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Estimates of initial conditions of parabolic equations and inequalities in infinite domains via lateral Cauchy data

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Abstract

A parabolic equation/inequality in an infinite domain is considered. The lateral Cauchy data are given at an arbitrary C^2-smooth lateral surface. The inverse problem of the interest of this paper consists in an estimate of the unknown initial condition via these Cauchy data.

Keywords: Inverse problem; Initial condition; Carleman estimate; Parabolic operator

1. Introduction

1.1. Statement of the main result

All functions considered in this paper are real-valued. Hence, Hilbert spaces considered here are spaces of real-valued functions. Let $\Omega \subseteq \mathbb{R}^n$ be a convex unbounded domain with the boundary $\partial\Omega \in C^1$. For any $T = \text{const} > 0$ denote $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$. For any function $s(x), x \in \mathbb{R}^n$ denote $s_i = \partial s/\partial x_i, i = 1, \ldots, n$, whenever the differentiation is appro-

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priate. Also, denote $\nabla s = (s_1, \ldots, s_n)$. Let $L = L(x,t,D)$ be an elliptic operator of the second order in Q_T,

$$Lu := L(x,t,D)u = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{ij} + \sum_{i,j=1}^{n} b^j(x,t)u_j + b^0(x,t)u,$$

with its principal part L_0,

$$L_0 u := L_0(x,t,D)u = \sum_{i,j=1}^{n} a^{ij}(x,t)u_{ij},$$

where coefficients

$$a^{ij} = a^{ji}, \quad a^{ij} \in C^1(\overline{Q}_T) \cap B(\overline{Q}_T); \quad a^{ij}_k, b^j, b^0 \in B(\overline{Q}_T),$$

where $B(\overline{Q}_T)$ is the set of functions bounded in $\overline{Q}_T$. Naturally, we assume the existence of two positive numbers $\sigma_1, \sigma_2, \sigma_1 \leqslant \sigma_2$, such that

$$\sigma_1 |\xi|^2 \leqslant \sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i \xi_j \leqslant \sigma_2 |\xi|^2, \quad \forall (x,t,\xi) \in \overline{Q}_T \times \mathbb{R}^n. \tag{1.1}$$

Let the function $u \in H^{2,1}(Q_T)$ be a solution of the parabolic equation

$$u_t = Lu + f(x,t), \quad \text{a.e. in } Q_T, \tag{1.2}$$

with the zero Dirichlet boundary condition

$$u|_{S_T} = 0 \tag{1.3}$$

and the initial condition $g(x)$

$$u(x,0) = g(x) \in H^1(\Omega), \tag{1.4}$$

where the function $f \in L_2(Q_T)$. In the case $\Omega = \mathbb{R}^n$ the boundary condition (1.3) is ignored and the classic Cauchy problem (1.2), (1.4) is considered. Along with Eq. (1.2) we will also consider a more general case of the parabolic inequality

$$|u_t - L_0 u| \leqslant M\big[|\nabla u| + |u| + \big|f(x,t)\big|\big], \quad \text{a.e. in } Q_T, \tag{1.5}$$

where the function $u \in H^{2,1}(Q_T)$ satisfies conditions (1.3), (1.4) and $M = \text{const} > 0$.

If the function $u \in H^{2,1}(Q_T)$ satisfies either conditions (1.2)–(1.4) or conditions (1.3)–(1.5), then the following classic estimates hold:

$$\|u\|^2_{H^{1,0}(Q_T)} \leqslant K\big(\|g\|^2_{L_2(\Omega)} + \|f\|^2_{L_2(Q_T)}\big), \tag{1.6}$$

$$\|u_t\|^2_{L_2(Q_T)} \leqslant K\big(\|g\|^2_{H^1(\Omega)} + \|f\|^2_{L_2(Q_T)}\big), \tag{1.7}$$

where the positive constant K depends on the domain Ω and numbers σ_1, σ_2, T, a and A; in the case of (1.2)–(1.4), and A should be replaced with M in the case of (1.3)–(1.5), see, e.g., Ladyzhenskaya, Solonnikov and Uraltceva [8]. Here

$$a = \max_{1 \leqslant i,j \leqslant n} \Bigl(\sup_{Q_T} \bigl| \nabla a^{ij} \bigr|, \sup_{Q_T} \bigl| a_t^{ij} \bigr| \Bigr) + 1 \tag{1.8}$$

and

$$A = \max_{0 \leqslant j \leqslant n} \Bigl(\sup_{Q_T} \bigl| b^j \bigr| \Bigr).$$

Let $P \subset \overline{\Omega}$, $P \in C^2$ be a finite hypersurface. In particular, in the case $\Omega \neq \mathbb{R}^n$ one might assume that $P \subset \partial\Omega$, although this is not necessary. Denote $P_T = P \times (0, T)$. Let $n = n(x)$, $x \in P$, be a unit normal vector on P. As to the direction of n: If $P \subset \partial\Omega$, then n is directed outwards of Ω. Alternatively, we choose any of two directions of n at an arbitrary point $x_0 \in P$ and since the function $n(x)$ is continuous on P, the direction of the vector $n(x)$ at all other points of P is uniquely defined.

Two inverse problems considered below have applications in such processes of diffusion, heat conduction and wave propagation, in which one is required to determine initial states using appropriate time dependent measurements at a surface.

Inverse Problem (IP). Assume that the following lateral Cauchy data $h^{(1)}(x,t)$ and $h^{(2)}(x,t)$ are given

$$u|_{P_T} = h^{(1)}(x,t), \qquad \left.\frac{\partial u}{\partial n}\right|_{P_T} = h^{(2)}(x,t), \tag{1.9}$$

where $P_T = P \times (0, T)$ and the function $u \in H^{2,1}(Q_T)$ satisfies either conditions (1.2)–(1.4) or conditions (1.3)–(1.5). Estimate the unknown initial condition g and the function u via functions $h^{(1)}, h^{(2)}$ and f.

This is an inverse problem of the determination of the initial condition in the parabolic equation using the lateral Cauchy data (1.9). Applications are in such diffusion and heat conduction processes, in which one is required to determine the initial state using time dependent measurements at a surface. We now describe a more specific applied example. Consider a cooling process of a solid. Suppose that its size of this solid is so large that one can assume that it coincides with an unbounded domain $\Omega \subseteq \mathbb{R}^3$. Assume that the initial temperature of this solid is high, unknown, and is a subject of ones interest. Suppose also that the major part of this solid is unavailable for the temperature measurements. Instead, one is measuring the time dependence of both the temperature u and the heat flux at a surface $P \subset \overline{\Omega}$ (the heat flux is proportional to the normal derivative $\partial u/\partial n$, at least in the case when the principal part of the operator L near the surface P is $L_0 = \Delta$). Hence, in this application the IP is the problem of the determination of the spatial distribution of the initial temperature $u(x, 0)$ of that solid from these surface measurements.

A particular use of this applied example is that it helps to understand the naturality of the assumption that in Theorem 1 *a priori* upper estimate is actually imposed on the norm $\| |\nabla g| \|_{L_2(\Omega)}$. Indeed, this assumption means *a priori* knowledge of the absence of high gradients in the initial temperature, which is quite natural in this application. A similar idea, although in a more general form, is one of the basic facts of the theory of ill-posed problems, and it was first introduced by Tikhonov in 1943 [12]; also see the book [13] for the Tikhonov fundamental theorem [12] about the continuity of the inverse operator on a compact set. In the applied literature, such a compact set is sometimes called "the set of admissible parameters." As to *a priori* bound of the norm $\|g\|^2_{L_2(\Omega\setminus\Phi)}$, it is often natural to assume in the above cooling process that an estimate of the initial temperature outside of a bounded domain of interest Φ is known.

Let $\Phi \subset \Omega$ be a convex bounded subdomain. We assume convexity for the sake of brevity only; it seems that results can be extended on the case of non-convex domains, although such an extension is outside of the scope of this paper. We shall say that Φ has the P-property, if the following two conditions are fulfilled: (1) For any point $x \in \Phi$ there exists a point $\tilde{x}(x) \in P$ such that the straight line connecting points x and $\tilde{x}$ does not lie in the hyperplane, which is tangent to the hypersurface P at the point $\tilde{x}$, and (2) $\operatorname{dist}[\overline{\Phi}, (\partial\Omega \setminus P)] > 0$, where $\operatorname{dist}[\overline{\Phi}, (\partial\Omega \setminus P)] := ds(\Phi)$ is the Hausdorff distance. An example of the P-property is the case when either $P \subseteq \partial\Phi$ or $P \subset \partial\Omega$ and $ds(\Phi) > 0$. Another example is when the hypersurface P is a part of a hyperplane, $P \subset \{x_1 = 0\}$, $ds(\Phi) > 0$ and either $\Phi \subset \{x_1 > 0\} \cap \Omega$ or $\Phi \subset \{x_1 < 0\} \cap \Omega$.

Theorem 1 is the main result of this publication. The new feature of this result is that the domain Ω is infinite, rather than finite, as in previous publications, see Section 1.2 for more details.

