Multiple positive solutions of superlinear elliptic problems with sign-changing weight

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Abstract

We study the existence of multiple positive solutions for a superlinear elliptic PDE with a sign-changing weight. Our approach is variational and relies on classical critical point theory on smooth manifolds. A special care is paid to the localization of minimax critical points.
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1. Introduction

We consider positive solutions of the boundary value problem,

$$\Delta u + (a_+(x) - \mu a_-(x))|u|^\gamma u = 0, \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $\mathcal{C}^1$, a_+ and a_- are continuous functions which are positive on non-overlapping domains and μ is a large parameter. Positive solutions u are defined to be such that $u(x) > 0$ for almost every $x \in \Omega$.

Problems with a sign varying coefficient were already studied in 1976 by Butler [5]. In this paper, he proved existence of infinitely many periodic solutions for the nonlinear Hill equation

$$u'' + a(t)|u|^{\gamma}u = 0$$

with a sign changing weight $a(t)$. Results concerning Dirichlet problems for ODE's were obtained in [12,14]. However, all these results concern multiplicity of oscillating solutions. For the ODE equivalent of (1.1) and for large values of μ, complete results were worked out in [6,7] concerning, respectively, the cases of the weight $a_+(t)$ being positive in two or three non-overlapping intervals. These results were obtained from an elementary shooting method. Although the argument becomes clumsy, it extends to the general case of $a_+(t)$ being positive in n non-overlapping intervals. In this case, $2^n - 1$ positive solutions can be obtained. For PDEs, related problems were studied by several authors using topological and variational methods [1,3,4,11]. In the present paper, using a variational approach we extend to the PDE problem (1.1) the results obtained in [6,7].

Notice first that finding positive solutions of problem (1.1) is equivalent to find non-trivial solutions of

$$\begin{aligned} \Delta u + (a_+(x) - \mu a_-(x))u_+^{\gamma+1} &= 0, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.2}$$

where $u_+ = \max\{u, 0\}$, since non-trivial solutions of (1.2) are positive. In the sequel we also write $u_- = \max\{-u, 0\}$.

We suppose throughout the paper the following assumptions:

(H) $\gamma > 0$, $\gamma + 2 < 2^* = \frac{2N}{N-2}$ if $N \geqslant 3$, $a_+, a_- : \overline{\Omega} \to \mathbb{R}$ are continuous functions and there exist n disjoint domains $\omega_i \subset \Omega$, with $i = 1, \ldots, n$, which are of class $\mathcal{C}^1$ and such that

(a) for all $x \in \Omega_+ := \bigcup_{i=1}^{n} \omega_i$, $a_-(x) = 0$, $a_+(x) > 0$ and

(b) for all $x \in \Omega_- := \Omega \setminus \overline{\Omega}_+$, $a_-(x) > 0$, $a_+(x) = 0$.

The existence of at least one positive solution for a superlinear equation like (1.1) follows easily from the Mountain Pass Theorem of Ambrosetti and Rabinowitz applied to (1.2), see e.g. [13]. Indeed, $u = 0$ is a local minimizer of the action functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) := \int_{\Omega} \left(\frac{1}{2}|\nabla u(x)|^2 - (a_+(x) - \mu a_-(x))\frac{u_+^{\gamma+2}(x)}{\gamma+2} \right) dx$$

and we can easily find a function $v \in H^1_0(\Omega)$ such that $I(v) < 0$. For the same reason, we also infer the existence of a local positive solution in each ω_i, i.e. a solution of the Dirichlet problem

$$\Delta u + (a_+(x) - \mu a_-(x))u^{\gamma+1}_+ = 0, \quad x \in \omega_i,$$
$$u(x) = 0, \quad x \in \partial\omega_i. \tag{1.3}$$

As the superlinear term is homogeneous of degree $\gamma + 1$, solutions of (1.1) can be alternatively obtained as critical points of the energy functional under a convenient constraint. Namely, we define the constraint functional $V_\mu : H^1_0(\Omega) \to \mathbb{R}$ by

$$V_\mu(u) := \int_\Omega (a_+(x) - \mu a_-(x)) \frac{u^{\gamma+2}_+(x)}{\gamma+2} \, dx. \tag{1.4}$$

From the continuous imbedding of $H^1_0(\Omega)$ into $L^{\gamma+2}(\Omega)$, it can be seen that $V_\mu(u)$ is of class $\mathcal{C}^{1,1}$. Next, we define the manifold

$$\mathfrak{B}_\mu := \{u \in H^1_0(\Omega) \mid V_\mu(u) = 1\} \tag{1.5}$$

and the energy functional $J : H^1_0(\Omega) \to \mathbb{R}, u \mapsto J(u)$, where

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx. \tag{1.6}$$

We consider then critical points of J under the constraint $u \in \mathfrak{B}_\mu$. It is a standard fact that such critical points satisfy the Euler–Lagrange equation

$$\nabla J(u) = \lambda \nabla V_\mu(u)$$

for some *Lagrange multiplier* $\lambda \in \mathbb{R}$, i.e. for any $w \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u(x) \nabla w(x) \, dx = \lambda \int_\Omega (a_+(x) - \mu a_-(x))u^{\gamma+1}_+(x)w(x) \, dx. \tag{1.7}$$

It then follows that u solves the problem

$$\Delta u + \lambda(a_+(x) - \mu a_-(x))u^{\gamma+1}_+ = 0, \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega.$$

Taking u as test function in (1.7), we obtain

$$\int_\Omega |\nabla u(x)|^2 \, dx = \lambda(\gamma + 2),$$

which implies $\lambda > 0$ since $0 \notin \mathfrak{B}_\mu$. Hence, any critical point of J on $\mathfrak{B}_\mu$ is such that the rescaled function $v = \lambda^{1/\gamma} u$ is a positive solution of (1.1). The existence of the above-mentioned local solutions in each ω_i now also follows from constraint minimization arguments in classes of functions with support in the adherence of one domain ω_i.

Our purpose in this paper is to prove the existence of multiple solutions for large values of the parameter μ. When μ is large, functions $u \in \mathfrak{B}_\mu$ with finite energy $J(u)$ have to be small in Ω_-. Indeed, the condition

$$\int_\Omega a_+(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx - 1 = \mu \int_\Omega a_-(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx$$

implies that for large μ

$$\int_{\Omega_-} a_-(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx$$

becomes small, i.e. the function u is small in Ω_-. Further if this function is a solution of (1.1), it is small on $\partial\omega_i \subset \partial\Omega \cup \partial\Omega_-$. Hence, it is reasonable to guess that such a solution is close to solutions of Dirichlet problems in the domains ω_i, i.e. to solutions of (1.3). Basically, the profile of these solutions consists then in various bumps concentrated in some of the sets ω_i. A solution which concentrates only in one of the ω_i will be referred to as a *single-bump* solution while it will be called *multi-bumps* solution if it has a significative contribution in more than one of the ω_i.

These intuitive observations suggest the existence of at least $2^n - 1$ positive solutions as on each domain ω_i, such solutions are close either to 0 or to a positive solution of (1.3). Though the existence of some of those solutions seem straightforward, it is quite tricky to catch all of them.

Let us now give a more precise definition of p-bumps solutions.

Definitions 1.1. A function $w \in H_0^1(\Omega)$ is a *p-bumps function* ($p = 1, \ldots, n$) if there exist p non-zero functions $e_j \in H_0^1(\Omega)$, with supp $e_j \subset \overline{\omega}_{i_j}$, $i_j \in \{1, \ldots, n\}$ and $i_j \neq i_k$ for $j, k \in \{1, \ldots, p\}$, such that $w = \sum_{j=1}^p e_j$. Given a set $\omega = \omega_{i_1} \cup \ldots \cup \omega_{i_p}$, a family of functions $\{u_\mu | \mu \geq \mu_0\} \subset H_0^1(\Omega)$ is said to be a *family of p-bumps solutions of* (1.1) *with limit support in* $\overline{\omega}$ if for each μ large enough u_μ solves (1.1), it has a cluster value for the weak topology in $H_0^1(\Omega)$ as μ goes to infinity, and any such value is a p-bumps function $w \in H_0^1(\Omega)$ with support in $\overline{\omega}$.

Notice that our definition implies that a family of p-bumps solutions of (1.1) $\{u_\mu \mid \mu \geq \mu_0\}$ with limit support in $\overline{\omega}$ is such that

$$u_\mu \to 0 \quad \text{in} \quad L^{2+\gamma}(\Omega \setminus \omega).$$

We can then state our main theorem.

Theorem 1.1. *Let assumptions* (H) *be satisfied. Then, for $\mu > 0$ large enough, there exist at least $2^n - 1$ positive solutions of* (1.1). *Moreover, for each set $\omega = \omega_{i_1} \cup \ldots \cup \omega_{i_p}$, one of those solutions defines a family of positive p-bumps solutions of* (1.1) *with limit support in $\overline{\omega}$.*

Our paper is organized as follows. In Section 2, we work out some preliminary results and define the key ingredients of our approach. The existence of p-bumps solutions for any $1 \leqslant p \leqslant n$ could follow from a unique proof. However, since much more intuitive arguments work fine for the single-bump solutions and the n-bumps solution, we treat them separately. In fact, the main difficulty is not really to distinguish the various type of solutions but rather those with the same number of bumps. For single-bump solutions, this is easily done as minimization arguments can be used to single out local minimizers in disjoint subsets of $\mathfrak{B}_\mu$. Section 3 deals with the existence of those local minimizers. Solutions with p bumps, $2 \leqslant p \leqslant n - 1$, correspond to critical values between the smallest energy of the minima and the energy of the n-bumps solution. However, these critical values are not necessarily ordered according to the number of bumps of the associated solutions. Nevertheless, a partial ordering holds. Let u_a and u_b be two families of solutions with limit support in $\overline{\omega}_a$ and $\overline{\omega}_b \subset \overline{\omega}_a$. Then for μ large enough, the energy of u_a is larger than the energy of u_b. We consider p-bumps solutions $(2 \leqslant p \leqslant n - 1)$ in Section 4. Here, the only use of classical minimax theorems is not sufficient to our purpose. In order to distinguish the solutions, we need a careful analysis and a localization of the deformation along the lines of the gradient flow used to obtain the desired minimax critical values. Basically, we identify disconnected regions from which these deformations cannot escape. At last, in Section 5, we prove the existence of a n-bumps solution using a quite standard minimax principle. This n-bumps solution has the greatest value of the energy among all the solutions we get.

