rangle |Q-P|^{-n} f(Q)\, d\sigma(Q) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\omega_n}\int_{|P-Q|>\varepsilon} \langle Q-P, n(Q)\rangle |Q-P|^{-n} f(Q)\, d\sigma(Q), \quad P \in \partial\Omega, \\ K^*f(P) &= p.v.\omega_n^{-1}\int_{\partial\Omega} \langle P-Q, n(P)\rangle |Q-P|^{-n} f(Q)\, d\sigma(Q), \quad P \in \partial\Omega. \end{aligned}$$

To prove Theorem 1.3, we utilize the following lemma which can be found in [W3].

**Lemma 4.1.** *Let $u$ be harmonic in Lipschitz domain $\Omega$, $0<p\leqslant 1$, $-1<s<1$, then $\delta^s u \in h^p(\Omega)$ iff $u \in h^p(\Omega, \delta^{sp}dx)$ iff $u \in L^p(\Omega, \delta^{sp}dx)$.*

**Proof of Theorem 1.3.** Let $s, p$ be as in Theorem 1.3, then $1<1+1/p-s<1+n/(n-1)<3$. There are three cases which should be considered.

*Case 1*: $2<1+1/p-s<3$. By (d) in Theorem 1.2, we should prove $\delta^{s+1-1/p}\partial_i\partial_j(\mathcal{S}f), \partial_i\mathcal{S}f, \mathcal{S}f \in h^p(\Omega)$ for $i,j = 1,\dots,n$. By the atomic characterization of Besov spaces (see Section 2) and Lemma 4.1, it is enough to prove

$$\int_\Omega \delta(X)^{sp+p-1}|(\partial_i\partial_j\mathcal{S}a_Q)(X)|^p + |\partial_i\mathcal{S}a_Q(X)|^p + |\mathcal{S}a_Q(X)|^p\, dx \leqslant c \tag{4.1}$$

for every atom function $a_Q$ of $B^p_{-s}(\partial\Omega)$. We first show

$$\int_\Omega \delta(X)^{sp+p-1}|(\partial_i\partial_i\mathcal{S}a_Q)(X)|^p\, dx \leqslant c. \tag{4.2}$$

Via a partition of unity, there is no loss of generality in assuming that the support of $a$ contained in a coordinate patch where $\partial\Omega$ is given by the graph of a Lipschitz function $\varphi : R^{n-1} \to R$. The problem localizes and, since the only nontrivial case is when $X$ is close to the boundary, it suffices to prove the estimates

$$\int_\Omega |g(X)|\delta(X)^{sp+p-1}|(\partial_i\partial_j\mathcal{S}a_Q)(X)|^p\, dX \leqslant c,$$

uniformly for $g \in L^\infty_{comp}(\Omega)$ with $||g||_{L^\infty(\Omega)} \leqslant 1$.

For the sake of the brevity, we set the center of the "cube" $Q$, $z=0$, and denote $a_Q$ by $a$. Let $Q_r = \{X \in \partial\Omega : |X| \leqslant cr\}$ and $\widetilde{Q_r} := \{(x,t) : |x| + t \leqslant 10c\sqrt{n}r\}$, where the constant $c$ is a large enough constant dependent in the Lipschitz constant $M$.

When $X \in \widetilde{Q_r}$, by the definition of the single layer (see (1.7)), we have

$$|(\partial_i\partial_j \mathcal{S}a)(X)| \leqslant cr^{-s-(n-1)/p} \int_{Q_r} |X-q|^{-n}\, d\sigma(q). \tag{4.3}$$