Theorem 1. *Suppose that above conditions imposed on coefficients of the operator $L(x,t,D)$, the domain Ω and the surface P are fulfilled. Let the function $u \in H^{2,1}(Q_T)$ satisfy conditions* (1.3)–(1.5). *Let $\Phi \subset \Omega$ be a convex bounded subdomain of the domain Ω, which has the P-property. Let the function $h^{(1)} \in H^{1,1}(P_T)$. Consider the vector-valued function $F = (h^{(1)}, h^{(2)}, f)$ and denote*

$$\|F\| = \left[\left\|h^{(1)}\right\|^2_{H^{1,1}(P_T)} + \left\|h^{(2)}\right\|^2_{L_2(P_T)} + \|f\|^2_{L_2(Q_T)}\right]^{1/2}.$$

Suppose that $\|F\| \leqslant B$, where B is a positive constant. Choose an arbitrary number $\mu \in (0,1)$. Then there exist constants $C_1 > 0$ and $\varepsilon_0 \in (0,1)$ such that the following stability estimate holds:

$$\|g\|^2_{L_2(\Phi)} \leqslant \frac{C_1}{\mu}\left[\ln\left(\frac{B}{\varepsilon_0\|F\|}\right)\right]^{-1}\left[\left\||\nabla g|\right\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\setminus\Phi)}\right] + C_1\left(\frac{B}{\varepsilon_0}\right)^{2\mu}\|F\|^{2(1-\mu)}. \tag{1.10}$$

Here the constant $C_1 = C_1(\sigma_1, \sigma_2, a, M, d(\Phi), ds(\Phi), P)$ depends on $\sigma_1, \sigma_2, a, T, M, d(\Phi)$, $ds(\Phi)$ and P, where $d(\Phi)$ is the diameter of the domain Φ. The constant ε_0 depends on the same parameters as ones listed for C_1, as well as on the number μ. In addition, the following estimate for the function u holds:

$$\|u\|_{H^{1,0}(Q_T)} \leqslant \frac{C_1}{\mu}\left[\ln\left(\frac{B}{\varepsilon_0\|F\|}\right)\right]^{-1}\left[\left\||\nabla g|\right\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\setminus\Phi)}\right] + C_1\left(\frac{B}{\varepsilon_0}\right)^{2\mu}\|F\|^{2(1-\mu)} + C_1\|g\|^2_{L_2(\Omega\setminus\Phi)}. \tag{1.11}$$

The major difficulty of the proof of this theorem is due to the fact that the idea of a combination of "lateral" and "backwards" Carleman estimates, which worked well in the case of finite domains in [7,14,15] (see some details in Section 1.2) cannot be applied to the case of an infinite domain Ω. This is because a lateral Carleman estimate would estimate the function $u(x,t_0)$ only in a finite subdomain $\Omega' \subset \Omega$, because lateral Carleman estimates in infinite domains are unknown. Here $t_0 \in (0,T)$ is a certain number. On the other hand, a backwards Carleman estimate would rely on an estimate of the function $u(x,t_0)$ in the entire domain Ω.

A lateral Carleman estimate is the one, which estimates solution of the parabolic equation (or inequality) via lateral Cauchy data. A backwards Carleman estimate is the one, which estimates the solution $u(x,t)$ of the parabolic equation for $t \in (b,c) \subset (0,T)$ via the function $u(x,c)$, i.e., it estimates solutions of parabolic equations with the reversed direction of time. In the above problem, a conventional lateral Carleman estimate of [9] would enable one to estimate the norms $\|u(x,t_0)\|_{L_2(\Phi)}$, $\||\nabla u(x,t_0)|\|_{L_2(\Phi)}$ via functions $h^{(1)}, h^{(2)}$ and f. At the same time, it is unclear how to *properly* estimate the norm $\|u(x,t_0)\|_{L_2(\Omega\setminus\Phi)}$ (at least in the case $n>1$), unless some stringent conditions would be imposed on the function $g(x)$. This prevents one from applying the backwards Carleman estimate on the second stage of the proof (unlike [7,14,15]), because the latter uses an estimate of the L_2-norm of the function $u(x,t_0)$ in the *entire* domain Ω [1,7,9,10].

To overcome this difficulty, we derive a new lateral Carleman estimate for the parabolic operator $\partial_t - L_0$ (Theorem 2 in Section 2). The level surface of the corresponding Carleman Weight Function (CWF) is contained in a thin strip $t \in \{|t-\delta| < \delta\sqrt{\omega_0}\}$, where $\delta > 0$ is sufficiently small and the number $\omega_0 \in (0,1)$. The main new feature of the estimate of Theorem 2 is that, unlike previously known Carleman estimates, this one does not break down when the width $2\delta\sqrt{\omega_0}$ of this strip approaches zero as $\delta = \delta(\|F\|) \to 0^+$ for $\|F\| \to 0$. This is achieved via an incorporation of the large parameter $1/\delta^2$ in the function $q(x,t)$ (Section 2). The authors believe that Theorem 2 might be of an interest in its own right for other possible applications.

Actually, we derive a pointwise Carleman estimate, see Chapter 4 of the book of Lavrent'ev, Romanov and Shishatskii [9]. The proof is cumbersome, which seems to be inevitable, see, e.g., Èmanuilov (Imanuvilov) [2], the book of Klibanov and Timonov [6] and Romanov [11] for some other examples of cumbersome proofs of Carleman estimates.

Since the function $u \in H^{2,1}(Q_T)$, functions $h^{(1)} \in H^{1,0}(P_T)$ and $h^{(2)} \in L_2(P_T)$ automatically. The condition $h^{(1)} \in H^{1,1}(P_T)$ means a little over-smoothness. It is fulfilled if, for example $u \in H^{2,1}(Q_T)$ and $u_t \in H^{1,0}(Q_T)$. If it is known *a priori* that $\operatorname{supp}(g) \subseteq \Phi$, then the term $\|g\|_{L_2(\Omega\setminus\Phi)}$ should be dropped in (1.10) and (1.11). The estimate (1.11) follows immediately from (1.10) and the standard estimate (1.6). Hence, we will concentrate on the proof of (1.10). It follows from (1.10) that the $L_2(\Phi)$-norm of the initial condition tends to zero with the speed proportional to the square root of the logarithm, as long as $\|F\| \to 0$. The above are the so-called "conditional stability estimates," see, e.g., [9] for the definition of conditional stability estimates. This is because these estimates rely on *a priori* upper bounds of the stronger norm $\||\nabla g|\|_{L_2(\Omega)}$ and the norm $\|g\|^2_{L_2(\Omega\setminus\Phi)}$.

Conditional rather than conventional (i.e., unconditional) stability estimates are inevitable in inverse problems, since they are ill-posed. In addition, because of the ill-posedness, it is natural in such an estimate to impose *a priori* bound on a certain norm of the data, e.g., $\|F\| \leqslant B$. In many works such a bound is replaced by the assumption that this norm is sufficiently small, because one is interested in the behavior of the solution when the error in the data tends to zero, see, e.g., Chapter 4 in [9]. One of basic facts of the theory of ill-posed problems, which follows from the above mentioned Tikhonov theorem is that a conditional stability estimate for an ill-posed problem enables one to obtain an *a priori* estimate of the difference between the approximate and the exact solutions of this problem, provided that the exact solution belongs to an *a priori* chosen compact (or, more generally, a bounded set), see, e.g., (2.6) in §1 of Chapter 2 of [9]. For example, in the case of the IP, that bounded set would be the set of all functions $g \in H^1(\Omega)$ whose norms $\| |\nabla g| \|_{L_2(\Omega)}, \|g\|_{L_2(\Omega \setminus \Phi)}$ are bounded by an *a priori* chosen constant D, and one would look to determine the initial condition (1.4) in an *a priori* chosen finite domain Φ. Such conditional stability estimates are usually quite helpful for establishing convergence rates of corresponding numerical methods, see, e.g., Section 2.5 in [6].

1.2. Published results

In the case $n > 1$ an analogue of Theorem 1 for an infinite domain Ω is unknown. Hölder stability estimates for solutions of parabolic equations and inequalities with the lateral Cauchy data are well known since the publication of the book [9]. They are obtained via Carleman estimates and hold in finite subdomains of cylinders Q_T which are bounded by lateral surfaces (of arbitrary shapes and sizes) and level surfaces of CWFs. These subdomains are finite and the domain G_0 in (3.9) is a typical example of such a subdomain, except that in previous publications the width of subdomains with respect to t was not "allowed" to tend to zero. Since those subdomains do not intersect with $\{t = 0, T\}$, then those Carleman estimates do not allow one to estimate initial conditions. Indeed, the break down at $t \to 0^+$. At the same time, the uniqueness of the IP follows from these results. The only known Carleman estimate which is valid in the entire cylinder Q_T is one of Èmanuilov (Imanuvilov) and Fursikov [2,3] and it is valid in the case of a finite domain Ω. The CWF of [2,3] vanishes exponentially at $\{t = 0, T\}$, which does not allow one to estimate the initial condition. Summarizing, the topic of stability estimates of initial conditions is more complicated one than its 'uniqueness counterpart.'

Stability estimates of initial conditions of parabolic equations via the lateral Cauchy data were obtained by Isakov and Kindermann [5], Xu and Yamamoto [14], Yamamoto and Zou [15], and Klibanov [7]. In [5] the equation $u_t = u_{zz}$, $z \in \mathbb{R}$, with the lateral Cauchy data at $\{z = 0, t \in (0, T)\}$ was considered, and the initial condition $u(z, 0)$ was estimated in a finite z-interval. The property of the analyticity of the function $u(z, t)$ with respect to $t > 0$ was used essentially in [5]. Note that this property cannot be guaranteed neither for a solution of Eq. (1.2) nor for a solution of the inequality (1.5). In [7,14,15] finite domains $\Omega \subset \mathbb{R}^n$ were considered. Proofs in these three references consist of two steps. First, the norms $\|u(x, t_0)\|_{L_2(\Omega)}, \| |\nabla u(x, t_0)| \|_{L_2(\Omega)}$ for a $t_0 \in (0, T)$ are estimated via the lateral Cauchy data using a Carleman estimate, which we call "lateral." On the second step the function $u(x, 0)$, $x \in \Omega$, is estimated via $u(x, t_0)$, using the backwards estimates. In [14] and [15] the heat equation $u_t = \Delta u$ was considered and the logarithmic stability method for the backwards estimate was used, see, e.g., books of Ames and Straugan [1] and Payne [10] for this method. Hence, results of [14] and [15] are also valid for parabolic equations with general self-adjoint operators L with x-dependent coefficients, since the logarithmic convexity method can be applied in this case. In [7] a certain newly observed feature of the backwards Carleman estimate for the parabolic operator led to the stability estimate for the inequality (1.5) for a general elliptic operator L with (x, t)-dependent coefficients.

In Section 2 we establish the above mentioned Carleman estimate. We prove Theorem 1 in Section 3.

