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Correspondence theorems for hierarchies of equations of pseudo-spherical type

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For my father Pedro Reyes Tello on his 80th birthday

Abstract

Hierarchies of evolution equations of pseudo-spherical type are introduced, thereby generalizing the notion of a single equation describing pseudo-spherical surfaces due to S.S. Chern and K. Tenenblat, and providing a connection between differential geometry and the study of hierarchies of equations which are the integrability condition of $sl(2, \mathbf{R})$-valued linear problems. As an application, it is shown that there exist local correspondences between *any two* (suitably generic) solutions of arbitrary hierarchies of equations of pseudo-spherical type.

Keywords: Equations of pseudo-spherical type; Pseudo-spherical surfaces; Bäcklund transformations; Integrable equations; Hierarchies of pseudo-spherical type; Generalized symmetries

1. Introduction

The class of differential equations of pseudo-spherical type (or "describing pseudo-spherical surfaces") was introduced by S.S. Chern and K. Tenenblat [7] in 1986, motivated by the following observation (R. Sasaki [27]): the domains of generic—in a sense to be made precise in Section 2—solutions $u(x, t)$ of equations integrable by the Ablowitz, Kaup, Newell, and Segur (AKNS) inverse scattering approach can be equipped, whenever their associated linear problems are real, with Riemannian metrics of constant Gaussian curvature equal to -1. Chern and Tenen-

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blat called a partial differential equation $\Xi = 0$ of *pseudo-spherical type* if generic solutions of $\Xi = 0$ determine Riemannian metrics of constant Gaussian curvature -1 on open subsets of their domains (a precise definition is given in Section 2). An example of such a PDE is of course the ubiquitous sine-Gordon equation [7,27,31].

The importance of this structure was recognized first in the realm of integrable systems: some fundamental properties of equations of pseudo-spherical type, such as the existence of conservation laws, symmetries, and Bäcklund transformations, can be studied by geometrical means [3, 7,20–23,27,31] and moreover, equations describing pseudo-spherical surfaces are the integrability condition of $sl(2, \mathbf{R})$-valued linear problems ([7,27] and Section 2 below) and therefore one can try to obtain solutions for them using a scattering/inverse scattering approach [1–3]. More recently [13,24] it has been realized that independently of their integrability properties, new analytical results about equations of pseudo-spherical type can be found motivated by geometrical considerations: inspired by the fact that two surfaces of constant Gaussian curvature equal to -1 are locally indistinguishable, N. Kamran and K. Tenenblat [13]—and then the present author [24]—have showed that if $\Xi = 0$ and $\widehat{\Xi} = 0$ describe pseudo-spherical surfaces, there exists a smooth mapping transforming a (generic) local solution $u(x,t)$ of $\Xi = 0$ into a (generic) local solution $\hat{u}(\hat{x}, \hat{t})$ of $\widehat{\Xi} = 0$. In some instances, but not always, one can even find an explicit formula for $\hat{u}(\hat{x}, \hat{t})$ in terms of $u(x,t)$. Examples of explicit and implicit transformations appear in [13,24,25].

Now, an aspect of the theory of integrable systems which appears to have been little studied from a geometrical point of view is the fact that, as stressed, for instance, in the treatises by Faddeev and Takhtajan [9] and Dickey [8], evolution equations $u_t = F$ which are the integrability condition of a nontrivial one-parameter family of linear problems

$$v_x = Xv, \qquad v_t = Tv \tag{1}$$

(in which X and T are, say, $sl(2, \mathbf{R})$-valued functions of u and a finite number of its derivatives with respect to x) are members of *infinite hierarchies* of evolution equations $u_{\tau_n} = F_n$ possessing the following features:

(a) the flows generated by the equations $u_{\tau_n} = F_n$ commute, and
(b) each equation $u_{\tau_n} = F_n$ is the integrability condition of a nontrivial one-parameter family of linear problems of the form

$$v_x = Xv, \qquad v_{\tau_n} = T_n v,$$

that is, the equations $u_{\tau_n} = F_n$ share with the given equation $u_t = F$ the "space" part of their associated linear problems.

One wonders if these observations have counterparts in the class of equations considered by Chern and Tenenblat: Can one define hierarchies of evolution equations describing pseudo-spherical surfaces so that these "hierarchies of pseudo-spherical type" possess characteristics (a) and (b)? If so, can one generalize to this new setting the correspondence theorems between generic solutions of single equations of pseudo-spherical type found in [13,24]? This paper is devoted to answering these two questions. Some basic aspects of the theory of equations of pseudo-spherical type are recalled in Section 2. Then, hierarchies of pseudo-spherical type are introduced in Section 3, and it is shown that they do satisfy (a) and (b) above. Finally, it is proven in Section 4 that indeed there exist correspondences between (suitably generic) solutions of *any two* hierarchies of equations of pseudo-spherical type.

Some of the results appearing in this paper were announced at the 2000 NSF–CBMS Conference on the Geometrical Study of Differential Equations [23] and more recently in [25].

2. Equations of pseudo-spherical type

Definition 1. A two-dimensional manifold M is called a pseudo-spherical surface if there exist one-forms $\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}^3$ on M that satisfy the independence condition $\overline{\omega}^1 \wedge \overline{\omega}^2 \neq 0$, and the structure equations

$$d\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \qquad d\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \qquad d\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2. \tag{2}$$

If M is a pseudo-spherical surface, it is a Riemannian manifold equipped with the metric $ds^2 = (\overline{\omega}^1)^2 + (\overline{\omega}^2)^2$, the differential form $\overline{\omega}^3$ is the corresponding torsion-free metric connection one-form, and its Gaussian curvature is $K = -1$ [12,13,31].

Following standard usage [18], one says that a *differential function* is a smooth function which depends on the independent variables x, t, the dependent variable u, and a finite number of derivatives of u. Hereafter, partial derivatives $\partial^{n+m}u/\partial x^n \partial t^m$, $n, m \geqslant 0$, will be also denoted by $u_{x^n t^m}$.

Definition 2. A scalar differential equation

$$\Xi(x, t, u, u_x, \ldots, u_{x^n t^m}) = 0 \tag{3}$$

in two independent variables x, t is of pseudo-spherical type (or, it describes pseudo-spherical surfaces) if there exist one-forms ω^α, $\alpha = 1, 2, 3$,

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \ldots, u_{x^r t^p})\,dx + f_{\alpha 2}(x, t, u, \ldots, u_{x^s t^q})\,dt, \tag{4}$$

whose coefficients $f_{\alpha\beta}$ are differential functions, such that the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$ satisfy the structure equations (2) whenever $u = u(x,t)$ is a solution to Eq. (3).

Consequently, if $\Xi = 0$ describes pseudo-spherical surfaces with associated one-forms ω^α, and $u(x,t)$ is a solution to $\Xi = 0$ such that $(\omega^1 \wedge \omega^2)(u(x,t)) \neq 0$, Definition 1 implies that the domain of $u(x,t)$ can be equipped with a Riemannian metric of constant Gaussian curvature $K = -1$.

The functions $f_{\alpha\beta}$ could, presumably, all depend only on x and t, but this would be a trivial case from the point of view of differential equations and it is therefore excluded from further considerations.

Example 3. Burgers' equation $u_t = u_{xx} + uu_x$ is an equation of pseudo-spherical type. Associated one-forms ω^α are

$$\omega^1 = \left(\frac{1}{2}u - \frac{\beta}{\eta}\right)dx + \frac{1}{2}\left(u_x + \frac{1}{2}u^2\right)dt, \qquad \omega^2 = \eta\,dx + \left(\frac{\eta}{2}u + \beta\right)dt,$$

and $\omega^3 = -\omega^2$, in which η is a nonzero parameter and β is a solution to the equation $\beta^2 - \eta\beta_x = 0$.

The expression "PSS equation" will be sometimes utilized below instead of the more formal phrase "equation of pseudo-spherical type."

Definition 4. Let $\Xi = 0$ be a PSS equation with associated one-forms ω^α, $\alpha = 1, 2, 3$. A solution $u(x,t)$ of $\Xi = 0$ is I-generic if $(\omega^3 \wedge \omega^2)(u(x,t)) \neq 0$; II-generic if $(\omega^1 \wedge \omega^3)(u(x,t)) \neq 0$; and III-generic if $(\omega^1 \wedge \omega^2)(u(x,t)) \neq 0$.

For instance (see Example 3) the travelling wave $u(x,t) = 2\mathrm{e}^{x+t}/(1+\mathrm{e}^{x+t})$ is a III-generic solution of Burgers equation, but it is not I-generic. Definition 4 allows one to refine the geometric interpretation of PSS equations given above:

Proposition 5. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^α, $\alpha = 1, 2, 3$; let $u(x,t)$ be a solution of $\Xi = 0$, and set $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$.*

(a) *If $u(x,t)$ is I-generic, $\overline{\omega}^2$ and $\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one-form given by $\overline{\omega}^1$.*
(b) *If $u(x,t)$ is II-generic, $\overline{\omega}^1$ and $-\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one-form given by $\overline{\omega}^2$.*
(c) *If $u(x,t)$ is III-generic, $\overline{\omega}^1$ and $\overline{\omega}^2$ determine a Riemannian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one-form given by $\overline{\omega}^3$.*

Proposition 5 is proven in [24]. It is a consequence of the structure equations of a surface equipped with a metric of signature $(1, \epsilon)$, $\epsilon = \pm 1$, which appear, for example, in [12,31].

The invariance properties of the structure equations (2) are spelled out in the following straightforward proposition.

Proposition 6. *Let ω^α, $\alpha = 1, 2, 3$, be one-forms whose coefficients are differential functions. Let $u(x,t)$ be a smooth function, and set $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$. The structure equations* (2) *are invariant under the transformations*

$$\widehat{\omega}^1 = \overline{\omega}^1 \cos\overline{\rho} + \overline{\omega}^2 \sin\overline{\rho}, \qquad \widehat{\omega}^2 = -\overline{\omega}^1 \sin\overline{\rho} + \overline{\omega}^2 \cos\overline{\rho}, \qquad \widehat{\omega}^3 = \overline{\omega}^3 + d\overline{\rho}; \tag{5}$$

$$\widehat{\omega}^1 = \overline{\omega}^1 \cosh\overline{\rho} - \overline{\omega}^3 \sinh\overline{\rho}, \qquad \widehat{\omega}^2 = \overline{\omega}^2 + d\overline{\rho}, \qquad \widehat{\omega}^3 = -\overline{\omega}^1 \sinh\overline{\rho} + \overline{\omega}^3 \cosh\overline{\rho}; \tag{6}$$

$$\widehat{\omega}^1 = \overline{\omega}^1 + d\overline{\rho}, \qquad \widehat{\omega}^2 = \overline{\omega}^2 \cosh\overline{\rho} + \overline{\omega}^3 \sinh\overline{\rho}, \qquad \widehat{\omega}^3 = \overline{\omega}^2 \sinh\overline{\rho} + \overline{\omega}^3 \cosh\overline{\rho}, \tag{7}$$

in which ρ is any differential function and $\overline{\rho} = \rho(u(x,t))$.

The geometric interpretation of this observation follows from Proposition 5: If $\Xi = 0$ is a PSS equation with associated one-forms ω^α and $u(x,t)$ is a III-generic solution, (5) is simply the transformation induced on the forms $\overline{\omega}^\alpha$ by a rotation of the moving frame dual to the coframe $\{\overline{\omega}^1, \overline{\omega}^2\}$. Analogously, if $u(x,t)$ is II-generic, (6) is the transformation induced on the forms $\overline{\omega}^\alpha$ by a Lorentz boost of the moving frame dual to the coframe $\{\overline{\omega}^1, -\overline{\omega}^3\}$, and if $u(x,t)$ is I-generic, (7) is the transformation induced on the forms $\overline{\omega}^\alpha$ by a Lorentz boost of the moving frame dual to the coframe $\{\overline{\omega}^2, \overline{\omega}^3\}$.

As pointed out in Section 1, conservation laws, symmetries, and classical Bäcklund transformations of PSS equations can be studied via the geometry of pseudo-spherical surfaces [3,7,20–22,27,31]. Of importance for this paper is the fact that if $\Xi(x,t,u,\dots)=0$ describes pseudo-spherical surfaces with associated one-forms ω^α, it is the integrability condition of a $sl(2,\mathbf{R})$-valued linear problem. Indeed, it is easy to see that the $sl(2,\mathbf{R})$-valued linear problem

$$\frac{\partial v}{\partial x}=Xv, \qquad \frac{\partial v}{\partial t}=Tv, \tag{8}$$

in which X and T are determined by the $sl(2,\mathbf{R})$-valued one-form

$$\Omega = X\,dx + T\,dt = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1-\omega^3 \\ \omega^1+\omega^3 & -\omega^2 \end{pmatrix}, \tag{9}$$

is integrable whenever $u(x,t)$ is a solution to $\Xi=0$. In other words, the structure equations (2) imply that the matrix equation

$$\frac{\partial X}{\partial t}-\frac{\partial T}{\partial x}+[X,T]=0 \tag{10}$$

is identically satisfied whenever $u(x,t)$ is a solution of $\Xi=0$, and so one may hope to study PSS equations via scattering/inverse scattering techniques [1,2]. An interesting example is provided by the equation

$$\bigl\{u_t-\bigl[\alpha g(u)+\beta\bigr]u_x\bigr\}_x=g'(u), \tag{11}$$

in which $g(u)$ satisfies the equation

$$g''+\mu g=\theta$$

and μ,θ,α,β are real numbers. M. Rabelo proved in [19] that Eq. (11) is of pseudo-spherical type with associated one-forms

$$\omega^1=\zeta u_x\,dx+\zeta(\alpha g+\beta)u_x\,dt, \qquad \omega^2=\eta\,dx+\bigl(\bigl(\zeta^2g-\theta\bigr)/\eta+\beta\eta\bigr)\,dt,$$
$$\omega^3=(\zeta g'/\eta)\,dt, \tag{12}$$

in which $\zeta^2=\alpha\eta^2-\mu$ and η is a real parameter. Beals, Rabelo and Tenenblat [3] then used the linear problem (8) determined by (9) and (12) to solve (11) by adapting to this case the rigorous scattering/inverse scattering method developed by Beals and Coifman in [1,2].

