Weakly Birkhoff recurrent switching signals, almost sure and partial stability of linear switched dynamical systems ☆

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ABSTRACT

Let $\{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ be arbitrary K matrices, where K and d both $\geqslant 2$. For any $0 < \Delta < \infty$, we denote by $\mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$ the set of all switching sequences $\mathfrak{u} = (\lambda_\cdot, t_\cdot) \colon \mathbb{N} \to \{1, \ldots, K\} \times \mathbb{R}_+$ satisfying $t_j - t_{j-1} \leqslant \Delta$ and

$$0 =: t_0 < t_1 < \cdots < t_{j-1} < t_j < \cdots \quad \text{with } t_j \to +\infty.$$

Differently from the classical weak-$*$ topology and L^1-norm, we equip $\mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$ with the topology so that the "one-sided Markov-type shift" $\vartheta_+ \colon \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K}) \to \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$, defined by

$$\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \mapsto \vartheta_+(\mathfrak{u}) = (\lambda_{j+1}, t_{j+1} - t_1)_{j=1}^{+\infty},$$

is continuous, which is different from and simpler than the classical continuous-time "translation". We study the stability of the linear switched dynamics (**A**):

$$\dot{x}(t) = A_{\mathfrak{u}(t)} x(t), \quad x(0) \in \mathbb{C}^d \text{ and } t > 0$$

where $\mathfrak{u}(t) \equiv \lambda_j$ if $t_{j-1} < t \leqslant t_j$, for any $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$. By introducing the concept "weakly Birkhoff recurrent switching signal", we show that if, under some norm $\|\cdot\|$, the principal matrix $\Phi_{\mathfrak{u}}(t)$ of (**A**) satisfies $\|\Phi_{\mathfrak{u}}(t)\| \leqslant 1$ for all $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$ and $t > 0$, then for any ϑ_+-ergodic probability $\mathbb{P}$ on $\mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$, either

$$\lim_{j \to +\infty} \frac{1}{j} \log \|\Phi_{\mathfrak{u}}(t_j)\| < 0 \quad \text{for } \mathbb{P}\text{-a.s. } \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty};$$

or

$$\left\| \Phi_{\vartheta^j_+(\mathfrak{u})}(t_{j+k} - t_j) \right\| = 1 \quad \forall k, j \geqslant 0 \text{ for } \mathbb{P}\text{-a.s. } \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}.$$

Some applications are presented, including: (i) equivalence of various stabilities; (ii) almost sure exponential stability of periodically switched stable systems; (iii) partial stability; and (iv) how to approach arbitrarily the stable manifold by that of periodically switched signals and how to select a stable switching signal for any initial data.

1. Introduction

A switched system is a dynamical system that consists of a family of subsystems and a logical rule, called "switching signal" in this paper, which orchestrates switching between these subsystems. One popular way to classify switched systems is based on the dynamics of their subsystems, for example, continuous-time or discrete-time, linear or nonlinear and so on. In this paper, we introduce new methods, different from the traditional Lyapunov functions, to the stability study of the continuous- and discrete-time linear switched dynamics. We use a unified treatment to both the continuous- and discrete-time cases by introducing new "indicator" and "discretization".

In this introductory section, we establish some basic notations and several lemmas needed and formulate precisely the fundamental theorems and then roughly describe their applications.

1.1. Fundamental theorems

We state our fundamental results in continuous-time case and discrete-time case, respectively.

1.1.1. Continuous-time switched dynamics and discretization

We first consider the continuous-time dynamics case. Let $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ be arbitrarily given K complex $d \times d$ matrices, where $K \geqslant 2$ and $d \geqslant 2$. Then associated to $\mathbf{A}$, a continuous-time, time-invariant, and linear switched dynamical system can be described as follows

$$\dot{x}(t) = A_{\mathfrak{u}(t)} x(t), \quad x(0) = x_0 \in \mathbb{C}^d \text{ and } t \in \mathbb{R}_+ \tag{$\mathbf{A}$}$$

where $x_0 \in \mathbb{C}^d$, viewed as a column vector, is the initial state and $\mathbb{R}_+ := (0, +\infty)$ the positive time-axis, and where the admissible switching signal

$$\mathfrak{u} : \mathbb{R}_+ \to \mathbb{K} := \{1, \ldots, K\}$$

is piecewise constant and left-hand side continuous, which has at most finite number of discontinuities in every interval of finite-length. Given any $0 < \Delta < +\infty$, such a switching signal $\mathfrak{u}(t)$ can be defined by a sequence $(\lambda_j, t_j)_{j=1}^{+\infty} \subset \mathbb{K} \times \mathbb{R}_+$, not necessarily unique, with

$$0 =: t_0 < t_1 < \cdots < t_{j-1} < t_j < \cdots \quad \text{with } t_j \to +\infty \text{ and } 0 < t_j - t_{j-1} \leqslant \Delta,$$

in this way:

$$\mathfrak{u}(t) \equiv \lambda_j \quad \forall t_{j-1} < t \leqslant t_j \text{ and } j \geqslant 1;$$

and vice versa. Notice here that we do not impose the restriction $\lambda_j \neq \lambda_{j+1}$ for all $j \geqslant 1$.

We will identify an admissible switching signal $\mathfrak{u}: \mathbb{R}_+ \to \mathbb{K}$ with such an associated sequence $(\lambda_j, t_j)_{j=1}^{+\infty}$. Let $\mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K})$ be the set of all such switching signals/sequences $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$. Then, we define the "one-sided Markov-type shift" transformation on it as follows:

$$\vartheta_+ : \mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K}) \to \mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K}); \quad \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \mapsto \vartheta_+(\mathfrak{u}) = (\lambda_{j+1}, t_{j+1} - t_1)_{j=1}^{+\infty}, \quad (1.1)$$

which is different from and simpler than the classical "translation" as in Footnote 1 below. To introduce the ergodic-theoretic and dynamics methods for the stability analysis of the switched dynamics (**A**), we need to equip $\mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K})$ with a reasonable topology. Although there already have been various classical topologies on it, for example, the weak-$*$ topology as done in [1,59] and the L^1-norm as in [11], yet we will introduce an other simpler topology for our convenience here, as follows.

For any pair $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$, $\mathfrak{u}' = (\lambda_j', t_j')_{j=1}^{+\infty} \in \mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K})$, we define

$$d_\varrho(\mathfrak{u}, \mathfrak{u}') = \sum_{j=1}^{+\infty} \frac{|\lambda_j - \lambda_j'| + |(t_j - t_{j-1}) - (t_j' - t_{j-1}')|}{\varrho^j (1 + |\lambda_j - \lambda_j'| + |(t_j - t_{j-1}) - (t_j' - t_{j-1}')|)} \quad (1.2)$$

where $\varrho > 1$ is an arbitrarily preassigned constant. Clearly, $d_\varrho(\cdot, \cdot)$ satisfies the standard metric axioms and so $(\mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K}), d_\varrho)$ is a metric space. Here, the induced topology by $d_\varrho(\cdot, \cdot)$ is much more simpler than the standard weak-$*$ topology as in [1,59] and L^1-norm as in [11].[1]

Now, letting $\Delta_j = t_j - t_{j-1}$ for $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K})$, there exists a 1-to-1 correspondence:

$$\mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K}) \ni \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$$

$$\longleftrightarrow \quad \sigma = (\lambda_j, \Delta_j)_{j=1}^{+\infty} \in \left(\mathbb{K} \times (0, \Delta] \right)^{\mathbb{N}} \quad \text{with} \quad \sum_{j=1}^{+\infty} \Delta_j = +\infty, \quad (1.3)$$

where and in the sequel $\mathbb{N} := \{1, 2, \ldots\}$ denotes the set of all natural numbers. This correspondence is topological; so, $(\mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K}), d_\varrho)$ is a separable metric space, since the infinite product topological space $(\mathbb{K} \times [0, \Delta])^{\mathbb{N}}$ is compact by the Tychonoff product theorem; see Lemma 1.1 below. Under this topology, it is easily seen that ϑ_+ is a continuous transformation; that is to say, $(\mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K}), \vartheta_+)$ is a topological dynamical system.

For any switching signal $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K})$, we denote by $\Phi_\mathfrak{u}(t)$ the "principal matrix" of the dynamics (**A**), namely, $\Phi_\mathfrak{u}(0) = \mathrm{Id}_{\mathbb{C}^d}$ and $\frac{d}{dt} \Phi_\mathfrak{u}(t) = A_{\mathfrak{u}(t)} \cdot \Phi_\mathfrak{u}(t)$ for all $t > 0$, where $\frac{d}{dt}|_{t=t_j}$ means the left-hand side derivative at jump discontinuities $t = t_j$ for each $j \geqslant 1$. So,

$$\Phi_\mathfrak{u}(t) = \begin{cases} e^{t A_{\lambda_1}} & \text{if } 0 < t \leqslant t_1; \\ e^{(t - t_{j-1}) A_{\lambda_j}} \cdots e^{\Delta_1 A_{\lambda_1}} & \text{if } t_{j-1} < t \leqslant t_j \text{ for } j \geqslant 2. \end{cases}$$

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[1] Traditionally, one needs to consider the two-sided switching signal $\mathfrak{u}: \mathbb{R} \to \mathbb{K}$ and the translation

$$\vartheta : (\tau, \mathfrak{u}) \mapsto \mathfrak{u}_\tau, \quad \text{where } \mathfrak{u}_\tau(t) = \mathfrak{u}(\tau + t) \, \forall \tau, t \in \mathbb{R}.$$

Under the weak-$*$ topology, ϑ is continuous [1,59]. On the other hand, since we will aim for the asymptotic stable behavior of the output $(x(t))_{t>0}$ of (**A**) associated to an input $(x_0, \mathfrak{u})$, here $\mathfrak{u} \in \mathcal{L}_\Delta^{pc}(\mathbb{R}_+, \mathbb{K})$ need not belong to $L^1(\mathbb{R}_+; \mathbb{K})$. So, the L^1-norm, as done in [11], does not work in our situation now.

Our aim of this paper is to analyze the stability of the output $(\Phi_{\mathfrak{u}}(t) \cdot x_0)_{t>0}$ of (**A**) corresponding to an input $(x_0, \mathfrak{u})$ in $\mathbb{C}^d \times \mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K})$. Since $0 < \Delta < +\infty$, we can see that for any $x_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ and any $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K})$, we have

$$\liminf_{t \to +\infty} \frac{1}{t} \log \|\Phi_{\mathfrak{u}}(t) \cdot x_0\| = \liminf_{j \to +\infty} \frac{1}{t_j} \log \|\Phi_{\mathfrak{u}}(t_j) \cdot x_0\| \qquad (1.4a)$$

and

$$\limsup_{t \to +\infty} \frac{1}{t} \log \|\Phi_{\mathfrak{u}}(t) \cdot x_0\| = \limsup_{j \to +\infty} \frac{1}{t_j} \log \|\Phi_{\mathfrak{u}}(t_j) \cdot x_0\|, \qquad (1.4b)$$

to be independent of the norm $\|\cdot\|$ on $\mathbb{C}^d$ used here. By the cocycle property

$$\Phi_{\vartheta_+^j(\mathfrak{u})}(t) \cdot \Phi_{\mathfrak{u}}(t_j) = \Phi_{\mathfrak{u}}(t + t_j)$$

for any $t > 0$ and all $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$, we can think of (**A**) as a linear skew-product semiflow driven by the one-sided Markov-type shift transformation $\vartheta_+ : \mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K}) \to \mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K})$.

However, transition from the switching instants $t = t_j$ to $t = t_{k+j}$, we will not be interesting to the dwelling period of time $T = t_{k+j} - t_j$; yet we will only care k, the times of switching or transition. For this reason, we now introduce a new quantity, called the *switching indicator of the dynamics* (**A**) *at the switching signal* $\mathfrak{u}$, as follows:

$$\zeta(\mathfrak{u}, \mathbf{A}) := \limsup_{j \to +\infty} \frac{1}{j} \log \|\Phi_{\mathfrak{u}}(t_j)\|, \qquad (1.5)$$

for any $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K})$. It is easily checked that $\zeta(\mathfrak{u}, \mathbf{A})$ is independent of the norm $\|\cdot\|$ used here.

Clearly, if $\mathfrak{u}$ is "slowly switching", i.e., $0 < \epsilon \leqslant t_j - t_{j-1} \leqslant \Delta$ uniformly for $j \geqslant 1$, then there holds one of the following relationships (1.6a) and (1.6b):

$$\epsilon \chi(\mathfrak{u}, \mathbf{A}) \leqslant \zeta(\mathfrak{u}, \mathbf{A}) \leqslant \Delta \chi(\mathfrak{u}, \mathbf{A}), \qquad (1.6a)$$

$$\epsilon \chi(\mathfrak{u}, \mathbf{A}) \geqslant \zeta(\mathfrak{u}, \mathbf{A}) \geqslant \Delta \chi(\mathfrak{u}, \mathbf{A}), \qquad (1.6b)$$

where the function $\chi(\mathfrak{u}, \mathbf{A})$, given by

$$\chi(\mathfrak{u}, \mathbf{A}) = \limsup_{j \to +\infty} \frac{1}{t_j} \log \|\Phi_{\mathfrak{u}}(t_j)\|, \qquad (1.7)$$

is just the traditional (maximal) "Lyapunov exponent" of the dynamics (**A**) at the switching signal $\mathfrak{u}$, for example, see [3,4,13].

This enables us to use ergodic-theoretic and dynamics methods to prove the following alternative result, which is fundamental for our applications later.