2. Carleman estimate

Let $\Omega' \subset \Omega$ be a certain bounded subdomain and the function $p \in C^2(\overline{\Omega}')$ has the following properties:

$$p(x) \in (\beta, \gamma), \quad \forall x \in \Omega',$$
$$\left|\nabla p(x)\right| \in \left(1, p^1\right), \quad \forall x \in \Omega',$$

where numbers $\beta, \gamma \in (0,1)$, $\beta < \gamma$, and the number $p^1 > 1$. Let the number $\delta \in (0, \min(1, T/2))$. Consider the function

$$q(x,t) = p(x) + \frac{(t-\delta)^2}{\delta^2}.$$

Consider the domain G_0,

$$G_0 = \left\{(x,t)\colon\ x \in \Omega',\ q(x,t) < \gamma < 1\right\}. \tag{2.1}$$

Since $p(x) \in (\beta, \gamma)$ in Ω', then $G_0 \cap \{t = \delta\} \neq \emptyset$, which means that $G_0 \neq \emptyset$. By (2.1)

$$G_0 \subset \left\{t \in \delta(1 - \sqrt{\gamma}, 1 + \sqrt{\gamma}\,)\right\} \subset (0,T). \tag{2.2}$$

Also,

$$q_i = p_i. \tag{2.3}$$

By (2.3),

$$|\nabla q| \in \left[1, p^1\right] \quad \text{in } G_0. \tag{2.4}$$

We have

$$q_t = \frac{2(t-\delta)}{\delta^2}. \tag{2.5}$$

Hence,

$$|q_t| < \frac{2\sqrt{\gamma}}{\delta} \quad \text{in } G_0. \tag{2.6}$$

Denote

$$G_\omega = \left\{(x,t) \in \Omega' \times (0,T)\colon\ q(x,t) < \gamma - \omega\right\}, \quad \forall \omega \in (0, \gamma - \beta). \tag{2.7}$$

Obviously

$$G_{\omega_2} \subset G_{\omega_1} \subset G_0, \quad \forall \omega_1, \omega_2 \in (0, \gamma - \beta) \text{ with } \omega_1 < \omega_2. \tag{2.8}$$

Denote

$$\overline{p} = \max_{\overline{G}_0} |p_{xx}|. \tag{2.9}$$

Let $\nu \geqslant 1$ be a parameter which will be chosen later. We define the CWF $\varphi(x,t)$ as

$$\varphi(x,t) = \exp\left(\frac{q^{-\nu}}{\delta}\right). \tag{2.10}$$

In this section we prove

Theorem 2 *(Pointwise Carleman estimate). There exist a sufficiently large constant $\nu_0 = \nu_0(\sigma_1, \sigma_2, p^1, a, \overline{p}, \beta, \gamma) > 1$, a sufficiently small constant $\delta_0 = \delta_0(\sigma_1, \sigma_2, p^1, a, \overline{p}, \beta, \gamma) \in (0, \min(1, T/2))$ and a constant $C = C(\sigma_1, \sigma_2) > 1$ such that for all $\nu \geqslant \nu_0$, $\delta \in (0, \delta_0)$ and for all functions $u \in C^{2,1}(\overline{G}_0)$ the following pointwise Carleman estimate holds in $\overline{G}_0$:*

$$(u_t - L_0 u)^2 \varphi^2 \geqslant Ca\frac{\nu}{\delta}|\nabla u|^2\varphi^2 + C\frac{\nu^4}{\delta^3}q^{-2\nu-2}u^2\varphi^2 + \nabla \cdot U + V_t, \tag{2.11}$$

where the vector function U and the function V satisfy

$$|U| \leqslant C\frac{\nu^3}{\delta^3}q^{-2\nu-1}\big(|\nabla u|^2 + u_t^2 + u^2\big)\varphi^2, \tag{2.12}$$

$$|V| \leqslant C\frac{\nu^3}{\delta^3}q^{-2\nu-1}\big(|\nabla u|^2 + u_t^2 + u^2\big)\varphi^2. \tag{2.13}$$

Remark 2.1. The dependence of the parameter ν_0 from the number a in (1.8) occurs in the course of the proof only in (2.47), (2.50), and (2.63). This observation might be useful for further research. The Carleman estimate (2.11) has the same pointwise character as the well-known Carleman estimate of §1 of Chapter 4 in [9]. The main difference compared with [9], however, is that the large parameter $1/\delta$ is now included in both the function q in (2.5) and in $\exp(q^{-\nu}/\delta)$. This causes additional difficulties in the proof. While (2.11) is a pointwise estimate, it seems that its intregral analog can be derived via the pseudoconvexity [4]. However, such a development is outside of the scope of this publication.

The proof of this theorem consists of proofs of three lemmata. Below in this section notations and conditions of Theorem 2 hold. Also, ν_0, δ_0 and C denote different positive constants depending on parameters listed in the formulation of this theorem.

2.1. Lemma 1

Lemma 1. *There exist constants ν_0, δ_0, C such that for all $\nu \geqslant \nu_0$, $\delta \in (0, \delta_0)$ and for all functions $u \in C^{2,1}(\overline{G}_0)$ the following estimate holds in $\overline{G}_0$:*

$$(u_t - L_0 u)u\varphi^2 \geqslant \frac{\sigma_1}{2}|\nabla u|^2\varphi^2 - C\frac{\nu^2}{\delta^2}q^{-2\nu-2} \cdot \left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right)u^2\varphi^2 + \nabla \cdot U^{(1)} + V_t^{(1)}, \tag{2.14}$$

where the vector function $(U^{(1)}, V^{(1)})$ *satisfies the estimate*

$$\left|U^{(1)}\right| \leqslant C\left(|\nabla u|^2 + \frac{\nu}{\delta} q^{-\nu-1} u^2\right)\varphi^2, \qquad V^{(1)} = \frac{u^2}{2}\varphi^2. \tag{2.15}$$

Proof. We have

$$(u_t - L_0 u) u\varphi^2 = u_t u\varphi^2 - \sum_{i,j=1}^{n} a^{ij} u_{ij} u\varphi^2 = M + N, \tag{2.16}$$

where

$$M = u_t u\varphi^2$$

and

$$N = -\sum_{i,j=1}^{n} a^{ij} u_{ij} u\varphi^2.$$

Estimate from the below terms M and N separately in five steps.

Step 1. Estimate M. By (2.5)

$$\begin{aligned} M = u_t u\varphi^2 &= \left(\frac{u^2}{2}\varphi^2\right)_t + \frac{2\nu}{\delta} q^{-\nu-1} \cdot \frac{2(t-\delta)}{\delta^2} u^2\varphi^2 \\ &= \frac{4\nu}{\delta} q^{-\nu-1} \cdot \frac{(t-\delta)}{\delta^2} u^2\varphi^2 + V_t^{(1)}. \end{aligned}$$

Since by (2.1) $q < \gamma < 1$, then (2.5) and (2.6) imply that

$$M \geqslant -C\frac{\nu q^{-\nu-1}}{\delta^2} u^2\varphi^2 + V_t^{(1)}. \tag{2.17}$$

Step 2. Estimate N from the below,

$$\begin{aligned} N &= -\sum_{i,j=1}^{n} a^{ij} u_{ij} u\varphi^2 \\ &= \sum_{j=1}^{n}\left(-\sum_{i=1}^{n} a^{ij} u_i u\varphi^2\right)_j + \sum_{i,j=1}^{n} \left(a^{ij} u\varphi^2\right)_j u_i = \sum_{j=1}^{n}\left(-\sum_{i=1}^{n} a^{ij} u_i u\varphi^2\right)_j \\ &\quad + \sum_{i,j=1}^{n} a^{ij} u_i u_j \varphi^2 + \sum_{i,j=1}^{n} a^{ij}_j u_i u\varphi^2 - \frac{2\nu}{\delta} q^{-\nu-1} \sum_{i,j=1}^{n} a^{ij} p_j u_i u\varphi^2. \end{aligned} \tag{2.18}$$

Estimate from the below each term in the last line of (2.18) separately.

Step 3. Estimate the first term. We have

$$\sum_{i,j=1}^{n} a^{ij} u_i u_j \varphi^2 \geqslant \sigma_1 |\nabla u|^2 \varphi^2. \tag{2.19}$$

Step 4. Estimate the second term. The Cauchy–Schwarz inequality implies that

$$\sum_{i,j=1}^{n} a_j^{ij} u_i u \varphi^2 \geqslant -\frac{\sigma_1}{2} |\nabla u|^2 \varphi^2 - C a^2 u^2 \varphi^2. \tag{2.20}$$

Step 5. Estimate the third term. We have

$$\begin{aligned}
&-\frac{2\nu}{\delta} q^{-\nu-1} \sum_{i,j=1}^{n} a^{ij} p_j u_i u \varphi^2 \\
&\quad = \sum_{i=1}^{n} \left(-\frac{\nu}{\delta} q^{-\nu-1} \sum_{j=1}^{n} a^{ij} p_j u^2 \varphi^2 \right)_i - \frac{\nu(\nu+1)}{\delta} q^{-\nu-2} \left(\sum_{i,j=1}^{n} a^{ij} p_i p_j \right) u^2 \varphi^2 \\
&\qquad + \frac{\nu}{\delta} q^{-\nu-1} \sum_{i,j=1}^{n} \left(a_i^{ij} p_j + a^{ij} p_{ij} \right) u^2 \varphi^2 - \frac{2\nu^2}{\delta^2} q^{-2\nu-2} \left(\sum_{i,j=1}^{n} a^{ij} p_i p_j \right) u^2 \varphi^2.
\end{aligned} \tag{2.21}$$

By (1.1) and (2.4)

$$\sum_{i,j=1}^{n} a^{ij} p_i p_j \geqslant \sigma_1 \quad \text{in } G_0. \tag{2.22}$$

Hence

$$-\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j \right)^{-1} \geqslant -\frac{1}{\sigma_1}. \tag{2.23}$$

Hence, (2.9) and (2.21)–(2.23) imply that

$$\begin{aligned}
-\frac{2\nu}{\delta} q^{-\nu-1} \sum_{i,j=1}^{n} a^{ij} p_j u_i u \varphi^2 &\geqslant -C \frac{\nu^2}{\delta^2} q^{-2\nu-2} \left(\sum_{i,j=1}^{n} a^{ij} p_i p_j \right) \left(1 + \delta a q^{\nu+1} + \delta \overline{p} q^{\nu+1} \right) u^2 \varphi^2 \\
&\quad + \sum_{i=1}^{n} \left(-\frac{\nu}{\delta} q^{-\nu-1} \sum_{j=1}^{n} a^{ij} p_j u^2 \varphi^2 \right)_i.
\end{aligned}$$

Since $\nu \geqslant 1$ and $q < 1$, then $\delta a q^{\nu+1} + \delta \overline{p} q^{\nu+1} < 1$ for $\delta \in (0, \delta_0)$. Hence,

$$-\frac{2\nu}{\delta}q^{-\nu-1}\sum_{i,j=1}^{n}a^{ij}p_ju_iu\varphi^2 \geqslant -C\frac{\nu^2}{\delta^2}q^{-2\nu-2}\left(\sum_{i,j=1}^{n}a^{ij}p_ip_j\right)u^2\varphi^2$$
$$+\sum_{i=1}^{n}\left(-\frac{\nu}{\delta}q^{-\nu-1}\sum_{j=1}^{n}a^{ij}p_ju^2\varphi^2\right)_i. \tag{2.24}$$