Remark 7. It is a classical observation [9,27] that—whenever $u(x,t)$ is a solution to $\Xi=0$—one can interpret Eqs. (8)–(10) in terms of connections: Let $\pi: U\times SL(2,\mathbf{R})\to U$, in which U is (an open subset of) the domain of the solution $u(x,t)$, be a principal fiber bundle, and consider the $sl(2,\mathbf{R})$-valued one-form $\Omega(u(x,t))$ determined by (9). This differential form can be thought of as a flat connection one-form on the bundle π, and the solution $v(x,t)$ to (8) as a covariantly constant section on a vector bundle $U\times V\to U$ associated to π, in which V is some two-dimensional vector space. This interpretation has been of use in the study of transformations of solutions of PSS equations [24].

A standard observation in integrable systems [9] is that an equation $\Xi = 0$ is not just the integrability condition of a linear problem such as (8), but that Eq. (10) is in fact *equivalent* to $\Xi = 0$: It is precisely under this assumption that nonlinear equations which are the integrability condition of linear problems have been found from geometrical or algebraic considerations, see, for example, Bobenko [4], Dickey [8], Lund and Regge [17], the monograph [31] by Tenenblat, and also [21,22]. In the case of kth order evolutionary PSS equations $u_t = F(x, t, u, \ldots, u_{x^k})$ one can formalize this equivalence as follows [10,13]:

Consider a manifold J equipped with coordinates $x, t, u, u_x, \ldots, u_{x^k}$, and the differential ideal I_F generated by the two-forms

$$du \wedge dx + F(x, t, u, \ldots, u_{x^k})\, dx \wedge dt, \qquad du_{x^l} \wedge dt - u_{x^{l+1}}\, dx \wedge dt,$$

in which $0 \leqslant l \leqslant k-1$, so that local solutions of $u_t = F$ correspond to integral submanifolds of the exterior differential system $\{I_F, dx \wedge dt\}$.

Definition 8. An evolution equation $u_t = F(x, t, u, \ldots, u_{x^k})$ is strictly pseudo-spherical if there exist one-forms $\omega^\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$, $\alpha = 1, 2, 3$, whose coefficients $f_{\alpha\beta}$ are smooth functions on J, such that the two-forms

$$\Omega_1 = d\omega^1 - \omega^3 \wedge \omega^2, \qquad \Omega_2 = d\omega^2 - \omega^1 \wedge \omega^3, \qquad \Omega_3 = d\omega^3 - \omega^1 \wedge \omega^2 \tag{13}$$

generate the differential ideal I_F.

It follows that if $u_t = F$ is strictly pseudo-spherical, it is the necessary and sufficient condition for the structure equations $\Omega_\alpha = 0$ to hold, as it was assumed in the seminal papers [7,27]. Definition 8 will provide important motivation for the constructions appearing in subsequent sections. The following lemma [24] will be also of interest.

Lemma 9. *Necessary and sufficient conditions for the kth order evolution equation $u_t = F$ to be strictly pseudo-spherical with associated differential functions $f_{\alpha\beta}$ are the conjunction of*

(a) *the functions $f_{\alpha\beta}$ satisfy the constraints*: $f_{\alpha 1, u_{x^a}} = 0$; $f_{\alpha 2, u_{x^k}} = 0$; $f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0$, *in which $a \geqslant 1$ and $\alpha = 1, 2, 3$; and*
(b) *the following three identities hold*:

$$-f_{11,u} F + \sum_{i=0}^{k-1} u_{x^{i+1}} f_{12, u_{x^i}} + f_{21} f_{32} - f_{31} f_{22} + f_{12,x} - f_{11,t} = 0, \tag{14}$$

$$-f_{21,u} F + \sum_{i=0}^{k-1} u_{x^{i+1}} f_{22, u_{x^i}} + f_{12} f_{31} - f_{11} f_{32} + f_{22,x} - f_{21,t} = 0, \tag{15}$$

$$-f_{31,u} F + \sum_{i=0}^{k-1} u_{x^{i+1}} f_{32, u_{x^i}} + f_{21} f_{12} - f_{11} f_{22} + f_{32,x} - f_{31,t} = 0. \tag{16}$$

3. Hierarchies of equations of pseudo-spherical type

Consider an affine space equipped with coordinates $(x, t, \tau_1, \tau_2, \dots)$. Hereafter, generalizing the convention adopted in Section 2, a *differential function* will be a real-valued smooth function depending on a finite number of independent variables x, t, τ_1, τ_2, $\dots$, the dependent variable u, and a finite number of x-derivatives of u. The independent variable t will be also denoted by τ_0 and, unless otherwise explicitly stated, the indices i, j, α, β will have ranges $i = 0, 1, 2, \dots$, $j = 1, 2, 3, \dots$, $\alpha = 1, 2, 3$, and $\beta = 1, 2$.

3.1. Basic definitions

Assume that $u_{\tau_i} = F_i$ is a countable sequence of evolution equations and set

$$D_x = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} u_{x^{j+1}} \frac{\partial}{\partial u_{x^j}} \quad \text{and} \quad D_{\tau_i} = \frac{\partial}{\partial \tau_i} + \sum_{j=0}^{\infty} \left(D_x^j F_i\right) \frac{\partial}{\partial u_{x^j}}. \tag{17}$$

Definition 10. Let $u_{\tau_i} = F_i$ be a countable number of evolutionary equations, in which F_i are differential functions. These equations form a hierarchy of equations describing pseudo-spherical surfaces (or $u_{\tau_i} = F_i$ is a hierarchy of pseudo-spherical type) if there exist differential functions $f_{\alpha\beta}$ and $h_{\alpha j}$ such that for each $n \geqslant 0$, the one-forms $\Theta_\alpha^{[n]}$ given by

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt + \sum_{k=1}^{n} h_{\alpha k}\, d\tau_k \tag{18}$$

satisfy the equations

$$d_H \Theta_1^{[n]} = \Theta_3^{[n]} \wedge \Theta_2^{[n]}, \qquad d_H \Theta_2^{[n]} = \Theta_1^{[n]} \wedge \Theta_3^{[n]}, \qquad d_H \Theta_3^{[n]} = \Theta_1^{[n]} \wedge \Theta_2^{[n]}, \tag{19}$$

in which $d_H \Theta_\alpha^{[n]}$ is computed by means of $d_H(dx) = d_H(d\tau_i) = 0$ and

$$d_H g = D_x g\, dx + \sum_{k=0}^{n} D_{\tau_k} g\, d\tau_k \tag{20}$$

for any differential function g.

If $u_{\tau_i} = F_i$ is a hierarchy of pseudo-spherical type, $u_t = F$ is called the seed equation, while the equations $u_{\tau_j} = F_j$ are referred to as the higher equations of the hierarchy. Examples are given at the end of Section 4.

Definition 11. A local smooth solution of a hierarchy of pseudo-spherical type $u_{\tau_i} = F_i$ is a sequence $\{u^{[n]}(x, \tau_0, \dots, \tau_n)\colon n \geqslant 0\}$ of smooth functions $u^{[n]} : V^{[n]} \subset \mathbf{R}^{n+2} \to \mathbf{R}$ such that for each $n \geqslant 0$ the following two conditions hold:

(a) $u^{[n]}$ is a local smooth solution of the equations $u_{\tau_i} = F_i$, $i = 0, \dots, n$,
(b) $u^{[n+1]}|_{V^{[n]}} = u^{[n]}$.

Equipped with this notion of solution one can study the geometrical content of Definition 10.

Theorem 12. *Let $u_{\tau_i} = F_i$ be a hierarchy of pseudo-spherical type with associated one-forms*

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt + \sum_{k=1}^{n} h_{\alpha k}\,d\tau_k, \quad n \geqslant 0, \tag{21}$$

and let the sequence of real-valued smooth functions $\{u^{[n]}:\ n \geqslant 0\}$ be an arbitrary solution of the hierarchy $u_{\tau_i} = F_i$. For each $n \geqslant 0$, let $V^{[n]} \subseteq \mathbf{R}^{n+2}$ be (an open subset of) the domain of the function $u^{[n]}$.

(1) *Assume that $n = 0$. If $\Theta_1^{[0]} \wedge \Theta_2^{[0]}(u^{[0]}(x,t)) \neq 0$, the tensor*

$$ds^2 = \Theta_1^{[0]}\big(u^{[0]}(x,t)\big) \otimes \Theta_1^{[0]}\big(u^{[0]}(x,t)\big) + \Theta_2^{[0]}\big(u^{[0]}(x,t)\big) \otimes \Theta_2^{[0]}\big(u^{[0]}(x,t)\big)$$

defines a Riemannian metric of constant Gaussian curvature $K = -1$ on $V^{[0]}$, and the one-form $\Theta_3^{[0]}(u^{[0]}(x,t))$ is the corresponding Levi–Civita connection one-form.

(2) *Assume that $n \geqslant 1$. Equip $V^{[n]}$ with a flat pseudo-Riemannian metric of index s, and let $\iota : D \subseteq \mathbf{R}^2 \to V^{[n]}$ be a smooth function from an open set $D \subseteq \mathbf{R}^2$ into $V^{[n]}$. Suppose that the forms $\Theta_\alpha^{[n]}(u^{[n]}(x,\tau_0,\ldots,\tau_n))$ satisfy the conditions*

$$\iota^*\big[\Theta_\alpha^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big] \neq 0 \quad \text{and} \quad \iota^*\big[\Theta_1^{[n]} \wedge \Theta_2^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big] \neq 0. \tag{22}$$

   (a) *The set D can be equipped with the structure of a pseudo-spherical surface*: *the pair of one-forms*

$$\iota^*\big[\Theta_1^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big] \quad \text{and} \quad \iota^*\big[\Theta_2^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big]$$

   are a moving coframe on D, and the one-form $\iota^*[\Theta_3^{[n]}(u^{[n]}(x,\tau_0,\ldots,\tau_n))]$ is the corresponding Levi-Civita connection one-form.

   (b) *If the index s is equal to n, there exists an isometric immersion of the pseudo-spherical surface (D, ds^2) constructed in* (a)*, in which*

$$ds^2 = \iota^*\big[\Theta_1^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big]^2 + \iota^*\big[\Theta_2^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big]^2,$$

   into the flat pseudo-Riemannian manifold $V^{[n]}$. Moreover, the normal bundle of the immersion is flat.

Proof. The following range of indices will be used in this proof:

$$1 \leqslant A, B \leqslant n+2; \qquad 1 \leqslant i, j \leqslant 2; \qquad 3 \leqslant \alpha, \beta \leqslant n+2.$$

Part 1 is a rephrasing of part (c) of Proposition 5, and it is also proven in [13].

Part 2(a) is a consequence of the basic properties of pull-back: Eqs. (19) and conditions (22) imply that the metric

$$ds^2 = \iota^*\big[\Theta_1^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big]^2 + \iota^*\big[\Theta_2^{[n]}\big(u^{[n]}(x,\tau_0,\ldots,\tau_n)\big)\big]^2$$

on D has Gaussian curvature $K = -1$, and that $\iota^*[\Theta_3^{[n]}(u^{[n]}(x, \tau_0, \ldots, \tau_n))]$ is the corresponding Levi–Civita connection one-form.

Part 2(b) is proven thus: Set $\sigma_A = 1$, except for s indices between 3 and $n+2$ for which $\sigma_A = -1$. The structure equations of a two-dimensional manifold isometrically immersed in the flat pseudo-Riemannian space $V^{[n]}$ are [12,31]

$$d\omega^1 = \omega^2 \wedge \omega_{21}, \qquad d\omega^2 = \omega^1 \wedge \omega_{12}, \tag{23}$$

together with the Gauss equation

$$d\omega_{12} = \sum_{\alpha=3}^{n+2} \sigma_\alpha \omega_{1\alpha} \wedge \omega_{\alpha 2}, \tag{24}$$

the Codazzi equations

$$d\omega_{i\alpha} = \sum_{j=1}^{2} \omega_{ij} \wedge \omega_{j\alpha} + \sum_{\beta=3}^{n+2} \sigma_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{25}$$

and the Ricci equations

$$d\omega_{\alpha\beta} = \sum_{\gamma=3}^{n+2} \sigma_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \tag{26}$$

in which the tensor

$$\Omega_{\alpha\beta} = \omega_{\alpha 1} \wedge \omega_{1\beta} + \omega_{\alpha 2} \wedge \omega_{2\beta} \tag{27}$$

is the normal curvature of the immersed surface. Moreover, the one-forms ω_{AB} satisfy the conditions

$$\omega^1 \wedge \omega_{1\alpha} + \omega^2 \wedge \omega_{2\alpha} = 0 \quad \text{and} \quad \omega_{AB} + \omega_{BA} = 0, \tag{28}$$

and the given immersed surface has constant Gaussian curvature K if and only if

$$\sum_{\alpha=3}^{n+2} \sigma_\alpha \omega_{1\alpha} \wedge \omega_{\alpha 2} = -K\omega^1 \wedge \omega^2. \tag{29}$$

Conversely, assume that there exist two linearly independent one-forms ω^i and $(n+2)^2$ one-forms ω_{AB} defined on an open subset D of $\mathbf{R}^2$ satisfying Eqs. (23)–(28), and equip D with the metric $\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$. The fundamental theorem for submanifolds of pseudo-Riemannian manifolds (see, for example, [12,30,31] and references therein) implies that for any $u_0 \in D$, $p_0 \in V^{[n]}$ and any orthonormal basis $\{e_1^0, \ldots, e_{n+2}^0\}$ of the tangent space of $V^{[n]}$ at p_0, the equations

$$dP = \omega^1 e_1 + \omega^2 e_2 \quad \text{and} \quad de_A = \sum \sigma_B \omega_{AB} e_B$$

determine a unique isometric immersion $P : D \to V^{[n]}$ and a unique local orthonormal moving frame $\{e_1, \dots, e_{n+2}\}$ near p_0 with $P(u_0) = p_0$ and $e_A(p_0) = e^0_A$.