After a change of variables based on the representations $X = (x, \varphi(x) + t)$, $q = (y, \varphi(y))$, by (4.3), we get $(X \in \widetilde{Q_r})$

$$\begin{aligned}|(\partial_i\partial_j \mathcal{S}a)(X)| &\leqslant cr^{-s-(n-1)/p} \int_{|y|+|\varphi(y)|\leqslant cr} (|x-y| + |t + \varphi(x) - \varphi(y)|)^{-n}\, dy \\ &\leqslant cr^{-s-(n-1)/p} \int_{|y|\leqslant cr} (|x-y|+t)^{-n}\, dy \leqslant cr^{-s-(n-1)/p} t^{-1}.\end{aligned}$$

And so by a change of variables again, we have

$$\begin{aligned}&\int_{\widetilde{Q_r}} |g(X)|\delta(X)^{sp+p-1} |(\partial_i\partial_j \mathcal{S}a_Q)(X)|^p\, dX \\ &\quad \leqslant c \int_{|x|+t\leqslant cr} t^{sp+p-1} r^{-sp-n+1} t^{-p}\, dx\, dt \leqslant c.\end{aligned} \tag{4.4}$$

When $X \notin \widetilde{Q_r}$, using the moment condition of $a$, we get

$$\begin{aligned}|(\partial_i\partial_j \mathcal{S}a)(X)| &= \left| \int_{\partial\Omega} (\partial_i\partial_j N(X-q) - \partial_i\partial_j N(X))a(q)\, d\sigma(q) \right| \\ &\leqslant c \int_{Q_r} \frac{r}{|X|^{n+1}} |a(q)|\, d\sigma(q) \leqslant cr^{n-s-(n-1)/p} |X|^{-n-1}.\end{aligned} \tag{4.5}$$

Therefore, we have

$$\begin{aligned}&\int_{(\widetilde{Q_r})^c} |g(X)|\delta(X)^{sp+p-1} |(\partial_i\partial_j \mathcal{S}a_Q)(X)|^p\, dX \\ &\leqslant c \int_{cr\leqslant|x|+|t|\leqslant c'} t^{sp+p-1} r^{p(n-s)-(n-1)} (|x|+t)^{-np-p}\, dx\, dt \\ &= c\left( \int_{cr\leqslant|x|\leqslant c', t\leqslant cr} + \int_{|x|\leqslant cr, cr\leqslant t\leqslant c'} + \int_{cr\leqslant|x|\leqslant c', cr\leqslant t\leqslant c'} \right) \\ &\quad \times t^{sp+p-1} r^{p(n-s)-(n-1)} (|x|+t)^{-np-p}\, dx\, dt \\ &= I_1 + I_2 + I_3.\end{aligned}$$

The first plan is to estimate $I_1$, $I_2$, $I_3$.

$$I_3 \leqslant \int_{cr \leqslant |x| \leqslant c'} r^{p(n-s)-(n-1)}\,dx \int_{cr \leqslant t \leqslant c'} t^{sp+p-1}(|x|+t)^{-np-p}\,dt$$
$$\leqslant cr^{p(n-s)-(n-1)} \int_{cr \leqslant |x| \leqslant c'} |x|^{sp-np}\,dx \leqslant cr^{p(n-s)-(n-1)} + c \leqslant c. \tag{4.6}$$

Here we use $s+n(1/p-1)<1/p$ and $r<1$.

Similarly, by $s+n(1/p-1)<1/p$, $r<1$ and $(n-1)/n<p\leqslant 1$, we have

$$I_2 \leqslant r^{p(n-s)-(n-1)} \int_{cr \leqslant t \leqslant c'} \int_{|x| \leqslant cr} t^{sp+p-1}(|x|+t)^{-np-p}\,dx\,dt$$
$$\leqslant cr^{p(n-s)-(n-1)} \int_{cr \leqslant t \leqslant c'} t^{sp-np+n-2}\,dt \int_{|y| \leqslant c} (|y|+1)^{-np-p}\,dy \leqslant c. \tag{4.7}$$