Note that

$$\frac{\nu}{\delta^2}q^{-\nu-1} < \frac{\nu^2}{\delta^2}q^{-2\nu-2}, \quad \forall\nu \geqslant 1,\ \forall\delta > 0.$$

This, (2.17) and (2.23) imply that

$$M \geqslant -C\frac{\nu^2}{\delta^2}q^{-2\nu-2}\left(\sum_{i,j=1}^{n}a^{ij}p_ip_j\right)u^2\varphi^2 + V_t^{(1)}.$$

Combining this with (2.16), (2.18)–(2.20), we obtain

$$(u_t - L_0u)u\varphi^2 \geqslant \frac{\sigma_1}{2}|\nabla u|^2\varphi^2 - C\frac{\nu^2}{\delta^2}q^{-2\nu-2}\left(\sum_{i,j=1}^{n}a^{ij}p_ip_j\right)u^2\varphi^2 + \nabla\cdot U^{(1)} + V_t^{(1)}, \tag{2.25}$$

where

$$\nabla\cdot U^{(1)} = \sum_{i=1}^{n}\left(-\sum_{j=1}^{n}a^{ij}u_iu\varphi^2 - \frac{\nu}{\delta}q^{-\nu-1}\sum_{j=1}^{n}a^{ij}p_ju^2\varphi^2\right)_i, \qquad V^{(1)} = \frac{u^2}{2}\varphi^2. \tag{2.26}$$

Hence,

$$\left|U^{(1)}\right| \leqslant C\left(|\nabla u|^2 + \frac{\nu}{\delta}q^{-\nu-1}u^2\right)\varphi^2. \tag{2.27}$$

Relations (2.24)–(2.27) imply (2.14) and (2.15). □

2.2. Lemma 2

Lemma 2. *There exist constants ν_0, δ_0, C such that for all $\nu \geqslant \nu_0$, $\delta \in (0,\delta_0)$ and for all functions $u \in C^{2,1}(\overline{G}_0)$ the following estimate holds in $\overline{G}_0$:*

$$(u_t - L_0u)^2q^{\nu+2}\varphi^2 \geqslant -Ca\frac{\nu}{\delta}|\nabla u|^2\varphi^2 + C\frac{\nu^4}{\delta^3}q^{-2\nu-2}\left(\sum_{i,j=1}^{n}a^{ij}p_ip_j\right)^2u^2\varphi^2$$
$$+\nabla\cdot U^{(2)} + V_t^{(2)}, \tag{2.28}$$

where the vector function $(U^{(2)}, V^{(2)})$ satisfies the estimate

$$\left|U^{(2)}\right| \leqslant C\frac{\nu^3}{\delta^3}q^{-2\nu-1}\left(|\nabla u|^2+u_t^2+u^2\right)\varphi^2, \tag{2.29}$$

$$\left|V^{(2)}\right| \leqslant C\frac{\nu^2}{\delta^2}q^{-\nu}u^2\varphi^2. \tag{2.30}$$

Proof. Denote

$$v = u\varphi = u\exp\left(\frac{q^{-\nu}}{\delta}\right).$$

Hence,

$$u = v\varphi^{-1} = v\exp\left(-\frac{q^{-\nu}}{\delta}\right).$$

Express derivatives of the function u through derivatives of the function v,

$$u_t = \left(v_t + \frac{2\nu q^{-\nu-1}}{\delta}\cdot\frac{(t-\delta)}{\delta^2}v\right)\varphi^{-1},$$

$$u_i = \left(v_i + \frac{\nu q^{-\nu-1}}{\delta}p_i v\right)\varphi^{-1},$$

$$u_{ij} = \left[v_{ij} + \frac{\nu q^{-\nu-1}}{\delta}(p_i v_j + p_j v_i)\right]\varphi^{-1} + \left[\frac{\nu^2 q^{-2\nu-2}}{\delta^2}p_i p_j - \frac{\nu(\nu+1)q^{-\nu-2}}{\delta}p_i p_j + \frac{\nu q^{-\nu-1}}{\delta}p_{ij}\right]v\varphi^{-1}.$$

Hence,

$$(u_t - L_0 u)^2 q^{\nu+2}\varphi^2 = (z_1+z_2+z_3+z_4+z_5)^2 q^{\nu+2} = \left[z_1+z_3+(z_2+z_4+z_5)\right]^2 q^{\nu+2}.$$

Hence,

$$(u_t - L_0 u)^2 q^{\nu+2}\varphi^2 \geqslant \left[z_1^2 + 2z_1z_2 + 2z_1z_3 + z_3^2 + 2z_2z_3 + 2z_1(z_4+z_5) + 2z_3(z_4+z_5)\right]q^{\nu+2}, \tag{2.31}$$

where

$$z_1 = v_t,$$

$$z_2 = -\sum_{i,j=1}^{n} a^{ij}v_{ij},$$

$$z_3 = -\frac{\nu q^{-\nu-1}}{\delta}\sum_{i,j=1}^{n} a^{ij}(p_i v_j + p_j v_i),$$

$$z_4=-\frac{\nu^2q^{-2\nu-2}}{\delta^2}\sum_{i,j=1}^{n}a^{ij}p_ip_j\big(1+O\big(\delta q^{\nu}\big)\big)v,$$

$$z_5=\frac{2\nu q^{-\nu-1}}{\delta}\cdot\frac{(t-\delta)}{\delta^2}v.$$

Here and below $O(\delta q^{\nu})$ denotes different $C^1(\overline{G}_0)$-functions independent on the function v and such that $\lim_{\delta\to 0}O(\delta q^{\nu})=0$, uniformly for all $\nu\geqslant 1$, $\gamma\in(0,1)$ and all $(x,t)\in G_0$. The same is true for their first x-derivatives. As to the t-derivative: by (2.6)

$$\left|\frac{\partial}{\partial t}O\big(\delta q^{\nu}\big)\right|\leqslant C\nu. \tag{2.32}$$

The major part of the proof of Lemma 2 consists in estimating from the below each term in the right-hand side of (2.31). This is done in five steps below.

Step 1. Estimate $2z_1z_2q^{\nu+2}$. We have

$$\begin{aligned}
2z_1z_2q^{\nu+2}={}&-\sum_{i,j=1}^{n}a^{ij}v_t(v_{ij}+v_{ji})q^{\nu+2}\\
={}&\sum_{j=1}^{n}\left(-\sum_{i=1}^{n}a^{ij}v_tv_iq^{\nu+2}\right)_j+\sum_{i,j=1}^{n}a^{ij}v_{tj}v_iq^{\nu+2}\\
&+v_t\sum_{i,j=1}^{n}a^{ij}_jv_iq^{\nu+2}+(\nu+2)q^{\nu+1}v_t\sum_{i,j=1}^{n}a^{ij}v_ip_j\\
&+\sum_{i=1}^{n}\left(-2\sum_{j=1}^{n}a^{ij}v_tv_jq^{\nu+2}\right)_i+\sum_{i,j=1}^{n}a^{ij}v_{ti}v_jq^{\nu+2}\\
&+v_t\sum_{i,j=1}^{n}a^{ij}_iv_jq^{\nu+2}+(\nu+2)q^{\nu+1}v_t\sum_{i,j=1}^{n}a^{ij}v_jp_i\\
={}&\sum_{j=1}^{n}\left(-2\sum_{j=1}^{n}a^{ij}v_tv_iq^{\nu+2}\right)_j+\sum_{i,j=1}^{n}a^{ij}(v_{tj}v_i+v_{ti}v_j)q^{\nu+2}\\
&+2v_t\left(\sum_{i,j=1}^{n}a^{ij}_jv_iq^{\nu+2}+(\nu+2)q^{\nu+1}\sum_{i,j=1}^{n}a^{ij}v_jp_i\right)\\
={}&\sum_{j=1}^{n}\left(-2\sum_{i=1}^{n}a^{ij}v_tv_iq^{\nu+2}\right)_j+2v_t\left(\sum_{i,j=1}^{n}a^{ij}_jv_iq^{\nu+2}+(\nu+2)q^{\nu+1}\sum_{i,j=1}^{n}a^{ij}v_jp_i\right)\\
&+\left(\sum_{i,j=1}^{n}a^{ij}v_iv_jq^{\nu+2}\right)_t-\sum_{i,j=1}^{n}a^{ij}_tv_iv_jq^{\nu+2}-2(\nu+2)q^{\nu+1}\frac{(t-\delta)}{\delta^2}\sum_{i,j=1}^{n}a^{ij}v_iv_j.
\end{aligned}$$

Hence,

$$2z_1z_2q^{\nu+2} = 2v_t\left(\sum_{i,j=1}^{n} a_j^{ij} v_i q^{\nu+2} + (\nu+2)q^{\nu+1}\sum_{i,j=1}^{n} a^{ij} v_j p_i\right) - \sum_{i,j=1}^{n} a_t^{ij} v_i v_j q^{\nu+2} - 2(\nu+2)q^{\nu+1}\frac{(t-\delta)}{\delta^2}\sum_{i,j=1}^{n} a^{ij} v_i v_j + \sum_{j=1}^{n}\left(-2\sum_{i=1}^{n} a^{ij} v_t v_i q^{\nu+2}\right)_j + \left(\sum_{i,j=1}^{n} a^{ij} v_i v_j q^{\nu+2}\right)_t. \tag{2.33}$$

Using (1.1), (2.1) and (2.6), we obtain

$$-\sum_{i,j=1}^{n} a_t^{ij} v_i v_j q^{\nu+2} - 2(\nu+2)q^{\nu+1}\frac{(t-\delta)}{\delta^2}\sum_{i,j=1}^{n} a^{ij} v_i v_j \geqslant -C\frac{\nu q^{\nu+1}}{\delta}(1+\delta a)|\nabla v|^2.$$

Since $\delta a < 1$, then $-C(1+\delta a) > -2C$. Hence, with a new constant C (2.33) leads to

$$2z_1z_2q^{\nu+2} \geqslant -C\frac{\nu q^{\nu+1}}{\delta}|\nabla v|^2 + 2z_1\left((\nu+2)q^{\nu+1}\sum_{i,j=1}^{n} a^{ij} v_j p_i + \sum_{i,j=1}^{n} a_j^{ij} v_i q^{\nu+2}\right) + \nabla\cdot U^{(2,1)} + V_t^{(2,1)}, \tag{2.34}$$

where

$$\left|U^{(2,1)}\right| \leqslant C\frac{\nu^2 q^{-\nu}}{\delta^2}\left(|\nabla u|^2 + u_t^2 + u^2\right)\varphi^2, \tag{2.35}$$

$$\left|V^{(2,1)}\right| \leqslant C\frac{\nu^2 q^{\nu}}{\delta^2}\left(|\nabla u|^2 + u^2\right)\varphi^2. \tag{2.36}$$