2. Preliminary results

We first complete the description of our functional framework. To this end, we endow $H_0^1(\Omega)$ with the usual inner product

$$\langle u, v \rangle := \int_\Omega \nabla u \nabla v \, dx$$

whose associated norm we denote by $\|u\| := [\int_\Omega |\nabla u(x)|^2 \, dx]^{1/2}$. Throughout the paper, orthogonality is understood in the sense of this inner product.

2.1. The manifold $\mathfrak{B}_\mu$

The following lemma gives the basic properties of $\mathfrak{B}_\mu$ and introduces a convenient projector on this manifold.

Lemma 2.1. *The set $\mathfrak{B}_\mu$ defined from* (1.5) *is a non-empty, weakly closed and arc connected manifold in $H_0^1(\Omega)$. Further the function Q_μ defined on dom $Q_\mu = \{u \in$*

$H_0^1(\Omega) \mid V_\mu(u) > 0\}$ *by*

$$(Q_\mu u)(x) := [V_\mu(u)]^{-\frac{1}{\gamma+2}} u(x)$$

is a continuous projector on $\mathfrak{B}_\mu$.

Proof. *Claim 1: The set* $\mathfrak{B}_\mu$ *is non-empty.* Consider a function $u \in H_0^1(\Omega)$ such that $u \geqslant 0$, $u \neq 0$ and supp $u \subset \overline{\Omega}_+$. Then $V_\mu(u) \neq 0$ so that $Q_\mu u \in \mathfrak{B}_\mu$.

Claim 2: The set $\mathfrak{B}_\mu$ *is weakly closed.* This is a direct consequence of the compact imbedding of $H_0^1(\Omega)$ into $L^{\gamma+2}(\Omega)$ and the continuity of V_μ.

Claim 3: Q_μ *is continuous.* This claim follows from the continuity of V_μ.

Claim 4: The set $\mathfrak{B}_\mu$ *is arc connected.* Let us first consider two functions u_1, $u_2 \in \mathfrak{B}_\mu$ with support in the same set $\overline{\omega}_i$. Each of them can be connected to its positive part since the paths $\gamma_s := (u_i)_+ - s(u_i)_-$ are in $\mathfrak{B}_\mu$ for all $s \in [0, 1]$. Then, the function

$$\xi_s := (1 - s)(u_1)_+ + s(u_2)_+$$

is such that

$$V_\mu(\xi_s) \geqslant \frac{1}{2^{\gamma+2}}$$

for all $s \in [0, 1]$. It follows that $Q_\mu \xi_s \in \mathfrak{B}_\mu$ for all $s \in [0, 1]$.

If u_1, $u_2 \in \mathfrak{B}_\mu$ have supports in different sets $\overline{\omega}_i$, then the path

$$s \mapsto (1 - s)^{\frac{1}{\gamma+2}} u_1 + s^{\frac{1}{\gamma+2}} u_2$$

stays in $\mathfrak{B}_\mu$ for all $s \in [0, 1]$.

Now, to complete the proof, we only need to show that any u in $\mathfrak{B}_\mu$ can be linked by a path in $\mathfrak{B}_\mu$ to some v with support in one of the $\overline{\omega}_i$. Observe that we necessarily have $a_+ u_+ \neq 0$ in some ω_i with $i \in (1, \ldots, n)$. Hence, we can choose an open set ω_0 in such a way that $\overline{\omega}_0 \subset \omega_i$ and for some $\delta > 0$,

$$1 > \int_{\omega_0} a_+(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx \geqslant \delta > 0 \quad \text{and} \quad \int_{\Omega \setminus \omega_0} (a_+(x) - \mu a_-(x)) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx \geqslant 0.$$

Next, we define a smooth function $h : [0, 1] \times \Omega \to [0, 1]$ such that

$$h(s, x) = \begin{cases} 1 & \text{if } x \in \overline{\omega}_0, \\ 1 - s & \text{if } x \in \Omega \setminus \overline{\omega}_i. \end{cases}$$

Then we compute

$$V_\mu(h(s, \cdot)u(\cdot)) = \int_\Omega (a_+(x) - \mu a_-(x)) \frac{(h(s,x)u_+(x))^{\gamma+2}}{\gamma+2} \, dx$$
$$\geqslant \delta + (1 - s)^{\gamma+2} \int_{\Omega \setminus \omega_0} (a_+(x) - \mu a_-(x)) \frac{u_+^{\gamma+2}}{\gamma+2} \, dx \geqslant \delta.$$

Now, observe that $h(1, \cdot)u(\cdot)$ has support in $\overline{\omega}_i$ and the path

$$s \mapsto Q_\mu(h(s, .)u(.))$$

stays in $\mathfrak{B}_\mu$ for all $s \in [0, 1]$. $\square$

2.2. Equivalence with the constraint problem

It is clear that positive solutions of (1.1) can be obtained from rescaling solutions of the constraint problem (1.7). As our main interest in this paper is to obtain multiplicity results, we check next that different critical points of J constrained by $\mathfrak{B}_\mu$ lead to distinct solutions of (1.1).

Lemma 2.2. *If u_1, u_2 are different critical points of J in $\mathfrak{B}_\mu$, then there exist λ_1, $\lambda_2 > 0$ such that $v_1 = \lambda_1^{1/\gamma} u_1$ and $v_2 = \lambda_2^{1/\gamma} u_2$ are two distinct positive solutions of (1.1).*

Proof. Let u_1 and u_2 be different critical points of J in $\mathfrak{B}_\mu$. The existence of the Lagrange multipliers λ_1, λ_2 follows from standard arguments and, as already observed, we have

$$\lambda_i = \frac{\|u_i\|^2}{\gamma + 2} > 0, \quad i = 1, 2.$$

Suppose $v_1 = v_2$, i.e. $u_1 = (\lambda_2/\lambda_1)^{1/\gamma} u_2$. We compute then

$$V_\mu(u_1) = \left(\frac{\lambda_2}{\lambda_1} \right)^{\frac{\gamma+2}{\gamma}} V_\mu(u_2).$$

As u_1, $u_2 \in \mathfrak{B}_\mu$, we deduce $\lambda_1 = \lambda_2$, which contradicts the fact that $u_1 \neq u_2$. $\square$

2.3. The functional J

Many of our arguments in the proof of the main theorems rely on an analysis of the functional J for functions with support in the $\overline{\omega}_i$'s. The following lemma is such a result.

Lemma 2.3. *The functional J has a non-negative minimum $\hat{u}_i$ on each of the disjoint manifolds*

$$\hat{\mathfrak{B}}_i := \{ u \in \mathfrak{B}_\mu \mid \operatorname{supp} u \subset \overline{\omega}_i \}, \quad i = 1, \ldots, n. \tag{2.1}$$

Remark 2.1. Notice that the sets $\hat{\mathfrak{B}}_i$ and the functions $\hat{u}_i$ are independent of μ since they only involve functions u so that $\operatorname{supp} u \subset \overline{\omega}_i$.

Proof. As $\hat{\mathfrak{B}}_i$ is weakly closed and J is coercive and weakly lower semi-continuous, we can minimize J in each manifold $\hat{\mathfrak{B}}_i$ and obtain n distinct non-trivial minimizers $\hat{u}_i \in \hat{\mathfrak{B}}_i$. These are non-negative. Indeed, if such a minimizer $\hat{u}_i$ is such that $(\hat{u}_i)_- \neq 0$, we have $(\hat{u}_i)_+ \in \hat{\mathfrak{B}}_i$ and $J((\hat{u}_i)_+) < J(\hat{u}_i)$ which is a contradiction. $\square$

We consider the gradient of J constrained to $\mathfrak{B}_\mu$

$$\nabla_\mu J(u) := \nabla J(u) - \frac{\langle \nabla J(u), \nabla V_\mu(u) \rangle}{\|\nabla V_\mu(u)\|^2} \nabla V_\mu(u). \tag{2.2}$$

It is well known that the Palais–Smale condition holds for this gradient. For completeness, we present here a proof of this property.

Lemma 2.4 (*The Palais–Smale condition*). *Let J and V_μ be defined from* (1.6) *and* (1.4). *Let $(v_n)_n$ be a sequence in $\mathfrak{B}_\mu$ so that*

$$J(v_n) \to c_1 \quad and \quad \nabla_\mu J(v_n) \overset{H_0^1}{\to} 0,$$

where $\nabla_\mu J(u)$ is defined in (2.2). *Then there exist a subsequence $(v_{n_i})_i$ and $v \in H_0^1(\Omega)$ such that*

$$v_{n_i} \overset{H_0^1}{\longrightarrow} v, \ J(v) = c_1 \quad and \quad \nabla_\mu J(v) = 0.$$

Proof. There exist $v \in H_0^1(\Omega)$ and some subsequence still denoted $(v_n)_n$ such that

$$v_n \overset{H_0^1}{\rightharpoonup} v \quad and \quad v_n \overset{L^{2+\gamma}}{\longrightarrow} v.$$

Let $\nabla_\mu J(v_n) = \nabla J(v_n) - \lambda_n \nabla V_\mu(v_n)$ and compute

$$\begin{aligned} |2J(v_n) - \lambda_n(\gamma + 2)| &= |\langle \nabla J(v_n), v_n \rangle - \lambda_n \langle \nabla V_\mu(v_n), v_n \rangle| \\ &\leqslant \|\nabla J(v_n) - \lambda_n \nabla V_\mu(v_n)\| \, \|v_n\| \to 0 \end{aligned}$$

so that $\lambda_n \to \frac{2c_1}{\gamma + 2}$. Next, we notice that

$$\langle \nabla J(v_n), v \rangle = \int_\Omega \nabla v_n \nabla v \, dx \to \int_\Omega |\nabla v|^2 \, dx$$

and

$$\langle \nabla V_\mu(v_n), v \rangle = \int_\Omega (a_+ - \mu a_-) v_n^{\gamma+1} v \, dx \to \int_\Omega (a_+ - \mu a_-) v^{\gamma+2} \, dx = \gamma + 2$$

so that

$$\langle \nabla J(v_n), v \rangle - \lambda_n \langle \nabla V_\mu(v_n), v \rangle \to 2J(v) - 2c_1 = 0.$$