And

$$I_1 \leqslant \int_{cr \leqslant |x| \leqslant c', t \leqslant cr} t^{sp+p-1} r^{p(n-s)-(n-1)}(|x|+t)^{-np-p}\,dx\,dt$$
$$\leqslant cr^{p(n-s)-(n-1)} \int_{cr \leqslant |x| \leqslant c'} |x|^{sp-np}\,dx \int_{\lambda \leqslant c} \lambda^{sp+p-1}(|\lambda|+1)^{-np-p}\,d\lambda \leqslant c. \tag{4.8}$$

By (4.6)–(4.8), we obtain the estimate

$$\int_{(\widetilde{Q}_r)^c} |g(X)|\delta(X)^{sp+p-1}|(\partial_i\partial_j \mathcal{S}a_Q)(X)|^p\,dx \leqslant c. \tag{4.9}$$

Combining (4.4) with (4.9), we obtain (4.2).

The methods of the estimates of

$$\int_\Omega |\partial_i \mathcal{S}a_Q(X)|^p + |\mathcal{S}a_Q(X)|^p\,dx \leqslant c$$

is similar to the above, and so we omit the details.

Therefore, we complete estimate (4.1).

*Case 2*: $1<1+1/p-s<2$. By Theorem 1.2, we should prove $\delta^{s-1/p}\partial_i(\mathcal{S}f)$, $\mathcal{S}f \in h^p(\Omega)$ with the estimate

$$\int_\Omega \delta(X)^{sp-1}|(\nabla \mathcal{S}a_Q)(X)|^p + |\mathcal{S}a_Q(X)|^p\,dx \leqslant c.$$

The methods of the proof is similar as in Case 1, and the details are omitted.

*Case 3*: $1+1/p-s=2$. This case can be proved by the interpolation theorem using Cases 1 and 2.

Note the inequality $s+n(1/p-1)<1/p$ is equivalent the inequality $(1+1/p-s)>(n-1)(1/p-1)+1/p$. By $0<s<1$ and $(n-1)/n<p\leqslant 1$, we know the trace

theorem remains true for the bounded Lipschitz domain. Therefore we complete the proof of Theorem 1.3. $\square$

In the next part of this section, we will show that the single layer potentials are invertible on various subspaces of $B_s^p(\partial\Omega)$ for $1/(1+\varepsilon)<p\leqslant 1$.

Recall $L^p(\partial\Omega)/\{1\} = \{f \in L^p(\partial\Omega) : \int_{\partial\Omega} f\, d\sigma = 0\}$, $H^p(\partial\Omega) = \dot{F}_0^{p,2}(\partial\Omega)$, $\hat{H}^p(\partial\Omega) = F_0^{p,2}(\partial\Omega)$ and $I$ is an identification operator, see [DK,Br] for more details.

**Lemma 4.2.** *Let $\Omega$ be a bounded Lipschitz domain, then there exists $\varepsilon = \varepsilon(M,n)$, such that*

(a.1) ([DK]) $I/2 \pm K^* : L^p(\partial\Omega)/\{1\} \to L^p(\partial\Omega)/\{1\}$ *and* $I/2 \pm K^* : L_{-1}^p(\partial\Omega)/\{1\} \to L_{-1}^p(\partial\Omega)/\{1\}$ *are invertible, whenever* $1<p<2+\varepsilon$.

(a.2) ([Br]) $I/2 - K^* : H^p(\partial\Omega) \to H^p(\partial\Omega)$ *and* $I/2 + K^* : \hat{H}^p(\partial\Omega) \to \hat{H}^p(\partial\Omega)$ *are invertible, whenever* $1/(1+\varepsilon)<p\leqslant 1$.

(b.1) ([DK]) $S : L^p(\partial\Omega) \to L_1^p(\partial\Omega)$ *and* $S : L_{-1}^p(\partial\Omega) \to L^p(\partial\Omega)$ *are invertible, whenever* $1<p<2+\varepsilon$.

(b.2) ([Br]) $S : H^p(\partial\Omega) \to H_1^p(\partial\Omega)$ *is invertible, whenever* $1/(1+\varepsilon)<p\leqslant 1$.