Step 2. Estimate $(z_1^2 + z_3^2 + 2z_1z_3 + 2z_1z_2)q^{\nu+2}$. Using (2.34)–(2.36), we obtain

$$\begin{aligned}
&\left(z_1^2 + z_3^2 + 2z_1z_3 + 2z_1z_2\right)q^{\nu+2} \\
&\quad\geqslant -C\frac{\nu q^{\nu+1}}{\delta}|\nabla v|^2 + \left(z_1^2 + z_3^2\right)q^{\nu+2} \\
&\qquad + 2z_1\left(z_3 + (\nu+2)q^{-1}\sum_{i,j=1}^{n} a^{ij} v_j p_i + \sum_{i,j=1}^{n} a_j^{ij} v_i\right)q^{\nu+2} + \nabla\cdot U^{(2,1)} + V_t^{(2,1)} \\
&\quad\geqslant -C\frac{\nu q^{\nu+1}}{\delta}|\nabla v|^2 + z_3^2 q^{\nu+2} - \left(z_3 + (\nu+2)q^{-1}\sum_{i,j=1}^{n} a^{ij} v_j p_i + 2\sum_{i,j=1}^{n} a_j^{ij} v_i\right)^2 q^{\nu+2} \\
&\qquad + \nabla\cdot U^{(2,1)} + V_t^{(2,1)}.
\end{aligned} \tag{2.37}$$

Estimate the sum of the second and third terms in the 4th line of (2.37). We have

$$\begin{aligned}
&z_3^2 q^{\nu+2} - \left(z_3 + (\nu+2)q^{-1}\sum_{i,j=1}^n a^{ij} v_j p_i + \sum_{i,j=1}^n a_j^{ij} v_i\right)^2 q^{\nu+2} \\
&\quad = -2(\nu+2)q^{\nu+1} z_3 \sum_{i,j=1}^n a^{ij} v_j p_i - 2q^{\nu+2} z_3 \sum_{i,j=1}^n a_j^{ij} v_i \\
&\quad - (\nu+2)^2 q^{\nu}\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)^2 - \left(\sum_{i,j=1}^n a_j^{ij} v_i\right)^2 q^{\nu+2} \\
&\quad - 2(\nu+2)q^{\nu+1}\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)\left(\sum_{k,s=1}^n a_s^{ks} v_k\right).
\end{aligned} \tag{2.38}$$

We have

$$-(\nu+2)q^{\nu+1} z_3 \sum_{i,j=1}^n a^{ij} v_j p_i = 2\frac{\nu(\nu+2)}{\delta}\left(\sum_{i,j=1}^n a^{ij} p_i v_j\right)^2. \tag{2.39}$$

Also, by the Cauchy–Schwarz inequality and (1.8)

$$\begin{aligned}
&-2(\nu+2)q^{\nu+1}\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)\left(\sum_{k,s=1}^n a_s^{ks} v_k\right) - \left(\sum_{i,j=1}^n a_j^{ij} v_i\right)^2 q^{\nu+2} \\
&\quad \geqslant -(\nu+2)^2 q^{\nu}\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)^2 - Ca^2 q^{\nu+2}|\nabla v|^2.
\end{aligned}$$

Hence, (2.39) implies that the right-hand side of the equality (2.38) can be estimated as

$$\begin{aligned}
&-2(\nu+2)q^{\nu+1} z_3 \sum_{i,j=1}^n a^{ij} v_j p_i - 2q^{\nu+2} z_3 \sum_{i,j=1}^n a_j^{ij} v_i - (\nu+2)^2 q^{\nu}\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)^2 \\
&\quad - \left(\sum_{i,j=1}^n a_j^{ij} v_i\right)^2 q^{\nu+2} - 2(\nu+2)q^{\nu+1}\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)\left(\sum_{k,s=1}^n a_s^{ks} v_k\right) \\
&\quad \geqslant 2\frac{\nu(\nu+2)}{\delta}\bigl(1 + O\bigl(\delta q^{\nu}\bigr)\bigr)\left(\sum_{i,j=1}^n a^{ij} v_j p_i\right)^2 - Ca^2 q^{\nu+2}|\nabla v|^2 \geqslant -Ca^2 q^{\nu+2}|\nabla v|^2.
\end{aligned}$$

Substituting this in (2.38), we obtain with a new constant C

$$z_3^2 q^{\nu+2} - \left(z_3 + 2(\nu+2)q^{-1}\sum_{i,j=1}^n a^{ij} v_j p_i + 2\sum_{i,j=1}^n a_j^{ij} v_i\right)^2 q^{\nu+2} \geqslant -Ca^2 q^{\nu+2}|\nabla v|^2.$$

Hence, (2.37) implies that

$$\left(z_1^2+z_3^2+2z_1z_3+2z_1z_2\right)q^{\nu+2}\geqslant -C\frac{\nu}{\delta}|\nabla v|^2+\nabla\cdot U^{(2,1)}+V_t^{(2,1)}. \tag{2.40}$$

Step 3. Estimate $2z_1(z_4+z_5)q^{\nu+2}$. We have

$$\begin{aligned}
2z_1(z_4+z_5)q^{\nu+2} &= -2\frac{\nu^2q^{-\nu}}{\delta^2}\sum_{i,j=1}^n a^{ij}p_ip_j\left(1+O\left(\delta q^\nu\right)\right)v_tv-\frac{4\nu q}{\delta}\cdot\frac{(t-\delta)}{\delta^2}v_tv\\
&= \left[-\frac{\nu^2q^{-\nu}}{\delta^2}\sum_{i,j=1}^n a^{ij}p_ip_j\left(1+O\left(\delta q^\nu\right)\right)v^2-\frac{2\nu q}{\delta}\cdot\frac{(t-\delta)}{\delta^2}v^2\right]_t\\
&\quad -\frac{2\nu^3q^{-\nu-1}}{\delta^2}\cdot\frac{(t-\delta)}{\delta^2}\sum_{i,j=1}^n a^{ij}p_ip_j\left(1+O\left(\delta q^\nu\right)\right)v^2\\
&\quad +2\frac{\nu^2q^{-\nu}}{\delta^2}\sum_{i,j=1}^n a^{ij}p_ip_j\left[O\left(\delta q^\nu\right)\right]_tv^2\\
&\quad +\frac{4\nu}{\delta}\cdot\frac{(t-\delta)^2}{\delta^4}v^2+\frac{2\nu q}{\delta^3}v^2+\frac{\nu^2q^{-\nu}}{\delta^2}\sum_{i,j=1}^n a_t^{ij}p_ip_j\left(1+O\left(\delta q^\nu\right)\right)v^2.
\end{aligned}$$

Hence, using (2.32), we obtain

$$2z_1(z_4+z_5)q^{\nu+2}\geqslant -C\frac{\nu^3q^{-\nu-1}}{\delta^3}\left(\sum_{i,j=1}^n a^{ij}p_ip_j\right)v^2+V_t^{(2,2)}, \tag{2.41}$$

where

$$\left|V^{(2,2)}\right|\leqslant C\frac{\nu^2q^{-\nu}}{\delta^2}u^2\varphi^2. \tag{2.42}$$

Step 4. Estimate $2z_3(z_4+z_5)q^{\nu+2}$. We have

$$\begin{aligned}
2z_3(z_4+z_5)q^{\nu+2} &= 4\frac{\nu q}{\delta}\left(\sum_{i,j=1}^n a^{ij}p_iv_j\right)\\
&\quad\times\left(\frac{\nu^2q^{-2\nu-2}}{\delta^2}\sum_{k,s=1}^n a^{ks}p_kp_s\left(1+O\left(\delta q^\nu\right)\right)v+\frac{2\nu q^{-\nu-1}}{\delta}\cdot\frac{(t-\delta)}{\delta^2}v\right).
\end{aligned} \tag{2.43}$$

First, estimate I_1, where

$$
\begin{aligned}
I_1 &= 4\frac{\nu^3 q^{-2\nu-1}}{\delta^3}\left(\sum_{i,j=1}^{n} a^{ij} p_i v_j v\right)\left(\sum_{k,s=1}^{n} a^{ks} p_k p_s\big(1+O\big(\delta q^{\nu}\big)\big)\right) \\
&= \sum_{j=1}^{n}\left[2\frac{\nu^3 q^{-2\nu-1}}{\delta^3}\left(\sum_{i=1}^{n} a^{ij} p_i\right)\left(\sum_{k,s=1}^{n} a^{ks} p_k p_s\big(1+O\big(\delta q^{\nu}\big)\big)\right)v^2\right]_j \\
&\quad + 2\frac{\nu^3(2\nu+1) q^{-2\nu-2}}{\delta^3}\left(\sum_{i=1}^{n} a^{ij} p_i p_j\right)\left(\sum_{k,s=1}^{n} a^{ks} p_k p_s\big(1+O\big(\delta q^{\nu}\big)\big)\right)v^2 \\
&\quad - 2\frac{\nu^3 q^{-2\nu-1}}{\delta^3}\left[\left(\sum_{i=1}^{n} a^{ij} p_i\right)\left(\sum_{k,s=1}^{n} a^{ks} p_k p_s\big(1+O\big(\delta q^{\nu}\big)\big)\right)\right]_j v^2.
\end{aligned} \tag{2.44}
$$

Since $1+O(\delta q^{\nu}) \geqslant 1/2$, we obtain

$$
\begin{aligned}
&2\frac{\nu^3(2\nu+1) q^{-2\nu-2}}{\delta^3}\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right)\left(\sum_{k,s=1}^{n} a^{ks} p_k p_s\big(1+O\big(\delta q^{\nu}\big)\big)\right)v^2 \\
&\quad \geqslant 2\frac{\nu^4 q^{-2\nu-2}}{\delta^3}\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right)^2 v^2.
\end{aligned} \tag{2.45}
$$

In addition, (1.1), (1.8) and (2.4) imply that

$$
\begin{aligned}
&-2\frac{\nu^3 q^{-2\nu-1}}{\delta^3}\left[\left(\sum_{i=1}^{n} a^{ij} p_i\right)\left(\sum_{k,s=1}^{n} a^{ks} p_k p_s\big(1+O\big(\delta q^{\nu}\big)\big)\right)\right]_j v^2 \\
&\quad \geqslant -Ca\big(p^1\big)^3\frac{\nu^3 q^{-2\nu-2}}{\delta^3} v^2.
\end{aligned} \tag{2.46}
$$