Now, Definition 10 implies that Eqs. (23) are satisfied if one sets

$$\omega^1 = \iota^*\big[\Theta_1^{[n]}\big(u^{[n]}(x, \tau_0, \dots, \tau_n)\big)\big], \qquad \omega^2 = \iota^*\big[\Theta_2^{[n]}\big(u^{[n]}(x, \tau_0, \dots, \tau_n)\big)\big], \tag{30}$$

$$\omega_{12} = \iota^*\big[\Theta_3^{[n]}\big(u^{[n]}(x, \tau_0, \dots, \tau_n)\big)\big]. \tag{31}$$

Moreover, since the first equation of (28) implies that

$$\omega_{i\alpha} = h^\alpha_{i1}\omega^1 + h^\alpha_{i2}\omega^2 \quad \text{and} \quad h^\alpha_{ij} = h^\alpha_{ji}, \tag{32}$$

one can take

$$\omega_{1\alpha} = \frac{1}{\sqrt{n}}\omega^1 \quad \text{and} \quad \omega_{2\alpha} = \frac{1}{\sqrt{n}}\omega^2, \tag{33}$$

define one-forms $\omega_{\alpha\beta}$ simply by

$$\omega_{\alpha\beta} = 0, \tag{34}$$

and set $\omega_{BA} = -\omega_{AB}$. Equation (34) implies that the Ricci equations (26) become $\Omega_{\alpha\beta} = 0$, and these equations are identically satisfied because of (33). Also, Eqs. (33) and (34) imply that the Codazzi equations (25) read

$$d\omega^1 = \omega_{12} \wedge \omega^2, \qquad d\omega^2 = \omega_{21} \wedge \omega^1,$$

which are precisely Eqs. (23). Lastly, the Gauss equation (24) becomes

$$d\omega_{12} = \sum_{\alpha=3}^{n+2} \sigma_\alpha \frac{1}{n}\omega^1 \wedge \big(-\omega^2\big) = -\frac{1}{n}\left(\sum_{\alpha=3}^{n+2} \sigma_\alpha\right)\omega^1 \wedge \omega^2, \tag{35}$$

and this equation holds if one takes $\sigma_\alpha = -1$ for all α. Thus, Eqs. (23)–(28) are satisfied, and therefore an isometric immersion $P : D \to V^{[n]}$ exists, if the index of the pseudo-Riemannian manifold $V^{[n]}$ is $s = n$. Equations (29) and (35) say that the Gaussian curvature of the immersed manifold $P(D)$ is $K = -1$, and finally, since the normal curvature tensor $\Omega_{\alpha\beta}$ is identically zero, the normal bundle of the immersion $P : D \to V^{[n]}$ is flat [31]. □

One can now show that each equation $u_{\tau_i} = F_i$ which belongs to a hierarchy of pseudo-spherical type describes pseudo-spherical surfaces. This natural result is a consequence of the following elementary lemma.

Lemma 13. *Let $u_{\tau_i} = F_i$ be a hierarchy of pseudo-spherical type with associated one-forms*

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt + \sum_{k=1}^{n} h_{\alpha k}\, d\tau_k, \quad n \geqslant 0. \tag{36}$$

Then, the following four sets of equations hold:

$$\begin{cases} -D_t f_{11} + D_x f_{12} = f_{31} f_{22} - f_{32} f_{21}, \\ -D_t f_{21} + D_x f_{22} = f_{11} f_{32} - f_{12} f_{31}, \\ -D_t f_{31} + D_x f_{32} = f_{11} f_{22} - f_{12} f_{21}; \end{cases} \tag{37}$$

$$\begin{cases} -D_{\tau_i} f_{11} + D_x h_{1i} = f_{31} h_{2i} - h_{3i} f_{21}, \\ -D_{\tau_i} f_{21} + D_x h_{2i} = f_{11} h_{3i} - h_{1i} f_{31}, \\ -D_{\tau_i} f_{31} + D_x h_{3i} = f_{11} h_{2i} - h_{1i} f_{21}; \end{cases} \tag{38}$$

$$\begin{cases} -D_{\tau_i} f_{12} + D_t h_{1i} = f_{32} h_{2i} - h_{3i} f_{22}, \\ -D_{\tau_i} f_{22} + D_t h_{2i} = f_{12} h_{3i} - h_{1i} f_{32}, \\ -D_{\tau_i} f_{32} + D_t h_{3i} = f_{12} h_{2i} - h_{1i} f_{22}; \end{cases} \tag{39}$$

$$\begin{cases} -D_{\tau_j} h_{1i} + D_{\tau_i} h_{1j} = h_{3i} h_{2j} - h_{3j} h_{2i}, \\ -D_{\tau_j} h_{2i} + D_{\tau_i} h_{2j} = h_{1i} h_{3j} - h_{1j} h_{3i}, \\ -D_{\tau_j} h_{3i} + D_{\tau_i} h_{3j} = h_{1i} h_{2j} - h_{1j} h_{2i}, \end{cases} \tag{40}$$

in which $i, j \geqslant 1$. Conversely, if $u_{\tau_i} = F_i$ is a sequence of evolution equations, the operators D_x and D_{τ_i} are defined as in (17)*, and there exist differential functions $f_{\alpha\beta}$, $h_{\alpha j}$ such that* (37)–(40) *are satisfied, then $u_{\tau_i} = F_i$ is a hierarchy of pseudo-spherical type with associated one-forms* (36).

Proposition 14. *Let $u_{\tau_i} = F_i$ be a hierarchy of pseudo-spherical type with associated one-forms*

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt + \sum_{i=1}^{n} h_{\alpha i}\, d\tau_i, \quad n \geqslant 0.$$

The equations $u_{\tau_i} = F_i$ describe pseudo-spherical surfaces with associated one-forms

$$\omega_0^\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt \quad (\text{if } i = 0) \quad \text{and} \quad \omega_i^\alpha = f_{\alpha 1}\, dx + h_{\alpha i}\, d\tau_i \quad (\text{if } i \geqslant 1), \tag{41}$$

and moreover, the one-forms $\sigma_j^\alpha = f_{\alpha 2}\, dt + h_{\alpha j}\, d\tau_j$ (j fixed) and $\sigma_{ij}^\alpha = h_{\alpha i}\, d\tau_i + h_{\alpha j}\, d\tau_j$ (i, j fixed, $i \neq j$) satisfy the structure equations (2) *of a pseudo-spherical surface.*

3.2. Hierarchies of PSS equations and integrability

In the remaining of this section it will be proven that Definition 10 encodes the properties of standard hierarchies of integrable equations recalled in Section 1. First of all, the fact that the higher equations of a hierarchy of pseudo-spherical type are the integrability condition of linear problems is straightforward.

Proposition 15. *Let $u_{\tau_i} = F_i$ be a hierarchy of pseudo-spherical type. There exist $sl(2, \mathbf{R})$-valued functions X and T_i such that for each $i \geqslant 0$, $u_{\tau_i} = F_i$ is the integrability condition of the linear problem*

$$v_x = X v, \qquad v_{\tau_i} = T_i v.$$

Proof. This result follows from Proposition 14 and Eqs. (8)–(10). □

Remark 16. Remark 7 on single PSS equations and $sl(2,\mathbf{R})$-valued connection forms generalizes to hierarchies: If $u_{\tau_i} = F_i$ is a hierarchy of pseudo-spherical type, it follows from (8), (9) and (19) that for each $n \geqslant 0$, the trivial bundle $V^{[n]} \times SL(2,\mathbf{R})$ can be equipped—whenever $\{u^{[n]}\colon\ n \geqslant 0\}$ is a solution to the hierarchy $u_{\tau_i} = F_i$—with a flat $sl(2,\mathbf{R})$-valued connection. Moreover, the foregoing discussion implies that for each $i \geqslant 0$ the pull-back of this connection to the submanifold of coordinates (x, τ_i) is a flat $sl(2,\mathbf{R})$-valued connection form associated with the equation $u_{\tau_i} = F_i$. This connection coincides with the one determined by the one-forms (41) associated to $u_{\tau_i} = F_i$.

Next, one shows that hierarchies of pseudo-spherical type generate hierarchies of pairwise commuting flows. In order to avoid some technicalities—mentioned in Remark 23 below—this property will be proven in the special case of *hierarchies of strictly pseudo-spherical type*, which are now defined in analogy with the single equation case reviewed in Section 2.

Consider a countable number of evolution equations $u_{\tau_i} = F_i$, and assume that the ith equation is of order k_i. For each $n \geqslant 0$, let $J^{(n)}$ be a manifold with coordinates $(x, t, \tau_1, \ldots, \tau_n, u, u_x, \ldots, u_{x^{M(n)}})$, where $M(n)$ is the maximum of the orders k_j, $0 \leqslant j \leqslant n$, and let $I^{(n)}$ be the differential ideal generated by the two-forms

$$du \wedge dx + F_0\, dx \wedge dt + \sum_{i=1}^{n} F_i\, dx \wedge d\tau_i; \tag{42}$$

$$du_{x^k} \wedge dt - u_{x^{k+1}}\, dx \wedge dt - \sum_{j=1}^{n} D_x^k F_j\, d\tau_j \wedge dt, \quad 0 \leqslant k \leqslant k_0 - 1; \tag{43}$$

$$du_{x^k} \wedge d\tau_j - u_{x^{k+1}}\, dx \wedge d\tau_j - D_x^k F_0\, dt \wedge d\tau_j - \sum_{l=1}^{n} D_x^k F_l\, d\tau_l \wedge d\tau_j, \tag{44}$$

in which for each $j = 1, \ldots, n$, the index k takes the values $k = 0, \ldots, k_j - 1$.

Definition 17. A countable collection of evolution equations $u_{\tau_i} = F_i$ is a hierarchy of strictly pseudo-spherical type if there exist differential functions $f_{\alpha\beta}$ and $h_{\alpha j}$ such that for every $n \geqslant 0$, the two-forms $\Omega_1^{[n]} = d\Theta_1^{[n]} - \Theta_3^{[n]} \wedge \Theta_2^{[n]}$, $\Omega_2^{[n]} = d\Theta_2^{[n]} - \Theta_1^{[n]} \wedge \Theta_3^{[n]}$, and $\Omega_3^{[n]} = d\Theta_3^{[n]} - \Theta_1^{[n]} \wedge \Theta_2^{[n]}$, in which

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt + \sum_{j=1}^{n} h_{\alpha j}\, d\tau_j,$$

generate the ideal $I^{(n)}$.

Instead of Lemma 9 on single strictly pseudo-spherical equations, one now proves the following:

Proposition 18. *Consider a sequence of equations $u_{\tau_i} = F_i$ of order k_i and for each $n \geqslant 0$ let $M(n)$ be the maximum of the orders k_j, $0 \leqslant j \leqslant n$. Necessary and sufficient conditions for* *the equations $u_{\tau_i} = F_i$ to define a hierarchy of strictly pseudo-spherical type with associated functions $f_{\alpha\beta}$ and $h_{\alpha j}$ are the conjunction of*:

(1) *The functions $f_{\alpha 1}$ satisfy the nontriviality inequality $f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0$;*
(2) *For each $n \geqslant 0$, the functions $f_{\alpha\beta}$ and $h_{\alpha j}$ satisfy*

$$f_{\alpha 1, u_{x^k}} = 0, \quad 1 \leqslant k \leqslant M(n); \qquad f_{\alpha 2, u_{x^k}} = 0, \quad k_0 \leqslant k \leqslant M(n);$$
$$h_{\alpha i, u_{x^k}} = 0, \quad k_i \leqslant k \leqslant M(n);$$

(3) *The functions F_i, $f_{\alpha\beta}$, and $h_{\alpha j}$ satisfy, for all $i, j \geqslant 0$,*

$$\begin{cases} -(f_{11,t} + f_{11,u} F_0) + D_x f_{12} = f_{31} f_{22} - f_{32} f_{21}, \\ -(f_{21,t} + f_{21,u} F_0) + D_x f_{22} = f_{11} f_{32} - f_{12} f_{31}, \\ -(f_{31,t} + f_{31,u} F_0) + D_x f_{32} = f_{11} f_{22} - f_{12} f_{21}; \end{cases} \tag{45}$$

$$\begin{cases} -(f_{11,\tau_i} + f_{11,u} F_i) + D_x h_{1i} = f_{31} h_{2i} - h_{3i} f_{21}, \\ -(f_{21,\tau_i} + f_{21,u} F_i) + D_x h_{2i} = f_{11} h_{3i} - h_{1i} f_{31}, \\ -(f_{31,\tau_i} + f_{31,u} F_i) + D_x h_{3i} = f_{11} h_{2i} - h_{1i} f_{21}; \end{cases} \tag{46}$$

$$\begin{cases} -D_{\tau_i} f_{12} + D_t h_{1i} = f_{32} h_{2i} - h_{3i} f_{22}, \\ -D_{\tau_i} f_{22} + D_t h_{2i} = f_{12} h_{3i} - h_{1i} f_{32}, \\ -D_{\tau_i} f_{32} + D_t h_{3i} = f_{12} h_{2i} - h_{1i} f_{22}; \end{cases} \tag{47}$$

$$\begin{cases} -D_{\tau_j} h_{1i} + D_{\tau_i} h_{1j} = h_{3i} h_{2j} - h_{3j} h_{2i}, \\ -D_{\tau_j} h_{2i} + D_{\tau_i} h_{2j} = h_{1i} h_{3j} - h_{1j} h_{3i}, \\ -D_{\tau_j} h_{3i} + D_{\tau_i} h_{3j} = h_{1i} h_{2j} - h_{1j} h_{2i}. \end{cases} \tag{48}$$

Proof. First, one expands the two-form $\Omega_1^{[n]}$ and considers the structure equation $\Omega_1^{[n]} = 0$. Definition 17 implies that

$$du \wedge dx + F_0 \, dx \wedge dt + \sum_{i=1}^{n} F_i \, dx \wedge d\tau_i = 0; \tag{49}$$

$$du_{x^k} \wedge dt - u_{x^{k+1}} \, dx \wedge dt - \sum_{j=1}^{n} D_x^k F_j \, d\tau_j \wedge dt = 0, \quad 0 \leqslant k \leqslant k_0 - 1, \tag{50}$$

and also that

$$du_{x^k} \wedge d\tau_j - u_{x^{k+1}} \, dx \wedge d\tau_j - D_x^k F_0 \, dt \wedge d\tau_j - \sum_{l=1}^{n} D_x^k F_l \, d\tau_l \wedge d\tau_j = 0, \tag{51}$$

in which for each $j = 1, \ldots, n$, $0 \leqslant k \leqslant k_j - 1$. Substituting (49)–(51) into $\Omega_1^{[n]} = 0$, and collecting terms, one finds the identities

$$-f_{11,t} - f_{11,u} F_0 + D_x f_{12} - f_{31} f_{22} + f_{32} f_{21} = 0, \tag{52}$$

$$-f_{11,\tau_i} - f_{11,u} F_i + D_x h_{1i} - f_{31} h_{2i} + h_{3i} f_{21} = 0, \tag{53}$$

$$-D_{\tau_i} f_{12} + D_t h_{1i} - f_{32} h_{2i} + h_{3i} f_{22} = 0, \tag{54}$$

$$-D_{\tau_j} h_{1i} + D_{\tau_i} h_{1j} - h_{3i} h_{2j} + h_{3j} h_{2i} = 0, \tag{55}$$

and the constraints $f_{11,u_{x^k}} = 0$ if $1 \leqslant k \leqslant M(n)$; $f_{12,u_{x^k}} = 0$ if $k_0 \leqslant k \leqslant M(n)$; and $h_{1i,u_{x^k}} = 0$ if $k_i \leqslant k \leqslant M(n)$.