Theorem A. *Consider the switched dynamics* (**A**) *based on* $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$. *If the principal matrix* $\Phi_{\mathfrak{u}}(t)$ *of* (**A**) *satisfies*

$$\|\Phi_{\mathfrak{u}}(t)\| \leqslant 1 \quad \forall \mathfrak{u} \in \mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K}) \text{ and } t > 0,$$

then there holds that for any ϑ_+-*ergodic probability* $\mathbb{P}$ *supported on* $\mathcal{L}_{\Delta}^{pc}(\mathbb{R}_+, \mathbb{K})$, *one has*

(1) *either the switching indicator*

$$\zeta(\mathfrak{u}, \mathbf{A}) = \lim_{j \to +\infty} \frac{1}{j} \log \|\Phi_{\mathfrak{u}}(t_j)\| < 0 \quad \text{for } \mathbb{P}\text{-a.s. } \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty},$$

(2) *or*

$$\left\|\Phi_{\vartheta_+^j(\mathfrak{u})}(t_{j+k} - t_j)\right\| = 1 \quad \forall k \in \mathbb{N} \text{ and } j \in \mathbb{Z}_+ \text{ for } \mathbb{P}\text{-a.s. } \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}.$$

Here such a $\mathfrak{u}$ is called a "$\|\cdot\|$-extremal switching signal" of the switched dynamics (**A**).

To prove this theorem, based on (1.4) and (1.6) we will deal with the dynamics (**A**) by considering its "discretization", borrowing a Markov-type shift symbolic dynamical system. This approach is different from what has been done by taking the 1-time transformation

$$\Phi_{\mathfrak{u}}(1) : \mathbb{C}^d \to \mathbb{C}^d$$

in available literature.

For lack of the "ϵ-slowly switching" condition, there only holds the right-hand side inequality in (1.6). However, under the situation of Theorem A, from Theorem B stated in Section 3 it follows that no existence of (2) is equivalent to that $\chi(\mathfrak{u}, \mathbf{A}) < 0$ holds for $\mathbb{P}$-a.s. $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$.

Let $\mathbb{K} \times [0, \Delta]$ be the compact product space of $\mathbb{K} = \{1, \ldots, K\}$ endowed with the discrete-topology and the interval $[0, \Delta]$, and we write the set of all discrete-time switching signals $\sigma : \mathbb{N} \to \mathbb{K} \times [0, \Delta]$ as $\Sigma_{\mathbb{K} \times [0, \Delta]}^+$, i.e.,

$$\Sigma_{\mathbb{K} \times [0, \Delta]}^+ = \left(\mathbb{K} \times [0, \Delta]\right)^{\mathbb{N}}, \quad \text{where } \mathbb{N} = \{1, 2, \ldots\} \text{ as before.}$$

Then, $\Sigma_{\mathbb{K} \times [0, \Delta]}^+$ is a compact topological space with the product topology that is compatible with the following metric

$$d_\varrho(\sigma, \varsigma) = \sum_{j=1}^{+\infty} \frac{d(\sigma(j), \varsigma(j))}{\varrho^j (1 + d(\sigma(j), \varsigma(j)))} \quad \forall \sigma, \varsigma \in \Sigma_{\mathbb{K} \times [0, \Delta]}^+,$$

where $d((\lambda, \tau), (\lambda', \tau')) = |\lambda - \lambda'| + |\tau - \tau'|$ for any $i = (\lambda, \tau)$, $i' = (\lambda', \tau') \in \mathbb{K} \times [0, \Delta]$ and $\varrho > 1$ is a preassigned constant as in (1.2). Then, the classical one-sided Markov shift transformation, setting by

$$\theta_+ : \Sigma_{\mathbb{K} \times [0, \Delta]}^+ \to \Sigma_{\mathbb{K} \times [0, \Delta]}^+; \quad \sigma = (i_j)_{j=1}^{+\infty} \mapsto \theta_+(\sigma) = (i_{j+1})_{j=1}^{+\infty}, \tag{1.8}$$

is continuous and surjective under this topology. Moreover, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K}) & \xrightarrow{\vartheta_+} & \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K}) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
\Sigma_{\mathbb{K} \times [0, \Delta]}^+ & \xrightarrow{\theta_+} & \Sigma_{\mathbb{K} \times [0, \Delta]}^+
\end{array}
\qquad \text{where } \pi : \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \mapsto \sigma_{\mathfrak{u}} = (\lambda_j, t_j - t_{j-1})_{j=1}^{+\infty},
$$

and π is continuous and injective, but it is not surjective; for example, $\pi^{-1}(\sigma) = \varnothing$ for any sequence $\sigma = (\lambda_j, \Delta_j)_{j=1}^{+\infty}$ in $\Sigma_{\mathbb{K} \times [0, \Delta]}^+$ with $\sum_{j=1}^{+\infty} \Delta_j < +\infty$.

However, one can simply observe the following useful fact, which tells us a clear topological structure of the admissible switching-signal space $\mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ for (**A**):

Lemma 1.1. $\pi(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}))$ *is dense in* $\Sigma^+_{\mathbb{K} \times [0, \Delta]}$, *i.e., for any* $\sigma' = (\lambda'_j, \tau'_j)^{+\infty}_{j=1} \in \Sigma^+_{\mathbb{K} \times [0, \Delta]}$ *and* $\varepsilon > 0$, *one can find some* $\sigma_u = (\lambda_j, \tau_j)^{+\infty}_{j=1} \in \Sigma^+_{\mathbb{K} \times [0, \Delta]}$ *with* $\sum^{+\infty}_{j=1} \tau_j = +\infty$ *such that* $d_\varrho(\sigma', \sigma_u) < \varepsilon$. *So,* $(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}), d_\varrho)$ *is a separable metric space, but not complete.*

Proof. It is easily seen that $\pi(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}))$ is a θ_+-invariant dense subspace of $\Sigma^+_{\mathbb{K} \times [0, \Delta]}$, neither closed nor open. So, we need to prove only the non-completeness. Let

$$\sigma_{u_k} = (\lambda_j, \tau^{(k)}_j)^{+\infty}_{j=1} \in \pi\left(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K})\right) \quad \text{with } \tau^{(k)}_j = \begin{cases} \frac{\Delta}{k} & \text{for } 1 \leqslant j \leqslant k; \\ \Delta & \text{for } j \geqslant k+1. \end{cases}$$

Then, $\sigma_{u_k} \to \sigma = (\lambda_j, 0)^{+\infty}_{j=1}$ as $k \to +\infty$, and so $\{u_k\}_{k \geqslant 1}$ is a Cauchy sequence in $(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}), d_\varrho)$. Since σ does not belong to $\pi(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}))$, $(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}), d_\varrho)$ is not complete, as claimed. □

Now, for any ϑ_+-ergodic probability measure $\mathbb{P}$ on $\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K})$, we can define a corresponding θ_+-ergodic probability measure $\mu_{\mathbb{P}}$ on $\Sigma^+_{\mathbb{K} \times [0, \Delta]}$ in this way:

$$\mu_{\mathbb{P}}(B) = \mathbb{P}\left(\pi^{-1}B\right) \quad \text{for each Borel set } B \subset \Sigma^+_{\mathbb{K} \times [0, \Delta]}.$$

Then, $\mu_{\mathbb{P}}(\pi(\mathcal{L}^{pc}_{\Delta}(\mathbb{R}_+, \mathbb{K}))) = 1$. Next, define

$$S_i = \exp(\tau A_\lambda) \quad \forall i = (\lambda, \tau) \in \mathbb{K} \times [0, \Delta].$$

Clearly,

$$\mathbf{S_A} : \mathbb{K} \times [0, \Delta] \to \mathbb{C}^{d \times d}; \quad i \mapsto S_i \tag{1.9}$$

is a bounded continuous matrix-valued function. We define the cocycle, denoted also by $\mathbf{S_A}$ for saving symbol,

$$\mathbf{S_A} : \mathbb{Z}_+ \times \Sigma^+_{\mathbb{K} \times [0, \Delta]} \to \mathbb{C}^{d \times d} \quad \text{by } (j, \sigma) \mapsto \begin{cases} \mathrm{Id}_{\mathbb{C}^d} & \text{for } j = 0, \\ S_{\sigma(j)} \cdots S_{\sigma(1)} & \text{for } j \geqslant 1. \end{cases} \tag{1.10}$$

Here $\mathrm{Id}_{\mathbb{C}^d}$ stands for the identity matrix. Clearly $\Phi_u(t_j) = \mathbf{S_A}(j, \sigma_u)$ for any $j \geqslant 1$ and we have

$$\mathbf{S_A}(j + k, \sigma) = \mathbf{S_A}\left(k, \theta^j_+(\sigma)\right) \cdot \mathbf{S_A}(j, \sigma) \quad \forall j, k \in \mathbb{Z}_+ \text{ and } \sigma \in \Sigma^+_{\mathbb{K} \times [0, \Delta]},$$

which is just the so-called "cocycle property" of $\mathbf{S_A}$, driven by the one-sided Markov shift transformation $\theta_+ : \Sigma^+_{\mathbb{K} \times [0, \Delta]} \to \Sigma^+_{\mathbb{K} \times [0, \Delta]}$.

Then, $\mathbf{S_A}$ or (1.10) can induce the following discrete-time linear switched dynamical system with subsystems $\{S_i\}_{i \in \mathbb{K} \times [0, \Delta]}$ and admissible switching signals $\sigma \in \Sigma^+_{\mathbb{K} \times [0, \Delta]}$

$$x_j = \mathbf{S_A}(j, \sigma) \cdot x_0, \quad x_0 \in \mathbb{C}^d \text{ and } j \geqslant 1, \tag{$\mathbf{S_A}$}$$

which is called the *discretization of the continuous-time dynamics* (**A**).

It should be noticed that our discretization ($\mathbf{S_A}$) of the continuous-time system (**A**) is different from the discrete-time Euler approximating system

$$x_j = (\mathrm{Id}_{\mathbb{C}^d} + \tau A_{u(t_j)}) \cdot x_{j-1}, \quad x_0 \in \mathbb{C}^d \text{ and } j \geqslant 1$$

for sufficiently small $\tau > 0$, considered in [6].

1.1.2. Discrete-time switched dynamics

In this paper, we will indeed consider a more general discrete-time linear switched dynamics than the discretization $\mathbf{S_A}$ corresponding to (**A**). From now on, assume

$$\mathbf{S}: \mathcal{I} \to \mathbb{C}^{d \times d}; \quad i \mapsto S_i \tag{1.11}$$

is an arbitrary continuous matrix-valued function, defined on an arbitrary separable metric space $\mathcal{I}$, not necessarily compact, with a metric $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}_+$. For example, $\mathcal{I}$ is a finite or countable symbolic space with the discrete-topology induced by the trivial metric $d(i, i') = 0$ if $i = i'$, 1 if $i \neq i'$, for all $i, i' \in \mathcal{I}$.

Then, the family $\{S_i\}_{i \in \mathcal{I}}$ generates a multiplicative semigroup, write $\mathbf{S}^+$. We say $\mathbf{S}$ is "product bounded" if $\mathbf{S}^+$ is bounded in $\mathbb{C}^{d \times d}$. This property is also called "absolute stability" of $\mathbf{S}$ in [35] and is independent of the norm $\|\cdot\|$ used here. Clearly, if $0 < \|\mathbf{S}\| := \sup\{\|S_i\|: i \in \mathcal{I}\} < +\infty$, then $\|\mathbf{S}\|^{-1}\mathbf{S}$ is product bounded.

Particularly, there follows immediately from Lemma 1.1 the following important result, for any finite family $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ and $0 < \Delta < +\infty$.

Lemma 1.2. *The switched dynamics* (**A**) *is uniformly Lyapunov stable, i.e.,*

$$\left\{ \Phi_\mathfrak{u}(t) \mid \mathfrak{u} \in \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K}) \text{ and } t > 0 \right\}$$

is bounded in $\mathbb{C}^{d \times d}$, *if and only if its discretization* $\mathbf{S_A}$ *is product bounded in* $\mathbb{C}^{d \times d}$.

For the general case where $\mathbf{S}$ is as in (1.11), let $(\Sigma_\mathcal{I}^+, \theta_+)$ and $\mathbf{S}(j, \sigma)$ be defined similar to $\mathbf{S_A}$ for the discretization of the dynamics (**A**), as done in (1.8) and (1.10) replacing $\mathbb{K} \times [0, \Delta]$ by $\mathcal{I}$; i.e.,

$$\Sigma_\mathcal{I}^+ = \{\sigma : \mathbb{N} \to \mathcal{I}\}, \quad \theta_+ : \Sigma_\mathcal{I}^+ \to \Sigma_\mathcal{I}^+; \quad (i_j)_{j=1}^{+\infty} \mapsto (i_{j+1})_{j=1}^{+\infty},$$

and

$$\mathbf{S}(j, \sigma) = \begin{cases} \mathrm{Id}_{\mathbb{C}^d} & \text{if } j = 0; \\ S_{i_j} \cdots S_{i_1} & \text{if } j \geqslant 1 \end{cases} \quad \forall \sigma = (i_j)_{j=1}^{+\infty}.$$

Then, $\mathbf{S}$ gives rise to the discrete-time linear switched dynamical system

$$x_j = \mathbf{S}(j, \sigma) \cdot x_0, \quad x_0 \in \mathbb{C}^d \text{ and } j \in \mathbb{N} \tag{$\mathbf{S}$}$$

where $\sigma = (i_j)_{j=1}^{+\infty} \in \Sigma_\mathcal{I}^+$ is also called a switching signal. We study the stability of the output $(x_j)_{j \geqslant 1}$ of (**S**). The following three concepts,

(I) σ-*pointwise asymptotic stability*: $\lim_{j \to +\infty} \|\mathbf{S}(j, \sigma) \cdot x_0\| = 0 \ \forall x_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$,

(II) σ-*asymptotic stability*: $\lim_{j \to +\infty} \|\mathbf{S}(j, \sigma)\| = 0$,

(III) σ-*exponential stability*: $\limsup_{j \to +\infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\| < 0$,

all are important for the fundamental theory and applications of linear switching systems, which all are independent of the norm $\|\cdot\|$ used here.