One can choose $\nu_0 = \nu_0(\sigma_1, \sigma_2, p^1, a)$ so large that

$$
\nu > \frac{Ca(p^1)^3}{2\sigma_1^2}, \quad \forall \nu \geqslant \nu_0. \tag{2.47}
$$

Hence, using (2.23) and (2.44)–(2.47), we obtain

$$
I_1 \geqslant \frac{\nu^4 q^{-2\nu-2}}{\delta^3}\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right)^2 v^2 + \nabla\cdot U^{(2,2)}, \quad \forall \nu \geqslant \nu_0, \tag{2.48}
$$

where

$$
\big|U^{(2,2)}\big| \leqslant C\frac{\nu^3 q^{-2\nu-1}}{\delta^3} u^2 \varphi^2. \tag{2.49}
$$

Because of (2.43), we now should estimate I_2, where

$$
\begin{aligned}
I_2 &= -8\frac{\nu^2 q^{-\nu}}{\delta^2}\left(\sum_{i,j=1}^{n} a^{ij} p_i v_j\right)\cdot\frac{(t-\delta)}{\delta^2} v \\
&= \sum_{j=1}^{n}\left[-4\frac{\nu^2 q^{-\nu}}{\delta^2}\cdot\frac{(t-\delta)}{\delta^2} v^2 \sum_{i=1}^{n} a^{ij} p_i\right]_j \\
&\quad - 4\frac{\nu^3 q^{-\nu-1}}{\delta^2}\cdot\frac{(t-\delta)}{\alpha^2\delta^2}\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right) v^2 + 4\frac{\nu^2 q^{-\nu}}{\delta^2}\cdot\frac{(t-\delta)}{\alpha^2\delta^2}\left(\sum_{i=1}^{n} a^{ij} p_i\right)_j v^2 \\
&\geqslant -C\frac{\nu^3 q^{-\nu-1}}{\delta^3}\left(1+\frac{a}{\nu}\right)\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right) v^2 + \nabla\cdot U^{(2,3)}.
\end{aligned}
$$

Hence, assuming that in addition to (2.47)

$$
\frac{a}{\nu} < 1, \quad \forall \nu \geqslant \nu_0, \tag{2.50}
$$

we obtain

$$
I_2 \geqslant -C\frac{\nu^3 q^{-\nu-1}}{\delta^3}\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right) v^2 + \nabla\cdot U^{(2,3)}, \tag{2.51}
$$

where

$$
\left|U^{(2,3)}\right| \leqslant C\frac{\nu^2 q^{-\nu}}{\delta^3} u^2 \varphi^2. \tag{2.52}
$$

By (2.1) $q^{-2\nu-2} > 2Cq^{-\nu-1}$. Also, by (2.43) $2z_3(z_4+z_5)q^{\nu+2} = I_1 + I_2$. Hence, (2.48)–(2.52) lead to

$$
2z_3(z_4+z_5)q^{\nu+2} \geqslant \frac{\nu^4 q^{-2\nu-2}}{2\delta^3}\left(\sum_{i,j=1}^{n} a^{ij} p_i p_j\right)^2 v^2 + \nabla\cdot U^{(2,4)}, \tag{2.53}
$$

where $U^{(2,4)} = U^{(2,2)} + U^{(2,3)}$ and

$$
\left|U^{(2,4)}\right| \leqslant C\frac{\nu^2 q^{-2\nu-1}}{\delta^3} u^2 \varphi^2. \tag{2.54}
$$

Step 5. We now estimate $2z_2z_3q^{\nu+2}$. We have

$$
\begin{aligned}
2z_2z_3q^{\nu+2} &= 2\frac{\nu}{\delta} q \sum_{i,j,k,s=1}^{n} a^{ij} a^{ks} v_{ij}(p_k v_s + p_s v_k) \\
&= \sum_{j=1}^{n}\left(2\frac{\nu}{\delta} q \sum_{i,k,s=1}^{n} a^{ij} a^{ks} v_i (p_k v_s + p_s v_k)\right)_j
\end{aligned}
$$

$$
\begin{aligned}
&-2\frac{\nu}{\delta}\sum_{i,j,k,s=1}^{n}\left[\left(a^{ij}a^{ks}q\right)_j v_i(p_k v_s+p_s v_k)+a^{ij}a^{ks}qv_i(p_{kj}v_s+p_{sj}v_k)\right]\\
&-2\frac{\nu}{\delta}\sum_{i,j,k,s=1}^{n}a^{ij}a^{ks}qv_i(p_k v_{sj}+p_s v_{kj})\\
&\geqslant -Ca\frac{\nu}{\delta}|\nabla v|^2-2\frac{\nu}{\delta}\sum_{i,j,k,s=1}^{n}a^{ij}a^{ks}qv_i(p_k v_{sj}+p_s v_{kj})+\nabla\cdot U^{(2,5)}, \qquad (2.55)
\end{aligned}
$$

where

$$
\nabla\cdot U^{(2,5)}=\sum_{j=1}^{n}\left(2\frac{\nu}{\delta}q\sum_{i,k,s=1}^{n}a^{ij}a^{ks}v_i(p_k v_s+p_s v_k)\right)_j. \qquad (2.56)
$$

Estimate from the below the second term in the right-hand side of the inequality (2.55). We have

$$
\begin{aligned}
&-2\frac{\nu}{\delta}\sum_{i,j,k,s=1}^{n}a^{ij}a^{ks}qv_i(p_k v_{sj}+p_s v_{kj})\\
&=-4\frac{\nu}{\delta}\sum_{i,j,k,s=1}^{n}a^{ks}a^{ij}qp_k v_i v_{sj}=-4\frac{\nu}{\delta}\sum_{k,s=1}^{n}a^{ks}qp_k\left(\sum_{i,j=1}^{n}a^{ij}v_i v_{sj}\right)\\
&=-2\frac{\nu}{\delta}\sum_{k,s=1}^{n}a^{ks}qp_k\left(\sum_{i,j=1}^{n}a^{ij}(v_i v_{sj}+v_j v_{si})\right).
\end{aligned}
\qquad (2.57)
$$

Since $(v_i v_{sj}+v_j v_{si})=(v_i v_j)_s$, then

$$
\begin{aligned}
&-2\frac{\nu}{\delta}\sum_{k,s=1}^{n}a^{ks}qp_k\left(\sum_{i,j=1}^{n}a^{ij}(v_i v_{sj}+v_j v_{si})\right)=-2\frac{\nu}{\delta}\sum_{i,j,k=1}^{n}a^{ij}qp_k\sum_{s=1}^{n}a^{ks}(v_i v_j)_s\\
&=\sum_{s=1}^{n}\left(-2\frac{\nu}{\delta}\sum_{i,j,k=1}^{n}a^{ks}a^{ij}qp_k v_i v_j\right)_s+2\frac{\nu}{\delta}\sum_{i,j,k=1}^{n}\left(\sum_{s=1}^{n}\left(a^{ks}a^{ij}qp_k\right)_s\right)v_i v_j\\
&\geqslant -Ca\frac{\nu}{\delta}|\nabla v|^2+\nabla\cdot U^{(2,6)}, \qquad (2.58)
\end{aligned}
$$

where

$$
\nabla\cdot U^{(2,6)}=\sum_{s=1}^{n}\left(-2\frac{k\nu}{\delta}\sum_{i,j,k=1}^{n}a^{ks}a^{ij}qp_k v_i v_j\right)_s. \qquad (2.59)
$$

Thus, (2.55)–(2.59) lead to

$$
2z_2z_3q^{\nu+2}\geqslant -Ca\frac{\nu}{\delta}|\nabla v|^2+\nabla\cdot U^{(2,7)}, \qquad (2.60)
$$

where $U^{(2,7)} = U^{(2,5)} + U^{(2,6)}$ where

$$\left|U^{(2,7)}\right| \leqslant C\frac{\nu^3}{\delta^3}q^{-2\nu-1}\left(|\nabla u|^2 + u^2\right)\varphi^2. \tag{2.61}$$

The estimate (2.61) is obtained via expressing the function v and its first derivatives through the function $u = v\varphi^{-1}$ and its first derivatives.

We are now ready to obtain estimates (2.28)–(2.30). Sum up (2.40), (2.41), (2.53) and (2.60) and compare with (2.31). Also, sum up expressions for divergent terms and use estimates (2.35), (2.36), (2.42), (2.54) and (2.61) for them. Then express the function v and its first derivatives through the function $u = v\varphi^{-1}$ and its first derivatives. Then we obtain estimates (2.28)–(2.30). $\square$

2.3. Proof of Theorem 2

Multiply the inequality (2.14) by $4Ca\nu/(\delta\sigma_1)$ and sum up with the inequality (2.28). Also, using (2.15) and (2.29) and (2.30), denote

$$U = 4Ca\frac{\nu}{\delta\sigma_1}U^{(1)} + U^{(2)}, \qquad V^{(3)} = 4Ca\frac{\nu}{\delta\sigma_1}V^{(1)} + V^{(2)}.$$

We obtain

$$\begin{aligned} &4Ca\frac{\nu}{\delta\sigma_1}(u_t - L_0u)u\varphi^2 + (u_t - L_0u)^2q^{\nu+2}\varphi^2 \\ &\quad\geqslant Ca\frac{\nu}{\delta}|\nabla u|^2\varphi^2 + C\frac{\nu^4}{\delta^3}q^{-2\nu-2}\left[1 - \frac{4Ca}{\sigma_1\nu}\cdot\left(\sum_{i,j=1}^n a^{ij}p_ip_j\right)^{-1}\right]\left(\sum_{i,j=1}^n a^{ij}p_ip_j\right)^2u^2\varphi^2 \\ &\qquad + \nabla\cdot U + V_t, \end{aligned} \tag{2.62}$$

where the vector function (U, V) satisfies the estimate (2.12). Choose $\nu_0 = \nu_0(\sigma_1, \sigma_2, p^1, a)$ so large that in addition to (2.47) and (2.50)

$$1 - \frac{4Ca}{\sigma_1^2}\cdot\frac{1}{\nu} < \frac{1}{2}, \quad \forall\nu \geqslant \nu_0. \tag{2.63}$$

Then (2.23) and (2.62) imply that for $\nu \geqslant \nu_0$

$$\begin{aligned} &4Ca\frac{\nu}{\delta\sigma_1}(u_t - L_0u)u\varphi^2 + (u_t - L_0u)^2q^{\nu+2}\varphi^2 \\ &\quad\geqslant Ca\frac{\nu}{\delta}|\nabla u|^2\varphi^2 + C\frac{\nu^4}{\delta^3}q^{-2\nu-2}u^2\varphi^2 + \nabla\cdot U + V_t. \end{aligned} \tag{2.64}$$

Note that

$$4Ca\frac{\nu}{\delta\sigma_1}(u_t - L_0u)u\varphi^2 + (u_t - L_0u)^2 q^{\nu+2}\varphi^2$$
$$\leqslant 3(u_t - L_0u)^2\varphi^2 + 2\left(\frac{Ca}{\sigma_1}\right)^2 \frac{\nu^2}{\delta^2}u^2\varphi^2.$$

Substituting this in (2.64), we obtain (2.11).