The other equations appearing in (37)–(40) are obtained from the equations $\Omega_2^{[n]} = \Omega_3^{[n]} = 0$. Finally, the inequality $f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0$ holds, for otherwise the equations $u_{\tau_i} = F_i$ would not be necessary and sufficient for the structure equations $\Omega_\alpha^{[n]} = 0$ to be satisfied. □

Corollary 19. *Let $u_{\tau_i} = F_i$ be a hierarchy of strictly pseudo-spherical type with associated functions $f_{\alpha\beta}$ and $h_{\alpha j}$. Then, the equation $u_t = F_0$ is strictly pseudo-spherical with associated one-forms $\omega_0^\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ and, for each value of j, the equation $u_{\tau_j} = F_j$ is strictly pseudo-spherical with associated one forms $\omega_j^\alpha = f_{\alpha 1}\,dx + h_{\alpha j}\,d\tau_j$.*

In order to prove that a hierarchy $u_{\tau_i} = F_i$ of strictly pseudo-spherical type generates a family of pairwise commuting flows, one needs to show [8,15,18] that the function F_p is a *generalized symmetry* of the equation $u_{\tau_q} = F_q$ for all $p, q \geqslant 0$.

Definition 20. A smooth function G depending on x, t, u and a finite number of derivatives of u is a generalized symmetry of an evolution equation $u_t = F(x, t, u, \ldots, u_{x^n})$ if for any local solution $u(x,t)$, the function $u(x,t) + \tau G(u(x,t))$ satisfies $u_t = F$ to first order in τ.

For any smooth function $f(x, t, u, \ldots, u_{x^k})$ define the operator f_* (the *formal linearization* of f) by means of [18]

$$f_* = \sum_{i=0}^{k} \frac{\partial f}{\partial u_{x^i}} D_x^i, \tag{56}$$

and consider the operator D_t defined as in (17). Then, a function G as in Definition 20 is a generalized symmetry of $u_t = F$ if and only if the equation

$$D_t G = F_* G$$

is identically satisfied. In other words, G is a generalized symmetry if the equation $D_t G = F_* G$, in which D_t is the usual total derivative with respect to t [18], holds identically once all derivatives with respect to t appearing in it are replaced by means of $u_t = F$. Equivalently [18] an evolution equation $u_\tau = G$ is a generalized symmetry of $u_t = F$ if the equation

$$\frac{\partial G}{\partial t} + F_* G - G_* F = 0 \tag{57}$$

holds whenever $u(x,t)$ is a solution of $u_t = F$. If F and G satisfy

$$\frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0, \tag{58}$$

Eq. (57) means that, at least formally, the flows of $u_t = F$ and $u_\tau = G$ commute.

The following characterization of generalized symmetries of strictly pseudo-spherical evolution equations holds [22].

Lemma 21. *Let $u_t = F(x, t, u, \dots, u_{x^m})$ be a strictly pseudo-spherical evolution equation with associated one-forms $\omega^\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$. Let G be a smooth function depending on x, t, u and a finite number of derivatives of u, and let $u(x, t)$ be a local solution of $u_t = F$. Consider the deformed one-forms*

$$\omega^\alpha\big(u(x,t)\big) + \tau \Lambda_\alpha\big(u(x,t)\big),$$

in which $\Lambda_\alpha(u(x,t))$ is given by

$$\Lambda_\alpha\big(u(x,t)\big) = f_{\alpha 1,u}\big(u(x,t)\big)G\big(u(x,t)\big)\,dx + \sum_{i=0}^{m-1} f_{\alpha 2,u_{x^i}}\big(u(x,t)\big)\frac{\partial^i G(u(x,t))}{\partial x^i}\,dt. \tag{59}$$

These forms satisfy the structure equations of a pseudo-spherical surface up to terms of order τ^2 if and only if the function G is a generalized symmetry of the equation $u_t = F$.

Motivated by condition (58), it will be assumed hereafter in this section that neither the functions F_j, nor the associated functions $f_{\alpha\beta}$ and $h_{\alpha j}$, depend explicitly on the independent "time" variables $\tau_0, \tau_1, \tau_2, \dots$.

Theorem 22. *Assume that $u_{\tau_i} = F_i$ is a hierarchy of strictly pseudo-spherical type with associated one-forms*

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt + \sum_{k=1}^{n} h_{\alpha k}\,d\tau_k, \quad n \geqslant 0. \tag{60}$$

The function F_j is a generalized symmetry of the equation $u_{\tau_i} = F_i$ for all $i, j \geqslant 0$.

Proof. Corollary 19 says that the equations $u_t = F$ and $u_{\tau_j} = F_j$ are strictly pseudo-spherical with associated one forms $\omega_0^\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ and $\omega_j^\alpha = f_{\alpha 1}\,dx + h_{\alpha j}\,d\tau_j$, respectively, and one can apply the last lemma.

One checks first that the equations $u_{\tau_i} = F_i$ determine generalized symmetries of the seed equation $u_t = F$. Consider the deformations of the one-forms ω_0^α induced by $u \mapsto u + \tau_i F_i$, and set $\Lambda_0^\alpha = f_{\alpha 1,\tau_i}\,dx + f_{\alpha 2,\tau_i}\,dt$, in which $f_{\alpha\beta,\tau_i} = D_{\tau_i} f_{\alpha\beta}$. These one-forms are of the type (59) if pulled-back by solutions $u(x,t)$ of $u_t = F$ because of Proposition 18. A straightforward computation shows that the deformed one-forms $\omega_0^\alpha + \tau_i \Lambda_0^\alpha$ describe pseudo-spherical surfaces to first order in τ_i if and only if the equations

$$d\Lambda_1 = \omega^3 \wedge \Lambda_2 + \Lambda_3 \wedge \omega^2, \tag{61}$$

$$d\Lambda_2 = \omega^1 \wedge \Lambda_3 + \Lambda_1 \wedge \omega^3, \quad \text{and} \tag{62}$$

$$d\Lambda_3 = \omega^1 \wedge \Lambda_2 + \Lambda_1 \wedge \omega^2 \tag{63}$$

hold whenever $u(x,t)$ is a solution of the equation $u_t = F$, in which the exterior derivative is taken in (x,t) space. Now, Eq. (61), for instance, holds if and only if

$$-f_{11,\tau_i t} + f_{12,\tau_i x} = f_{31} f_{22,\tau_i} - f_{32} f_{21,\tau_i} + f_{31,\tau_i} f_{22} - f_{21} f_{32,\tau_i}. \tag{64}$$

But, because of Eqs. (46) and (47), Eq. (64) is equivalent to

$$-h_{2i}(-f_{31,t} + f_{32,x}) + h_{3i}(-f_{21,t} + f_{22,x}) = -h_{2i}(f_{22} f_{11} - f_{21} f_{12}) + h_{3i}(f_{32} f_{11} - f_{31} f_{12}),$$

and this equation does hold whenever $u(x,t)$ is a solution of $u_t = F$, since $u_t = F$ describes pseudo-spherical surfaces with associated one-forms ω_0^α. Equations (62) and (63) are treated in the same way.

Now one can check, in an analogous fashion, that the equations $u_{\tau_j} = F_j$ determine generalized symmetries of the equations $u_{\tau_i} = F_i$, $i \neq j$. Deform the associated one-forms $\omega_i^\alpha = f_{\alpha 1}\,dx + h_{\alpha i}\,d\tau_i$ by means of $u \mapsto u + \tau_j F_j$. As before, the "infinitesimal deformations" $\Lambda_i^\alpha = f_{\alpha 1,\tau_j}\,dx + h_{\alpha i,\tau_j}\,d\tau_i$ are of type (59) if pulled-back by solutions $u(x,\tau_i)$ of $u_{\tau_i} = F_i$. One then needs to prove that the one-forms $\omega_i^\alpha + \tau_i \Lambda_i^\alpha$ describe pseudo-spherical surfaces to first order in τ_j, or, in other words, that the equations

$$d\Lambda_i^1 = \omega_i^3 \wedge \Lambda_i^2 + \Lambda_i^3 \wedge \omega_i^2, \tag{65}$$

$$d\Lambda_i^2 = \omega_i^1 \wedge \Lambda_i^3 + \Lambda_i^1 \wedge \omega_i^3, \tag{66}$$

$$d\Lambda_i^3 = \omega_i^1 \wedge \Lambda_i^2 + \Lambda_i^1 \wedge \omega_i^2 \tag{67}$$

are satisfied whenever $u(x,\tau_i)$ is a solution of the equation $u_{\tau_i} = F_i$, in which the exterior derivative is taken in (x,τ_i) space. The proof goes as before, using this time Eqs. (47) and (48). □

Remark 23. It is possible to generalize Theorem 22 to arbitrary hierarchies of pseudo-spherical type $u_{\tau_i} = F_i$, but extra technical difficulties appear: one needs to consider generalized symmetries of arbitrary PSS equations, and state an analog of Lemma 21 for them. Such a result can be proven by taking into account the gauge invariance of the theory of equations describing pseudo-spherical surfaces, as in [24].

4. Correspondence results

4.1. Introduction

Kamran and Tenenblat in their seminal paper [13]—and then the present author in [24]—observed that the following two facts: (i) each suitably generic solution $u(x,t)$ of a PSS equation determines a (pseudo-)Riemannian metric of constant Gaussian curvature -1 on (open subsets of) the domain of $u(x,t)$ and (ii) (pseudo-)Riemannian surfaces of constant Gaussian curvature are locally isometric, allow one to establish the existence of local transformations between (suitably generic) solutions of PSS equations. Kamran and Tenenblat's result was proven by considering the Riemannian interpretation of the structure equations (2) (item (c) of Proposition 5) and, accordingly, Eq. (5) of Proposition 6. The pseudo-Riemannian point of view was first studied in [24].

These correspondences are very different from classical Bäcklund transformations [31]: they require explicit changes of independent variables; they appear to be unrelated to symmetry considerations (as it has been observed to happen in the Bäcklund case [3,7,31]); and they are not restricted to transforming solutions of a same equation, or even equations of the same order. For instance, the following three results are proven in [24]:

Theorem 24. *Let $\Xi(x,t,u,\ldots)=0$ and $\widehat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots)=0$ be two PSS equations with associated one-forms $\omega^\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ and $\widehat{\omega}^\alpha = \hat{f}_{\alpha 1}\,d\hat{x} + \hat{f}_{\alpha 2}\,d\hat{t}$, respectively, and assume that $\omega^1 \wedge \omega^3 \not\equiv 0$, and $\widehat{\omega}^1 \wedge \widehat{\omega}^3 \not\equiv 0$. Then, for any II-generic solutions $u(x,t)$ of $\Xi = 0$ and $\hat{u}(\hat{x},\hat{t})$ of $\widehat{\Xi}=0$, there exist a local diffeomorphism $\Upsilon : V \to \widehat{V}$ in which V and $\widehat{V}$ are open subsets of the domains of $u(x,t)$ and $\hat{u}(\hat{x},\hat{t})$, respectively, and a smooth function $\nu : V \to \mathbf{R}$, such that the one-forms $\omega^\alpha(u(x,t))$ and $\widehat{\omega}^\alpha(\hat{u}(\hat{x},\hat{t}))$ satisfy the equations*

$$
\begin{aligned}
\Upsilon^*\widehat{\omega}^1 &= \omega^1\cosh\nu - \omega^3\sinh\nu,\\
\Upsilon^*\widehat{\omega}^2 &= \omega^2 + d\nu,\\
\Upsilon^*\widehat{\omega}^3 &= -\omega^1\sinh\nu + \omega^3\cosh\nu.
\end{aligned}
\tag{68}
$$

That the maps Υ and ν exist is a way of stating that (pseudo-)Riemannian surfaces of constant Gaussian curvature are locally isometric. Now write $\Upsilon(x,t) = (\gamma(x,t), \delta(x,t))$. The functions γ, δ and ν depend on *both* solutions $u(x,t)$ and $\hat{u}(\hat{x},\hat{t})$, of course. However, a closer analysis of (68) allows one to find a system of equations for γ, δ and ν, for which local existence of solutions can be proven without previous knowledge of $\hat{u}(\hat{x},\hat{t})$.