As a consequence $J(v_n) \to c_1 = J(v)$, i.e. $\|v_n\| \to \|v\|$. This implies $v_n \xrightarrow{H_0^1} v$, $\nabla_\mu J(v_n) \xrightarrow{L^2} \nabla_\mu J(v)$ and $\nabla_\mu J(v) = 0$. $\square$

2.4. Decomposition of $H_0^1(\Omega)$

The solutions we are interested in are near multi-bumps functions, i.e. large within the set Ω_+ and almost zero on Ω_-. It is then natural to decompose such a function as a sum of a multi-bumps function and some small perturbation. Using this idea, we introduce the following orthogonal decomposition of $H_0^1(\Omega)$. Let $\overline{H} := \{u \in H_0^1(\Omega) \mid \operatorname{supp} u \subset \overline{\Omega}_+\}$ be the space of the multi-bumps functions and $\tilde{H} := (\overline{H})^\perp$ its orthogonal complement. Given $u \in H_0^1(\Omega)$, we define then $\overline{u} \in \overline{H}$ from the following lemma.

Lemma 2.5. *Let $u \in H_0^1(\Omega)$. Then the problem*

$$\int_{\Omega_+} \nabla \overline{u}(x) \nabla \varphi(x) \, dx = \int_{\Omega_+} \nabla u(x) \nabla \varphi(x) \, dx \quad for \ all \ \varphi \in H_0^1(\Omega_+)$$

has a unique solution $\overline{u} \in \overline{H}$. Further the function

$$\overline{R} : H_0^1(\Omega) \to \overline{H} \subset H_0^1(\Omega), u \mapsto \overline{R}u = \overline{u}$$

is a continuous projector for the weak topologies, i.e.

$$u_n \xrightarrow{H_0^1} u \quad implies \quad \overline{R}u_n \xrightarrow{H_0^1} \overline{R}u.$$

Also, we have

$$J(\overline{R}u) \leqslant J(u). \tag{2.3}$$

At last, the function $\tilde{u} := u - \overline{u}$ is in $\tilde{H}$ and satisfies

$$\int_{\Omega_+} \nabla \tilde{u}(x) \nabla \varphi(x) \, dx = 0 \quad for \ all \ \varphi \in H_0^1(\Omega_+). \tag{2.4}$$

Proof. Existence and uniqueness of the solution $\bar{u} \in H_0^1(\Omega_+)$ follow from Lax–Milgram Theorem. Next, we extend $\bar{u}$ by

$$\bar{u}(x) = 0 \quad \text{if } x \in \Omega \setminus \Omega_+.$$

It is clear that $\bar{u} \in \bar{H}$. Further, it also follows from Lax–Milgram Theorem that $\bar{u}$ depends continuously on $u \in H_0^1(\Omega)$.

Notice then that $\tilde{u} = u - \bar{u} \in \tilde{H}$ since

$$\int_{\Omega} \nabla \bar{u} \nabla \tilde{u} \, dx = \int_{\Omega_+} \nabla \bar{u} \nabla \tilde{u} \, dx = \int_{\Omega_+} \nabla \bar{u} (\nabla u - \nabla \bar{u}) \, dx = 0.$$

We also have (2.4). To proof (2.3), we compute

$$\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} |\nabla \tilde{u}|^2 \, dx + \int_{\Omega} |\nabla \bar{u}|^2 \, dx,$$

which implies $J(\overline{R}u) \leqslant J(u)$.

To complete the proof we check the continuity of $\overline{R}$ for the weak topologies. Let $(u_n)_n \subset H_0^1(\Omega)$ be such that $u_n \overset{H_0^1}{\rightharpoonup} u$ and write $u_n = \bar{u}_n + \tilde{u}_n$ and $u = \bar{u} + \tilde{u}$, where $\bar{u}_n = \overline{R}u_n$ and $\bar{u} = \overline{R}u$. We know that for any $\varphi \in H_0^1(\Omega_+)$ we have

$$\int_{\Omega_+} \nabla \bar{u}(x) \nabla \varphi(x) \, dx = \int_{\Omega_+} \nabla u(x) \nabla \varphi(x) \, dx$$

and

$$\int_{\Omega_+} \nabla \bar{u}_n(x) \nabla \varphi(x) \, dx = \int_{\Omega_+} \nabla u_n(x) \nabla \varphi(x) \, dx.$$

Hence, the weak convergence of the sequence $(u_n)_n$ implies $\bar{u}_n \overset{H_0^1(\Omega_+)}{\rightharpoonup} \bar{u}$, that is $\bar{u}_n \overset{H_0^1}{\rightharpoonup} \bar{u}$. $\quad \square$

Let us write

$$\hat{V}(u) := \int_{\Omega_+} a_+(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2} \, dx$$

and denote for any $r > 0$, the ball

$$\mathfrak{B}_{\mu,r} := \{ u \in \mathfrak{B}_\mu \mid \|u\| < r \}.$$

The following lemma controls $\hat{V}(\bar{u})$ if $u \in \mathfrak{B}_{\mu,r}$ and μ is large enough. Notice that in this lemma $\hat{V}(\bar{u})$ depends on μ from the fact that we choose u in the set $\mathfrak{B}_{\mu,r}$ which depends on this parameter.

Lemma 2.6. *Let $r > 0$ and $\varepsilon > 0$ be given. Then, for all $\mu > 0$ large enough and $u \in \mathfrak{B}_{\mu,r}$,*

$$\hat{V}(\bar{u}) \geqslant 1 - \varepsilon.$$

Proof. Let $r > 0$ and $\varepsilon > 0$ be given and define $\eta > 0$ to be such that

$$(1 + \eta)^{\gamma+2}(1 - \varepsilon) < 1.$$

If the claim does not hold, there exist sequences $(\mu_j)_j$, $(u_j)_j \subset \mathfrak{B}_{\mu_j,r}$ such that $\lim_{j \to \infty} \mu_j = +\infty$ and

$$\hat{V}(\bar{u}_j) = \tfrac{1}{\gamma+2} \int_\Omega a_+(\bar{u}_j)_+^{\gamma+2}\, dx < 1 - \varepsilon,$$

for j large enough.

As the sequence $(u_j)_j$ is bounded in $H_0^1(\Omega)$, going to subsequence we can assume

$$u_j \overset{H_0^1}{\rightharpoonup} u$$

and therefore using Lemma 2.5 we infer that

$$\tilde{u}_j \overset{H_0^1}{\rightharpoonup} \tilde{u} \quad \text{and} \quad \tilde{u}_j \overset{L^{2+\gamma}}{\longrightarrow} \tilde{u}.$$

Claim 1: $\tilde{u}_+(x) = 0$ a.e. in Ω_-. We compute for some $C > 0$,

$$1 + \mu_j \int_{\Omega_-} a_-(\tilde{u}_j)_+^{\gamma+2}\, dx = 1 + \mu_j \int_{\Omega_-} a_-(u_j)_+^{\gamma+2}\, dx$$

$$= 1 + \mu_j \int_\Omega a_-(u_j)_+^{\gamma+2}\, dx = \int_\Omega a_+(u_j)_+^{\gamma+2}\, dx \leqslant C r^{\gamma+2}.$$

It follows that

$$\int_{\Omega_-} a_- \tilde{u}_+^{\gamma+2}\, dx = \lim_{j \to \infty} \int_{\Omega_-} a_-(\tilde{u}_j)_+^{\gamma+2}\, dx = 0$$

and as $a_-(x) > 0$ in Ω_-, the claim follows.

Claim 2: $\tilde{u}_+(x) = 0$ *a.e. in* Ω. As Ω_+ is of class $\mathcal{C}^1$, we deduce from Claim 1 that $\tilde{u}_+ \in H_0^1(\Omega_+)$ (see [10, Chapter 1-8.2]). Further, using (2.4) and the maximum principle (see [8, Theorem 8.1]), we obtain $\sup\limits_{\Omega_+} \tilde{u} \leqslant \sup\limits_{\partial\Omega_+} \tilde{u} \leqslant 0$.

Conclusion: Notice that

$$(u_j)_+ \leqslant (\overline{u}_j)_+ + (\tilde{u}_j)_+ \leqslant \max\{(1+\eta)(\overline{u}_j)_+, (1+\tfrac{1}{\eta})(\tilde{u}_j)_+\}$$

so that

$$(u_j)_+^{\gamma+2} \leqslant (1+\eta)^{\gamma+2}(\overline{u}_j)_+^{\gamma+2} + (1+\tfrac{1}{\eta})^{\gamma+2}(\tilde{u}_j)_+^{\gamma+2}.$$

It follows that

$$1 \leqslant \tfrac{1}{\gamma+2} \int_\Omega a_+(u_j)_+^{\gamma+2}\, dx \leqslant \tfrac{(1+\eta)^{\gamma+2}}{\gamma+2} \int_\Omega a_+(\overline{u}_j)_+^{\gamma+2}\, dx + \tfrac{(1+\frac{1}{\eta})^{\gamma+2}}{\gamma+2} \int_\Omega a_+(\tilde{u}_j)_+^{\gamma+2}\, dx.$$

Using Claim 2, we obtain then the contradiction

$$1 \leqslant \tfrac{(1+\eta)^{\gamma+2}}{\gamma+2} \lim_{j\to\infty} \int_\Omega a_+(\overline{u}_j)_+^{\gamma+2}\, dx \leqslant (1+\eta)^{\gamma+2}(1-\varepsilon) < 1. \qquad \square$$

2.5. The nonlinear simplex $\mathfrak{S}$

Let $\hat{u}_i$ be the local minimizers of J in $\hat{\mathfrak{B}}_i$ defined by Lemma 2.3 and consider the nonlinear simplex

$$\mathfrak{S} := \left\{ u = \sum_{i=1}^n s_i^{\frac{1}{\gamma+2}} \hat{u}_i \mid (s_1, \ldots, s_n) \in \varDelta \right\} \subset \mathfrak{B}_\mu,$$

where

$$\Delta := \left\{ (s_1, \ldots, s_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n s_i = 1 \right\}.$$

We can evaluate J along functions of $\mathfrak{S}$ and introduce

$$f(s) := J\left(\sum_{i=1}^n s_i^{\frac{1}{\gamma+2}} \hat{u}_i \right) = \sum_{i=1}^n s_i^{\frac{2}{\gamma+2}} J(\hat{u}_i), \quad s \in \Delta. \tag{2.5}$$

Notice that the set $\mathfrak{S}$ will be a key ingredient in the minimax characterization of the multi-bumps solutions as the geometry of f on Δ is a good model of the geometry of J on $\mathfrak{B}_\mu$ for large μ. The following lemmas study this geometry. As their proofs are elementary we omit them.