3. Proof of Theorem 1

Change variables

$$(x', t') = \left(\frac{x}{2d(\Phi)}, \frac{t}{4d^2(\Phi)}\right), \tag{3.1}$$

leaving for new variables, domains and coefficients of the operator L the same notations as before, for brevity. By (3.1)

$$|x| \leqslant \frac{1}{2}, \quad \forall x \in \overline{\Phi}. \tag{3.2}$$

The number a in (1.7) is replaced with

$$a_1 = (a-1)d + 1, \quad d := \max\left[d(\Phi), d^2(\Phi)\right]. \tag{3.3}$$

Also, the number $ds(\Phi)$ is replaced with

$$ds_1(\Phi) = \frac{ds(\Phi)}{2d(\Phi)}. \tag{3.4}$$

Denote $x = (x_1, y_1, \dots, y_{n-1}) = (x_1, y)$, $y^2 = y_1^2 + \dots + y_{n-1}^2$. Consider an arbitrary point $x_0 \in \Phi$ and let $\tilde{x}(x_0)$ be a point at the hypersurface P such that the straight line connecting x_0 and $\tilde{x}(x_0)$ does not lie in the hyperplane, which is tangent to P at the point $\tilde{x}(x_0)$. Without loss of generality we assume that $\tilde{x}(x_0) = 0$. Consider a piece of the straight line $l'(x_0) \subset \Phi$ passing through points $\{0\}$ and x_0. Extend $l'(x_0)$ beyond the point x_0 until its intersection with the boundary $\partial\Phi$ at the point $x_0' \in \partial\Phi$. Denote $l(x_0)$ the part of the straight line connecting points $\{0\}$ and x_0'. Rotate the coordinate system in such a way that $l(x_0)$ becomes $l(x_0) = \{x = (x_1, y)\colon x_1 \in (0, x_{10}'),\ y = 0\}$. Hence $x_0 = (x_{10}, 0, \dots, 0)$, $x_0' = (x_{10}', 0, \dots, 0)$ and $x_{10}' > x_{10}$.

We can represent the equation of a small part P', $0 \in P'$, of the hypersurface P as $x_1 = \eta(y)$, $|y| < \theta$, $\eta(0) = 0$, where θ is a small positive number and the function $\eta \in C^2(|y| \leqslant \theta)$. Change variables as $(x, y) \leftrightarrow (x', y) = (x - \eta(y), y)$ for $y \in \{|y| \leqslant \theta\}$, leaving again "old" notations for these new variables, for brevity. Hence, in new variables

$$P' = \left\{x_1 = 0, |y| < \theta\right\}. \tag{3.5}$$

Points x_0 and x_0' remain the same and the operator L still remains elliptic, with the same constants σ_1, σ_2. However, the constant a_1 in (3.3) will change depending on the $C^1(|y| \leqslant \theta)$-norm of the function $\eta(y)$, and this is why the constant C_1 in Theorem 1 depends on the hypersurface P. Next, choose a number α_0 such that

$$0 < \alpha_0 = \alpha\big(x_0', ds_1(\Phi)\big) < \frac{1}{2}\min\left(\frac{1}{4}, ds_1(\Phi)\right)$$

and denote

$$PR(x_0) = \left\{x:\ x_1 + \frac{y^2}{\theta^2} + \alpha_0 < x_{10}' + 2\alpha_0,\ x_1 > 0\right\}. \tag{3.6}$$

Hence, by (3.1) and (3.2)

$$x_0, x_0' \in PR(x_0) \subset \Omega \quad \text{and} \quad \overline{PR(x_0)} \cap (\partial\Omega \setminus P) = \emptyset. \tag{3.7}$$

We now specify the function $q(x,t)$ (beginning of Section 2) as

$$q(x,t) = x_1 + \frac{y^2}{\theta^2} + \frac{(t-\delta)^2}{\delta^2} + \alpha_0. \tag{3.8}$$

Because of (3.8), we specify the domain G_0 as (see (2.1))

$$\begin{aligned} G_0 &= \big\{(x,t):\ q(x,t) < x_{10}' + 2\alpha_0,\ x_1 > 0\big\} \\ &= \left\{(x,t):\ x_1 + \frac{y^2}{\theta^2} + \frac{(t-\delta)^2}{\delta^2} + \alpha_0 < x_{10}' + 2\alpha_0,\ x_1 > 0\right\}. \end{aligned} \tag{3.9}$$

Hence, by (3.6) and (3.7)

$$x_0, x_0' \in G_0 \cap \{t = \delta\} = PR(x_0) \subset \Omega. \tag{3.10}$$

Note that by (3.2)

$$\alpha_0 < q(x,t) < x_{10}' + 2\alpha_0 < 3/4, \quad (x,t) \in G_0. \tag{3.11}$$

Following (2.7), (3.9) and (3.11), we specify the domain G_ω as follows

$$G_\omega = \big\{(x,t):\ q(x,t) < x_{10}' + 2\alpha_0 - \omega,\ x_1 > 0\big\}, \quad \forall\omega \in \big(0, x_{10}' - x_{10} + \alpha_0\big).$$

By (3.10) there exists a small $\omega_0 = \omega_0(x_0) \in (0, x_{10}' - x_{10} + \alpha_0)$ such that

$$x_0, x_0' \in \big\{G_{5\omega_0} \cap \{t = \delta\}\big\}. \tag{3.12}$$

Consider a cut-off function $\chi(x,t) \in C^2(\overline{G}_0)$ such that $0 \leqslant \chi \leqslant 1$ and

$$\chi(x,t) = \left\{\begin{array}{ll} 1 & \text{for } (x,t) \in G_{2\omega_0} \\ 0 & \text{for } (x,t) \in G_0 \setminus G_{\omega_0} \end{array}\right\}.$$

Note that by (2.8)

$$G_{5\omega_0} \subset G_{3\omega_0} \subset G_{2\omega_0} \subset G_{\omega_0} \subset G_0.$$

Consider the function $v(x,t) = (\chi u)(x,t)$. Then $u = \chi u + (1-\chi)u = v + (1-\chi)u$. Hence, by (1.5) and (3.3)

$$|v_t - L_0 v| \leqslant M_1\big[|\nabla v| + |v| + (1-\chi)|\nabla u| + (1-\chi)|u| + |f|\big], \quad \text{a.e. in } G_0, \tag{3.13}$$

$$v|_{P_T} = \chi h^{(1)}, \quad \left.\frac{\partial v}{\partial n}\right|_{P_T} = \chi h^{(2)}(x,t) + h^{(1)}\frac{\partial \chi}{\partial n}. \tag{3.14}$$

Here $M_1 = M_1(M,d)$ is a positive constant depending on constants M in (1.5) and d in (3.3). Since the constant C_1 in the formulation of Theorem 1 also depends on these parameters (as well as on some others), then M_1 is "absorbed" by C_1 in the proof below. Consider an arbitrary function $w \in C^{2,1}(\overline{G}_0)$ such that $w = 0$ in $G_0 \setminus G_{\omega_0}$. Substitute w in (2.11) and integrate that formula over the domain G_0 using (2.12), (2.13) and the Gauss' formula. Using (3.2) and (3.4), set in Theorem 2

$$\nu := \nu_0 = \nu_0\big(\sigma_1, \sigma_2, \theta, a_1, ds_1(\Phi)\big) := \nu_0\big(\sigma_1, \sigma_2, P, a, d(\Phi), ds(\Phi)\big). \tag{3.15}$$

We obtain from (2.11)–(2.13)

$$\int\limits_{G_0} (w_t - L_0 w)^2 \varphi^2\,dx\,dt \geqslant \frac{C_1}{\delta}\int\limits_{G_0} |\nabla w|^2\varphi^2\,dx\,dt + \frac{C_1}{\delta^3}\int\limits_{G_0} w^2\varphi^2\,dx\,dt$$
$$- \frac{C_1}{\delta^3}\exp\left(\frac{2}{\delta}\alpha^{-\nu_0}\right)\int\limits_{P_T}\big(|\nabla w|^2 + w_t^2 + w^2\big)\,dS, \quad \forall \delta \in (0,\delta_0),\ \forall w \in C^{2,1}(\overline{G}_0).$$

The standard density arguments imply that this inequality is also valid for the function $v \in H^{2,1}(G_0)$ since $v = 0$ in $G_0 \setminus G_{\omega_0}$. Hence, (3.11), (3.13) and (3.14) imply that

$$\int\limits_{G_0}\big[|\nabla v|^2 + v^2 + f^2\big]\varphi^2\,dx\,dt + \int\limits_{G_0\setminus G_{2\omega_0}}\big[|\nabla u|^2 + u^2\big]\varphi^2\,dx\,dt$$
$$+ \frac{C_1}{\delta^3}\exp\left(\frac{2}{\delta}\alpha_0^{-\nu}\right)\int\limits_{P_T}\big[\big|\nabla h^{(1)}\big|^2 + \big(h_t^{(1)}\big)^2 + \big(h^{(2)}\big)^2\big]\,dS$$
$$\geqslant \frac{C_1}{\delta}\int\limits_{G_0}|\nabla v|^2\varphi^2\,dx\,dt + \frac{C_1}{\delta^3}\int\limits_{G_0} v^2\varphi^2\,dx\,dt, \quad \forall \delta \in (0,\delta_0).$$