Lemma 25. *Let $\Xi(x,t,u,\ldots)=0$ and $\widehat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots)=0$ be two equations describing pseudo-spherical surfaces with associated one-forms $\omega^\alpha = f_{\alpha 1}\,dx + f_{\alpha 2}\,dt$ and $\widehat{\omega}^\alpha = \hat{f}_{\alpha 1}\,d\hat{x} + \hat{f}_{\alpha 2}\,d\hat{t}$, respectively. Suppose that $\hat{f}_{21} = \hat{u}$, and also that $\omega^2\wedge\omega^3 \not\equiv 0$, and $\widehat{\omega}^2 \wedge \widehat{\omega}^3 \not\equiv 0$. Let $\Upsilon(x,t) = (\gamma(x,t),\delta(x,t))$ be a smooth map from (an open subset of) the space of independent variables x, t to (an open subset of) the space of independent variables $\hat{x},\hat{t}$. Set $J = \gamma_x\delta_t - \gamma_t\delta_x$, and let $\nu(x,t)$ be a smooth real-valued function. The system of equations*

$$\big(\Upsilon^*\hat{f}_{11}\big)\gamma_x + \big(\Upsilon^*\hat{f}_{12}\big)\delta_x = f_{11}\cosh\nu - f_{31}\sinh\nu, \tag{69}$$

$$\big(\Upsilon^*\hat{f}_{11}\big)\gamma_t + \big(\Upsilon^*\hat{f}_{12}\big)\delta_t = f_{12}\cosh\nu - f_{32}\sinh\nu, \tag{70}$$

$$J\big(\Upsilon^*\hat{f}_{22}\big) = -\big[\gamma_t(f_{21}+\nu_x) - \gamma_x(f_{22}+\nu_t)\big], \tag{71}$$

$$\big(\Upsilon^*\hat{f}_{31}\big)\gamma_x + \big(\Upsilon^*\hat{f}_{32}\big)\delta_x = -f_{11}\sinh\nu + f_{31}\cosh\nu, \tag{72}$$

$$\big(\Upsilon^*\hat{f}_{31}\big)\gamma_t + \big(\Upsilon^*\hat{f}_{32}\big)\delta_t = -f_{12}\sinh\nu + f_{32}\cosh\nu, \tag{73}$$

in which the pull-backs of $\hat{u}$ and its derivatives appearing in the functions $(\Upsilon^*\hat{f}_{\alpha\beta})(x,t)$ have been evaluated by means of the equation

$$\hat{u}\circ\Upsilon = \frac{1}{J}\big(\delta_t(f_{21}+\nu_x) - \delta_x(f_{22}+\nu_t)\big), \tag{74}$$

admits—whenever $u(x,t)$ is a II-generic solution of $\Xi = 0$—a local solution $\gamma(x,t)$, $\delta(x,t)$, $\nu(x,t)$ defined on (an open subset of) the domain of $u(x,t)$, such that $\Upsilon(x,t) = (\gamma(x,t), \delta(x,t))$ is a local diffeomorphism.

The condition $\hat{f}_{21} = \hat{u}$ appearing in this lemma (and also in Theorem 26 below) can be removed, see [24]. Lemma 25 implies that there exist transformations carrying a II-generic solution to $\Xi(x,t,u,\ldots) = 0$ into a II-generic solution to $\widehat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots) = 0$.

Theorem 26. *Let $\Xi(x,t,u,\ldots) = 0$ and $\widehat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots) = 0$ be two equations describing pseudo-spherical surfaces with associated one-forms $\omega^\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$ and $\widehat{\omega}^\alpha = \hat{f}_{\alpha 1}\, d\hat{x} + \hat{f}_{\alpha 2}\, d\hat{t}$, respectively. Assume that $\hat{f}_{21} = \hat{u}$ and that $\omega^2 \wedge \omega^3 \not\equiv 0$ and $\widehat{\omega}^2 \wedge \widehat{\omega}^3 \not\equiv 0$. For any II-generic solution $u(x,t)$ to $\Xi = 0$, take a real-valued function $\nu(x,t)$ and a local diffeomorphism $\Upsilon(x,t) = (\gamma(x,t), \delta(x,t))$ which solve the system of equations appearing in Lemma 25. Then, the function $\hat{u}(\hat{x},\hat{t})$ defined by means of*

$$\hat{u} \circ \Upsilon = \frac{1}{J}\big(\delta_t(f_{21} + \nu_x) - \delta_x(f_{22} + \nu_t)\big), \tag{75}$$

in which J is the Jacobian of Υ, is a II-generic solution to $\widehat{\Xi} = 0$.

One certainly expects correspondence theorems for hierarchies of pseudo-spherical type. As explained above, the results in the single equation case are based solely on the local geometry of (pseudo-)Riemannian surfaces of Gaussian curvature equal to -1—as encoded in the structure equations (2)—and analogous equations appear when one is dealing with hierarchies!

4.2. Correspondence theorems for hierarchies

If $u_{\tau_i} = F_i$ is a hierarchy of pseudo-spherical type with associated one-forms $\Theta_\alpha^{[n]}$, the pull–backs of the differential forms $\Theta_\alpha^{[n]}$ by solutions $u^{[n]}$ will be denoted again by $\Theta_\alpha^{[n]}$, no confusion should arise.

Theorem 27. *Let $u_{\tau_i} = F_i(x,t,u,\ldots)$ and $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i(\hat{x},\hat{t},\hat{u},\ldots)$ be two hierarchies of pseudo-spherical type with associated one-forms given by*

$$\Theta_\alpha^{[n]} = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt + \sum_{i=1}^{n} h_{\alpha i}\, d\tau_i \quad \text{and} \quad \widehat{\Theta}_\alpha^{[n]} = \hat{f}_{\alpha 1}\, d\hat{x} + \hat{f}_{\alpha 2}\, d\hat{t} + \sum_{i=1}^{n} \hat{h}_{\alpha i}\, d\hat{\tau}_i. \tag{76}$$

Let $\{u^{[n]}\}$ and $\{\hat{u}^{[n]}\}$ be solutions of $u_{\tau_i} = F_i$ and $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$, respectively, and assume that $u^{[0]}(x,t)$ and $\hat{u}^{[0]}(\hat{x},\hat{t})$ are III-generic. For each $n \geqslant 0$ there exist a local diffeomorphism $\Upsilon^{[n]} : V^{[n]} \to \widehat{V}^{[n]}$—in which $V^{[n]}$ and $\widehat{V}^{[n]}$ are open subsets of the domains of $u^{[n]}$ and $\hat{u}^{[n]}$—and a smooth function $\mu^{[n]} : V^{[n]} \to \mathbf{R}$ such that

$$\Upsilon^{[n]*}\widehat{\Theta}_1^{[n]} = \Theta_1^{[n]} \cos\mu^{[n]} + \Theta_2^{[n]} \sin\mu^{[n]}, \tag{77}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_2^{[n]} = -\Theta_1^{[n]} \sin\mu^{[n]} + \Theta_2^{[n]} \cos\mu^{[n]}, \tag{78}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_3^{[n]} = \Theta_3^{[n]} + d\mu^{[n]}. \tag{79}$$

Moreover, the maps $\Upsilon^{[n]}$ and $\mu^{[n]}$ can be chosen so that for each $n \geqslant 0$,

$$\Upsilon^{[n+1]}\big|_{V^{[n]}} = \Upsilon^{[n]} \quad \text{and} \quad \mu^{[n+1]}\big|_{V^{[n]}} = \mu^{[n]}. \tag{80}$$

Proof. For each $n \geqslant 0$ consider the one-forms $\sigma_\alpha^{[n]}$ given by

$$\sigma_1^{[n]} = \frac{1}{\hat{t}}\, d\hat{x} + \sum_{i=1}^{n} \frac{1}{\hat{t}}\, d\hat{\tau}_i, \qquad \sigma_2^{[n]} = \frac{1}{\hat{t}}\, d\hat{t}, \qquad \sigma_3^{[n]} = \sigma_1^{[n]}. \tag{81}$$

Note that $\sigma_1^{[0]} \otimes \sigma_1^{[0]} + \sigma_2^{[0]} \otimes \sigma_2^{[0]}$ is exactly the standard hyperbolic metric on the Poincaré upper-half plane, and that $\sigma_3^{[0]}$ is the corresponding connection one-form. One easily checks that for any $n \geqslant 0$ the one-forms $\sigma_\alpha^{[n]}$ satisfy the structure equations

$$d\sigma_1^{[n]} = \sigma_3^{[n]} \wedge \sigma_2^{[n]}, \qquad d\sigma_2^{[n]} = \sigma_1^{[n]} \wedge \sigma_3^{[n]}, \qquad d\sigma_3^{[n]} = \sigma_1^{[n]} \wedge \sigma_2^{[n]} \tag{82}$$

identically. Now one can prove that for each $n \geqslant 0$ there exists a local diffeomorphism $\Gamma^{[n]} : (x, t, \tau_1, \dots, \tau_n) \mapsto (\hat{x}, \hat{t}, \hat{\tau}_1, \dots, \hat{\tau}_n)$ and a real-valued smooth function $\theta^{[n]}(x, t, \tau_1, \dots, \tau_n)$ such that

$$\Gamma^{[n]*}\sigma_1^{[n]} = \Theta_1^{[n]} \cos\theta^{[n]} + \Theta_2^{[n]} \sin\theta^{[n]}, \tag{83}$$

$$\Gamma^{[n]*}\sigma_2^{[n]} = -\Theta_1^{[n]} \sin\theta^{[n]} + \Theta_2^{[n]} \cos\theta^{[n]}, \tag{84}$$

$$\Gamma^{[n]*}\sigma_3^{[n]} = \Theta_3^{[n]} + d\theta^{[n]}. \tag{85}$$

Indeed, write $\Gamma^{[n]} = (\alpha, \beta, T_1, \dots, T_n)$, in which $\alpha, \beta, T_i : V^{[n]} \to \mathbf{R}$. Equations (83)–(85) are equivalent to the following system of equations:

$$\frac{1}{\beta}\left(\alpha_x + \sum_{i=1}^{n} T_{i,x}\right) = f_{11} \cos\theta^{[n]} + f_{21} \sin\theta^{[n]}, \tag{86}$$

$$\frac{1}{\beta}\left(\alpha_t + \sum_{i=1}^{n} T_{i,t}\right) = f_{12} \cos\theta^{[n]} + f_{22} \sin\theta^{[n]}, \tag{87}$$

$$\frac{1}{\beta}\left(\alpha_{\tau_j} + \sum_{i=1}^{n} T_{i,\tau_j}\right) = h_{1j} \cos\theta^{[n]} + h_{2j} \sin\theta^{[n]}, \quad j = 1, \dots, n, \tag{88}$$

$$\frac{1}{\beta}\beta_x = -f_{11} \sin\theta^{[n]} + f_{21} \cos\theta^{[n]}, \tag{89}$$

$$\frac{1}{\beta}\beta_t = -f_{12} \sin\theta^{[n]} + f_{22} \cos\theta^{[n]}, \tag{90}$$

$$\frac{1}{\beta}\beta_{\tau_j} = -h_{1j} \sin\theta^{[n]} + h_{2j} \cos\theta^{[n]}, \quad j = 1, \dots, n, \tag{91}$$

$$\frac{1}{\beta}\left(\alpha_x + \sum_{i=1}^{n} T_{i,x}\right) = f_{31} + \theta_x^{[n]}, \tag{92}$$

$$\frac{1}{\beta}\left(\alpha_t+\sum_{i=1}^{n}T_{i,t}\right)=f_{32}+\theta_t^{[n]}, \tag{93}$$

$$\frac{1}{\beta}\left(\alpha_{\tau_j}+\sum_{i=1}^{n}T_{i,\tau_j}\right)=h_{3j}+\theta_{\tau_j}^{[n]}, \quad j=1,\dots,n. \tag{94}$$

Substituting Eqs. (86)–(88) into (92)–(94) one can write the system (86)–(94) as

$$d\left(\alpha+\sum_{i=1}^{n}T_i\right)=\beta\left(\Theta_1^{[n]}\cos\theta^{[n]}+\Theta_2^{[n]}\sin\theta^{[n]}\right), \tag{95}$$

$$d(\ln\beta)=-\Theta_1^{[n]}\sin\theta^{[n]}+\Theta_2^{[n]}\cos\theta^{[n]}, \tag{96}$$

$$\Theta_1^{[n]}\cos\theta^{[n]}+\Theta_2^{[n]}\sin\theta^{[n]}=\Theta_3^{[n]}+d\theta^{[n]}, \tag{97}$$

in which the operator d indicates exterior derivative on $V^{[n]}$. The Pfaffian system (97) is completely integrable for $\theta^{[n]}(x,t,\tau_1,\dots,\tau_n)$, since that for each $n\geqslant 0$ the equations

$$d\Theta_1^{[n]}=\Theta_3^{[n]}\wedge\Theta_2^{[n]}, \qquad d\Theta_2^{[n]}=\Theta_1^{[n]}\wedge\Theta_3^{[n]}, \qquad d\Theta_3^{[n]}=\Theta_1^{[n]}\wedge\Theta_2^{[n]}, \tag{98}$$

are satisfied on solutions of the hierarchy $u_{\tau_i}=F_i$. Using (97) it is then straightforward to check that the right-hand sides of (95) and (96) are closed one-forms. Thus, Eqs. (95) and (96) determine β and $\alpha+\sum_{i=1}^{n}T_i$. The functions T_i, $i=1,\dots,n$, are almost arbitrary: they are constrained only by the fact that $\Gamma^{[n]}=(\alpha,\beta,T_1,\dots,T_n)$ be a local diffeomorphism. A natural choice is to take $T_i=\tau_i$. It then follows that the Jacobian determinant of $\Gamma^{[n]}$ is

$$\alpha_x\beta_t-\alpha_t\beta_x=\beta^2(f_{11}f_{22}-f_{12}f_{21}),$$

and therefore, since $u^{[0]}(x,t)$ is III-generic, $\Gamma^{[n]}$ is a local diffeomorphism.