Lemma 2.7. *The function $f : \Delta \to \mathbb{R}$ defined from (2.5) is such that the apexes $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of Δ are strict local minima of f.*

Lemma 2.8. *Let $E := \{i_1, \ldots, i_k\}, 2 \leqslant k \leqslant n, \Delta_k := \{s = (s_1, \ldots, s_k) \in \mathbb{R}_+^k \mid \sum_{j=1}^k s_j = 1\}$ and let $\hat{u}_i$ be the local minimizers in $\hat{\mathfrak{B}}_i$ defined by Lemma 2.3. Then the function $f_E : \Delta_k \to \mathbb{R}$ defined from*

$$f_E(s) := \sum_{j=1}^k s_j^{\frac{2}{\gamma+2}} J(\hat{u}_{i_j})$$

has a unique global maximum c_E at some point $s^* = (s_1^*, \ldots, s_k^*) \in \Delta_k$ such that $s_j^* > 0$ for all $j = 1, \ldots, k$. Further, if $F \subsetneqq E$, then $c_F < c_E$.

2.6. Projection on $\mathfrak{S}$

The following lemmas define a continuous projector on the nonlinear simplex $\mathfrak{S}$ that increases the energy as little as we wish.

Lemma 2.9. *The mapping $R_\mu : H_0^1(\Omega) \to H_0^1(\Omega)$, defined by*

$$R_\mu u := Q_\mu((\overline{u})_+),$$

is continuous. Further if $r > 0$ and $\delta > 0$ are given, then for $\mu > 0$ large enough and $u \in \mathfrak{B}_{\mu,r}$,

$$J(R_\mu u) \leqslant J(u) + \delta. \tag{2.6}$$

Proof. Notice that the mapping $u \in H_0^1(\Omega) \to u_+ \in H_0^1(\Omega)$ is continuous. This follows from [8, Lemma 7.6]. The continuity of R_μ follows.

To prove (2.6), we first fix $r > 0$ and $\delta > 0$. Next, we choose $\varepsilon \in]0, 1/2]$ such that

$$\frac{Cr^2\varepsilon}{2} \leqslant \delta,$$

where $C > 0$ verifies $(1 - t)^{-\frac{2}{\gamma+2}} \leqslant 1 + Ct$ for $t \in]0, 1/2]$. From Lemma 2.6 we know that for $\mu > 0$ large enough, $\hat{V}((\overline{u})_+) = \hat{V}(\overline{u}) \geqslant 1 - \varepsilon$. Recalling that

$$J((\overline{u})_+) \leqslant J(\overline{u}) \leqslant J(u)$$

we compute

$$J(R_\mu u) = [\hat{V}((\overline{u})_+)]^{-\frac{2}{\gamma+2}} J((\overline{u})_+) \leqslant (1 + C\varepsilon) J(u) \leqslant J(u) + \frac{Cr^2\varepsilon}{2} \leqslant J(u) + \delta. \quad \square$$

For the next lemma, it is convenient to define the local constraints

$$\hat{V}_i(u) := \int_{\omega_i} a_+(x) \frac{u_+^{\gamma+2}(x)}{\gamma+2}\, dx.$$

These are such that if $v \in \mathfrak{B}_\mu$ and $\operatorname{supp} v \subset \overline{\Omega}_+$ then

$$V_\mu(v) = \hat{V}(u) = \sum_{i=1}^n \hat{V}_i(v) = 1.$$

Lemma 2.10. *The mapping* $P_\mu : H_0^1(\Omega) \to \mathfrak{S} \subset H_0^1(\Omega)$, *defined by*

$$P_\mu u := \sum_{i=1}^n [\hat{V}_i(R_\mu u)]^{\frac{1}{\gamma+2}} \hat{u}_i,$$

where $\hat{u}_i$ *are the local minimizers in* $\hat{\mathfrak{B}}_i$, *is continuous. Further if* $r > 0$ *and* $\delta > 0$ *are given, then for* $\mu > 0$ *large enough and* $u \in \mathfrak{B}_{\mu,r}$

$$J(P_\mu u) \leqslant J(u) + \delta.$$

Proof. Clearly, P_μ is continuous.

We next denote the local components of $R_\mu u$ by $R_{i\mu}u := (R_\mu u)\chi_{\omega_i}$, where χ_{ω_i} is the characteristic function of the set ω_i. Whenever $\hat{V}_i(R_\mu u) \neq 0$, we have $Q_\mu R_{i\mu}u \in \hat{\mathfrak{B}}_i$ and therefore $J(Q_\mu R_{i\mu}u) \geqslant J(\hat{u}_i)$ or equivalently

$$J(R_{i\mu}u) \geqslant \hat{V}_i(R_\mu u)^{\frac{2}{\gamma+2}} J(\hat{u}_i).$$

We now come out with the estimate

$$J(R_\mu u) = \sum_{i=1}^n J(R_{i\mu}u) \geqslant \sum_{i=1}^n \hat{V}_i(R_\mu u)^{\frac{2}{\gamma+2}} J(\hat{u}_i) = J(P_\mu u)$$

and the proof follows from Lemma 2.9. $\quad\square$

3. Single-bump solutions

In this section, we prove the existence of positive solutions that concentrate mainly on a single domain ω_i. To obtain such solutions we penalize in some sense the

action in the other ω_j's by assuming that the contribution to the constraint mainly occurs in ω_i.

Theorem 3.1. *Let Assumptions* (H) *be satisfied. Then for every* $i = 1, \ldots, n$ *there is a family of positive single-bump solutions of* (1.1) *with limit support in* $\overline{\omega}_i$.

Proof. *(1) Existence of n positive solutions.* Let $\hat{u}_i$ be the minimizers defined in Lemma 2.3. From Lemma 2.7, we can choose $\rho_1, \ldots, \rho_n \in [\frac{2}{3}, 1]$ and $\theta_1, \ldots, \theta_n > 0$ such that for all $i = 1, \ldots, n$ and $s \in \Delta$ with $s_i = \rho_i$,

$$f(s_1, \ldots, \rho_i, \ldots, s_n) > J(\hat{u}_i) + \theta_i,$$

where f is defined by (2.5), and

$$f(s) > J(\hat{u}_i) \tag{3.1}$$

for any $(s_1, \ldots, s_n) \in \Delta$ with $s_i \in [\rho_i, 1[$.

We fix then $r = \max_i J(\hat{u}_i) + 1$, $\delta \in]0, \min_i \theta_i[$, and $\mu_0 > 0$ large enough so that the conclusion of Lemma 2.10 holds for $\mu \geqslant \mu_0$. Define then for each $i = 1, \ldots, n$ the sets

$$\mathfrak{F}_{i\mu} := \{u \in \mathfrak{B}_\mu \mid \hat{V}_i(P_\mu u) = \hat{V}_i(R_\mu u) \geqslant \rho_i\}$$

and notice that since $\rho_i \geqslant \frac{2}{3}$, these sets are disjoint.

As in Lemma 2.3, we can prove that J has minimizers $v_{i\mu}$ in each set $\mathfrak{F}_{i\mu}$. This implies

$$J(v_{i\mu}) \leqslant J(\hat{u}_i). \tag{3.2}$$

Assume now that $v_{i\mu} \in \partial \mathfrak{F}_{i\mu}$, i.e. $\hat{V}_i(P_\mu v_{i\mu}) = \rho_i$. We deduce from Lemma 2.10 the estimate

$$J(P_\mu v_{i\mu}) \leqslant J(v_{i\mu}) + \theta_i.$$

Now, observe that there exists $s \in \Delta$ with $s_i = \rho_i$ such that

$$P_\mu v_{i\mu} = \sum_{j=1}^n s_j^{\frac{1}{\gamma+2}} \hat{u}_j.$$

Hence

$$J(P_\mu v_{i\mu}) = f(s_1, \ldots, \rho_i, \ldots, s_{n-1}) > J(\hat{u}_i) + \theta_i.$$

We deduce then that

$$J(v_{i\mu}) \geqslant J(P_\mu v_{i\mu}) - \theta_i > J(\hat{u}_i)$$

which contradicts the fact that $v_{i\mu}$ is a minimizer. It follows that $v_{i\mu}$ is in the interior of $\mathfrak{F}_{i\mu}$ so that $v_{i\mu}$ is a critical point of J in $\mathfrak{B}_\mu$. Solutions of (1.1) are then obtained from rescaling as in Lemma 2.2.