Denoting $\delta_1 = \min[\delta_0, 1/(2C_1)]$, we obtain with a new constant C_1

$$\int\limits_{G_0\setminus G_{2\omega_0}} \left[|\nabla u|^2 + u^2\right]\varphi^2\,dx\,dt + \frac{C_1}{\delta^3}\exp\left(\frac{2}{\delta}\alpha_0^{-\nu}\right)\|F\|^2$$
$$\geqslant \frac{C_1}{2\delta}\int\limits_{G_0} |\nabla v|^2\varphi^2\,dx\,dt + \frac{C_1}{2\delta^3}\int\limits_{G_0} v^2\varphi^2\,dx\,dt, \quad \forall\delta\in(0,\delta_1). \tag{3.16}$$

We have

$$\varphi^2(x,t)\leqslant \exp\left[\frac{2}{\delta}\left(x'_{10}+2\alpha_0-2\omega_0\right)^{-\nu}\right] \quad \text{in } G_0\setminus G_{2\omega_0}.$$

Also,

$$\begin{aligned}
&\frac{C_1}{2\delta}\int\limits_{G_0} |\nabla v|^2\varphi^2\,dx\,dt + \frac{C_1}{2\delta^3}\int\limits_{G_0} v^2\varphi^2\,dx\,dt\\
&\quad\geqslant \frac{C_1}{2\delta}\int\limits_{G_{3\omega_0}} |\nabla v|^2\varphi^2\,dx\,dt + \frac{C_1}{2\delta^3}\int\limits_{G_{3\omega_0}} v^2\varphi^2\,dx\,dt\\
&\quad= \frac{C_1}{2\delta}\int\limits_{G_{3\omega_0}} |\nabla u|^2\varphi^2\,dx\,dt + \frac{C_1}{2\delta^3}\int\limits_{G_{3\omega_0}} u^2\varphi^2\,dx\,dt\\
&\quad\geqslant \frac{C_1}{2\delta^3}\exp\left[\frac{2}{\delta}(x_{10}+2\alpha_0-3\omega_0)^{-\nu}\right]\int\limits_{G_{3\omega_0}} u^2\varphi^2\,dx\,dt.
\end{aligned}$$

Hence, (3.16) implies that

$$\int\limits_{G_{3\omega_0}} u^2\,dx\,dt \leqslant C_1\exp\left(\frac{2}{\delta}\alpha_0^{-\nu}\right)\|F\|^2 + C_1\exp\left(-\frac{\rho_0}{\delta}\right)\|u\|^2_{H^{1,0}(Q_T)}, \tag{3.17}$$

where

$$\rho_0=\rho_0(x_0)=(x_{10}+2\alpha_0-3\omega_0)^{-\nu}-(x_{10}+2\alpha_0-2\omega_0)^{-\nu}>0. \tag{3.18}$$

Consider the domain $D(x_0,\omega_0)$,

$$D(x_0,\omega_0)=\left\{x\colon\ x_1+\frac{y^2}{\theta^2}<x'_{10}+\alpha_0-5\omega_0,\ x_1>0\right\}.$$

By (3.6), (3.10) and (3.12) $D(x_0,\omega_0)\subset PR(x_0)$ and $x_0, x'_0\in D(x_0,\omega_0)$. Also, the time cylinder

$$\left\{(x,t)\colon\ x\in D(x_0,\omega_0),\ |t-\delta|<\delta\sqrt{\omega_0}\right\}\subset G_{3\omega_0}.$$

There exists a neighborhood $N(x_0,\omega_0)=\{x\colon |x-x_0|<\xi(x_0,\omega_0),\ \xi(x_0,\omega_0)>0\}$ of the point x_0 such that $N(x_0,\omega_0)\subset D(x_0,\omega_0)$. Hence, the time cylinder

$$N(x_0,\omega_0,\delta)=N(x_0,\omega_0)\times\left\{t\colon |t-\delta|<\delta\sqrt{\omega_0}\right\}\subset G_{3\omega_0}.$$

Hence, (3.17) and (3.18) imply that

$$\int\limits_{N(x_0,\omega_0,\delta)} u^2\,dx\,dt\leqslant C_1\exp\left(\frac{2}{\delta}\alpha_0^{-\nu}\right)\|F\|^2+C_1\exp\left(-\frac{\rho_0}{\delta}\right)\|u\|^2_{H^{1,0}(Q_T)}. \tag{3.19}$$

Consider a finite number of points $\{x_0^{(i)}\}_{i=1}^s\subset\Phi$ such that

$$\Phi\subset\bigcup_{i=1}^s N\left(x_0^{(i)},\omega_0^{(i)}\right):=N$$

and $\operatorname{dist}(N,(\partial\Omega\setminus P))\geqslant ds_1(\Phi)/2$, where $N(x_0^{(i)},\omega_0^{(i)})$ is a neighborhood of the point $x_0^{(i)}$ which is constructed similarly with the neighborhood $N(x_0,\omega_0)$. Note that the number ν in (3.15) is independent on the point x_0, and, therefore, we chose it the same for all points $x_0^{(i)}$. Let $\{x_0^{\prime(i)}\}_{i=1}^s\subset\partial\Phi$ be the set of corresponding point x_0' and $\{\omega_0^{(i)}\}_{i=1}^s$ be the set of corresponding numbers ω_0. Denote

$$\rho=\min_{1\leqslant i\leqslant s}\rho_0\left(x_0^{(i)}\right),\qquad \alpha=\min_{1\leqslant i\leqslant s}\left[\alpha_0=\alpha_0\left(x_0^{\prime(i)},ds_1(\Phi)\right)\right],\qquad \omega_1=\min_{1\leqslant i\leqslant s}\omega_0^{(i)}.$$

Then (3.19) implies that

$$\int\limits_{\Phi_\delta} u^2\,dx\,dt\leqslant C_1\exp\left(\frac{2}{\delta}\alpha^{-\nu}\right)\|F\|^2+C_1\exp\left(-\frac{\rho}{\delta}\right)\|u\|^2_{H^{1,0}(Q_T)},\quad \forall\delta\in(0,\delta_1), \tag{3.20}$$

where $\Phi_\delta=\Phi\times\{t\colon |t-\delta|<\delta\sqrt{\omega_1}\}$. Note that since $\omega_1\in(0,1/2)$ then $\{|t-\delta|<\delta\sqrt{\omega_1}\}\subset(0,T)$. By the mean value theorem there exists a number $t^*\in\delta(1-\sqrt{\omega_1},1+\sqrt{\omega_1})$ such that

$$\int\limits_{\Phi} u^2(x,t^*)\,dx\leqslant\frac{1}{2\delta\sqrt{\omega_1}}\int\limits_{\Phi_\delta} u^2\,dx\,dt.$$

Hence, using (3.20) and (1.6), we obtain

$$\int\limits_{\Phi} u^2(x,t^*)\,dx\leqslant C\exp\left(\frac{3}{\delta}\alpha^{-\nu}\right)\|F\|^2+C\exp\left(-\frac{\rho}{2\delta}\right)\|g\|^2_{L_2(\Omega)},\quad \forall\delta\in(0,\delta_1). \tag{3.21}$$

Now,

$$g(x)=u(x,0)=u(x,t^*)-\int\limits_0^{t^*} u_t(x,t)\,dt.$$

Hence,

$$\|g\|^2_{L_2(\Phi)} \leqslant 2\|u(x,t^*)\|^2_{L_2(\Phi)} + 2\delta(1+\sqrt{\omega_1})\|u_t(x,t)\|^2_{L_2(Q_T)}, \quad \forall\delta \in (0,\delta_1).$$

Hence, (1.7) implies that

$$\|g\|^2_{L_2(\Phi)} \leqslant K\|u(x,t^*)\|^2_{L_2(\Phi)} + \delta K\big(\|g\|^2_{H^1(\Omega)} + \|f\|^2_{L_2(\Omega)}\big).$$

Let

$$\delta_2 = \min\left(\delta_1, \frac{1}{2K}\right). \tag{3.22}$$

Then

$$\|g\|^2_{L_2(\Phi)} \leqslant K\|u(x,t^*)\|^2_{L_2(\Phi)} + \delta K\big(\||\nabla g|\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\setminus\Phi)} + \|f\|^2_{L_2(\Omega)}\big), \quad \forall\delta \in (0,\delta_2).$$

Substituting this in (3.21), we obtain

$$\|g\|^2_{L_2(\Phi)} \leqslant C_1\delta\big[\||\nabla g|\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\setminus\Phi)}\big] + C_1\exp\left(\frac{3}{\delta}\alpha^{-\nu}\right)\|F\|^2, \quad \forall\delta \in (0,\delta_2). \tag{3.23}$$

Denote

$$\tilde{g} = \frac{\varepsilon_0}{B}g, \qquad \widetilde{F} = \frac{\varepsilon_0}{B}F,$$

where the number $\varepsilon_0 > 0$ will be chosen later. Then $\|\tilde{g}\|_{H^1(\Omega)} \leqslant \varepsilon_0$, $\|\widetilde{F}\| \leqslant \varepsilon_0$ and (3.23) holds for functions $\tilde{g}$ and $\widetilde{F}$. Take an arbitrary number $\mu \in (0,1)$ and choose $\delta = \delta(F)$ such that

$$\exp\left(\frac{3}{\delta}\alpha^{-\nu}\right)\|\widetilde{F}\|^2 = \|\widetilde{F}\|^{2(1-\mu)}.$$

Hence,

$$\delta = \frac{3}{2\mu\alpha^{\nu}}\left[\ln\left(\frac{B}{\varepsilon_0\|F\|}\right)\right]^{-1}. \tag{3.24}$$

Since we should have $\delta \in (0,\delta_2)$ and

$$\ln\left(\frac{B}{\varepsilon_0\|F\|}\right) \geqslant \ln\left(\frac{1}{\varepsilon_0}\right),$$

then (3.24) implies the following requirement for the number ε_0

$$\varepsilon_0 \leqslant \exp\left(-\frac{3}{2\mu\delta_2\alpha^{\nu}}\right),$$

where the number δ_2 is defined in (3.22). Hence, we choose

$$\varepsilon_0 = \exp\left(-\frac{3}{2\mu\delta_2\alpha^{\nu}}\right). \tag{3.25}$$

Therefore, (3.23) and (3.24) lead to

$$\begin{aligned}\|g\|^2_{L_2(\Phi)} \leqslant \frac{C_1}{\mu}\left[\ln\left(\frac{B}{\varepsilon_0\|F\|}\right)\right]^{-1}\left[\| |\nabla g| \|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\setminus\Phi)}\right] \\ + C_1\left(\frac{B}{\varepsilon_0}\right)^{2\mu}\|F\|^{2(1-\mu)}.\end{aligned} \tag{3.26}$$

Relations (3.22), (3.25) and (3.26) complete the proof of Theorem 1.

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