Next, arguing as above one finds a diffeomorphism $\widehat{\Gamma}^{[n]}$ and a function $\hat{\theta}^{[n]}$ satisfying Eqs. (83)–(85) with $\Theta_\alpha^{[n]}$ replaced by $\widehat{\Theta}_\alpha^{[n]}$. It is then straightforward to check that the maps

$$\Upsilon^{[n]}=\left(\widehat{\Gamma}^{[n]}\right)^{-1}\circ\Gamma^{[n]} \quad\text{and}\quad \mu^{[n]}=\theta^{[n]}-\hat{\theta}^{[n]}\circ\Upsilon^{[n]}$$

satisfy Eqs. (77)–(79).

It remains to prove that the functions $\Upsilon^{[n]}$ and $\mu^{[n]}$, $n\geqslant 0$, can be chosen so that they satisfy the compatibility condition (80). The fact that $\Upsilon^{[n+1]}|_{V^{[n]}}=\Upsilon^{[n]}$ follows trivially from the construction of the local diffeomorphisms $\Upsilon^{[n]}$. On the other hand, in order to check that $\mu^{[n+1]}|_{V^{[n]}}=\mu^{[n]}$, it is of course enough to see that

$$\theta^{[n+1]}\big|_{V^{[n]}}=\theta^{[n]} \quad\text{and}\quad \hat{\theta}^{[n+1]}\big|_{\widehat{V}^{[n]}}=\hat{\theta}^{[n]}, \quad n\geqslant 0. \tag{99}$$

Consider the first equation appearing in (99). Expanding Eq. (97) with $\theta^{[n]}$ replaced by $\theta^{[n+1]}$, one sees that the function $\theta^{[n+1]}$ is determined by the Pfaffian system

$$\Theta_1^{[n]}\cos\theta^{[n+1]}+\Theta_2^{[n]}\sin\theta^{[n+1]}=\Theta_3^{[n]}+d\theta^{[n+1]}, \tag{100}$$

$$h_{1,n+1}\cos\theta^{[n+1]}+h_{2,n+1}\sin\theta^{[n+1]}=h_{3,n+1}+\frac{\partial\theta^{[n+1]}}{\partial\tau_{n+1}}, \tag{101}$$

in which the operator d appearing in (100) is the exterior derivative on $V^{[n]}$. Now consider the Cauchy problem

$$\begin{cases}\frac{\partial z}{\partial\tau_{n+1}}=h_{1,n+1}\cos z+h_{2,n+1}\sin z-h_{3,n+1},\\ z(x,t,\tau_1,\dots,\tau_n,0)=\theta^{[n]}(x,t,\tau_1,\dots,\tau_n).\end{cases} \tag{102}$$

It is proven in [11], for instance, that Cauchy problems of the form (102) have unique solutions. Let $z(x,t,\tau_1,\dots,\tau_n,\tau_{n+1})$ be such a solution, and set $\theta^{[n+1]}=z$. Since $\theta^{[n]}$ satisfies Eq. (100) by construction, the function $\theta^{[n+1]}$ satisfies (100), (101), and $\theta^{[n+1]}|_{V^{[n]}}=\theta^{[n]}$. The second condition appearing in (99) is treated in a similar way. This ends the proof. □

Corollary 28. *Let $u_{\tau_i}=F_i(x,t,u,\dots)$ and $\hat u_{\hat\tau_i}=\widehat{F}_i(\hat x,\hat t,\hat u,\dots)$ be two hierarchies of pseudo-spherical type with associated one-forms given by* (76). *Let $\{u^{[n]}\}$ and $\{\hat u^{[n]}\}$ be solutions of $u_{\tau_i}=F_i$ and $\hat u_{\hat\tau_i}=\widehat{F}_i$, respectively, and assume that $u^{[0]}(x,t)$ and $\hat u^{[0]}(\hat x,\hat t)$ are III-generic. There exist sequences of maps $\{\Upsilon^{[n]}\}$ and $\{\mu^{[n]}\}$ such that*:

(a) *for each $n\geqslant 0$, $\Upsilon^{[n]}=(\psi^{[n]},\varphi^{[n]},\phi_1^{[n]},\dots,\phi_n^{[n]})$ is a local diffeomorphism from an open subset $V^{[n]}$ of the domain of $u^{[n]}$ to an open subset $\widehat{V}^{[n]}$ of the domain of $\hat u^{[n]}$ and $\Upsilon^{[n+1]}|_{V^{[n]}}=\Upsilon^{[n]}$;*
(b) *for each $n\geqslant 0$, $\mu^{[n]}:V^{[n]}\to\mathbf{R}$ is a smooth real-valued function and $\mu^{[n+1]}|_{V^{[n]}}=\mu^{[n]}$; and*
(c) *for each $n\geqslant 0$, one can in fact choose $\Upsilon^{[n]}=(\psi^{[n]},\varphi^{[n]},\tau_1,\dots,\tau_n)$, and the pull-backs of the functions $f_{\alpha\beta}$, $h_{\alpha i}$, $\hat f_{\alpha\beta}$, and $\hat h_{\alpha i}$ by $u^{[n]}$ and $\hat u^{[n]}$, respectively, satisfy*

$$\hat f_{11}\circ\Upsilon^{[n]}=\frac{\Delta_1^{[n]}}{\Delta^{[n]}},\qquad \hat f_{12}\circ\Upsilon^{[n]}=\frac{\Delta_2^{[n]}}{\Delta^{[n]}},\qquad \hat h_{1j}\circ\Upsilon^{[n]}=\frac{\Delta_{j+2}^{[n]}}{\Delta^{[n]}}, \tag{103}$$

in which $\Delta^{[n]}$ is given by

$$\Delta^{[n]}=\begin{vmatrix}\psi_x^{[n]} & \varphi_x^{[n]} & 0 & \dots & 0\\ \psi_{\tau_0}^{[n]} & \varphi_{\tau_0}^{[n]} & 0 & \dots & 0\\ \psi_{\tau_1}^{[n]} & \varphi_{\tau_1}^{[n]} & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \psi_{\tau_n}^{[n]} & \varphi_{\tau_n}^{[n]} & 0 & \dots & 1\end{vmatrix}=\psi_x^{[n]}\varphi_t^{[n]}-\varphi_x^{[n]}\psi_t^{[n]}, \tag{104}$$

and for each $i=1,2,\dots,n+2$, the determinant Δ_i is equal to Δ, except for the ith column which is replaced by the column vector

$$\begin{pmatrix}f_{11}\cos\mu^{[n]}+f_{21}\sin\mu^{[n]}\\ f_{12}\cos\mu^{[n]}+f_{22}\sin\mu^{[n]}\\ h_{11}\cos\mu^{[n]}+h_{21}\sin\mu^{[n]}\\ \vdots\\ h_{1n}\cos\mu^{[n]}+h_{2n}\sin\mu^{[n]}\end{pmatrix}. \tag{105}$$

Proof. Theorem 27 implies that there exist sequences $\{\Upsilon^{[n]}\}$ and $\{\mu^{[n]}\}$ of smooth maps satisfying (a) and (b) above, and such that for each $n \geqslant 0$,

$$\Upsilon^{[n]*}\widehat{\Theta}_1^{[n]} = \Theta_1^{[n]} \cos\mu^{[n]} + \Theta_2^{[n]} \sin\mu^{[n]}, \tag{106}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_2^{[n]} = -\Theta_1^{[n]} \sin\mu^{[n]} + \Theta_2^{[n]} \cos\mu^{[n]}, \tag{107}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_3^{[n]} = \Theta_3^{[n]} + d\mu^{[n]}. \tag{108}$$

Write $\Upsilon^{[n]} = (\psi^{[n]}, \varphi^{[n]}, \phi_1^{[n]}, \ldots, \phi_n^{[n]})$, $\hat{g}_{\alpha\beta} = \hat{f}_{\alpha\beta} \circ \Upsilon^{[n]}$, and $\widehat{H}_{\alpha j} = \hat{h}_{\alpha j} \circ \Upsilon^{[n]}$. The construction of $\Upsilon^{[n]}$ appearing in the proof of Theorem 27 implies that $\phi_j^{[n]} = \tau_j$, $j = 1, 2, \ldots, n$, and therefore system (106)–(108) becomes

$$\hat{g}_{11}\psi_x^{[n]} + \hat{g}_{12}\varphi_x^{[n]} = f_{11}\cos\mu^{[n]} + f_{21}\sin\mu^{[n]}, \tag{109}$$

$$\hat{g}_{11}\psi_t^{[n]} + \hat{g}_{12}\varphi_t^{[n]} = f_{12}\cos\mu^{[n]} + f_{22}\sin\mu^{[n]}, \tag{110}$$

$$\hat{g}_{11}\psi_{\tau_j}^{[n]} + \hat{g}_{12}\varphi_{\tau_j}^{[n]} + \widehat{H}_{1j} = h_{1j}\cos\mu^{[n]} + h_{2j}\sin\mu^{[n]}, \tag{111}$$

$$\hat{g}_{21}\psi_x^{[n]} + \hat{g}_{22}\varphi_x^{[n]} = -f_{11}\sin\mu^{[n]} + f_{21}\cos\mu^{[n]}, \tag{112}$$

$$\hat{g}_{21}\psi_t^{[n]} + \hat{g}_{22}\varphi_t^{[n]} = -f_{12}\sin\mu^{[n]} + f_{22}\cos\mu^{[n]}, \tag{113}$$

$$\hat{g}_{21}\psi_{\tau_j}^{[n]} + \hat{g}_{22}\varphi_{\tau_j}^{[n]} + \widehat{H}_{2j} = -h_{1j}\sin\mu^{[n]} + h_{2j}\cos\mu^{[n]}, \tag{114}$$

$$\hat{g}_{31}\psi_x^{[n]} + \hat{g}_{32}\varphi_x^{[n]} = f_{31} + \mu_x^{[n]}, \tag{115}$$

$$\hat{g}_{31}\psi_t^{[n]} + \hat{g}_{32}\varphi_t^{[n]} = f_{32} + \mu_t^{[n]}, \tag{116}$$

$$\hat{g}_{31}\psi_{\tau_j}^{[n]} + \hat{g}_{32}\varphi_{\tau_j}^{[n]} + \widehat{H}_{3j} = h_{3j} + \mu_{\tau_j}^{[n]}, \tag{117}$$

in which $j = 1, 2, \ldots, n$. One can consider the $n+2$ equations (109)–(111) as a linear system for the functions $\hat{g}_{11}$, $\hat{g}_{12}$, and $\widehat{H}_{1j}$, $j = 1, 2, \ldots, n$. It is easy to see that the determinant of the left-hand side of (109)–(111) is given by (104), and therefore Cramer's rule implies that

$$\hat{g}_{11} = \hat{f}_{11} \circ \Upsilon^{[n]} = \frac{\Delta_1^{[n]}}{\Delta^{[n]}}, \qquad \hat{g}_{12} = \hat{f}_{12} \circ \Upsilon^{[n]} = \frac{\Delta_2^{[n]}}{\Delta^{[n]}}, \quad \text{and} \quad \widehat{H}_{1j} = \hat{h}_{1j} \circ \Upsilon^{[n]} = \frac{\Delta_{j+2}^{[n]}}{\Delta^{[n]}},$$

in which Δ_i is determined by (104) and (105). □

Now one would like to use Theorem 27 and Corollary 28 to show the existence of a local smooth mapping transforming a solution $\{u^{[n]}\}$ of the hierarchy $u_{\tau_i} = F_i$ into a solution $\{\hat{u}^{[n]}\}$ of the hierarchy $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$. In order to avoid some technicalities related to the gauge freedom one has to determine one-forms associated with differential equations describing pseudo-spherical surfaces [24], only hierarchies of strictly pseudo-spherical type will be considered in what follows. Motivated by Corollary 28, let

$$\Upsilon^{[n]} = \left(\psi^{[n]}, \varphi^{[n]}, \phi_1^{[n]}, \ldots, \phi_n^{[n]}\right), \quad n \geqslant 0, \tag{118}$$

be a smooth map from (an open subset of) $\mathbf{R}^{n+2}$ equipped with coordinates $(x, t, \tau_1, \ldots, \tau_n)$ to (an open subset of) $\mathbf{R}^{n+2}$ equipped with coordinates $(\hat{x}, \hat{t}, \hat{\tau}_1, \ldots, \hat{\tau}_n)$, and let $\mu^{[n]}(x, t, \tau_1, \ldots, \tau_n)$

be a smooth real-valued function on (an open subset of) $\mathbf{R}^{n+2}$. Also, define the determinants $\Delta^{[n]}$ and $\Delta_i^{[n]}$ as

$$\Delta^{[n]} = \text{Jacobian}\big(\Upsilon^{[n]}\big) \tag{119}$$

and $\Delta_i^{[n]} = \Delta^{[n]}$, except for the ith column, which is replaced by the vector (105).