When conditions (I), (II) and (III) hold for all $\sigma \in \Sigma_{\mathcal{I}}^+$, (**S**) is said to be *absolutely pointwise asymptotically stable*, *absolutely asymptotically stable* and *absolutely exponentially stable*, respectively. Because a real-world control system often obeys some switching constraints imposed by uncertainty about the model or about environment in which the object operates, usually one needs to consider a θ_+-invariant probability μ on $\Sigma_{\mathcal{I}}^+$; for example, $\mu_{\mathbb{P}}$ on $\Sigma_{\mathcal{I}}^+$ pushed out by $\mathbb{P}$ on $\mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ where $\mathcal{I} = \mathbb{K} \times [0, \Delta]$. Then, (**S**) is called, respectively, to be μ-*almost surely* pointwise asymptotically, asymptotically, and exponentially stable, provided that conditions (I), (II) and (III) hold, respectively, for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$.

From (1.6), we can see that the $\mathfrak{u}$-exponential stability of (**A**) need not be equivalent to the $\sigma_{\mathfrak{u}}$-exponential stability of (**S$_A$**), unless $0 < \epsilon \leqslant t_j - t_{j-1} \leqslant \Delta$ for all $j \geqslant 1$, for an arbitrary $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$. Yet, since $\Phi_{\mathfrak{u}}(t_j) = \mathbf{S}_A(j, \sigma_{\mathfrak{u}})$ for all $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$ in $\mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$, we can easily obtain the following equivalence relationships for any $\mathbf{A} = \{A_1, \dots, A_K\}$ and $0 < \Delta < +\infty$.

Lemma 1.3. *For the switched dynamics* (**A**) *and its discretization* (**S$_A$**), *there hold the following equivalence relationships*: *for any* $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$,

(1) *the $\mathfrak{u}$-pointwise asymptotic stability of* (**A**), *i.e.,* $\lim_{t \to +\infty} \Phi_{\mathfrak{u}}(t) \cdot x_0 = \mathbf{0}$ $\forall x_0 \in \mathbb{C}^d$, *is equivalent to the $\sigma_{\mathfrak{u}}$-pointwise asymptotic stability of* (**S$_A$**);

(2) *the $\mathfrak{u}$-asymptotic stability of* (**A**), *i.e.,* $\lim_{t \to +\infty} \|\Phi_{\mathfrak{u}}(t)\| = 0$, *is equivalent to the $\sigma_{\mathfrak{u}}$-asymptotic stability of* (**S$_A$**);

(3) *the $\mathfrak{u}$-exponential switching-stability of* (**A**), *i.e.,* $\zeta(\mathfrak{u}, \mathbf{A}) < 0$, *is equivalent to the $\sigma_{\mathfrak{u}}$-exponential stability of* (**S$_A$**).

We note here that if (**A**) is $\mathfrak{u}$-exponentially stable, i.e., $\|\Phi_{\mathfrak{u}}(t)\|$ converges exponentially fast to 0 as $t \to +\infty$, then its discrete-time Euler approximating system, associated to $\mathfrak{u}$ defined as before, is exponentially stable for sufficiently small $\tau > 0$.[2] However, there is no an equivalence available in literature. Importantly, parallel to the statement of Theorem A, we can obtain its discrete-time version as follows:

Theorem A'. *If the family* $\mathbf{S} = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$ *is product bounded, then one can define a norm* $\|\cdot\|_*$ *on* $\mathbb{C}^d$ *such that for any* θ_+-*ergodic Borel probability* μ *supported on* $\Sigma_{\mathcal{I}}^+$, *one has*

(1) *either the Lyapunov exponent*

$$\chi(\sigma, \mathbf{S}) = \lim_{j \to \infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\|_* < 0 \quad \text{for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+;$$

(2) *or*

$$\|\mathbf{S}(k, \theta_+^j(\sigma))\|_* = 1 \quad \forall k \geqslant 1 \text{ and } j \geqslant 0 \text{ for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+.$$

So in this case, μ-a.e. σ is $\|\cdot\|_*$-*extremal of* **S**.

Note 1. In fact, if a $\|\cdot\|_*$ satisfies $\|S_i\|_* \leqslant 1$ $\forall i \in \mathcal{I}$, then there holds the same statement. Such a norm $\|\cdot\|_*$ is called to be "pre-extremal" of **S** in [22]. We notice that a pre-extremal norm $\|\cdot\|_*$ of **S** need not be an "extremal norm" of **S** defined in literature, for example, in [2,5,30,58,17,22]. However, if the above case (2) happens, then $\|\cdot\|_*$ is exactly an extremal norm of **S**.

Note 2. If **S** has joint/generalized spectral radius 1 and is irreducible, then extremal norms always exist for **S** from Barabanov's extremal norm theorem [2].

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[2] This assertion is communicated by professor Zhendong Sun.

We notice here that it is already known from [8,7,36,37,31,19,22] that the system (**S**) need not be asymptotically stable almost surely even if **S** is product bounded.

1.2. Applications of Theorems A and A′

The above Theorem A/A′ is inspired by the following three important aspects, which have attracted in recent years the interests of researchers from quite different fields.

(1) From the viewpoint of numerical analysis, one always expects that $\|\mathbf{S}(j,\sigma)\|$ converges *exponentially fast* to 0 as $j \to \infty$ when (**S**) is σ-asymptotically stable. But, *is this the case*? In addition, *if* (**S**) *is σ-pointwise asymptotically stable then, is it σ-exponentially stable*?

(2) For a periodically switched signal $\sigma = (w, w, \ldots)$ composed by a word $w = (i_1, \ldots, i_\kappa)$ in $\mathcal{I}^k$, the stability of the deterministic switching system

$$x_j = \mathbf{S}(j, \sigma) \cdot x_0, \quad x_0 \in \mathbb{C}^d \text{ and } j \geqslant 1$$

is easily determined by the spectral radius $\rho(S_{i_\kappa} \cdots S_{i_1})$. *If* (**S**) *is periodically switched stable, i.e.,* (**S**) *is σ-asymptotically stable for all periodically switched signals σ, can it be concluded that* (**S**) *is exponentially stable almost surely in the sense of some typical probability μ?*

(3) Let $\mathbf{S} \subset \mathrm{GL}(d, \mathbb{C})$ for which the Euclidean vector norm $\| \cdot \|_2$ on $\mathbb{C}^d$ is pre-extremal, i.e., $\|S_i\|_2 \leqslant 1$ $\forall i \in \mathcal{I}$. If its admissible switching-signal set $\Lambda \subsetneqq \Sigma_{\mathcal{I}}^+$ possesses the dynamics property – minimality, then we will verify that
(a) either (**S**) is Λ-absolutely exponentially stable,
(b) or $\|\mathbf{S}(j, \sigma) \cdot x\|_2 = \|x\|_2$ for all $x \in \mathbb{C}^d$, $\sigma \in \Lambda$ and $j \geqslant 1$,
(c) or there exists a continuous, invariant splitting of $\mathbb{C}^d$ into subspaces

$$\mathbb{C}^d = \mathbb{E}^s(\sigma) \oplus \mathbb{E}^c(\sigma), \qquad 1 \leqslant \dim \mathbb{E}^s(\sigma) \equiv \mathbf{i} < d \quad \forall \sigma \in \Lambda$$

satisfying for any $j \geqslant 1$

$$\big\|\mathbf{S}(j, \sigma) \cdot x_0\big\|_2 = \|x_0\|_2 \quad \forall x_0 \in \mathbb{E}^c(\sigma),$$

$$\big\|\mathbf{S}(j, \sigma) \cdot y_0\big\|_2 \leqslant C\xi^j \|y_0\|_2 \quad \forall y_0 \in \mathbb{E}^s(\sigma),$$

where $C > 0$ and $0 < \xi < 1$ are constants that both are independent of the choices of inputs $(y_0, \sigma) \in \mathbb{E}^s(\sigma) \times \Lambda$.
See Theorems D and D′ shown in Section 5.
When the case (c) appears, the stability analysis of the switched system (**S**) with admissible switching-signal set Λ becomes very complicated. For any $\sigma \in \Lambda$ non-periodic, the stable manifold $\mathbb{E}^s(\sigma)$ depends completely upon the infinite switching sequence $\sigma = (i_j)_{j=1}^{+\infty}$, not upon any sub-word $(i_1, \ldots, i_\ell)$ of finite-length of σ. So, in engineering, the question is this: *Whether or not there are suitable ways to approximate arbitrarily the stable manifold $\mathbb{E}^s(\sigma)$ by that of periodically switched signals*?
On the other hand, *can one, for any fixed initial state $y_0 \in \mathbb{C}^d$, design an exponentially stable switching signal, i.e., does there exist any $\sigma \in \Lambda$ satisfying the output/trajectory $\{\mathbf{S}(j, \sigma) \cdot y_0\}_{j=1}^{+\infty}$ to be exponentially stable*? For example, a launcher of rockets could be regarded as an initial data. This stabilization problem is one of the fundamental problems for linear switched systems and has been widely addressed in the literature; for example, see the survey papers [54,39] for some recent development.

For these problems above and their continuous-time versions, we need to study the weak recurrence of switching signals in the sense of Z. Zhou [60] and to study the rotation number of switching

signals. Our approaches presented in this paper are new and completely different from those addressed in the current available literature.

1.3. Outline

This paper is organized as follows. In Section 2, by introducing some dynamics and ergodic-theoretic approaches, particularly "weakly Birkhoff recurrent switching signals", we will prove mainly Theorems A and A′. The reason why we will introduce the weakly Birkhoff recurrent signal there is that, according to the classical theory [43], by the recurrence, switching signals can be described by the layers:

$$\{\text{periodic signals}\} \subset \{\text{almost periodic signals}\} \subset \{\text{Birkhoff recurrent signals}\}$$

$$\subset \{\text{Poisson stable signals}\}.$$

If a switching signal is Birkhoff recurrent, then it has a positive recurrent frequency; but the recurrent frequency of a Poisson stable signal might be zero. However, the set of all the Poisson stable signals is of total measure 1; but this is not true for the Birkhoff recurrent signals. Under our context below, we will need both the properties of total measure 1 and the positive recurrent frequency. So, we will need to insert a new recurrent layer between the Birkhoff recurrence and the Poisson stability. This is the most important point of the present paper.

In the rest sections, we will apply Theorem A (resp. A′) to the stability analysis of a linear, continuous-time (resp. discrete-time), switched system driven by the one-sided shift transformations ϑ_+ (resp. θ_+).

In Section 3, as a consequence of Theorem A, if the "joint spectral radius $\hat{\rho}(\mathbf{A})$ of $(\mathbf{A})$" is equal to 1 (we note that $\hat{\rho}(\mathbf{A}) \geqslant 1$, in general, from Lemma 3.1 by our definition in the statement of Theorem B below), then its pointwise asymptotic stability is equivalent to its exponential stability almost surely; see Theorem B and Corollary 3.2 stated in Section 3, which seem to be important for the numerical analysis of linear switched systems.

We shall apply Theorem A to a linear switched system that is periodically switching-stable in Section 4; see Theorem C stated there, which asserts that if a ϑ_+-ergodic probability $\mathbb{P}$ can approach arbitrarily to a periodical switching signal, then $(\mathbf{A})$ is exponentially stable $\mathbb{P}$-almost surely.

In Section 5 we will study the partial stability of a continuous-time, linear, switched system driven by a recurrent switching signal, using Theorem A. The main result Theorem D proved there is a continuous-time version of corresponding theorems of Ian D. Morris [40, Theorems 2.1 and 2.2] for invertible driving dynamics, using different methods.

Moreover, we will further consider, in the case driven by minimal dynamics, how to approximate arbitrarily a stable initial data by ones of periodically switched signals and how to pick a stable switching signal for any given initial data; see Theorems E and F stated in Section 6. These seem to be very interesting for one to design desired switches in engineering. To prove Theorems D and E, we will introduce two known theorems respectively in Sections 5.2.1 and 6.1 from [14,16]. The rotation number has been well defined and studied for a random orientation-preserving circle homeomorphism driven by a quasi-periodically dynamical system. To prove Theorem F, we will introduce it into the more general framework of switched dynamics in Section 6.2.1, using an approach that different from the traditional methods, presented in [33,32] for example. Here we will employ a quasi-additive ergodic theorem, which is an improvement of the classical Birkhoff ergodic theorem and itself of interest independently for the study of rotation numbers; see Theorem 6.10 stated in Section 6.3 below.

We will end this paper with concluding remarks in Section 7.

2. Weak recurrence of switching signals and exponential stability of switched dynamics

This section is devoted to proving Theorem A and Theorem A′ stated in Section 1.1, using topological dynamics and ergodic-theoretic approaches. Particularly, we will introduce the important notation – weakly Birkhoff recurrent switching signals.

2.1. Weakly Birkhoff recurrent points

In this subsection, we will consider first an abstract topological dynamical system defined on a separable metric space Ω with a metric $\mathrm{dist}(\cdot,\cdot)$. Let

$$T : \Omega \to \Omega$$

be a continuous surjective transformation of the space Ω, where Ω is not necessarily compact.

Recall from [56] that a probability measure $\mathbb{P}$ on the Borel measurable space $(\Omega, \mathscr{B}_\Omega)$ is said to be T-*invariant*, if $\mathbb{P} = \mathbb{P} \circ T^{-1}$, i.e., $\mathbb{P}(B) = \mathbb{P}(T^{-1}B) \; \forall B \in \mathscr{B}_\Omega$. Further, a T-invariant probability measure $\mathbb{P}$ is called T-*ergodic*, provided that for any $B \in \mathscr{B}_\Omega$, $\mathbb{P}(B \triangle T^{-1}B) = 0$ implies that $\mathbb{P}(B) = 1$ or 0, where $A \triangle B$ is the symmetric difference of two subsets A, B of Ω.