Using the above lemma, we have the following interpolation theorem, and the similar results can also be found in [MM1].

**Proposition 4.3.** *Let $\Omega$ be a bounded Lipschitz domain, then there exists $\varepsilon = \varepsilon(M,n) \in (0, 1/n]$, for $1/(1+\varepsilon)<p\leqslant 1$, $-1<s<0$, $1/p<1+\varepsilon(s+1)$, such that $I/2 \pm K^* : \dot{B}_s^p(\partial\Omega) \to \dot{B}_s^p(\partial\Omega)$ and $S : B_s^p(\partial\Omega) \to B_{1+s}^p(\partial\Omega)$ are invertible.*

**Proof.** Firstly, we can prove that, wherever the domains of the operators $S$, $I/2 \pm K^*$ of definition intersect, any two of these operators coincide, so that we will obtain our theorem by interpolating between a rather wide selection of pairs of these operators, see [FMM,Z] for more details, here we omit the details. As an analogue with [FMM], by (a.l) and (a.2), we can use the interpolation theorem $(H^{p_0}(\partial\Omega), L_{-1}^{p_1}(\partial\Omega)/\{1\})_{p,\theta} = \dot{B}_s^p(\Omega)$ (see [Z]), where $p_0<1$, $p_1>1$, $s=-\theta$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. And then we obtain $I/2 \pm K^* : \dot{B}_s^p(\partial\Omega) \to \dot{B}_s^p(\partial\Omega)$ is invertible. The rest can similarly be obtained by (b.1) and (b.2).

**Remarks.** The above results in Lemma 4.2 and Proposition 4.3 are seemly sharp in the class of Lipschitz domains. Moreover, when $\partial\Omega \in C^1$, we can take $\varepsilon = 1/n$, see [FMM,JK] for more details.

Lastly, we give a note here. Let $\Omega$ be a bounded Lipschitz domain of $R^n$, and let $s,p$ satisfy $n/(n+1)<p\leqslant\infty$, $n(1/p-1)+1/p<s<1+1/p$, then the operator $\frac{\partial}{\partial\nu}$, originally defined in $C^\infty(\bar{\Omega})$, can be extended from $B_s^p(\Omega)$ to $\dot{B}_{s-1-1/p}^p(\partial\Omega)$. This fact can be deduced by the atomic characterization of $B_s^p(\Omega)$. By the atom definition of $B_s^p(\Omega)$ (see [TW, p. 658]) and the atom decomposition theorem (see [TW, p. 664]), for every atom function $a(x)$ which belongs to $B_s^p(\Omega)$, we can get that $\frac{\partial a}{\partial v}$ satisfies the formula

$$supp\frac{\partial a}{\partial v}\subset suppa\cap\partial\Omega,\quad \left|\frac{\partial a}{\partial v}\right|\leqslant|suppa\cap\partial\Omega|^{(s-1-1/p)/(n-1)-1/p}\quad\text{and}\quad\int_{\partial\Omega}\frac{\partial a}{\partial v}d\sigma=0.$$

That is, $\frac{\partial a}{\partial v}$ is an atom $\dot{B}^p_{s-1-1/p}(\partial\Omega)$. Here we omit the details.

## 5. Proof of Theorem 1.1

In this section, we will prove our main Theorem 1.1, i.e., the Neumann problem.

**Proof of Theorem 1.1.** *Existence*: Firstly, we recall the atomic characterization of the boundary spaces (see Section 2). We know that every $f\in\dot{B}^p_{s-1-1/p}(\partial\Omega)$ has the form $f=\sum_Q s_Q a_Q$, where $\sum_Q |s_Q|^p<\infty$ and $a_Q$ is atom of $\dot{B}^p_{s-1-1/p}(\partial\Omega)$ with $\int a_Q\,d\sigma=0$.