(2) *Claim: $v_{i\mu}$ is a family of single-bump solution with limit support in $\overline{\omega}_i$.* We write $v_{i\mu} = \tilde{v}_{i\mu} + \overline{v}_{i\mu}$, where $\tilde{v}_{i\mu} \in \tilde{H}$ and $\overline{v}_{i\mu} \in \overline{H}$. It follows from (3.2) that the family $\{v_{i\mu} \mid \mu \geqslant \mu_0\}$ is uniformly bounded in $H_0^1(\Omega)$. Hence the family $v_{i\mu}$ has cluster values for the weak topology in $H_0^1(\Omega)$. Let v be such a value, i.e. there exists a sequence $(v_{i\mu_j})_j$ with $\mu_j \to \infty$ so that

$$v_{i\mu_j} \overset{H_0^1}{\rightharpoonup} v \quad \text{and} \quad v_{i\mu_j} \overset{L^{2+\gamma}}{\longrightarrow} v.$$

We deduce from Lemma 2.5 that

$$\overline{v}_{i\mu_j} \overset{H_0^1}{\rightharpoonup} \overline{v}.$$

Hence, we have

$$\overline{v}_{i\mu_j} \overset{L^{2+\gamma}}{\longrightarrow} \overline{v} \quad \text{and} \quad \tilde{v}_{i\mu_j} \overset{L^{2+\gamma}}{\longrightarrow} \tilde{v}.$$

Arguing as in Claim 1 in the proof of Lemma 2.6, we infer that $\tilde{v}(x) \leqslant 0$ almost everywhere in Ω_-. As

$$\overline{v}_{i\mu_j} = v_{i\mu_j} - \tilde{v}_{i\mu_j} \overset{L^{2+\gamma}}{\longrightarrow} v - \tilde{v}$$

we deduce that $0 = v(x) - \tilde{v}(x) \geqslant v(x)$ almost everywhere in Ω_-. Notice also that $v_{i\mu_j} \geqslant 0$ for any j so that $v(x) \geqslant 0$ almost everywhere in Ω. Consequently, $v(x) = \tilde{v}(x) = 0$ on Ω_-. Arguing now as in Claim 2 in the proof of Lemma 2.6 (with $\tilde{v}$ instead of $\tilde{u}_+$), we obtain $\tilde{v}(x) = 0$ on Ω so that $v = \overline{v} \in \overline{H}$. We also have for any μ

$$V_\mu(v) = \hat{V}(v) = \frac{1}{\gamma+2} \int_\Omega a_+(x) v_+^{\gamma+2}(x) \, dx.$$

On the other hand, we have the estimate

$$\frac{1}{\gamma+2} \int_\Omega a_+(x)(v_{i\mu_j})_+^{\gamma+2}(x) \, dx = 1 + \frac{1}{\gamma+2} \int_\Omega \mu_j a_-(x)(v_{i\mu_j})_+^{\gamma+2}(x) \, dx \geqslant 1$$

so that

$$\hat{V}(v) = \lim_{j\to\infty} \tfrac{1}{\gamma+2} \int_\Omega a_+(x)(v_{i\mu_j})_+^{\gamma+2}(x)\, dx \geqslant 1.$$

Observe that $Q_\mu v = [\hat{V}(v)]^{-\frac{1}{\gamma+2}} v$ is independent of μ so that $Q_\mu v \in \mathfrak{B}_\mu$ for any $\mu > 0$ and

$$J(Q_\mu v) = [\hat{V}(v)]^{-\frac{2}{\gamma+2}} J(v) \leqslant J(v).$$

We now deduce from the lower semi-continuity of J and (3.2) that

$$J(Q_\mu v) \leqslant J(v) \leqslant \lim_{j\to\infty} J(v_{i\mu_j}) \leqslant J(\hat{u}_i). \tag{3.3}$$

Assume by contradiction that the support of v is not included in $\overline{\omega}_i$. Hence, we have for some $k \neq i$ that $\chi_k v \neq 0$, where χ_k is the characteristic function of the set ω_k. From the definition of P_μ we compute $\hat{V}_i(P_\mu v_{i\mu_j}) = \hat{V}_i(R_\mu v_{i\mu_j})$ so that

$$\rho_i \leqslant \hat{V}_i(P_\mu v_{i\mu_j}) = \hat{V}_i(R_\mu v_{i\mu_j}) = \tfrac{1}{\gamma+2} \int_{\omega_i} a_+ Q_\mu((\overline{v}_{i\mu_j})_+)^{\gamma+2}\, dx$$

and

$$\frac{\hat{V}(\chi_i v)}{\hat{V}(v)} = \frac{\int_{\omega_i} a_+ v_+^{\gamma+2}\, dx}{\int_{\Omega_+} a_+ v_+^{\gamma+2}\, dx} = \lim_{j\to\infty} \tfrac{1}{\gamma+2} \int_{\omega_i} a_+ Q_\mu((\overline{v}_{i\mu_j})_+)^{\gamma+2}\, dx \geqslant \rho_i. \tag{3.4}$$

As further $\displaystyle\sum_{j=1}^n \frac{\hat{V}(\chi_j v)}{\hat{V}(v)} = 1$, using (3.4) and (3.1) we obtain the estimate

$$J(Q_\mu v) = \sum_{j\in F} \frac{\hat{V}^{\frac{2}{\gamma+2}}(\chi_j v)}{\hat{V}^{\frac{2}{\gamma+2}}(v)} J(Q_\mu(\chi_j v)) \geqslant \sum_{j=1}^n \frac{\hat{V}^{\frac{2}{\gamma+2}}(\chi_j v)}{\hat{V}^{\frac{2}{\gamma+2}}(v)} J(\hat{u}_j) > J(\hat{u}_i),$$

where $F = \{i = 1, \ldots, n \mid \chi_i v \neq 0\}$. This contradicts (3.3). $\square$

4. Multi-bumps solutions

We already know that there exist n families of positive single-bump solutions. We prove in this section that for any p with $1 < p < n$ we can find C_n^p families of positive p-bumps solutions of (1.1). For that purpose we introduce the following notations.

Let us fix p of the functions $\hat{u}_i$ defined by Lemma 2.3. To simplify the notations, we assume that these functions are numbered in such a way that they correspond to $\hat{u}_1, \ldots, \hat{u}_p$. We denote by $E = \{1, \ldots, p\}$ the set of corresponding indices. Define then the corresponding nonlinear simplex $\mathfrak{S}_E$ constructed on the function $\hat{u}_1, \ldots, \hat{u}_p$,

$$
\mathfrak{S}_E := \left\{ u = \sum_{j=1}^{p} s_j^{\frac{1}{\gamma+2}} \hat{u}_j \mid (s_1, \ldots, s_p) \in \Delta_p \right\},
$$

where Δ_p is defined in Lemma 2.8. It follows from this lemma that J has a unique global maximum on $\mathfrak{S}_E$ at some interior point $w = (s_1^*)^{\frac{1}{\gamma+2}} \hat{u}_1 + \ldots (s_p^*)^{\frac{1}{\gamma+2}} \hat{u}_p$. We therefore expect, for large μ, the existence of a critical point of J in $\mathfrak{B}_\mu$ whose projection in $\mathfrak{S}$ is close to w. The corresponding solution would define a family of positive p-bumps solutions. In order to obtain this, a standard tool would be a general minimax principle. Define the class

$$
H_E := \{ h \in C(\mathfrak{S}_E, \mathfrak{B}_\mu) \mid \forall u \in \partial \mathfrak{S}_E, h(u) = u \text{ and } \forall u \in \mathfrak{S}_E, J(h(u)) \leqslant J(u) \}
$$

which can be seen as the class of continuous deformations of $Id_{\mathfrak{S}_E}$ which fix

$$
\partial \mathfrak{S}_E := \left\{ u = \sum_{j=1}^{p} s_j^{\frac{1}{\gamma+2}} \hat{u}_j \mid s_i = 0 \text{ for some } i = 1, \ldots, p \right\} \tag{4.1}
$$

and decrease the energy. It is then rather easy to check that the minimax value

$$
\inf_{h \in H} \max_{u \in \mathfrak{S}_E} J(h(u))
$$

is a critical value of J in $\mathfrak{B}_\mu$ if μ is sufficiently large. This follows from a general linking theorem, see e.g. [15]. However, it is not clear that for different sets E this minimax characterization produces different critical points and even that the corresponding solutions are p-bumps solutions. This comes from a lack of information about the localization of the critical points. To overcome this difficulty, we base our approach on deformation arguments and localize the deformation along the lines of the gradient flow.

In the next lemma, we identify disconnected regions where the gradient of J constrained to $\mathfrak{B}_\mu$ is bounded away from zero. As in Lemma 2.8, we write $c_E = J(w)$, where $w = \sum_{j=1}^{p} (s_j^*)^{\frac{1}{\gamma+2}} \hat{u}_j$ is the maximizer of J on the corresponding nonlinear simplex $\mathfrak{S}_E$, and we define for $\rho \in]0, 1/4[$,

$$
\mathfrak{E}_\mu(\rho) := \{ u \in \mathfrak{B}_\mu \mid J(u) \leqslant c_E \text{ and }
$$
$$
\forall i = 1, \ldots, p, \ s_i = \hat{V}_i(R_\mu u) \geqslant \rho, \ |s_i - s_i^*| \geqslant \rho \}.
$$

Lemma 4.1. *There exists $\theta > 0$ such that for any $\mu > 0$ large enough and all $u \in \mathfrak{E}_\mu(\rho)$, $\|\nabla_\mu J(u)\| \geqslant \theta$, where*

$$\nabla_\mu J(u) = \nabla J(u) - \frac{\langle \nabla J(u), \nabla V_\mu(u) \rangle}{\|\nabla V_\mu(u)\|^2} \nabla V_\mu(u). \tag{4.2}$$

Proof. Let us assume by contradiction that there exist $(\mu_j)_j \subset \mathbb{R}$ and $(u_j)_j \subset \mathfrak{E}_{\mu_j}(\rho)$ such that

$$\lim_{j \to \infty} \mu_j = \infty \quad \text{and} \quad \lim_{j \to \infty} \|\nabla_{\mu_j} J(u_j)\| = 0.$$

As the sequence $(u_j)_j$ is bounded in $H_0^1(\Omega)$, going to a subsequence if necessary, we can assume there exists $u \in H_0^1(\Omega)$ such that

$$u_j \overset{H_0^1}{\rightharpoonup} u \quad \text{and} \quad u_j \overset{L^{2+\gamma}}{\longrightarrow} u.$$