From the topological structure of our switching-signal space $\Sigma_{\mathcal{I}}^+$ defined as in Section 1.1.2, there always exist θ_+-ergodic Borel probability measures $\mathbb{P}$ supported on it. For example, a periodically switched signal, that will be precisely defined in Section 4, can induce an atomic θ_+-ergodic probability measure on $\Sigma_{\mathcal{I}}^+$.

To prove Theorem A/A′, we need to study the recurrence of the switching signals $\mathfrak{u}$ in $\mathcal{L}_\triangle^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ or σ in $\Sigma_{\mathcal{I}}^+$. A point $\omega \in \Omega$ is said to be "Poisson stable" of T, if there is a sequence of positive integers $n_k \nearrow +\infty$ such that $T^{n_k}(\omega) \to \omega$ as $k \to +\infty$; ω is called "Birkhoff recurrent" of T, if for any $\varepsilon > 0$ one can find a relatively dense subset $L(\varepsilon)$ of $\mathbb{N} = \{1, 2, \ldots\}$ such that $\mathrm{dist}(T^k(\omega), \omega) < \varepsilon$ for all $k \in L(\varepsilon)$. See [43]. However, the recurrence of a Birkhoff recurrent motion $T^n(\omega)$ is so strong that it is too minor to capture. Although the Poisson motions $T^n(\omega)$ are abundant, their recurrence is too weak to satisfy our requirement here. So, we need to insert a new recurrence.

The following important concept is due to Z. Zhou:

Definition 2.1. (See [60,61].) A point $\omega \in \Omega$ is called a "*weakly Birkhoff recurrent point*" of T, provided that for any $\varepsilon > 0$, there exists an integer $N \geqslant 1$ such that

$$\sum_{k=0}^{jN-1} \mathrm{I}_{\mathbb{B}(\omega,\varepsilon)}\big(T^k(\omega)\big) \geqslant j \quad \forall j \in \mathbb{N},$$

where $\mathrm{I}_{\mathbb{B}(\omega,\varepsilon)} : \Omega \to \{0, 1\}$ stands for the indicator function of the open ball $\mathbb{B}(\omega, \varepsilon) \subset \Omega$ of radius ε centered at ω.

In Z. Zhou's paper [60], such a point is called a "weakly almost periodic point" of T. Here we rename it "weakly Birkhoff recurrent point", this is because it lies, by recurrence of the motion $T^n(\omega)$, between the Birkhoff recurrent motion and the Poisson stable motion. And the recurrence of an "almost periodic motion", however, is stronger than a Birkhoff recurrent motion, see [43], also Definition 5.1 below.

We denote by $W(T)$ the set of all weakly Birkhoff recurrent points of T. Clearly, from the continuity of T it follows that the set $W(T)$ is T-invariant; namely, $T(W(T)) \subseteq W(T)$; or, equivalently, $T^{-1}(W(T)) \supseteq W(T)$. On the other hand, $W(T)$ is independent of the compatible metric $\mathrm{dist}(\cdot,\cdot)$ on Ω used here.

Notice here that if T is situated in a compact metric space Ω, then $W(T)$ is of total measure 1 from Z. Zhou [60]. In his talks, Z. Zhou has asked the following question: *Is $W(T)$ a Borel subset of Ω?* A positive answer to this question is also convenient for our arguments later. For that, we define

$$\mathfrak{d}(\omega, \varepsilon) = \liminf_{\ell \to +\infty} \frac{1}{\ell} \sum_{k=0}^{\ell-1} \mathrm{I}_{\mathbb{B}(\omega,\varepsilon)}\big(T^k(\omega)\big).$$

It is easily seen that for any $\varepsilon > 0$, the function $\mathfrak{d}(\cdot, \varepsilon) : \Omega \to [0, 1]$ is Borel measurable.

Then, the following lemma gives an affirmative answer to Zhou's question above:

Lemma 2.2. $W(T) = \bigcap_{k=1}^{\infty} \{\omega \in \Omega \mid \mathfrak{d}(\omega, \frac{1}{k}) > 0\}$ and so $W(T)$ is a Borel subset of Ω.

Proof. Let

$$W_k(T) = \left\{\omega \in \Omega \mid \mathfrak{d}\left(\omega, \frac{1}{k}\right) > 0\right\}.$$

Since $\mathfrak{d}(\cdot, \frac{1}{k})$ is Borel measurable on Ω, we need to prove only that $W(T) = \bigcap_{k=1}^{\infty} W_k(T)$. From Definition 2.1, there follows immediately that $W(T) \subseteq \bigcap_{k=1}^{\infty} W_k(T)$, noting

$$\mathfrak{d}(\omega, \varepsilon) = \liminf_{j \to +\infty} \frac{1}{jN} \sum_{k=0}^{jN-1} I_{\mathbb{B}(\omega, \varepsilon)}\left(T^k(\omega)\right)$$

for any $N \geqslant 1$; the other direction inclusion is obvious from the definition of the function $\mathfrak{d}(\omega, \frac{1}{k})$ as well.

This proves the lemma. $\quad\square$

We notice that in the case where Ω is noncompact, for example, $T : x \mapsto x + 1$ that preserves the Lebesgue measure defined on $\Omega = \mathbb{R}$ the 1-dimensional real Euclidean space, $W(T)$ might be empty. However, we could obtain the following result, which shows that if the dynamics (Ω, T) has an invariant probability measure, then there always exist weakly Birkhoff recurrent motions.

Theorem 2.3. If μ is a T-ergodic Borel probability on Ω, then $\mu(W(T)) = 1$.

Proof. Let $\mathrm{supp}(\mu)$ be the *support* of μ, which is defined by

$$\mathrm{supp}(\mu) = \left\{\omega \in \Omega \mid \mu\left(\mathbb{B}(\omega, \varepsilon)\right) > 0 \; \forall \varepsilon > 0\right\};$$

it is just the minimal, closed, T-invariant subset of Ω with μ-measure 1, since Ω is separable.[3] On the other hand, according to [15, Lemma 3] one could find a T-invariant Borel subset $G_{\mu}(T) \subset \Omega$ of μ-measure 1 such that for any continuous, bounded function $\varphi : \Omega \to \mathbb{R}$ and any $\omega \in G_{\mu}(T)$, there holds

$$\lim_{\ell \to +\infty} \frac{1}{\ell} \sum_{j=0}^{\ell-1} \varphi\left(T^j(\omega)\right) = \int_{\Omega} \varphi \, d\mu.$$

So, to prove the statement of Theorem 2.3, it is sufficient to prove that $\mathrm{supp}(\mu) \cap G_{\mu}(T) \subseteq W(T)$.

In fact, let $\omega \in \mathrm{supp}(\mu) \cap G_{\mu}(T)$ and $\varepsilon > 0$ both be arbitrarily given. Let E be the closure of the open ball $\mathbb{B}(\omega, \varepsilon/2)$ in Ω and $F = \Omega \setminus \mathbb{B}(\omega, \varepsilon)$. Then, by Urysohn's lemma there is a continuous function $\psi : \Omega \to [0, 1]$ with $\psi(x) = 1$ for all $x \in E$, $\psi(y) = 0$ for all $y \in F$. From

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[3] The separable property implies that for μ-a.s. $\omega \in \Omega$ its forward T-orbit,

$$\mathrm{Orb}_T^+(\omega) = \left\{T^n(\omega) \mid n = 0, 1, 2, \ldots\right\},$$

is dense in $\mathrm{supp}(\mu)$. In the proof of Theorem A/A′ and Section 5 we will need this.

$$\liminf_{\ell \to +\infty} \frac{1}{\ell} \sum_{j=0}^{\ell-1} \mathbb{I}_{\mathbb{B}(\omega,\varepsilon)}\big(T^j(\omega)\big) \geqslant \lim_{\ell \to +\infty} \frac{1}{\ell} \sum_{j=0}^{\ell-1} \psi\big(T^j(\omega)\big)$$

$$= \int_{\Omega} \psi \, d\mu \geqslant \mu\big(\mathbb{B}(\omega, \varepsilon/2)\big)$$

$$> 0,$$

it follows that ω belongs to $W(T)$.

Thus, this completes the proof of Theorem 2.3. □

From this theorem proved, it is easy to see that $W(T)$ is of total measure 1 whenever Ω is compact.

2.2. Weakly Birkhoff recurrent switching signals and exponential stability

We now turn to the study of the exponential stability of the linear, discrete-time, switched dynamical system of the form

$$x_j = \mathbf{S}(j, \sigma) \cdot x_0 \quad \big(x_0 \in \mathbb{C}^d, \ j \geqslant 1 \text{ and } \sigma \in \Sigma_{\mathcal{I}}^+\big), \tag{2.1}$$

where the set $\mathcal{I}$ of control values is a separable metric space with a metric $d : \mathcal{I} \times \mathcal{I} \to [0, +\infty)$, and where

$$\mathbf{S} : \mathcal{I} \to \mathbb{C}^{d \times d}; \quad i \mapsto S_i$$

is continuous, which defines the cocycle $\mathbf{S}(j, \sigma)$ in this way: for any $\sigma = (i_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$,

$$\mathbf{S}(j, \sigma) = \begin{cases} \mathrm{Id}_{\mathbb{C}^d} & \text{for } j = 0, \\ S_{i_j} \cdots S_{i_1} & \text{for } j \geqslant 1. \end{cases} \tag{2.1}'$$

Hereafter, write $W(\theta_+)$ as the weakly Birkhoff recurrent point set of the one-sided Markov shift

$$\theta_+ : \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$$

defined in the manner as in Section 1.1.2. Let μ be a θ_+-ergodic Borel probability measure on $\Sigma_{\mathcal{I}}^+$. Then, $W(\theta_+)$ is a nonempty Borel set such that $\mu(W(\theta_+)) = 1$ from Theorem 2.3 above.

Next, for any switching signal $\sigma \in W(\theta_+)$, we will study the stability of the corresponding linear switched dynamical system

$$x_j = \mathbf{S}(j, \sigma) \cdot x_0 \quad \big(x_0 \in \mathbb{C}^d \text{ and } j \geqslant 1\big). \tag{S}$$

Using the recurrence of a switching signal, the following criterion of stability is the key step towards the proof of Theorem A/A′:

Theorem 2.4. *Let $\mathbf{S} = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$ be continuous in $i \in \mathcal{I}$ and assume $\boldsymbol{\sigma} = (\mathbf{i}_j)_{j=1}^{+\infty} \in W(\theta_+)$ is arbitrarily given. If there exists a pre-extremal norm $\| \cdot \|_*$ on $\mathbb{C}^d$ (i.e. $\|S_i\|_* \leqslant 1 \ \forall i \in \mathcal{I}$) for which $\|S_{\mathbf{i}_\ell} \cdots S_{\mathbf{i}_1}\|_* < 1$ for some $\ell \geqslant 1$, then (S) is $\boldsymbol{\sigma}$-exponentially stable, i.e.,*

$$\lim_{j \to +\infty} \frac{1}{j} \log \big\| \mathbf{S}(j, \boldsymbol{\sigma}) \big\|_* < 0.$$

Proof. Let $\boldsymbol{\sigma} = (\mathbf{i}_j)_{j=1}^{+\infty}$, $\|\cdot\|_*$ and $\ell \geqslant 1$ all be given as in the hypothesis of the statement. Then, from the continuity of $\mathbf{S}$ it follows that there exist two constants $\epsilon > 0$ and $\xi \in (0, 1)$ such that for any $\sigma' = (i'_j)_{j=1}^{+\infty} \in \mathbb{B}(\boldsymbol{\sigma}, \epsilon)$ there holds the following inequality:

$$\|S_{i'_\ell} \cdots S_{i'_1}\|_* \leqslant \xi, \quad \text{i.e.,} \quad \|S_{i'_\ell} \cdots S_{i'_1} \cdot x\|_* \leqslant \xi \|x\|_* \quad \forall x \in \mathbb{C}^d.$$

Here the open ball $\mathbb{B}(\boldsymbol{\sigma}, \epsilon) \subset \Sigma_{\mathcal{I}}^+$ of radius ϵ centered at the given $\boldsymbol{\sigma}$ is defined under the metric $\mathrm{d}_\varrho(\cdot, \cdot)$ on $\Sigma_{\mathcal{I}}^+$ as in Section 1.1. Since $\boldsymbol{\sigma}$ is a weakly Birkhoff recurrent point of the one-sided Markov shift transformation $\theta_+ : \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$, there exists an integer $N \gg \ell$ such that

$$\sum_{k=0}^{jN-1} \mathrm{I}_{\mathbb{B}(\boldsymbol{\sigma}, \epsilon)}\big(\theta_+^k(\boldsymbol{\sigma})\big) \geqslant j \quad \forall j \geqslant 1.$$

This implies that for any $j \geqslant 1$ one could find j integers, say $\check{k}_1, \ldots, \check{k}_j$, such that

$$0 \leqslant \check{k}_1 < \check{k}_2 < \cdots < \check{k}_j \leqslant jN - 1 \quad \text{and} \quad \theta_+^{\check{k}_s}(\boldsymbol{\sigma}) \in \mathbb{B}(\boldsymbol{\sigma}, \epsilon) \quad \text{for } 1 \leqslant s \leqslant j.$$

As j is big sufficiently, there exist at least $[j/\ell]$ integers in $\{\check{k}_1, \ldots, \check{k}_j\}$, say $\check{k}_{j_1}, \ldots, \check{k}_{j_{[j/\ell]}}$, such that

$$0 \leqslant \check{k}_{j_1} < \check{k}_{j_2} < \cdots < \check{k}_{j_{[j/\ell]}} \leqslant jN - 1,$$
$$\check{k}_{j_{s-1}} + \ell \leqslant \check{k}_{j_s} \quad \text{for } 2 \leqslant s \leqslant [j/\ell]$$

and

$$\theta_+^{\check{k}_{j_s}}(\boldsymbol{\sigma}) \in \mathbb{B}(\boldsymbol{\sigma}, \epsilon) \quad \text{for } 1 \leqslant s \leqslant [j/\ell].$$

Therefore,

$$\begin{aligned}
\limsup_{n \to +\infty} \frac{1}{n} \log \|\mathbf{S}(n, \boldsymbol{\sigma})\|_* &= \limsup_{j \to +\infty} \frac{1}{jN} \log \|\mathbf{S}(jN, \boldsymbol{\sigma})\|_* \\
&= \limsup_{j \to +\infty} \frac{1}{jN + \ell} \log \|S_{\mathbf{i}_{jN+\ell}} \cdots S_{\mathbf{i}_1}\|_* \\
&\leqslant \limsup_{j \to +\infty} \frac{1}{jN} \log \xi^{[j/\ell]} \\
&= \frac{1}{N\ell} \log \xi \\
&< 0
\end{aligned}$$

which shows that $(\mathbf{S})$ is $\boldsymbol{\sigma}$-exponentially stable, as desired.