Lemma 29. *Let $u_{\tau_i} = F_i$ and $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$ be two hierarchies of strictly pseudo-spherical type with associated functions $f_{\alpha\beta}$, $h_{\alpha j}$, and $\hat{f}_{\alpha\beta}$, $\hat{h}_{\alpha j}$ respectively, and assume that $\hat{f}_{11} = G(\hat{u})$ with $G' \neq 0$. For each $n \geqslant 0$, let $\Upsilon^{[n]}$ be a smooth map as in* (118)*, let $\mu^{[n]}(x, t, \tau_1, \ldots, \tau_n)$ be a smooth real-valued function on an open subset of $\mathbf{R}^{n+2}$, and set $\hat{g}_{\alpha\beta} = \hat{f}_{\alpha\beta} \circ \Upsilon^{[n]}$ and $\widehat{H}_{\alpha j} = \hat{h}_{\alpha j} \circ \Upsilon^{[n]}$. The system of equations*

$$\Delta^{[n]} \hat{g}_{12} = \Delta_2^{[n]}, \tag{120}$$

$$\Delta^{[n]} \widehat{H}_{1j} = \Delta_{j+2}^{[n]}, \tag{121}$$

$$\hat{g}_{21}\psi_x^{[n]} + \hat{g}_{22}\varphi_x^{[n]} + \sum_{i=1}^{n} \widehat{H}_{2i}\phi_{i,x} = f_{21}\cos\mu^{[n]} - f_{11}\sin\mu^{[n]}, \tag{122}$$

$$\hat{g}_{21}\psi_t^{[n]} + \hat{g}_{22}\varphi_t^{[n]} + \sum_{i=1}^{n} \widehat{H}_{2i}\phi_{i,t} = f_{22}\cos\mu^{[n]} - f_{12}\sin\mu^{[n]}, \tag{123}$$

$$\hat{g}_{21}\psi_{\tau_j}^{[n]} + \hat{g}_{22}\varphi_{\tau_j}^{[n]} + \sum_{i=1}^{n} \widehat{H}_{2j}\phi_{i,\tau_j} = h_{2j}\cos\mu^{[n]} - h_{1j}\sin\mu^{[n]}, \tag{124}$$

$$\hat{g}_{31}\psi_x^{[n]} + \hat{g}_{32}\varphi_x^{[n]} + \sum_{i=1}^{n} \widehat{H}_{3i}\phi_{i,x} = f_{31} + \mu_x^{[n]}, \tag{125}$$

$$\hat{g}_{31}\psi_t^{[n]} + \hat{g}_{32}\varphi_t^{[n]} + \sum_{i=1}^{n} \widehat{H}_{3i}\phi_{i,t} = f_{32} + \mu_t^{[n]}, \tag{126}$$

$$\hat{g}_{31}\psi_{\tau_j}^{[n]} + \hat{g}_{32}\varphi_{\tau_j}^{[n]} + \sum_{i=1}^{n} \widehat{H}_{3j}\phi_{i,\tau_j} = h_{3j} + \mu_{\tau_j}^{[n]}, \tag{127}$$

in which the index j runs from 1 *to n and the left-hand sides of* (120)–(127) *have been evaluated by means of the equation*

$$\hat{f}_{11} \circ \Upsilon^{[n]} = G(\hat{u}) \circ \Upsilon^{[n]} = \frac{\Delta_1^{[n]}}{\Delta^{[n]}}, \tag{128}$$

admits—whenever $\{u^{[n]}\}$ is a solution of the hierarchy $u_{\tau_i} = F_i$ such that $u^{[0]}(x,t)$ is III-generic—a local solution $\psi^{[n]}$, $\varphi^{[n]}$, $\phi_1^{[n]}, \ldots, \phi_n^{[n]}$ and $\mu^{[n]}$, defined on an open subset $V^{[n]}$

of $\mathbf{R}^{n+2}$*, such that* $\Upsilon^{[n]} = (\psi^{[n]}, \varphi^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]})$ *is a local diffeomorphism. Moreover, the maps* $\Upsilon^{[n]}$ *and* $\mu^{[n]}$ *can be chosen so that*

$$\Upsilon^{[n+1]}\big|_{V^{[n]}} = \Upsilon^{[n]} \quad \text{and} \quad \mu^{[n+1]}\big|_{V^{[n]}} = \mu^{[n]}, \quad n \geqslant 0. \tag{129}$$

Proof. Let $\{\hat{u}^{[n]}\}$ be any solution of the hierarchy $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$ such that $\hat{u}^{[0]}(x,t)$ is III-generic. By Theorem 27, there exist functions $\psi^{[n]}$, $\varphi^{[n]}$, $\phi_i^{[n]}$ and $\mu^{[n]}$ satisfying (77)–(79), (129), and such that $\Upsilon^{[n]} = (\psi^{[n]}, \varphi^{[n]}, \phi_1^{[n]}, \dots, \phi_n^{[n]})$ is a local diffeomorphism. One can show that these functions also satisfy the system of equations (120)–(127). Indeed, let $\widetilde{G}_{\alpha\beta}$, and $\widetilde{H}_{\alpha j}$ be the functions depending on x, t, $\psi^{[n]}$, $\varphi^{[n]}$, $\phi_1^{[n]}, \dots, \phi_n^{[n]}$, and their derivatives, which are obtained from $\hat{f}_{\alpha\beta}$ and $\hat{h}_{\alpha j}$ by computing the pull-backs $\Upsilon^* \hat{f}_{\alpha\beta}$ and $\Upsilon^* \hat{h}_{\alpha j}$ by means of Eq. (128), as in the enunciate of the lemma. On the solutions $\psi^{[n]}$, $\varphi^{[n]}$, $\phi_i^{[n]}$ and $\mu^{[n]}$ of system (77)–(79) one has, for any α and β,

$$\widetilde{G}_{\alpha\beta} = \hat{f}_{\alpha\beta} \circ \Upsilon^{[n]} = \hat{g}_{\alpha\beta}, \qquad \widetilde{H}_{\alpha j} = \hat{h}_{\alpha j} \circ \Upsilon^{[n]} = \widehat{H}_{\alpha j},$$

since on these solutions, Eq. (128) is an identity. Thus, system (120)–(127) reduces to the first order system (77)–(79), and the result follows by writing this system in components, as in the proof of Corollary 28. □

Now one can prove a correspondence result generalizing the single equation case treated in [13,24].

Theorem 30. *Let* $u_{\tau_i} = F_i$ *and* $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$ *be two hierarchies of strictly pseudo-spherical type with associated functions* $f_{\alpha\beta}$*,* $h_{\alpha j}$*, and* $\hat{f}_{\alpha\beta}$*,* $\hat{h}_{\alpha j}$*, respectively. Assume that* $\hat{f}_{11} = G(\hat{u})$ *with* $G' \neq 0$*. For any solution* $\{u^{[n]}\}$ *of the hierarchy* $u_{\tau_i} = F_i$ *such that* $u^{[0]}(x,t)$ *is III-generic, there exist local diffeomorphisms* $\Upsilon^{[n]}$ *and real-valued functions* $\mu^{[n]}$*,* $n \geqslant 0$*, on open subsets* $V^{[n]}$ *of* $\mathbf{R}^{n+2}$*, satisfying the conditions*

$$\Upsilon^{[n+1]}\big|_{V^{[n]}} = \Upsilon^{[n]}, \qquad \mu^{[n+1]}\big|_{V^{[n]}} = \mu^{[n]}, \tag{130}$$

and such that the equation

$$\hat{f}_{11} \circ \Upsilon^{[n]} = \frac{\Delta_1^{[n]}}{\Delta^{[n]}} \tag{131}$$

determines a solution $\{\hat{u}^{[n]}\}$ *of the hierarchy* $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$ *for which* $\hat{u}^{[0]}(x,t)$ *is also III-generic.*

Proof. Let $\{u^{[n]}\}$ be a solution of the hierarchy $u_{\tau_i} = F_i$ such that $u^{[0]}(x,t)$ is III-generic. By Lemma 29, the system of Eqs. (120)–(127) possesses local solutions $\psi^{[n]}, \varphi^{[n]}$, $\phi_1, \dots, \phi_n$ and $\mu^{[n]}$, such that $\Upsilon^{[n]} = (\psi^{[n]}, \varphi^{[n]}, \phi_1, \dots, \phi_n)$ is a local diffeomorphism with domain $V^{[n]} \subseteq \mathbf{R}^{n+2}$ and such that restriction (130) holds. Define $\hat{u}^{[n]} \circ \Upsilon^{[n]}$ by means of Eq. (131). Then, for each $n \geqslant 0$ one finds a system of equations equivalent to

$$\Upsilon^{[n]*}\widehat{\Theta}_1^{[n]} = \Theta_1^{[n]}\cos\mu^{[n]} + \Theta_2^{[n]}\sin\mu^{[n]}, \tag{132}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_2^{[n]} = -\Theta_1^{[n]}\sin\mu^{[n]} + \Theta_2^{[n]}\cos\mu^{[n]}, \tag{133}$$

$$\Upsilon^{[n]*}\widehat{\Theta}_3^{[n]} = \Theta_3^{[n]} + d\mu^{[n]}, \tag{134}$$

and moreover $\hat{u}^{[n+1]} \circ \Upsilon^{[n+1]}|_{V^{[n]}} = \hat{u}^{[n]} \circ \Upsilon^{[n]}$. Since $\Upsilon^{[n]}$ is a local diffeomorphism and the one-forms $\Theta_\alpha^{[n]}$ satisfy Eqs. (19), so do the one-forms $\widehat{\Theta}_\alpha^{[n]}$. This means that (131) determines a solution of the hierarchy $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$, as claimed. Finally, note that, in particular, Eqs. (132)–(134) hold for $n = 0$. But then, (132) and (133) imply that (notation as in Proposition 14)

$$\Upsilon^{[0]*}\big(\widehat{\omega}_0^1 \wedge \widehat{\omega}_0^2\big) = \omega_0^1 \wedge \omega_0^2,$$

that is,

$$\text{Jacobian}\big(\Upsilon^{[0]}\big)\big(\hat{f}_{11}\hat{f}_{22} - \hat{f}_{12}\hat{f}_{21}\big) = f_{11}f_{22} - f_{12}f_{21},$$

and one concludes that $\hat{u}^{[0]}(\hat{x}, \hat{t})$ is indeed a III-generic solution since $\Upsilon^{[0]}$ is a diffeomorphism. □

Thus, the existence of local correspondences between solutions to hierarchies of strictly pseudo-spherical type has been proven in this paper basically by "dressing"—via Eq. (5)—the hierarchy of pseudo-spherical structures (81) naturally induced by the Poincaré metric. Obviously, one expects that results analogous to the ones appearing in this section can be obtained if one takes a "pseudo-Riemannian" point of view, as in [24], and dresses—via Eqs. (6) and (7)—standard hierarchies of pseudo-Riemannian surfaces of constant Gaussian curvature.

It is also important to stress the fact that, as it happens in the single equation case [13,24,25], the transformation (131) appearing in Theorem 30 depends on the particular solution $\{u^{[n]}\}$ of the hierarchy $u_{\tau_i} = F_i$ one starts with, and that the proof of existence of the functions $\Upsilon^{[n]}$ and $\mu^{[n]}$ of Lemma 29 relies on the Frobenius theorem. Since the natural arena for the study of the formal geometry of differential equations is the theory of infinite-dimensional jet spaces ([15,29] and references therein) and at this level there is no Frobenius theorem available [15,16,29], Theorem 30 does not imply that one can "transfer" information on, for instance, conservation laws or symmetries from one hierarchy to another by means of (131). However, one could think of (131) intuitively as determining a "mapping" from the space of (smooth, local) solutions to a hierarchy $u_{\tau_i} = F_i$ to the space of (smooth, local) solutions to a hierarchy $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$. Then one could ask whether information about the structure of the former (for example, information about its topology, along the lines of [5,16]) gives some insight into the structure of the latter. These matters deserve further investigation.

4.3. Examples

This paper ends with some simple applications of Theorems 27 and 30 to the ubiquitous Korteweg–de Vries hierarchy. It follows from the seminal paper [6] by Chern and Peng that this hierarchy is of strictly pseudo-spherical type with associated functions

$$f_{11} = 1 - u, \qquad f_{12} = \lambda u_x - u_{xx} - 2u^2 + 2u - \lambda^2 u + \lambda^2; \tag{135}$$

$$f_{21} = \lambda, \qquad f_{22} = \lambda^3 + 2\lambda u - 2u_x; \tag{136}$$

$$f_{31} = -1 - u, \qquad f_{32} = \lambda u_x - u_{xx} - \lambda^2 u - 2u^2 - \lambda^2 - 2u \tag{137}$$

and

$$h_{1i} = \frac{1}{2}\lambda B_x^{(i+1)} - \frac{1}{2}B_{xx}^{(i+1)} - uB^{(i+1)} + B^{(i+1)}; \tag{138}$$

$$h_{2i} = \lambda B^{(i+1)} - B_x^{(i+1)}; \tag{139}$$

$$h_{3i} = \frac{1}{2}\lambda B_x^{(i+1)} - \frac{1}{2}B_{xx}^{(i+1)} - uB^{(i+1)} - B^{(i+1)}, \tag{140}$$

in which

$$B^{(i)} = \sum_{j=0}^{i} B_j \lambda^{2(i-j)},$$

and the functions B_j are defined recursively by means of the equations

$$B_{0,x} = 0, \tag{141}$$

$$B_{j+1,x} = B_{j,xxx} + 4uB_{j,x} + 2u_x B_j, \quad j \geqslant 0. \tag{142}$$

The functions F_i, $i \geqslant 0$, are given by

$$F_i = \frac{1}{2}B_{i+1,xxx} + u_x B_{i+1} + 2uB_{i+1,x} = \frac{1}{2}B_{i+2,x} \tag{143}$$

or, equivalently in terms of the functions $B^{(i)}$, by

$$F_i = \frac{1}{2}B_{xxx}^{(i+1)} + u_x B^{(i+1)} + 2uB_x^{(i+1)} - \frac{1}{2}\lambda^2 B_x^{(i+1)}.$$

For instance, one can easily check that if $B_0 = 1$ and all integration constants are set to zero, the equation $u_{\tau_0} = F_0$ is the standard K–dV equation $u_t = u_{xxx} + 6uu_x$, and $u_{\tau_1} = F_1$ and $u_{\tau_2} = F_2$ are, respectively,

$$\begin{aligned} u_{\tau_1} &= u_{xxxxx} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2 u_x, \\ u_{\tau_2} &= u_{xxxxxxx} + 70u_{xx}u_{xxx} + 42u_x u_{xxxx} + 14uu_{xxxxx} + 70u_x^3 \\ &\quad + 280uu_x u_{xx} + 70u^2 u_{xxx} + 140u^3 u_x. \end{aligned}$$

A detailed study of the initial value problem for the K–dV hierarchy appears in Schwarz Jr.'s paper [28]. An elementary solution to this hierarchy is the sequence of functions $\{u^{[n]}\colon n \geqslant 0\}$ given by

$$u^{[n]}(x, t, \tau_1, \dots, \tau_n) = 2\,\mathrm{sech}^2\big(x + 4t + 16\tau_1 + 64\tau_2 + \dots + 4^{n+1}\tau_n\big), \tag{144}$$

as it can be easily checked by induction. The physical content of solution (144) has been investigated by Kraenkel, Manna and Pereira in [14]. These authors have also shown in [14] that the higher K–dV equations themselves play an important role in the description of the propagation of long-surface waves in a shallow inviscid fluid.