We introduce now the manifold

$$\hat{\mathfrak{B}} := \{u \in H_0^1(\Omega) \mid \operatorname{supp} u \subset \overline{\Omega}_+, \quad \hat{V}(u) = 1\},$$

which is such that $\hat{\mathfrak{B}} \subset \mathfrak{B}_\mu$ for any $\mu > 0$. We denote the tangent space to $\hat{\mathfrak{B}}$ at u by

$$T_u(\hat{\mathfrak{B}}) := \{v \in H_0^1(\Omega) \mid \operatorname{supp} v \subset \overline{\Omega}_+, \quad \int_\Omega a_+ u_+^{\gamma+1} v \, dx = 0\}.$$

Claim 1: $\langle \nabla J(u), v \rangle = 0$ *for all* $v \in T_u(\hat{\mathfrak{B}})$. Let $v \in T_u(\hat{\mathfrak{B}})$. We first observe that we can choose λ_j such that $v - \lambda_j u_j \in T_{u_j}(\mathfrak{B}_{\mu_j})$, where

$$T_{u_j}(\mathfrak{B}_{\mu_j}) := \{v \in H_0^1(\Omega) \mid \int_\Omega (a_+ - \mu_j a_-)(u_j)_+^{\gamma+1} v \, dx = 0\}$$

is the tangent space to $\mathfrak{B}_{\mu_j}$ at u_j. Indeed, as v is supported in Ω_+, we just need to take

$$\lambda_j = \frac{1}{\gamma + 2} \int_\Omega a_+ (u_j)_+^{\gamma+1} v \, dx.$$

We then notice that since $(u_j)_+ \overset{L^{2+\gamma}}{\longrightarrow} u_+$ and $v \in T_u(\hat{\mathfrak{B}})$, we have

$$\int_\Omega a_+ (u_j)_+^{\gamma+1} v \, dx \to \int_\Omega a_+ u_+^{\gamma+1} v \, dx = 0.$$

Hence, we deduce that $\lambda_j \to 0$. Computing

$$\langle \nabla J(u), v \rangle = \int_\Omega \nabla u \, \nabla v \, dx$$

$$= \int_\Omega (\nabla u - \nabla u_j) \nabla v \, dx + \int_\Omega \nabla u_j \nabla (v - \lambda_j u_j) \, dx + \lambda_j \int_\Omega |\nabla u_j|^2 \, dx$$

and using the fact that

$$u_j \overset{H^1}{\rightharpoonup} u, \quad v - \lambda_j u_j \in T_{u_j}(\mathfrak{B}_{\mu_j}), \quad \nabla_{\mu_j} J(u_j) \to 0 \quad \text{and} \quad \lambda_j \to 0,$$

the claim follows as

$$\int_\Omega (\nabla u - \nabla u_j) \nabla v \, dx \to 0,$$

$$\int_\Omega \nabla u_j \nabla (v - \lambda_j u_j) \, dx = \langle \nabla_{\mu_j} J(u_j), v - \lambda_j u_j \rangle \to 0$$

and

$$\lambda_j \int_\Omega |\nabla u_j|^2 \, dx \to 0.$$

Claim 2: $u \in \overline{H}$ *so that* $u = \overline{u}$. We write $u_j = \overline{u}_j + \tilde{u}_j$ and $u = \overline{u} + \tilde{u}$, where $\overline{u}_j, \overline{u} \in \overline{H}$ and $\tilde{u}_j, \tilde{u} \in \tilde{H}$. We first deduce from Lemma 2.5 that

$$\overline{u}_j \overset{H^1_0}{\rightharpoonup} \overline{u} \quad \text{and} \quad \tilde{u}_j \overset{H^1_0}{\rightharpoonup} \tilde{u}$$

so that

$$\overline{u}_j \overset{L^{2+\gamma}}{\longrightarrow} \overline{u} \quad \text{and} \quad \tilde{u}_j \overset{L^{2+\gamma}}{\longrightarrow} \tilde{u}.$$

The arguments of Claims 1 and 2 in the proof of Lemma 2.5 then imply that $\tilde{u}_+ = 0$. Let us prove that $\tilde{u}_- = 0$ a.e. in Ω_-. Since

$$\lim_{j \to \infty} \|\nabla_{\mu_j} J(u_j)\| = 0,$$

and $(u_j)_- \in T_{u_j}(\mathfrak{B}_{\mu_j})$, we deduce that

$$\int_\Omega |\nabla(u_j)_-|^2 \, dx = \langle \nabla_{\mu_j} J(u_j), (u_j)_- \rangle \to 0.$$

This means $(u_j)_- \to 0$ in $H_0^1(\Omega)$ and therefore $u_- = 0$. This in turn implies that $\tilde{u}_-(x) = 0$ for a.e. $x \in \Omega_-$. It follows that u is supported in $\overline{\Omega}_+$ which means $u \in \overline{H}$.

Define $w_i := \chi_i u$, where χ_i is the characteristic function of the set ω_i, and let $F := \{i = 1, \ldots, n \mid w_i \neq 0\}$. Observe that $w_i \neq 0$ for all $i = 1, \ldots, p$. Indeed, this follows from the convergence of u_j in $L^{2+\gamma}(\Omega)$ and the definition of $\mathfrak{E}_\mu(\rho)$. Changing the order of the indices of the subdomains ω_i's for $i > p$ if necessary, we may assume without loss of generality that $F = \{1, 2, \ldots, m\}$ for some $p \leqslant m \leqslant n$. Next, we introduce the function

$$\phi(s) := \sum_{i=1}^{m} s_i^{\frac{1}{\gamma+2}} Q_\mu w_i \in \hat{\mathfrak{B}},$$

where $s \in \Delta_m$ and Δ_m is defined in Lemma 2.8. Observe that $Q_\mu w_i$ is independent of μ since the w_i's are, respectively, supported in $\overline{\omega}_i$. We also define $\overline{s}$ to be such that $\phi(\overline{s}) = Q_\mu u$, i.e. $\overline{s}_i = \frac{\hat{V}(w_i)}{\hat{V}(u)}$, and we write

$$g(s) := J(\phi(s)) = \sum_{i=1}^{m} s_i^{\frac{2}{\gamma+2}} J(Q_\mu w_i).$$

Claim 3: $\forall i \in E = \{1, \ldots, p\}$, $\overline{s}_i \geqslant \rho$ and $|\overline{s}_i - s_i^*| \geqslant \rho$. This follows from the convergence of u_j in $L^{2+\gamma}(\Omega)$.

Claim 4: $\hat{V}(u) \geqslant 1$. For all $j \in \mathbb{N}$, we have

$$\hat{V}(u_j) = \int_\Omega a(u_j)_+^{\gamma+2} \, dx = 1 + \int_\Omega \mu_j a_-(u_j)_+^{\gamma+2} \, dx \geqslant 1.$$

Using the convergence of u_j in $L^{2+\gamma}(\Omega)$, we deduce that $\hat{V}(u) = \int_\Omega a_+ u_+^{\gamma+2} \, dx \geqslant 1$.

Claim 5: $g(\overline{s}) = J(Q_\mu u) \leqslant c_E$. Using the convexity of J and the weak convergence of the sequence $(u_j)_j$, we can write

$$c_E \geqslant \lim_{j \to \infty} J(u_j) \geqslant J(u).$$

It then follows from Claim 4 that

$$J(u) = \hat{V}^{\frac{2}{\gamma+2}}(u) J(Q_\mu u) \geqslant J(Q_\mu u).$$

Claim 6: $g(\overline{s}) < \max_{s \in \Delta_m} g(s)$. In case $m > p$, we have

$$g(s) \geqslant \sum_{i=1}^{m} s_i^{\frac{2}{\gamma+2}} J(\hat{u}_i) = f_F(s)$$

and therefore we infer from Lemma 2.8 that

$$\max_{s \in \Delta_m} g(s) \geqslant c_F > c_E = c \geqslant g(\overline{s}).$$

On the other hand, if $m = p$ and for some $i_0 \in E$, $J(Q_\mu w_{i_0}) \neq J(\hat{u}_{i_0})$, we have

$$g(s) \geqslant \sum_{i=1}^{p} s_i^{\frac{2}{\gamma+2}} J(\hat{u}_i) + s_{i_0}^{\frac{2}{\gamma+2}} (J(Q_\mu w_{i_0}) - J(\hat{u}_{i_0})) = f_E(s) + s_{i_0}^{\frac{2}{\gamma+2}} (J(Q_\mu w_{i_0}) - J(\hat{u}_{i_0})),$$

where f_E is defined in Lemma 2.8, and

$$\max_{s \in \Delta_m} g(s) \geqslant c + (s_{i_0}^*)^{\frac{2}{\gamma+2}} (J(Q_\mu w_{i_0}) - J(\hat{u}_{i_0})) > c \geqslant g(\overline{s}).$$

At last, if $m = p$ and for all $i = 1, \ldots, m$, $J(Q_\mu w_i) = J(\hat{u}_i)$, then $g(s) = f_E(s)$ so that the claim follows from Claim 3 and Lemma 4.1 as $|\overline{s}_i - s_i^*| \geqslant \rho$.

Conclusion: As $\phi(s) \in \hat{\mathfrak{B}}$, we deduce $\phi'(\overline{s}) \in T_{\phi(s)}(\hat{\mathfrak{B}})$, and it follows from Claim 1 that

$$g'(\overline{s}) = \langle \nabla J(\phi(\overline{s})), \phi'(\overline{s}) \rangle = \langle \nabla J(Q_\mu u), \phi'(\overline{s}) \rangle = 0.$$

Since the only stationary point of g is its maximum, this contradicts Claim 6. $\quad\square$

We now turn to the proof of the existence of C_n^p families of positive p-bumps solutions of (1.1) for any p with $2 \leqslant p \leqslant n - 1$.

Theorem 4.2. *Let assumptions* (H) *be satisfied. Let* $\omega = \omega_{i_1} \cup \ldots \cup \omega_{i_p}$ *with* $2 \leqslant p \leqslant n - 1$. *Then, for* μ *sufficiently large, there exists a family of positive* p-*bumps solutions of* (1.1) *with limit support in* $\overline{\omega}$.