This proves the theorem. $\quad\square$

From the above proof, one can find the importance of the positive frequency of recurrence of a switching sequence to ensure the exponential stability.

This theorem possesses the same flavor as the celebrated Pliss lemma [45] and Liao sifting lemma [38] which both are powerful tools in the hyperbolicity theory of differentiable dynamical systems.

2.3. Proof of Theorem A′

For proving Theorem A′, we need a basic important result due to R.K. Brayton and C.H. Tong:

Lemma 2.5. *(See [10, Theorem 1], also [35, Theorem 3].) If* $\mathbf{S} = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$ *is product bounded, then one can define a pre-extremal norm* $\| \cdot \|_*$ *on* $\mathbb{C}^d$ *for it; that is to say, it holds that* $\|S_i\|_* \leqslant 1$ *for all* $i \in \mathcal{I}$.

It should be noted that this pre-extremal norm $\| \cdot \|_*$ given by Lemma 2.5 is not necessarily an extremal norm of **S**; see [22] for a counterexample. To keep things as simple as possible, we give the proof here.

Proof. Let $S_i \not\equiv \mathbf{0}_{d \times d}$ for all $i \in \mathcal{I}$; otherwise any norm is pre-extremal for **S**. Define

$$\|x\|_* = \sup\big\{\big\|\mathbf{S}(j, \sigma) \cdot x\big\|_2 \colon \sigma \in \Sigma_{\mathcal{I}}^+, \ j \geqslant 1\big\},$$

where $\| \cdot \|_2$ denotes the standard Euclidean vector norm on $\mathbb{C}^d$. Then, $\|S_i\|_* \leqslant 1$ for all $i \in \mathcal{I}$. This completes the proof of the lemma. $\square$

Now, with Theorems 2.3 and 2.4 at hands, we can readily prove our discrete-time main result Theorem A′.

Proof of Theorem A′. Let $\mathbf{S} \colon \mathcal{I} \ni i \mapsto S_i \in \mathbb{C}^{d \times d}$ be product bounded and continuous, and let μ be a θ_+-ergodic Borel probability on $\Sigma_{\mathcal{I}}^+$. From Lemma 2.5, it follows that there is a vector norm $\| \cdot \|_*$ on $\mathbb{C}^d$ such that $\|S_i\|_* \leqslant 1$ for all $i \in \mathcal{I}$. Let

$$\mathcal{E} = \big\{\sigma = (i_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+; \ \|S_{i_\ell} \cdots S_{i_1}\|_* < 1 \text{ for some } \ell \geqslant 1\big\}.$$

Since $\mathbf{S} \colon i \mapsto S_i$ is continuous with respect to $i \in \mathcal{I}$, $\mathcal{E}$ is an open subset of $\Sigma_{\mathcal{I}}^+$ and so Borel measurable. Then, either $\mu(\mathcal{E}) > 0$ or $\mu(\mathcal{E}) = 0$.

Case (1). If $\mu(\mathcal{E}) > 0$, then from Theorem 2.3, it follows that $\mu(W(\theta_+) \cap \mathcal{E}) > 0$. And moreover, from Theorem 2.4, it follows that for any $\sigma \in W(\theta_+) \cap \mathcal{E}$, (**S**) is σ-exponentially stable. Thus in this case, it holds that

$$\chi(\mu, \mathbf{S}) := \lim_{j \to \infty} \frac{1}{j} \log\big\|\mathbf{S}(j, \sigma)\big\|_* < 0 \quad \text{for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+$$

from the θ_+-ergodicity of μ and the classical multiplicative ergodic theorem [27,44]. We notice that the quantity $\chi(\mu, \mathbf{S})$ is independent of the norm $\| \cdot \|_*$ used here.

Therefore, in this case the statement (1) of Theorem A′ holds.

Case (2). We now assume $\mu(\mathcal{E}) = 0$. This concludes that

$$\big\|\mathbf{S}(j, \sigma)\big\|_* = 1 \quad \forall j \geqslant 1 \text{ for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+;$$

that is to say, there is a Borel subset B_0 of $\Sigma_{\mathcal{I}}^+$ with $\mu(B_0) = 1$ such that $\|\mathbf{S}(j, \sigma)\|_* = 1 \ \forall j \geqslant 1$ for any $\sigma \in B_0$.

Now, let $B = \bigcap_{k=0}^{+\infty} \theta_+^{-k}(B_0)$. Since μ is θ_+-ergodic, there holds that B is θ_+-invariant (i.e., $\theta_+(B) \subseteq B$) with $\mu(B) = 1$. Thus, for any $\sigma \in B$

$$\big\|\mathbf{S}\big(k, \theta_+^j(\sigma)\big)\big\|_* = 1 \quad \forall k \geqslant 1 \text{ and } j \geqslant 0.$$

So, in this case the statement (2) of Theorem A′ holds.

Thus, the proof of Theorem A′ is completed. $\square$

From Theorem A′, we could immediately obtain the following results.

Theorem 2.6. *Assume $\{S_i \mid i \in \mathcal{I}\} \subset \mathbb{C}^{d \times d}$ satisfies $\|S_i\| \leqslant 1$ for all $i \in \mathcal{I}$ under a vector norm $\|\cdot\|$ of $\mathbb{C}^d$. Let*

$$E = \left\{ \sigma \in \Sigma_{\mathcal{I}}^+; \ \lim_{j \to \infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\| < 0 \right\},$$

$$U = \left\{ \sigma \in \Sigma_{\mathcal{I}}^+; \ \|\mathbf{S}\big(k, \theta_+^j(\sigma)\big)\| = 1 \ \forall k \geqslant 1 \text{ and } j \geqslant 0 \right\}.$$

Then, E is a θ_+-invariant Borel subset and U a θ_+-invariant closed subset; and $E \cup U$ is of total measure 1, i.e., $\mu(E \cup U) = 1$ for any θ_+-ergodic Borel probability μ on $\Sigma_{\mathcal{I}}^+$.

Note here that $E \cap U = \varnothing$. So, either $\mu(E) = 1$ and $\mu(U) = 0$ or $\mu(U) = 1$ and $\mu(E) = 0$, from the θ_+-ergodicity of μ.

Theorem 2.7. *Let $\{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$ satisfy $0 < \|\mathbf{S}\| := \sup_{i \in \mathcal{I}} \|S_i\| < \infty$ under a vector norm $\|\cdot\|$ of $\mathbb{C}^d$. Then, for any θ_+-ergodic Borel probability μ supported on $\Sigma_{\mathcal{I}}^+$, one has*

(1) *either*

$$\lim_{j \to \infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\| < \log \|\mathbf{S}\| \quad \text{for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+;$$

(2) *or*

$$\|\mathbf{S}\big(k, \theta_+^j(\sigma)\big)\| = \|\mathbf{S}\|^k \quad \forall k \geqslant 1 \text{ and } j \geqslant 0 \text{ for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+.$$

Proof. We need to consider only $\|\mathbf{S}\|^{-1}\mathbf{S}$ instead of $\mathbf{S}$ using Theorem A′. $\square$

We notice that, for any θ_+-ergodic Borel probability μ, if its (maximal) Lyapunov exponent

$$\lambda(\mu, \mathbf{S}) := \lim_{j \to +\infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\| \quad \text{for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+ \tag{2.2}$$

is not less than $\log \|\mathbf{S}\|$, then

$$\lim_{j \to +\infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\| = \log \|\mathbf{S}\| \quad \text{for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^+. \tag{2.3}$$

However, the statement (2) of Theorem 2.7 is more stronger than the above (2.3), this is because it implies that μ-almost every $\sigma = (i_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$ are $\|\cdot\|$-extremal of $\mathbf{S}$.

2.4. The continuous-time version

Based on Theorem A′ proved in Section 2.3, we now can prove Theorem A by considering its discretization introduced in Section 1.1.

Proof of Theorem A. Let $\mathbb{P}$ be an arbitrary ϑ_+-ergodic Borel probability supported on $\mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$. Let $\mathbf{S_A}$ be defined in the manner as in (1.9) or (1.10). By Lemma 1.2, it follows that $\|\mathbf{S_A}(j, \sigma)\| \leqslant 1$ for all $j \geqslant 1$ and any $\sigma \in \Sigma_{\mathcal{I}}^+$, where $\mathcal{I} = \mathbb{K} \times [0, \Delta]$. Then, Theorem A follows immediately from Theorem A′ with $\mu = \mu_{\mathbb{P}}$.

The proof of Theorem A is thus completed. $\square$

2.5. A remark on the dynamics model (2.1)

The switching dynamics model (2.1) considered here includes the very interesting case of linear skew-product dynamical systems.

Let $T : \Omega \to \Omega$ be a continuous map of a Polish space Ω and $\mathbf{S} : \Omega \to \mathbb{C}^{d \times d}$; $\omega \mapsto S_\omega$ a continuous matrix-valued function. Then, one can define a linear skew-product dynamical system:

$$\mathbf{S}_T : \Omega \times \mathbb{C}^d \to \Omega \times \mathbb{C}^d; \quad (\omega, x) \mapsto \big(T(\omega), S_\omega(x)\big) \tag{2.4}$$

driven by $T : \Omega \to \Omega$.

Let $\Omega_T = \{\bar{\omega} = (\omega, T(\omega), T^2(\omega), \ldots) \mid \omega \in \Omega\}$. Then, Ω_T is a θ_+-invariant closed subset of the Markov symbolic space Σ_Ω^+. Moreover, if μ is a T-ergodic Borel probability measure on Ω, then one could define a Borel probability $\mathbb{P}_\mu$ which is θ_+-ergodic on Σ_Ω^+ such that $\mathbb{P}_\mu(\Omega_T) = 1$ from [15]. Clearly, the μ-stability of $\mathbf{S}_T$ is equivalent to the stability of the switched system ($\mathbf{S}$) in terms of $\mathbb{P}_\mu$. Conversely, corresponding to the switched system (2.1), it will be convenient to write

$$\mathbf{S}_{\theta_+} : \Sigma_\mathcal{I}^+ \times \mathbb{C}^d \to \Sigma_\mathcal{I}^+ \times \mathbb{C}^d; \quad (\sigma, x) \mapsto \big(\theta_+(\sigma), S_{\sigma(1)}(x)\big), \tag{2.5}$$

it is a discrete-time linear skew-product dynamical system driven by the one-sided Markov shift $\theta_+ : \Sigma_\mathcal{I}^+ \to \Sigma_\mathcal{I}^+$, induced by the random matrix $\mathbf{S} : \sigma \mapsto S_{\sigma(1)}$. Then, for any $j \geqslant 0$, $x \in \mathbb{C}^d$ and σ, the cocycle $\mathbf{S}(j, \sigma) \cdot x$ is defined by the equation $\mathbf{S}_{\theta_+}^j(\sigma, x) = (\theta_+^j(\sigma), \mathbf{S}(j, \sigma) \cdot x)$.

3. Equivalence relationships of pointwise asymptotic and exponential stabilities

As the background, let us consider first the continuous-time linear switched dynamical system of the form

$$\dot{x}(t) = A_{\mathfrak{u}(t)} x(t), \quad x(0) = x_0 \in \mathbb{C}^d \text{ and } t \in \mathbb{R}_+ \tag{3.1}$$

where $\mathfrak{u}(t) \in \mathbb{K} = \{1, \ldots, K\}$, $K \geqslant 2$, is the switching signal to be designed, and where $A_k \in \mathbb{C}^{d \times d}$ are known matrices for all $k \in \mathbb{K}$. The switching signal $\mathfrak{u} : \mathbb{R}_+ \to \mathbb{K}$ is a piecewise constant and left-hand side continuous function of positive time t such that the number of switches is finite in any finite time interval. In other words, $\mathfrak{u}$ belongs to $\mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$, for some $0 < \Delta < +\infty$. Let $\{\Phi_\mathfrak{u}(t) \cdot x_0\}_{t \in \mathbb{R}_+}$ denote the state trajectory – output – initiated by $\Phi_\mathfrak{u}(0) \cdot x_0 = x_0$ via the switching signal $\mathfrak{u}$. Recall from [53] that system (3.1) is said to be

(1) *switched convergent*, if for each $x_0 \in \mathbb{C}^d$ there corresponds a switching signal, say $\mathfrak{u}_{x_0}'$, that makes $\Phi_{\mathfrak{u}_{x_0}'}(t) \cdot x_0$ convergent to $\mathbf{0}$ as $t \to +\infty$, that is, $\lim_{t \to +\infty} \|\Phi_{\mathfrak{u}_{x_0}'}(t) \cdot x_0\| = 0$;

(2) *exponentially stabilizable*, if there exist two real numbers $\alpha > 0$, $\beta > 0$ such that to any $x_0 \in \mathbb{C}^d$ there corresponds a switching signal, say $\mathfrak{u}_{x_0}''$, satisfying

$$\big\| \Phi_{\mathfrak{u}_{x_0}''}(t) \cdot x_0 \big\| \leqslant \beta \|x_0\| \exp(-\alpha t) \quad \forall t \in \mathbb{R}_+.$$

In [53, Theorem 1], it has been proved by Z. Sun that for system (3.1), the switched convergence is equivalent to the exponential stabilizability. From Sun's proof presented in [53], however, it is easily seen that although (1) and (2) are equivalent to each other, yet there $\mathfrak{u}_{x_0}' \neq \mathfrak{u}_{x_0}''$ that results in different outputs, for any given same initial data $x_0 \in \mathbb{C}^d$. This is very limited in applications, because a real-world situation often obeys some constraints; that is to say, the admissible switching signals only form a proper subset of all the switching signals. So, the question is this: *When $\mathfrak{u}_{x_0}' \in \mathrm{U}$ satisfies* (1), *could one guarantee that $\mathfrak{u}_{x_0}'' \in \mathrm{U}$ satisfies* (2) *and further $\mathfrak{u}_{x_0}' = \mathfrak{u}_{x_0}''$?* Here $\mathrm{U} \subsetneq \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ is a preassigned set.