Straightening-out the K–dV hierarchy. In this example is shown how one can transform a solution to the K–dV hierarchy into a solution of a hierarchy of linear equations. Consider the hierarchy $\hat{u}_{\hat{\tau}_i} = \widehat{F}_i$ with seed equation $\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}\hat{x}} + \hat{u}_{\hat{x}}$, and higher equations

$$\hat{u}_{\hat{\tau}_i} = a_{i+1}^{i+2}\hat{u}_{\hat{x}^{i+2}} + \sum_{l=1}^{i} a_l^{i+2}\hat{u}_{\hat{x}^{l+1}} + \sum_{l=1}^{i+1} a_l^{i+2}\hat{u}_{\hat{x}^l}, \tag{145}$$

in which the constants a_s^r are arbitrary except that $a_1^r = 1$, $r \geqslant 1$. It is straightforward to check that the hierarchy (145) is of strictly pseudo-spherical type with associated functions

$$\hat{f}_{11} = \hat{u}, \qquad \hat{f}_{12} = \hat{u}_{\hat{x}}, \qquad \hat{f}_{21} = 1, \qquad \hat{f}_{22} = 0, \qquad \hat{f}_{31} = \hat{u}, \qquad \hat{f}_{32} = \hat{u}_{\hat{x}}; \tag{146}$$

$$\hat{h}_{1i} = \sum_{k=1}^{i+1} a_k^{i+2}\hat{u}_{\hat{x}^k}, \qquad \hat{h}_{2i} = 0, \qquad \hat{h}_{3i} = \hat{h}_{1i}. \tag{147}$$

Now, the proof of Theorem 27 implies that if the sequence $\{u^{[n]}\colon n \geqslant 0\}$ is a solution of the K–dV hierarchy such that $u^{[0]}(x,t)$ is III-generic (for instance, solution (144) satisfies this condition) there exist functions $\alpha(x,t,\ldots,\tau_n)$, $\beta(x,t,\ldots,\tau_n)$ and $\theta^{[n]}(x,t,\ldots,\tau_n)$, $n \geqslant 0$, such that Eqs. (95)–(97) hold. Define diffeomorphisms $\Upsilon^{[n]} : (x,t,\tau_1,\ldots,\tau_n) \mapsto (\hat{x},\hat{t},\hat{\tau}_1,\ldots,\hat{\tau}_n)$ and functions $\mu^{[n]} : V^{[n]} \to \mathbf{R}$, $n \geqslant 0$, by means of the formulae

$$\hat{x} = -\ln\left|\beta + \frac{1}{\beta}\left(\alpha + \sum_{i=1}^{n}\tau_i\right)^2\right|; \tag{148}$$

$$\hat{t} = -\left(\frac{\alpha + \sum_{i=1}^{n}\tau_i}{\beta}\right) + 1 - \sum_{i=1}^{n}\tau_i + \ln\left|\beta + \frac{1}{\beta}\left(\alpha + \sum_{i=1}^{n}\tau_i\right)^2\right|; \tag{149}$$

$$\hat{\tau}_i = \tau_i, \quad 1 \leqslant i \leqslant n, \tag{150}$$

and

$$\mu^{[n]}(x,t,\tau_1,\ldots,\tau_n) = \theta^{[n]}(x,t,\tau_1,\ldots,\tau_n) - \tilde{\theta}^{[n]} \circ \Upsilon^{[n]}(x,t,\tau_1,\ldots,\tau_n), \tag{151}$$

in which $\tilde{\theta}^{[n]}$ is determined by the relations

$$\cos\tilde{\theta}^{[n]} = \frac{-1+K^2}{1+K^2}, \qquad \sin\tilde{\theta}^{[n]} = \frac{2K}{1+K^2}, \qquad K = -\frac{1}{\beta}\left(\alpha + \sum_{i=1}^{n}\tau_i\right).$$

The functions $\Upsilon^{[n]}$ and $\mu^{[n]}$ solve Eqs. (120)–(127) of Lemma 29, and therefore Theorem 30 can be applied. One finds that $\{\hat{u}^{[n]}\colon n \geqslant 0\}$, in which

$$\hat{u}^{[n]} = \hat{x} + \hat{t} + \hat{\tau}_1 + \cdots + \hat{\tau}_n,$$

is a solution to the hierarchy (145).

From linear equations to the K–dV hierarchy. In this example the "hatted" variables correspond to the K–dV hierarchy. It will be shown how a simple solution to the K–dV equation, which also solves the K–dV hierarchy, can be obtained starting from the solution $u^{[n]} = x + t + \tau_1 + \cdots + \tau_n$ to (145).

Consider the following function from the space $\widehat{V}^{[n]}$ into itself:

$$\hat{x} = \frac{1}{3\widehat{D}}\left(-12\hat{t}\hat{x}^2 + \hat{x}^5 - 2\hat{x}^3 - 12\hat{t}\right) - \sum_{i=1}^{n}\hat{\tau}_i; \tag{152}$$

$$\hat{t} = \frac{1}{\widehat{D}}\hat{x}^4; \tag{153}$$

$$\hat{\tau}_i = \hat{\tau}_i, \quad i = 1, \ldots, n, \tag{154}$$

in which $\widehat{D} = 4(6\hat{t} + \hat{x}^3)^2 + \hat{x}^2(\hat{x}^3 - 12\hat{t})^2$. The Jacobian of (152)–(154) is $-8\hat{x}^3(\hat{x}^2 + 2)/\widehat{D}^2$, and therefore this transformation is a local diffeomorphism away from $\hat{x} = 0$. Let $T^{[n]}$ be its inverse transformation.

Now pull-back the one-forms $\Theta_\alpha^{[n]}$ determined by functions (146)–(147) by means of the solution $u^{[n]} = x + t + \tau_1 + \cdots + \tau_n$. Then, the transformation $\Gamma^{[n]}$ given by

$$\hat{x} = \frac{-(x + t + \tau_1 + \cdots + \tau_n - 1)\mathrm{e}^{-x}}{1 + (x + t + \tau_1 + \cdots + \tau_n - 1)^2} - \sum_{i=1}^{n}\tau_i; \tag{155}$$

$$\hat{t} = \frac{\mathrm{e}^{-x}}{1 + (x + t + \tau_1 + \cdots + \tau_n - 1)^2}; \tag{156}$$

$$\hat{\tau}_i = \tau_i, \quad i = 1, \ldots, n, \tag{157}$$

and the function $\theta^{[n]} : V^{[n]} \to \mathbf{R}$, determined by

$$\cos\theta^{[n]} = \frac{-1 + (x + t + \tau_1 + \cdots + \tau_n - 1)^2}{1 + (x + t + \tau_1 + \cdots + \tau_n - 1)^2}, \qquad \sin\theta^{[n]} = \frac{2(x + t + \tau_1 + \cdots + \tau_n - 1)}{1 + (x + t + \tau_1 + \cdots + \tau_n - 1)^2},$$

solve Eqs. (83)–(85), thereby determining a correspondence between the Poincaré family of pseudo-spherical structures (81) and the family of pseudo-spherical structures defined by the one-forms $\Theta_\alpha^{[n]}$. Let $\Upsilon^{[n]}(x, t, \tau_1, \ldots, \tau_n)$ be the composition of $T^{[n]}$ and (155)–(157), and set

$$\mu^{[n]}(x, t, \tau_1, \ldots, \tau_n) = \theta^{[n]}(x, t, \tau_1, \ldots, \tau_n) - \left(\hat{\theta}^{[n]} \circ \Upsilon^{[n]}\right)(x, t, \tau_1, \ldots, \tau_n),$$

in which

$$\hat{\theta}^{[n]} = 2\arctan\left(\frac{\hat{x}(\hat{x}^3 - 12\hat{t})}{2(6\hat{t} + \hat{x}^3)}\right).$$

Then, the diffeomorphism $\Upsilon^{[n]}$ and the function $\mu^{[n]}$ satisfy the system of Eqs. (120)–(127) of Lemma 29, and Theorem 30 implies that the sequence $\{\hat{u}^{[n]}\colon n \geqslant 0\}$, given by

$$\hat{u}^{[n]}(x, t, \tau_1, \dots, \tau_n) = -\frac{2}{\hat{x}^2}, \tag{158}$$

is a solution to the K–dV hierarchy.

It is possible to find a transformation from the solution $u^{[n]} = x + t + \tau_1 + \dots + \tau_n$ to *arbitrary* stationary solutions to the (higher) K–dV equations, but computations become more involved. Details will appear elsewhere [26].

Bäcklund transformation for the K–dV hierarchy. As a final example, it will be shown that the Bäcklund transformation for the K–dV hierarchy can be recovered from the proof of Theorem 27. Indeed, this transformation appears if one considers Eq. (97). Recall that this equation reads

$$\Theta_1^{[n]} \cos\theta^{[n]} + \Theta_2^{[n]} \sin\theta^{[n]} = \Theta_3^{[n]} + d\theta^{[n]}, \tag{159}$$

and note that this Pfaffian system is integrable *if and only if* the structure equations (19) are satisfied. Changing variables by means of $\Gamma^{[n]} = \cot(\theta^{[n]}/2)$ and using formulae (135)–(140), one finds that (159) is equivalent to the Riccati system

$$\left(\Gamma^{[n]}\right)^2 + \lambda\Gamma^{[n]} + u = -\Gamma_x^{[n]}, \tag{160}$$

$$B^{(n+1)}\left(\Gamma^{[n]}\right)^2 + \left(\lambda B^{(n+1)} - B_x^{(n+1)}\right)\Gamma^{[n]} - \left(\frac{1}{2}\lambda B_x^{(n+1)} - \frac{1}{2}B_{xx}^{(n+1)} - uB^{(n+1)}\right) = -\Gamma_{\tau_n}^{[n]}. \tag{161}$$

Setting $\lambda = 0$, so that $B^{(i)} = B_i$ for all $i \geqslant 0$, one obtains the important Riccati system

$$\left(\Gamma^{[n]}\right)^2 + u = -\Gamma_x^{[n]}, \tag{162}$$

$$B_{n+1}\left(\Gamma^{[n]}\right)^2 - B_{n+1,x}\Gamma^{[n]} + \frac{1}{2}B_{n+1,xx} + uB_{n+1} = -\Gamma_{\tau_n}^{[n]}. \tag{163}$$

For instance, one can note that if $n = 0$, Eq. (162) is precisely the Miura transformation [6,8] and that replacing (162) into Eq. (163) yields the modified K–dV equation $\Gamma_t = (\Gamma_{xx} - 2\Gamma^3)_x$. In order to generalize this observation to arbitrary n one proceeds as follows, inspired once more by the beautiful paper [6]: Define a sequence of functions $R_j(\Gamma)$ by means of

$$R_{0,x} = 0, \tag{164}$$

$$\Gamma^{-1}R_{j+1,x} = \left[\Gamma^{-1}R_{j,x}\right]_{xx} - 4(\Gamma R_j)_x, \tag{165}$$

and also set

$$M_n(\Gamma) = \frac{1}{2}\Gamma^{-1}R_{n+1,x}. \tag{166}$$

The following technical lemma can be proven by a straightforward computation.

Lemma 31. *Let $\widetilde{B}_j = M_{j-1} + R_j$ and set $R_0 = 1$. The functions $\widetilde{B}_j$ satisfy the recursion relation* (141) *and* (142) *if u is replaced by $-\Gamma_x - \Gamma^2$.*

Thus, since Eqs. (141) and (142) determine the polynomials B_j, one concludes that if one substitutes $u = -\Gamma_x - \Gamma^2$ into B_j one obtains precisely $M_{j-1} + R_j$. Making this substitution in (163) and simplifying, one concludes that system (162), (163) is equivalent to

$$u = -\Gamma_x - \Gamma^2, \qquad -\frac{1}{2} M_{n+1} = \Gamma_{\tau_n}. \tag{167}$$

The second equation in (167) is the nth order modified K–dV. In order to obtain the Bäcklund transformation for the K–dV hierarchy one now follows the arguments of [6,7]: The functions R_j are invariant under the change from Γ to $-\Gamma$ and therefore the functions M_j satisfy $M_j(-\Gamma) = -M_j(\Gamma)$. It follows that the second equation of (167) is invariant under the change $\Gamma \mapsto -\Gamma$. The crucial observation is that the first equation of (167) *is not* invariant under this change of variables. It becomes

$$\tilde{u} = \Gamma_x - \Gamma^2,$$

which, by construction, determines a new solution to the K–dV hierarchy. Writing $u = w_x$ and $\tilde{u} = \widetilde{w}_x$ one easily obtains

$$-\frac{1}{2}(w - \widetilde{w}) = \Gamma,$$

and it follows that w and $\widetilde{w}$ satisfy the system

$$w_x + \widetilde{w}_x = -\frac{1}{2}(w - \widetilde{w}), \tag{168}$$

$$w_{\tau_i} - \widetilde{w}_{\tau_i} = M_{i+1}\left(-\frac{1}{2}(w - \widetilde{w})\right). \tag{169}$$

This is the classical Bäcklund transformation for the K–dV hierarchy.

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