Proof. *Choice of r:* For any set $F = \{i_1, \ldots, i_k\}$ with $2 \leqslant k \leqslant n$, we define from Lemma 2.8 the point w_F which maximizes J on the set

$$\mathfrak{S}_F = \left\{ u = \sum_{j=1}^{k} s_j^{\frac{1}{\gamma+2}} \hat{u}_{i_j} \mid (s_1, \ldots, s_k) \in \Delta_k \right\},$$

where

$$\Delta_k := \left\{ (s_1, \ldots, s_k) \in \mathbb{R}_+^k \mid \sum_{j=1}^{k} s_j = 1 \right\}.$$

We choose then $r > 0$ to be such that the neighbourhoods $B(w_F, 2r)$ do not intersect.

Choice of ρ: Consider now p sets ω_i and assume these are numbered $\omega_1, \ldots, \omega_p$. Define

$$\omega := \bigcup_{i=1}^{p} \omega_i.$$

Let $\hat{u}_1, \ldots, \hat{u}_p$ be the respective minimizers as in Lemma 2.3. We then denote by $\mathfrak{S}_E$ the nonlinear simplex with $E = \{1, \ldots, p\}$ and by $w_E := (s_1^*)^{\frac{1}{\gamma+2}} \hat{u}_1 + \ldots + (s_p^*)^{\frac{1}{\gamma+2}} \hat{u}_p$ the point which maximizes J on $\mathfrak{S}_E$. We write $c_E := J(w_E)$ and we fix $\rho \in]0, \frac{1}{4}[$ in such a way that $u \in \mathfrak{B}_\mu$ and $r \leqslant \|P_\mu u - w_E\| \leqslant 2r$ implies that for all $i = 1, \ldots, p$, we have $s_i = \hat{V}_i(R_\mu u) \geqslant \rho$ and $|s_i - s_i^*| \geqslant \rho$.

Claim 1: There exists $s > 0$ such that for all $\mu > 0$, if $u, v \in \mathfrak{B}_\mu$ satisfy $J(u) \leqslant c_E$, $J(v) \leqslant c_E$ and $\|u - v\| \leqslant s$, then $\|P_\mu u - P_\mu v\| \leqslant r$. If the claim is false, there exist $(\mu_n)_n \subset \mathbb{R}^+$, $(u_n)_n \subset \mathfrak{B}_{\mu_n}$ and $(v_n)_n \subset \mathfrak{B}_{\mu_n}$ such that $J(u_n) \leqslant c_E$, $J(v_n) \leqslant c_E$, $\|u_n - v_n\| \to 0$ and $\|P_{\mu_n} u_n - P_{\mu_n} v_n\| \geqslant r$. We then infer up to a subsequence that

$$u_n \overset{H_0^1}{\rightharpoonup} u, \quad v_n \overset{H_0^1}{\rightharpoonup} u, \quad u_n \overset{L^{2+\gamma}}{\longrightarrow} u \quad \text{and} \quad v_n \overset{L^{2+\gamma}}{\longrightarrow} u.$$

On the other hand, we also have

$$\overline{u}_n \overset{H_0^1}{\rightharpoonup} \overline{u}, \quad \overline{v}_n \overset{H_0^1}{\rightharpoonup} \overline{u}, \quad (\overline{u}_n)_+ \overset{L^{2+\gamma}}{\longrightarrow} (\overline{u})_+ \quad \text{and} \quad (\overline{v}_n)_+ \overset{L^{2+\gamma}}{\longrightarrow} (\overline{u})_+.$$

Hence, we deduce that

$$R_{\mu_n} u_n \overset{L^{2+\gamma}}{\longrightarrow} \frac{1}{\hat{V}((\overline{u})_+)} (\overline{u})_+ \quad \text{and} \quad R_{\mu_n} v_n \overset{L^{2+\gamma}}{\longrightarrow} \frac{1}{\hat{V}((\overline{u})_+)} (\overline{u})_+.$$

This in turn implies that for any $i = 1, \ldots, n$, $\hat{V}_i(R_{\mu_n} u_n) - \hat{V}_i(R_{\mu_n} v_n) \to 0$ so that finally $\|P_{\mu_n} u_n - P_{\mu_n} v_n\| \to 0$ which is a contradiction.

The set $\tilde{Q}$: Let θ be given by Lemma 4.1 and define $\gamma > 0$ to be such that $2\gamma < \frac{s\theta^2}{1+\theta}$ and

$$\forall u \in \mathfrak{S}_E \setminus B(w_E, r), \quad J(u) < c_E - \gamma.$$

We also choose $\delta \in]0, \gamma]$ and $\varepsilon > 0$ small enough to verify

$$(1 - (n - p)\varepsilon)^{\frac{2}{\gamma+2}} c_E \geqslant c_E - \gamma$$

and

$$(1 - (n - p)\varepsilon)^{\frac{2}{\gamma+2}} c_E + \varepsilon^{\frac{2}{\gamma+2}} \min_{i=p+1,\ldots,n} J(\hat{u}_i) > c_E + \delta.$$

Let us write

$$\tilde{\mathfrak{S}} := \{ u = \tilde{s}^{\frac{1}{\gamma+2}} w_E + \sum_{i=p+1}^{n} \tilde{s}_i^{\frac{1}{\gamma+2}} \hat{u}_i \mid \tilde{s} \geqslant 0, \ \tilde{s}_i \geqslant 0, \ \tilde{s} + \sum_{i=p+1}^{n} \tilde{s}_i = 1 \},$$

$$\tilde{Q} := \{ u \in \tilde{\mathfrak{S}} \mid \max_{i=p+1,\ldots,n} \tilde{s}_i \leqslant \varepsilon \},$$

$$\partial \tilde{Q} := \{ u \in \tilde{\mathfrak{S}} \mid \max_{i=p+1,\ldots,n} \tilde{s}_i = \varepsilon \}.$$

Notice that if $u \in \partial \tilde{Q}$, we can compute

$$
\begin{aligned}
J(u) &= \tilde{s}^{\frac{2}{\gamma+2}} c_E + \sum_{i=p+1}^{n} \tilde{s}_i^{\frac{2}{\gamma+2}} J(\hat{u}_i) \\
&\geqslant (1 - (n-p)\varepsilon)^{\frac{2}{\gamma+2}} c_E + \varepsilon^{\frac{2}{\gamma+2}} \min_{i=p+1,\ldots,n} J(\hat{u}_i) > c_E + \delta
\end{aligned}
\tag{4.3}
$$

and if $u \in \tilde{Q}$, we have

$$J(u) \geqslant (1 - (n-p)\varepsilon)^{\frac{2}{\gamma+2}} c_E \geqslant c_E - \gamma. \tag{4.4}$$

Choose next $\mu_0 > 0$ large enough so that Lemma 2.10 applies (with $r = c_E + 1$ and δ as above) and so that Lemma 4.1 holds. From now on we assume $\mu \geqslant \mu_0$.

The deformation: Consider the Cauchy problem

$$\eta' = -\Phi(J(\eta)) \frac{\nabla_\mu J(\eta)}{1 + \|\nabla_\mu J(\eta)\|}, \quad \eta(0) = u_0, \tag{4.5}$$

where $\nabla_\mu J(u)$ is defined in (4.2) and $\Phi : \mathbb{R} \to [0, 1]$ is a smooth function such that

$$\Phi(r) = \begin{cases} 0 & \text{if } r < c_0, \\ 1 & \text{if } r > \dfrac{c_0 + c_E - \gamma}{2}, \end{cases}$$

where $c_0 := \max\{ J(u) \mid u = \sum_{i=1}^{p} s_i^{\frac{1}{\gamma+2}} \hat{u}_i, \ (s_1, \ldots, s_p) \in \Delta_p \text{ and } \exists s_i = 0 \}$. The problem (4.5) has a unique solution $\eta(\cdot; u_0)$ defined on $\mathbb{R}$ and continuous in (t, u_0).

Claim 2: For all $t \geqslant 0$ and $u_0 \in \mathfrak{S}_E$, $J(P_\mu(\eta(t; u_0))) \leqslant c_E + \delta$. To prove this claim, we have to notice that

$$J(\eta(0; u_0)) = J(u_0) \leqslant c_E$$

and J decreases along solutions of (4.5) as

$$\frac{d}{dt} J(\eta(t; u_0)) = -\Phi(J(\eta(t; u_0))) \frac{\|\nabla_\mu J(\eta(t; u_0))\|^2}{1 + \|\nabla_\mu J(\eta(t; u_0))\|} \leqslant 0.$$

It then follows using Lemma 2.10 that

$$J(P_\mu(\eta(t; u_0))) \leqslant J(\eta(t; u_0)) + \delta \leqslant c_E + \delta.$$

Claim 3: For all $t \geqslant 0$, there exists $u_t \in \mathfrak{S}_E$ such that $P_\mu(\eta(t; u_t)) \in \tilde{Q}$. Let us write

$$P_\mu(\eta(t; u_t)) = \sum_{i=1}^{n} y_i^{\frac{1}{\gamma+2}}(t; u_t) \hat{u}_i$$

$$= \overline{Y}^{\frac{1}{\gamma+2}} \sum_{i=1}^{p} \left[\left(\frac{y_i(t; u_t)}{\overline{Y}} \right)^{\frac{1}{\gamma+2}} - (s_i^*)^{\frac{1}{\gamma+2}} \right] \hat{u}_i$$

$$+ \overline{Y}^{\frac{1}{\gamma+2}} w_E + \sum_{i=p+1}^{n} y_i^{\frac{1}{\gamma+2}}(t; u_t) \hat{u}_i,$$

where $\overline{Y} = \sum_{i=1}^{p} y_i(t; u_t)$. It is clear that $P_\mu(\eta(t; u_t)) \in \tilde{\mathfrak{S}}$ if and only if

$$f_i(t, u_t) = y_i(t; u_t) - \overline{Y} s_i^* = 0, \quad i = 1, \ldots, n. \tag{4.6}$$

It follows now from a degree argument (see [2, Lemma 1.2]) that there exists a connected set $\Sigma \subset \mathbb{R}_+ \times \mathfrak{S}_E$ of solutions (t, u_t) of (4.6) so that for all $t \geqslant 0$ there exists $u_t \in \mathfrak{S}_E$ with $(t, u_t) \in \Sigma$. Hence the set

$$\tilde{\Sigma} = \{ P_\mu(\eta(t; u_t)) \mid (t, u_t) \in \Sigma \} \subset \tilde{\mathfrak{S}}$$

is connected. As $(0, w_E)$ is the only solution of (4.6) with $t = 0$, we know that $P_\mu(\eta(0; w_E)) = w_E \in \tilde{Q}$. Also, it follows from (4.3) and Claim 2 that there is no $(t; u_0) \in \Sigma$ so that $P_\mu(\eta(t; u_0)) \in \partial \tilde{Q}$. Hence, the connected set $\tilde{\Sigma}$ is in $\tilde{Q}$ which proves the claim.