For every atom $a_Q\in L^\infty(\partial\Omega)\subset L^2(\partial\Omega)/\{1\}$, we know that there exists a unique $u_Q=\mathcal{S}(1/2-K^*)^{-1}a_Q\in L^2_{3/2}(\Omega)$ (see [DK]) be a solution to (1.5) with the boundary data $a_Q\in L^2(\Omega)/\{1\}$.

Therefore, if we recall the invertibility of $I/2-K^*$ (see Proposition 4.3) and the mapping properties of $\mathcal{S}$ (see Theorem 1.3), the operator $\mathcal{S}(I/2-K^*)^{-1}$, originally defined in $L^2(\partial\Omega)/\{1\}$, can be extended from $\dot{B}^p_{s-1-1/p}(\partial\Omega)$ to $B^p_s(\Omega)$. Moreover, we have the estimate

$$||u_Q||_{B^p_s(\Omega)}=||\mathcal{S}(1/2-K^*)^{-1}a_Q||_{B^p_s(\Omega)}\leqslant c,\tag{5.1}$$

where $s,p$ satisfy the following properties. There exists a positive numbers $\varepsilon_1=\varepsilon_1(M,n)\in(0,1/n]$, such that

(i) $n/(n+1)<1/(1+\varepsilon_1)<p\leqslant1$ and $1/p<1+\varepsilon_1[(s-1-1/p)+1]$ by Proposition 4.3,
(ii) $(-s+1+1/p)+n(1/p-1)<1/p$ and $0<-s+1+1/p<1$ by Theorem 1.3.

If letting $\varepsilon=n\varepsilon_1$, these imply
(i) $n/(n+\varepsilon)<p\leqslant1$, (ii) $\varepsilon^{-1}n(1/p-1)+1/p<s<1+1/p$.

After careful computation, we can obtain $n(1/p-1)+1-\varepsilon+1/p>\frac{1}{\varepsilon}n(1/p-1)+1/p$.

Therefore, if $s,p$ satisfy (1.4), we have estimate (5.1). Combining this fact with the atomic characterization of B-spaces, we can complete the proof of existence and their estimates.

*Uniqueness*: The uniqueness of Theorem 1.1 means that if $u \in B^p_s(\Omega)$ is the solution to the equation

$$\Delta u = 0 \text{ in } B^p_{s-2}(\Omega), \quad \frac{\partial u}{\partial v} = 0 \text{ in } \dot{B}^p_{s-1-1/p}(\partial\Omega), \tag{5.2}$$

then $u = const$.

Assume that $s, p$ satisfy (1.4), we will choose $s_1, p_1$ such that $s_1, p_1$ satisfy the condition (b) in Theorem A and that the embedding results $B^p_s(\Omega) \subset L^{p_1}_{s_1}(\Omega)$, $B^p_{s-1-1/p}(\partial\Omega) \subset B^{p_1,p_1}_{s_1-1-1/p_1}(\partial\Omega)$ hold. Therefore by (5.2) we can obtain

$$\Delta u = 0 \text{ in } B^{p_1}_{s_1-2}(\Omega), \quad \frac{\partial u}{\partial v} = 0 \text{ in } B^{p_1}_{s_1-1-1/p}(\partial\Omega). \tag{5.3}$$

Then $u = const$ by Theorem A.

Remember the embedding theorems

$$B^p_s(\Omega) = B^{p,p}_s(\Omega) = F^{p,p}_s(\Omega) \subset F^{p_1,2}_{s_1}(\Omega) = L^{p_1}_{s_1}(\Omega),$$

$$B^p_{s-1-1/p}(\partial\Omega) = B^{p,p}_{s-1-1/p}(\partial\Omega) \subset B^{p_1,p}_{s_1-1-1/p_1}(\partial\Omega) = B^{p_1,p_1}_{s_1-1-1/p_1}(\partial\Omega)$$

where $s > s_1$, $s_1 = s - n(1/p - 1/p_1)$ and $p \leqslant 1 < p_1$.