On the other hand, it is easy to see, from definitions, that asymptotic stability is weaker than exponential stability for an individual function; for example, to $\varphi(t) = \frac{1}{\sqrt{t}}$

$$\big|\varphi(t)\big| \to 0 \quad \text{but} \quad \big|\varphi(t)\big| \xrightarrow{\text{not exponentially fast}} 0, \quad \text{i.e.} \quad \frac{1}{t}\log\big|\varphi(t)\big| \to 0 \not< 0 \quad \text{as } t \to +\infty;$$

so, $\varphi(t)$ converges asymptotically to 0, but does not exponentially fast. It shows that the difference of the two concepts, asymptotic and exponential stabilities, is essential for a deterministic linear switching system.

Let $\mathbf{S}: \mathcal{I} \to \mathbb{C}^{d \times d}$ be an arbitrary continuous function. In [19,23], it has been proved that if the space $\mathcal{I}$ of control values is compact and if to any θ_+-ergodic probability μ of $\Sigma_{\mathcal{I}}^+$, there holds that

$$x_j = \mathbf{S}(j, \sigma) \cdot x_0, \quad x_0 \in \mathbb{C}^d \text{ and } j \in \mathbb{N} \tag{S}$$

is exponentially stable for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$, then the discrete-time dynamics (**S**) is absolutely exponentially stable.

In general, for an arbitrary θ_+-ergodic probability μ on $\Sigma_{\mathcal{I}}^+$, the μ-a.s. asymptotic stability is essentially weaker than the μ-a.s. exponential stability for (**S**). However, here we ask the following question: *If to any θ_+-ergodic probability μ on $\Sigma_{\mathcal{I}}^+$ there (**S**) is asymptotically stable μ-almost surely then, is (**S**) absolutely asymptotically stable?*

In this section, using Theorem A, we will provide an affirmative answer to this question; see Corollary 3.2 below. In fact, we can obtain a more general continuous-time result, stated as follows:

Theorem B. *Let* $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ *and* $0 < \Delta < \infty$ *be arbitrarily given and assume the* joint spectral radius $\hat{\rho}(\mathbf{A})$ *of* (**A**) *is equal to 1, where*[4]

$$\hat{\rho}(\mathbf{A}) := \limsup_{j \to +\infty} \left\{ \sup_{\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})} \big\| \Phi_{\mathfrak{u}}(t_j) \big\|^{1/j} \right\}.$$

Then, for any ϑ_+-*ergodic Borel probability* $\mathbb{P}$ *on* $\mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$*, the following statements are equivalent to each other*:

(a) *The dynamics* (**A**) *is* $\mathfrak{u}$-*pointwise asymptotically stable, i.e.,*

$$\lim_{t \to +\infty} \big\| \Phi_{\mathfrak{u}}(t) \cdot x_0 \big\| = 0 \quad \forall x_0 \in \mathbb{C}^d,$$

for $\mathbb{P}$-*a.s.* $\mathfrak{u} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$.
(b) *The dynamics* (**A**) *is* $\mathfrak{u}$-*asymptotically stable, i.e.,*

$$\lim_{t \to +\infty} \big\| \Phi_{\mathfrak{u}}(t) \big\| = 0,$$

for $\mathbb{P}$-*a.s.* $\mathfrak{u} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$.

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[4] Here $\hat{\rho}(\mathbf{A})$ is defined very differently from the traditional one in the continuous-time case in available literature, for example, in [13,58,59,4], there the joint spectral radius of (**A**) was defined by

$$\hat{\rho}(\mathbf{A}) = \limsup_{t \to +\infty} \left\{ \sup_{\mathfrak{u} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})} \big\| \Phi_{\mathfrak{u}}(t) \big\|^{1/t} \right\}.$$

(c) *The dynamics* (**A**) *is* $\mathfrak{u}$-*exponentially switching-stable, i.e.,*

$$\zeta(\mathfrak{u}, \mathbf{A}) = \lim_{j \to +\infty} \frac{1}{j} \log \| \Phi_{\mathfrak{u}}(t_j) \| < 0,$$

for $\mathbb{P}$-*a.s.* $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$.
(d) *The dynamics* (**A**) *is* $\mathfrak{u}$-*exponentially stable, i.e.,*

$$\chi(\mathfrak{u}, \mathbf{A}) = \lim_{j \to +\infty} \frac{1}{t_j} \log \| \Phi_{\mathfrak{u}}(t_j) \| < 0,$$

for $\mathbb{P}$-*a.s.* $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$.

The statements of Theorem B are all independent of the norm $\| \cdot \|$ used. We notice that our $\mathfrak{u}$-pointwise asymptotic stability is similar to the notion "consistent asymptotic stabilizability" in Z. Sun [51, Definition 2] where $\Phi_{\mathfrak{u}}(t)$ is required to be uniformly bounded for $t \in \mathbb{R}_+$. We also notice that the condition that $\hat{\rho}(\mathbf{A}) = 1$ is weaker than the uniform boundedness of $\Phi_{\mathfrak{u}}(t)$; let us see a discrete-time simple example:

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

for which $\hat{\rho}(S) = 1$, but $\|S^m\| \to +\infty$ as m tends to $+\infty$. Under his situation, Z. Sun [51] showed that $\mathfrak{u}$-consistent asymptotic stabilizability implies $\mathfrak{u}'$-exponential stabilizability; however, $\mathfrak{u}$ need not be equal to $\mathfrak{u}'$ there. But in our statement, $\mathfrak{u}$ is exactly equal to $\mathfrak{u}'$ $\mathbb{P}$-almost surely.

Let $\mathbf{S_A}$ be the discretization of the switched dynamics (**A**) as in (1.9) or (1.10) in Section 1.1. Then from Lemma 1.1, we can easily obtain the following

Lemma 3.1. *The joint spectral radius* $\hat{\rho}(\mathbf{A})$, *defined as in Theorem* B, *of* (**A**) *equals the joint spectral radius of its discretization* (**S$_\mathbf{A}$**), *i.e.,* $\hat{\rho}(\mathbf{A}) = \hat{\rho}(\mathbf{S_A})$, *where*

$$\hat{\rho}(\mathbf{S_A}) := \lim_{j \to +\infty} \left\{ \sup_{\sigma \in \Sigma_{\mathcal{I}}^+} \| \mathbf{S_A}(j, \sigma) \|^{1/j} \right\}.$$

So, $\hat{\rho}(\mathbf{A}) \geqslant 1$ *always holds.*

Proof. This follows obviously from

$$\sup_{\sigma \in \Sigma_{\mathcal{I}}^+} \| \mathbf{S_A}(j, \sigma) \|^{1/j} = \sup_{\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_{\Delta}^{\text{pc}}(\mathbb{R}_+, \mathbb{K})} \| \Phi_{\mathfrak{u}}(t_j) \|^{1/j} \quad \forall j \in \mathbb{N}$$

by Lemma 1.1. □

The joint spectral radius was firstly introduced by G.-C. Rota and G. Strang in [47] for the discrete-time case. It is well known from [2] that a discrete-time linear switched system (**S**) with a compact control-value set $\mathcal{I}$ is absolutely exponentially stable if and only if $\hat{\rho}(\mathbf{S}) < 1$. However, Lemma 3.1 implies that this is not true for continuous-time case under the sense of our definition given in Theorem B; see Footnote 4 before. For example, when $\{A_1, \ldots, A_K\}$ are commutative and each of the subsystems is exponentially stable, the switched dynamics (**A**) with admissible set $\mathcal{L}_{\Delta}^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$ is absolutely exponentially stable from [42]; but $\hat{\rho}(\mathbf{A}) \geqslant 1$ by Lemma 3.1. This is caused partially by the topological structure of $\mathcal{L}_{\Delta}^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$, noncompactness; see [18].

As a result of the statements of Lemmas 3.1 and 1.3, Theorem B comes immediately from the following discrete-time version with $\mathbf{S} = \mathbf{S_A}$ and $\mu = \mu_{\mathbb{P}}$.

Theorem B′. *Let* $\mathbf{S}: i \mapsto S_i \in \mathbb{C}^{d \times d}$ *be continuous in* $i \in \mathcal{I}$ *and bounded, where* $\mathcal{I}$ *is a separable metric space, not necessarily compact or countable. Assume* $\hat{\rho}(\mathbf{S}) = 1$. *Then, for a* θ_+-*ergodic Borel probability* μ *on* $\Sigma_{\mathcal{I}}^+$, *the following statements are equivalent to each other*:

(a) *The switched dynamics* (**S**) *is* σ-*pointwise asymptotically stable, for* μ-*a.s.* $\sigma \in \Sigma_{\mathcal{I}}^+$.
(b) *The switched dynamics* (**S**) *is* σ-*asymptotically stable, for* μ-*a.s.* $\sigma \in \Sigma_{\mathcal{I}}^+$.
(c) *The switched dynamics* (**S**) *is* σ-*exponentially stable, for* μ-*a.s.* $\sigma \in \Sigma_{\mathcal{I}}^+$.

This theorem shows some equivalence relationships under the hypothesis $\hat{\rho}(\mathbf{S}) = 1$. They are important for creating upper bounds, finding convergence rates and exploiting other basic system properties for switched linear systems, see Remark 4.5 below. For example, as a consequence of Theorem B′ above, we can obtain the following statement:

Corollary 3.2. *Let* $\mathbf{S}: i \mapsto S_i \in \mathbb{C}^{d \times d}$ *be continuous in* $i \in \mathcal{I}$, *where* $\mathcal{I}$ *is compact. If to any* θ_+-*ergodic Borel probability* μ *on* $\Sigma_{\mathcal{I}}^+$ *there* (**S**) *is asymptotically stable* μ-*almost surely, then* (**S**) *is absolutely exponentially stable and moreover,*

$$\left\| \mathbf{S}(j, \sigma) \right\| \to 0 \quad as\ j \to +\infty$$

uniformly for $\sigma \in \Sigma_{\mathcal{I}}^+$.

Here the uniformity follows from the semi-uniform subadditive ergodic theorem independently due to [48,55]; see [19] for an elementary simple proof.

The rest of this section will be devoted to proving Theorem B′.

3.1. Equivalence theorem in the product-bounded case

To prove Theorem B′, based on Theorem A′, we will first prove the following result, which implies the pointwise asymptotic stability is equivalent to the exponential stability almost surely for product bounded systems. It is somewhat interesting itself.

Theorem 3.3. *Let* $\mathbf{S}: i \mapsto S_i \in \mathbb{C}^{d \times d}$ *be continuous in* $i \in \mathcal{I}$ *and product bounded, where* $\mathcal{I}$ *is a separable metric space, not necessarily compact or countable. If for a* θ_+-*ergodic Borel probability* μ *on* $\Sigma_{\mathcal{I}}^+$, (**S**) *is* σ-*pointwise asymptotically stable for* μ-*a.s.* $\sigma \in \Sigma_{\mathcal{I}}^+$, *then* (**S**) *is* σ-*exponentially stable for* μ-*a.s.* $\sigma \in \Sigma_{\mathcal{I}}^+$.

Proof. Since the two types of stabilities involved in this theorem both are independent of the choice of a norm of $\mathbb{C}^d$, the statement comes immediately from Theorem A′.

In fact, for an arbitrary θ_+-ergodic Borel probability μ on $\Sigma_{\mathcal{I}}^+$, we only need to show that (**S**) is exponentially stable μ-almost surely if it is σ-pointwise asymptotically stable for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$. Assume the statement (1) of Theorem A′ were not true. Then for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$, we have

$$\left\| \mathbf{S}(n, \sigma) \right\|_* = 1 \quad \forall n \geqslant 1 \text{ for some fixed pre-extremal norm } \|\cdot\|_* \text{ of } \mathbf{S}.$$

Let

$$E_n^c(\sigma) = \left\{ x \in \mathbb{C}^d \colon \left\| \mathbf{S}(k, \sigma) \cdot x \right\|_* = 1 \text{ for } 0 \leqslant k \leqslant n \right\}$$

for all $n \geqslant 0$. It is easily checked that $E_n^c(\sigma)$ is a compact subset of $\mathbb{C}^d$ such that $E_n^c(\sigma) \supseteq E_{n+1}^c(\sigma) \neq \varnothing$ for all $n \geqslant 0$, for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$. Then

$$E^c(\sigma) := \bigcap_{n \geqslant 0} E_n^c(\sigma) \neq \varnothing$$

for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$. Now, for any $x_0 \in E^c(\sigma)$, there follows that $\|\mathbf{S}(n, \sigma) \cdot x_0\|_* = 1 \nrightarrow 0$ as $n \to +\infty$, a contradiction to the pointwise asymptotic stability μ-almost sure.