Existence of a Palais–Smale sequence $(v_n)_n$: From the preceding claim, we can find a sequence $(u_n)_n \subset \mathfrak{S}_E$ so that $P_\mu(\eta(n; u_n)) \in \tilde{Q}$ and using (4.4) we have

$$J(P_\mu(\eta(n; u_n))) \geqslant c_E - \gamma.$$

A subsequence $(u_{n_i})_i$ converges to $u_0 \in \mathfrak{S}_E$ which is such that for all $t \geqslant 0$, $J(P_\mu(\eta(t; u_0))) \geqslant c_E - \gamma$. Hence

$$J(\eta(t; u_0)) \geqslant J(P_\mu(\eta(t; u_0))) - \delta \geqslant c_E - \gamma - \delta \geqslant c_E - 2\gamma, \tag{4.7}$$

which implies that there exist some c_1 and a sequence $(t_n)_n$ with $t_n \to \infty$ such that

$$J(\eta(t_n; u_0)) \to c_1$$

and

$$\langle \nabla_\mu J(\eta(t_n; u_0)), \eta'(t_n; u_0) \rangle = -\frac{\|\nabla_\mu J(\eta(t_n; u_0))\|^2}{1 + \|\nabla_\mu J(\eta(t_n; u_0))\|} \to 0,$$

i.e.

$$\nabla_\mu J(\eta(t_n; u_0)) \to 0.$$

Hence, we can choose $v_n := \eta(t_n; u_0)$.

Claim 4: We claim that for all n, $\|P_\mu(v_n) - w_E\| \leqslant 2r$. Suppose the claim is false. In this case, it follows from the definition of γ that $P_\mu(\eta(0; u_0)) = u_0 \in B(w_E, r)$. Therefore, we can find $t_1, t_2 > 0$ such that

$$\|P_\mu(\eta(t_1; u_0)) - w_E\| = r, \quad \|P_\mu(\eta(t_2; u_0)) - w_E\| = 2r$$

and

$$r \leqslant \|P_\mu(\eta(t; u_0)) - w_E\| \leqslant 2r$$

for all $t \in [t_1, t_2]$. It follows from Claim 1 and the definition of the deformation that

$$|t_2 - t_1| \geqslant \|\eta(t_2; u_0) - \eta(t_1; u_0)\| \geqslant s.$$

On the other hand, using (4.7), we have $c_E - 2\gamma \leqslant J(\eta(t; u_0)) \leqslant c_E$ which implies $|J(\eta(t_2; u_0)) - J(\eta(t_1; u_0))| \geqslant 2\gamma$. Further, we infer from the choice of ρ that $\eta(t; u_0) \in \mathfrak{E}_\mu(\rho)$ for any $t \in [t_1, t_2]$. Using

$$|J(\eta(t_2; u_0)) - J(\eta(t_1; u_0))| = \int_{t_1}^{t_2} \frac{\|\nabla_\mu J(\eta(s; u_0))\|^2}{1 + \|\nabla_\mu J(\eta(s; u_0))\|} \, ds,$$

we deduce then from Lemma 4.1 that

$$2\gamma \geqslant \frac{\theta^2}{1+\theta}|t_2 - t_1| \geqslant \frac{s\theta^2}{1+\theta},$$

which contradicts the choice of γ.

Conclusion: Since we proved in Lemma 2.4 that the Palais–Smale condition holds, there exist $v_\mu \in \mathfrak{S}_\mu$ and a subsequence we still denote by $(v_n)_n$ such that

$$v_n \to v_\mu, \quad \nabla_\mu J(v_n) \to \nabla_\mu J(v_\mu) \ \text{and} \ \nabla_\mu J(v_\mu) = 0.$$

To complete the proof, it remains to show that $\{v_\mu \mid \mu \geqslant \mu_0\}$ is a family of p-bumps solutions with limit support in $\overline{\omega}$. Let v be a cluster value for the weak convergence in $H_0^1(\Omega)$, i.e. there exists a sequence $(\mu_j)_j \subset \mathbb{R}^+$ such that

$$\mu_j \to \infty \ \text{and} \ v_{\mu_j} \overset{H_0^1}{\rightharpoonup} v.$$

Arguing as in the proof of Theorem 3.1 we infer that v is positive and has support in $\overline{\Omega}_+$. It also follows from Claim 4 that $P_{\mu_j} v_{\mu_j} \in B(w_E, 2r)$ for any j. We claim $P_{\mu_j} v_{\mu_j} \to w_E$ as $j \to \infty$. Otherwise, there exist $\hat{\rho} > 0$ and a subsequence μ_{j_k} such that $v_{\mu_{j_k}} \in \mathfrak{C}_{\mu_{j_k}}(\hat{\rho})$ for any k but then for μ_{j_k} large enough, $\nabla_{\mu_{j_k}} J(v_{\mu_{j_k}}) \neq 0$ by Lemma 4.1. On the other hand, using by now familiar arguments, it can be checked that

$$P_{\mu_j} v_{\mu_j} \to \sum_{i=1}^n \left(\frac{\hat{V}_i(v)}{\hat{V}(v)} \right)^{\frac{1}{\gamma+2}} \hat{u}_i.$$

We therefore conclude that $\hat{V}_i(v) \neq 0$ for $i \in E$ and $\hat{V}_i(v) = 0$ for $i \notin E$ so that v is a p-bumps function with support in $\overline{\omega}$. This completes the proof. $\quad\square$

5. A *n*-bumps solution

We have proven in the preceding sections the existence of $2^n - 2$ positive solutions of (1.1) for sufficiently large μ. Indeed, the families of solutions we obtained have different limit supports so that they certainly differ for large μ. In this last section, we state the existence of a solution whose energy is greater than all the previous ones. For that purpose we consider the class

$$H := \{h \in C(\mathfrak{S}, \mathfrak{B}_\mu) \mid \forall u \in \partial\mathfrak{S}, \ h(u) = u \ \text{and} \ \forall u \in \mathfrak{S}, \quad J(h(u)) \leqslant J(u)\},$$

where $\partial \mathfrak{S}$ is the boundary of $\mathfrak{S}$ defined as in (4.1). We then define for each μ the minimax value

$$c_n := \inf_{h \in H} \max_{u \in \mathfrak{S}} J(h(u))$$

and claim that c_n is a critical value if μ is sufficiently large. Observe that this minimax characterization corresponds to the choice $E = \{1, \dots, n\}$ and the critical value defined in Section 4. However in this case, the energy level of the solution allows to distinguish it from the others.

Let w_E be the point in $\mathfrak{S}$ which maximizes J and write $c_E = J(w_E)$. Let $\delta > 0$ be such that

$$\max_{u \in \partial \mathfrak{S}} J(u) + 2\delta \leqslant c_E.$$

Let $\mu > 0$ be sufficiently large so that Lemma 2.10 holds with this choice of δ and $r = c_E + 1$. We then define the closed set

$$\tilde{\mathfrak{S}} := \{u \in \mathfrak{B}_\mu \mid P_\mu(u) = w \text{ and } J(u) \leqslant r\}.$$

We claim that $\tilde{\mathfrak{S}}$ has the intersection property by which we mean that for every $h \in H$, $h(\mathfrak{S}) \cap \tilde{\mathfrak{S}} \neq \emptyset$. Indeed, the function $P_\mu \circ h$ is a continuous deformation of $Id_\mathfrak{S}$ so that for all $u \in \partial \mathfrak{S}$, $P_\mu(h(u)) - w = u - w \neq 0$. It follows that

$$\deg(P_\mu \circ h - w, \mathfrak{S}) = \deg(Id_\mathfrak{S} - w, \mathfrak{S}) = 1$$

and the claim easily follows as for every $h \in H$ and all $u \in \mathfrak{S}$, $J(h(u)) \leqslant \max_{\mathfrak{S}} J \leqslant c_E$. We therefore deduce that the min–max value c_n is well defined as

$$\max_{u \in \mathfrak{S}} J(h(u)) \geqslant \min_{u \in \tilde{\mathfrak{S}}} J(u) \geqslant \min_{u \in \tilde{\mathfrak{S}}} J(P_\mu u) - \delta = c_E - \delta.$$

Notice that for $u \in \partial \mathfrak{S}$, we have

$$J(u) \leqslant c_E - 2\delta$$

so that we easily conclude that c_n is a critical value of J in $\mathfrak{B}_\mu$. Moreover, $c_n \geqslant c_E - \delta \geqslant \max_{u \in \partial \mathfrak{S}} J(u) + \delta$ so that for μ large enough the corresponding solution is different from any p-bumps solution with $1 \leqslant p \leqslant n - 1$.

It seems natural that the above minimax principle leads to a n-bumps solution. However, this additional information requires a localization of the Palais–Smale sequence. Using the arguments of Section 4, we can derive a precise result.

Theorem 5.1. *For μ large enough, there exists a family of positive n-bumps solutions of* (1.1) *with limit support in* $\overline{\Omega}_+$.

Since the proof consists in slight modifications of the arguments used in the proof of Theorem 4.2, we leave it to the reader.

Remark 5.1. We would like to emphasize that our approach only requires the quadraticity, the coercivity and the weak lower semi-continuity of *J*. Therefore, the method can be used for more general equations than (1.1). One could add for example a linear term $-V(x)u$ in the equation provided that *V* is above $-\lambda_1(\Omega)$, the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω.

Acknowledgments

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