For $s, p$ as in (1.4), now we choose $p_1, s_1 > 0$, such that

$$1 < p_1 < 2 \quad \text{and} \quad 3/p_1 - 1 - \varepsilon < s_1 < 1 + 1/p_1$$

as stated in Theorem A.

In fact, we choose $p_1$ with $1 < p_1 \leqslant (n-3)/(n-2-1/p) < 2$ by $n \geqslant 3$, then we have

$$\begin{aligned} s_1 &> n(1/p - 1) - \varepsilon + 1 + 1/p - n(1/p - 1/p_1) \\ &= 1/p + n(1/p_1 - 1) + 1 - \varepsilon > 3/p_1 - 1 - \varepsilon. \end{aligned}$$

On the other hand, by $p \leqslant 1 < p_1$, then we have

$$s_1 < 1 + 1/p - n(1/p - 1/p_1) = 1 + (1-n)/p + n/p_1 < 1 + 1/p_1.$$

So we complete the proof of uniqueness.

Combining the existence and the uniqueness, we complete the proof of Theorem 1.1. $\square$

## References

[Br] M.R. Brown, The Neumann problem on Lipschitz domains in Hardy spaces of order less than one, Pacific J. Math. 171.2 (1995) 389–407.

[CKS] D.C. Chang, S.G. Krantz, E.M. Stein, $H^p$ theory on a smooth domain in $R^n$ elliptic boundary value problems, J. Funct. Anal. 114 (1993) 286–347.

[DK] B.E.J. Darlberg, C.E. Kenig, Hardy spaces and the Neumann problem in $L^p$ for Laplace's equation in Lipschitz domains, Ann. Math. 125 (1987) 437–465.

[ET] D.E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers, Differential Operators, Cambridge University Press, Cambridge, 1996.

[FJ] M. Frazier, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34.4 (1985) 777–798.

[FMM] E. Fabes, O. Mendez, M. Mitrea, Boundary layers on Sobolev–Besov spaces and poisson's equation for the Laplacian in Lipschitz domains, J. Funct. Anal. 159 (1998) 323–368.

[JK] D. Jerison, C.E. Kenig, The inhomogeneous Dirichlet problems in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.

[MM1] S. Mayboroda, M. Mitrea, Layer potentials and boundary value problems for laplacian in Lipschitz domains with data in Quasi-Banach Besov spaces, preprint.

[MM2] S. Mayboroda, M. Mitrea, Square functions estimates for singular integrals and applications to partial differential equations, preprint.

[Nec] J. Necas, Sur les normes equivalentes dans $W^{k,p}(\Omega)$ et sur la coercivite des formes formellement positives. Les Presses de l'Universite de Montreal, 1967, 102–128.

[Se] S. Semmes, A primer on Hardy spaces and some remarks on a theorem of Evans and Muller, Commun. Partial Differential Equations 19 (1&2) (1994) 277–319.

[St] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1979.

[Tr1] H. Triebel, Theory of Function Spaces, Birkhauser, Basel, Boston, Berlin, 1983.

[TW] H. Tribel, H. Winkelvoß, Intrisic atomic characterization spaces on domains, Math. Z. 221 (1996) 647–673.

[W1] H.G. Wang, Atomic decomposition and restriction theorems of Triebel–Lizorkin spaces on domains, Chinese Ann. Math. 24.1 (2003) 73–80 (in Chinese).

[W2] H.G. Wang, An extension of the Poincare inequality in F-spaces and its application, preprint.

[W3] H.G. Wang, The characterization of weighted local Hardy spaces on domains and its application, in J. Zhejiang Univ. Sci. (November 2004), to appear.

[Z] D.Z. Zanger, The inhomogeneous Neumann problem in Lipschtiz domains, C. P. D. E., (9&10) (2000) 1771–1808.