This proves the theorem. □

We notice here that the product boundedness of $\mathbf{S}$ implies $\hat{\rho}(\mathbf{S}) \leqslant 1$, yet the converse is not the case as is shown by the simple example presented in the notices behind Theorem B. So, Theorem B′ is more general than Theorem 3.3.

3.2. Three important lemmas

For the proof of Theorem B′, we need three important lemmas besides Theorem 3.3.

Recall that the *generalized spectral radius* of $(\mathbf{S})$ (cf. I. Daubechies and J.C. Lagarias [24], also see [59,18] for systems with constraints), is defined by

$$\rho(\mathbf{S}) = \limsup_{\ell \to +\infty} \left\{ \sup_{w \in \mathcal{I}^\ell} \rho(\mathbf{S}(w))^{1/\ell} \right\} \quad \text{or equivalently} \quad \rho(\mathbf{S}) = \sup_{\ell \geqslant 1} \left\{ \sup_{w \in \mathcal{I}^\ell} \rho(\mathbf{S}(w))^{1/\ell} \right\},$$

where $\rho(A)$ denotes the usual spectral radius of a matrix $A \in \mathbb{C}^{d \times d}$ and where

$$\mathbf{S}(w) = S_{i_\ell} \cdots S_{i_1}$$

for any word $w = (i_1, \dots, i_\ell) \in \mathcal{I}^\ell$ of length ℓ.

We need the famous Gel'fand-type spectral-radius formula due to M.A. Berger and Y. Wang [5], stated as follows:

Lemma 3.4 (*Generalized Gel'fand spectral-radius formula*). *(See [5].) If $\mathbf{S} : \mathcal{I} \to \mathbb{C}^{d \times d}$ is bounded in $\mathbb{C}^{d \times d}$, then there holds the identity $\rho(\mathbf{S}) = \hat{\rho}(\mathbf{S})$.*

In addition, we need a reduction theorem due to L. Elsner [25], stated as follows:

Lemma 3.5. *(See [25].) Let $\mathbf{S} : \mathcal{I} \ni i \mapsto S_i \in \mathbb{C}^{d \times d}$ be bounded in $\mathbb{C}^{d \times d}$. If $\hat{\rho}(\mathbf{S}) = 1$ and $\mathbf{S}$ is not product bounded, there is a nonsingular $B \in \mathbb{C}^{d \times d}$ and $1 \leqslant n_1 < d$ such that*

$$B^{-1} S_i B = \begin{bmatrix} S_i^{(1,1)} & S_i^{(1,2)} \\ \mathbf{0}_{(d-n_1) \times n_1} & S_i^{(2,2)} \end{bmatrix} \quad \forall i \in \mathcal{I}$$

where $S_i^{(1,1)} \in \mathbb{C}^{n_1 \times n_1}$ and $S_i^{(2,2)} \in \mathbb{C}^{(d-n_1) \times (d-n_1)}$ for all $i \in \mathcal{I}$.

One can find simple proofs for the above two theorems in the recent work [17].

The third lemma needed is on the growth of the spectral radius, due to Ian D. Morris, which is proved based on the multiplicative ergodic theorem (cf. [27,44,26]) using invariant cone.

Lemma 3.6. *(See [41].) Let $T : (\Omega, \mathscr{B}_\Omega, \mu) \to (\Omega, \mathscr{B}_\Omega, \mu)$ be a measure-preserving continuous transformation over a metrizable space Ω and $\psi : \mathbb{Z}_+ \times \Omega \to \mathbb{C}^{d \times d}$ a Borel measurable linear cocycle driven by T, i.e.,*

$$\psi(0, \omega) = \mathrm{Id}_{\mathbb{C}^d} \quad \text{and} \quad \psi(\ell + m, \omega) = \psi(\ell, T^m(\omega)) \cdot \psi(m, \omega) \quad \forall \omega \in \Omega \text{ and } \ell, m \geqslant 1.$$

If $\int_\Omega \log^+ \|\psi(1, \omega)\| \, d\mu(\omega) < \infty$ where $\log 0 = -\infty$ and $\log^+ x = \max\{0, \log x\}$ for any $x \geqslant 0$ then, one can find a T-invariant Borel subset Υ of Ω with $\mu(\Upsilon) = 1$ and a T-invariant measurable function $\chi(\omega)$ such that

$$\chi(\omega) = \limsup_{\ell \to +\infty} \frac{1}{\ell} \log \rho\big(\psi(\ell, \omega)\big) = \lim_{\ell \to +\infty} \frac{1}{\ell} \log \big\| \psi(\ell, \omega) \big\|$$

for all $\omega \in \varUpsilon$. Particularly, if μ is ergodic, then $\chi(\omega)$ is constant for all $\omega \in \varUpsilon$.

Here $\chi(\omega)$ is just the (maximal) Lyapunov exponent of ψ at the base point $\omega \in \varUpsilon$.

Next for proving Theorem B′ we will apply this lemma to the case where $\Omega = \Sigma_{\mathcal{I}}^+$, $T = \theta_+$, $\psi(0, \sigma) = \mathrm{Id}_{\mathbb{C}^d}$, and $\psi(j, \sigma) = \mathbf{S}(j, \sigma)$ for all $j \geqslant 1$ and $\sigma \in \Sigma_{\mathcal{I}}^+$.

3.3. Proof of Theorem B′

After these preliminaries, we now are in a position to prove Theorem B′ by induction on the dimension d of the state-space $\mathbb{C}^d$:

Proof of Theorem B′. Let μ be a θ_+-ergodic Borel probability on $\Sigma_{\mathcal{I}}^+$ such that (**S**) is σ-pointwise asymptotically stable for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$. In addition, by the hypothesis of the statement we have $\hat{\rho}(\mathbf{S}) = 1$. So, $\rho(\mathbf{S}) = 1$ by the Berger–Wang formula (Lemma 3.4).

Step 1. In the case $d = 1$, condition $\rho(\mathbf{S}) = 1$ implies that **S** is product bounded because of the following identity

$$\rho(\mathbf{S}) = \sup_{\ell \geqslant 1} \Big\{ \sup_{w \in \mathcal{I}^\ell} \rho\big(\mathbf{S}(w)\big)^{1/\ell} \Big\}.$$

So, from Theorem 3.3 it follows that (**S**) is exponentially stable μ-almost surely.

Step 2. Now, let $\mathfrak{m} \geqslant 2$ be an arbitrary integer and assume the assertion holds for any dimension $d < \mathfrak{m}$.

Step 3. We need to prove only that (**S**) is exponentially stable μ-almost surely in the case of $d = \mathfrak{m}$.

Indeed, if **S** is product bounded then Theorem 3.3 implies that the statement holds. So, we next assume **S** is product unbounded. Thus, by Lemma 3.5 there exists a nonsingular $B \in \mathbb{C}^{\mathfrak{m} \times \mathfrak{m}}$ and an integer $1 \leqslant n_1 < \mathfrak{m}$ such that

$$B^{-1} S_i B = \begin{bmatrix} S_i^{(1)} & S_i^{(1,2)} \\ \mathbf{0}_{n_2 \times n_1} & S_i^{(2)} \end{bmatrix} \quad \forall i \in \mathcal{I}$$

where $S_i^{(1)} \in \mathbb{C}^{n_1 \times n_1}$, $S_i^{(2)} \in \mathbb{C}^{n_2 \times n_2}$ for all $i \in \mathcal{I}$, where $n_2 = \mathfrak{m} - n_1$. Set

$$\mathbf{S}^{(r)} : \mathcal{I} \ni i \mapsto S_i^{(r)}, \quad r = 1, 2.$$

Then, both systems ($\mathbf{S}^{(1)}$) and ($\mathbf{S}^{(2)}$) are σ-pointwise asymptotically stable for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$. Since n_1, n_2 both $< \mathfrak{m}$, from the induction assumption, both ($\mathbf{S}^{(1)}$) and ($\mathbf{S}^{(2)}$) are σ-exponentially stable for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$. Since for any $\ell \geqslant 1$ and any word $w \in \mathcal{I}^\ell$ of length ℓ there holds that

$$\rho\big(\mathbf{S}(w)\big) = \max\big\{ \rho\big(\mathbf{S}^{(1)}(w)\big), \rho\big(\mathbf{S}^{(2)}(w)\big) \big\},$$

from Lemma 3.6 there follows that one can find a constant $0 < \alpha < 1$ which is such that for μ-a.s. $\sigma = (i_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$

$$\rho(S_{i_\ell} \cdots S_{i_1})^{1/\ell} \leqslant \alpha$$

as ℓ sufficiently large. Then, by using Lemma 3.6 once again, it follows that

$$\lim_{\ell \to +\infty} \frac{1}{\ell} \log \|\mathbf{S}(\ell, \sigma)\| \leqslant \log \alpha < 0 \quad \text{for } \mu\text{-a.s. } \sigma \in \Sigma_{\mathcal{I}}^{+}.$$

Thus, (**S**) is σ-exponentially stable for μ-a.s. $\sigma \in \Sigma_{\mathcal{I}}^{+}$, as desired.

This therefore proves the statement of Theorem $\tilde{\text{B}}'$. $\quad\square$

4. Linear switched systems periodically switched stable

Let us first consider the continuous-time case. For any given set $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ and any $0 < \Delta < +\infty$, $\mathbf{A}$ gives rise to as before the switched dynamics

$$\dot{x} = A_{u(t)} x, \quad x \in \mathbb{C}^d \text{ and } t \in \mathbb{R}_+ \tag{A}$$

where the admissible switching signals $u : \mathbb{R}_+ \to \mathbb{K} = \{1, \ldots, K\}$ belong to $\mathcal{L}_\Delta^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$ that is defined in the same manner as in Section 1.1. An $u = (\lambda_j, t_j)_{j=1}^{+\infty}$ is said to be *periodically switched*, provided that there is an integer $\mathbf{k} \geqslant 1$ so that

$$\lambda_{j+\mathbf{k}} = \lambda_j \quad \text{and} \quad t_{j+\mathbf{k}} - t_{j+\mathbf{k}-1} = t_j - t_{j-1} \quad \forall j \geqslant 1.$$

In other words, u is periodic if and only if $\vartheta_+^{\mathbf{k}}(u) = u$ for some integer $\mathbf{k} \geqslant 1$. The dynamics (**A**) is called to be *periodically switched stable* if (**A**) is u-asymptotically stable for any periodically switched signal $u \in \mathcal{L}_\Delta^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$.

It is easily seen that the periodically switched stability implies $\hat{\rho}(\mathbf{A}) = 1$ from Lemma 3.1 in the continuous-time case. For this kind of switched system, E.S. Pyatnitskiĭ has asked this important question:

Problem. (E.S. Pyatnitskiĭ, cf. [46,49].) Does periodically switched stability imply absolute asymptotic stability, and further exponential stability, for the linear switched dynamics (**A**)?

In this section, we will present a weak positive solution to this problem using Theorem A.

Theorem C. *Let* $0 < \Delta < +\infty$, $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ *satisfy* $\hat{\rho}(\mathbf{A}) = 1$ *and assume that* $\mathbb{P}$ *is a* ϑ_+-*ergodic probability on* $\mathcal{L}_\Delta^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$. *If* supp$(\mathbb{P})$ *contains a periodically switched signal* u' *for which* (**A**) *is* u'-*asymptotically stable, then* (**A**) *is exponentially switching-stable* $\mathbb{P}$-*almost surely, i.e.,* $\zeta(u, \mathbf{A}) < 0$ *for* $\mathbb{P}$-*a.s.* $u \in \mathcal{L}_\Delta^{\text{pc}}(\mathbb{R}_+, \mathbb{K})$.

This result implies that if the dynamics (**A**) is periodically switched stable and $\mathbb{P}$ has a periodical density point u', then (**A**) is asymptotically and exponentially stable $\mathbb{P}$-almost surely from Theorem B.

Next, we consider the discrete-time case. Let $\mathbf{S} : \mathcal{I} \ni i \mapsto S_i \in \mathbb{C}^{d \times d}$ be continuous, where the set $\mathcal{I}$ of control values is a separable metric space. Then, it defines naturally the discrete-time linear switched system

$$x_j = \mathbf{S}(j, \sigma) \cdot x_0, \quad x_0 \in \mathbb{C}^d \text{ and } j \geqslant 1 \tag{S}$$

where the admissible switching signals $\sigma : \mathbb{N} \to \mathcal{I}$ belong to $\Sigma_{\mathcal{I}}^{+}$, which can be thought of as a linear cocycle driven by the one-sided Markov shift $\theta_+ : \Sigma_{\mathcal{I}}^{+} \to \Sigma_{\mathcal{I}}^{+}$. Similarly, σ is said to be *periodical* if $\sigma(j + \mathbf{k}) = \sigma(j)$ for all $j \geqslant 1$ for some integer $\mathbf{k} \geqslant 1$. (**S**) is called to be *periodically switched stable*, provided that (**S**) is σ-asymptotically stable for each periodically switched signal $\sigma \in \Sigma_{\mathcal{I}}^{+}$. This implies that $\hat{\rho}(\mathbf{S}) \leqslant 1$ and that particularly, **S** is product bounded in the cases $d \leqslant 3$ from [22]. Here, it is

interesting to note that the discretization ($\mathbf{S_A}$) is not itself periodically switched stable, since $\mathrm{Id}_{\mathbb{C}^d}$ belongs to $\mathbf{S_A}$.

In the discrete-time case, it is a known fact that periodically switched stability need not imply the absolute asymptotic stability even in the case that $\mathcal{I}$ is finite as is shown by [8,7,36,37]. In [21], the authors induced the exponential stability almost surely from periodically switched stability for canonical Markovian θ_+-ergodic measures in the case that $\mathcal{I}$ is finite. Now, we shall induce the exponential stability almost surely for some more general θ_+-ergodic measures. The following is the discrete-time version of Theorem C.

Theorem C′. *Let* $\mathbf{S} : \mathcal{I} \ni i \mapsto S_i \in \mathbb{C}^{d \times d}$ *be continuous and bounded with* $\hat{\rho}(\mathbf{S}) = 1$, *where* $\mathcal{I}$ *is not necessarily compact. Assume that* μ *is a* θ_+-*ergodic probability on* $\Sigma_{\mathcal{I}}^+$. *If* $\mathrm{supp}(\mu)$ *contains a periodically switched signal that is asymptotically stable, then* ($\mathbf{S}$) *is* σ-*exponentially stable for* μ-*a.s.* $\sigma \in \Sigma_{\mathcal{I}}^+$.

We notice that since $\mathcal{I}$ is not necessarily compact in the situation of Theorem C′, the statement is nontrivial even in the special case $d = 1$.

We can prove Theorem C by applying Theorem C′ with $\mathbf{S} = \mathbf{S_A}$ and $\mu = \mu_{\mathbb{P}}$. So, here we need only to prove Theorem C′.

In [21], the proof relies sharply on a stability criterion established in [20]. Here the main new ingredient of proving Theorem C′ is Theorem A′ proved before.

The rest of this section will be devoted to proving Theorem C′ stated above.

4.1. Stability in the product-bounded case

To prove Theorem C′ above, we need a lemma using Theorem A′.

Lemma 4.1. *Let* $\mathbf{S} : i \mapsto S_i \in \mathbb{C}^{d \times d}$ *be continuous in* $i \in \mathcal{I}$ *and product bounded, where* $\mathcal{I}$ *is not necessarily compact. Assume* μ *is a* θ_+-*ergodic Borel probability on* $\Sigma_{\mathcal{I}}^+$. *If* $\mathrm{supp}(\mu)$ *contains a periodically switched signal that is asymptotically stable, then* ($\mathbf{S}$) *is exponentially stable* μ-*almost surely.*

Proof. Since $\mathbf{S}$ is product bounded, from Lemma 2.5 we could assume that there exists a norm, denoted by $\| \cdot \|$, on $\mathbb{C}^d$ such that $\|S_i\| \leqslant 1$ for all $i \in \mathcal{I}$.

Assume, by contradiction, that the statement of the lemma were not true. Then, from Theorem A′, it follows that there exists a Borel subset Γ with $\mu(\Gamma) = 1$ such that

$$\|S_{i_{n+k}} \cdots S_{i_n}\| = 1 \quad \forall n \geqslant 1 \text{ and } k \geqslant 0$$

for all $\sigma = (i_j)_{j=1}^{+\infty} \in \Gamma$. By the assumption of the lemma, one can pick a periodically switched signal of period $\mathbf{k} \geqslant 1$, say

$$\sigma' = \big(\underbrace{i'_1, \ldots, i'_{\mathbf{k}}}, \underbrace{i'_1, \ldots, i'_{\mathbf{k}}}, \ldots \big) \in \Sigma_{\mathcal{I}}^+, \quad \text{write } w' = \big(\underbrace{i'_1, \ldots, i'_{\mathbf{k}}} \big),$$

such that $\sigma' \in \mathrm{supp}(\mu)$ and ($\mathbf{S}$) is σ'-asymptotically stable. Then, one can find some switching signal $\sigma \in \mathrm{supp}(\mu) \cap \Gamma$ such that σ' is an ω-limit point of σ under the action of the one-sided Markov shift θ_+, that is to say, there is a sequence $j_k \nearrow +\infty$ so that $\theta_+^{j_k}(\sigma) \to \sigma'$ as $k \to +\infty$. This implies that

$$\big\| \big(\mathbf{S}(w') \big)^{\ell} \big\| = 1 \quad \forall \ell \geqslant 1,$$

which contradicts that ($\mathbf{S}$) is σ'-asymptotically stable.

This proves the lemma. $\quad \square$

4.2. Proof of Theorem C′

Using Lemma 4.1 proved and Lemmas 3.4, 3.5, 3.6, one could prove Theorem C′ by induction on the dimension of the state-space $\mathbb{C}^d$ similar to the proof of Theorem B′. But we now will prove it following the framework of [21].

First, the following lemma comes immediately from Barabanov's norm theorem [2], which is simply proven in the recent work [17].

Lemma 4.2. *If the bounded family* $\mathbf{S} = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$ *is irreducible (i.e., there is no a common, nontrivial, and proper linear subspace of* $\mathbb{C}^d$ *for each* S_i*) with* $\hat{\rho}(\mathbf{S}) = 1$*, then it is product bounded.*

The following is a standard reduction lemma in the theory of linear algebras.

Lemma 4.3. *(See [2].) For any family* $\mathbf{S} = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$*, there exists a nonsingular (unitary) matrix* $\mathcal{O} \in \mathbb{C}^{d \times d}$ *and* r *positive integers* $n_1, \ldots, n_r$ *with* $n_1 + \cdots + n_r = d$ *such that for each* $i \in \mathcal{I}$

$$\mathcal{O} S_i \mathcal{O}^{-1} = \begin{bmatrix} \widetilde{S}_i^{(1,1)} & \widetilde{S}_i^{(1,2)} & \cdots & \widetilde{S}_i^{(1,r)} \\ \mathbf{0}_{n_2 \times n_1} & \widetilde{S}_i^{(2,2)} & \cdots & \widetilde{S}_i^{(2,r)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_r \times n_1} & \mathbf{0}_{n_r \times n_2} & \cdots & \widetilde{S}_i^{(r,r)} \end{bmatrix},$$

where

$$\widetilde{\mathbf{S}}^{(k)} := \left\{ \widetilde{S}_i^{(k,k)} \right\}_{i \in \mathcal{I}} \subset \mathbb{C}^{n_k \times n_k}$$

is irreducible for each $1 \leqslant k \leqslant r$*.*

Based on this triangularization, we further have got the following useful result.

Lemma 4.4. *(See [21].) Let* μ *be an arbitrary* θ_+*-ergodic Borel probability on* $\Sigma_{\mathcal{I}}^+$*. Then, for the continuous bounded family* $\mathbf{S} = \{S_i\}_{i \in \mathcal{I}} \subset \mathbb{C}^{d \times d}$*, under the block-triangular decomposition of Lemma 4.3, one has*

$$\lambda(\mu, \mathbf{S}) = \max_{1 \leqslant k \leqslant r} \lambda\big(\mu, \widetilde{\mathbf{S}}^{(k)}\big).$$

Here the Lyapunov exponents

$$\lambda(\mu, \mathbf{S}) = \lim_{j \to +\infty} \frac{1}{j} \log \|S_{i_j} \cdots S_{i_1}\| \quad \text{and} \quad \lambda\big(\mu, \widetilde{\mathbf{S}}^{(k)}\big) = \lim_{j \to +\infty} \frac{1}{j} \log \big\| \widetilde{S}_{i_j}^{(k,k)} \cdots \widetilde{S}_{i_1}^{(k,k)} \big\|$$

for μ*-a.e.* $(i_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$*, all are independent of the norm* $\|\cdot\|$ *used.*

Remark 4.5. This lemma shows that if every sub-blocks $\widetilde{\mathbf{S}}^{(k)}$ are exponentially stable μ-almost surely, then $\mathbf{S}$ is exponentially stable μ-almost surely. However, if every sub-blocks $\widetilde{\mathbf{S}}^{(k)}$ are only asymptotically stable μ-almost surely, we cannot guarantee the μ-almost sure asymptotic stability for $\mathbf{S}$. This proves the importance of Theorems B and B′.

We now can finish the proof of Theorem C′.

Proof of Theorem C′. By the hypothesis of the statement, because of Lemma 4.2 the sub-block systems $\widetilde{\mathbf{S}}^{(k)}$, $1 \leqslant k \leqslant r$, defined by Lemma 4.3, all are product bounded. Then from Lemma 4.1, they

are exponentially stable μ-almost surely. This together with Lemma 4.4 completes the proof of Theorem C′.   □

5. A trichotomy result of continuous-time linear switched systems

For a continuous-time linear switched dynamics, if its admissible switching signal $\mathfrak{u}$ is recurrent, for example, almost periodic, we can obtain a trichotomy result that is more subtler than the statement of Theorem A.

Throughout this section, let $0 < \Delta < +\infty$ and $\mathbf{A} = \{A_1, \dots, A_K\} \subset \mathbb{C}^{d \times d}$, $d \geqslant 2$ be arbitrarily given, which give rise to the switched dynamics

$$\dot{x} = A_{\mathfrak{u}(t)} x, \quad x \in \mathbb{C}^d \text{ and } t \in \mathbb{R}_+ \tag{A}$$

where the admissible switching signals $\mathfrak{u} \colon \mathbb{R}_+ = (0, +\infty) \to \mathbb{K} = \{1, \dots, K\}$ belong to $\mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$ that is defined in the same manner as in Section 1.1 and on which there is the topological dynamical system

$$\vartheta_+ \colon \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K}) \to \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K}); \quad \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \mapsto \vartheta_+(\mathfrak{u}) = (\lambda_{j+1}, t_{j+1} - t_1)_{j=1}^{+\infty}.$$

Definition 5.1. (See [43].) For a switching signal $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$, it is called, under the dynamics ϑ_+, to be

1. *almost periodic* if for any $\varepsilon > 0$ there exists an integer $L(\varepsilon)$ defining an $L(\varepsilon)$-relatively dense set of integers $\{\tau_k\}$ in $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, such that

$$\mathsf{d}_\varrho\big(\vartheta_+^j(\mathfrak{u}), \vartheta_+^{j+\tau_k}(\mathfrak{u})\big) < \varepsilon \quad \forall j \in \mathbb{Z}_+;$$

2. *Birkhoff recurrent* if for any $\varepsilon > 0$ the set of values of $j \in \mathbb{N} = \{1, 2, \dots\}$ for which

$$\mathsf{d}_\varrho\big(\mathfrak{u}, \vartheta_+^j(\mathfrak{u})\big) < \varepsilon$$

be relatively dense in $\mathbb{Z}_+$.

Clearly, if $\mathfrak{u}$ is periodical, it is almost periodic. From the definitions, there hold the following inclusions:

$$\{\text{almost periodic } \mathfrak{u}\} \subseteq \{\text{Birkhoff recurrent } \mathfrak{u}\} \subseteq \{\text{weakly Birkhoff recurrent } \mathfrak{u}\}.$$

For any $\mathfrak{u}$, let

$$\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u}) = \big\{\vartheta_+^j(\mathfrak{u}) \mid j = 0, 1, \dots\big\},$$

which is called the forward ϑ_+-*orbit of* $\mathfrak{u}$. By $\mathrm{Cl}(\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u}))$ we denote the closure of $\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u})$ in the space $(\mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K}), \mathsf{d}_\varrho)$.

Let $\|\cdot\|_2$ denote the usual Euclidean vector norm on $\mathbb{C}^d$. We will prove the following result.

Theorem D. *Let* $\mathbf{A} = \{A_1, \dots, A_K\} \subset \mathbb{C}^{d \times d}$ *be such that* $\|\Phi_{\mathfrak{u}}(t)\|_2 \leqslant 1$ *for all* $t > 0$ *and any* $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$. *If* $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$ *in* $\mathcal{L}_\Delta^{\mathsf{pc}}(\mathbb{R}_+, \mathbb{K})$ *is Birkhoff recurrent under* ϑ_+, *then there holds one of the following three statements.*

(1) *The dynamics* (**A**) *is* $\mathrm{Cl}(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u}))$-*uniformly exponentially switching-stable*, *namely, there are two constants* $C > 0$ *and* $0 < \xi < 1$ *so that*

$$\big\| \Phi_{\mathfrak{u}'}\big(t'_j\big) \big\|_2 \leqslant C\xi^j \quad \forall j \in \mathbb{N} \text{ and } \mathfrak{u}' = \big(\lambda'_j, t'_j\big)^{+\infty}_{j=1} \in \mathrm{Cl}\big(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u})\big).$$

(2) *The dynamics* (**A**) *is of* $\mathrm{Cl}(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u}))$-*rigid motion*, *that is to say,*

$$\big\| \Phi_{\mathfrak{u}'}\big(t'_j\big) \cdot x \big\|_2 = \|x\|_2 \quad \forall j \in \mathbb{N}, \ x \in \mathbb{C}^d \text{ and } \mathfrak{u}' = \big(\lambda'_j, t'_j\big)^{+\infty}_{j=1} \in \mathrm{Cl}\big(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u})\big).$$

(3) *The dynamics* (**A**) *is* $\mathrm{Cl}(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u}))$-*partially switching-stable*, *i.e., for some integer* $1 \leqslant \mathbf{i} < d$, *some constants* $0 < \xi < 1$, $C > 0$, *and a continuous splitting*

$$\mathrm{Cl}\big(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u})\big) \ni \mathfrak{u}' \mapsto \mathbb{C}^d = \mathbb{E}^c\big(\mathfrak{u}'\big) \oplus \mathbb{E}^s\big(\mathfrak{u}'\big), \quad \dim \mathbb{E}^s\big(\mathfrak{u}'\big) = \mathbf{i}$$

with the invariance $\Phi_{\mathfrak{u}'}(t'_j) \cdot \mathbb{E}^{c/s}(\mathfrak{u}') = \mathbb{E}^{c/s}(\vartheta^j_+(\mathfrak{u}'))$ *for all* $j \geqslant 1$ *and any* $\mathfrak{u}' = (\lambda'_j, t'_j)^{+\infty}_{j=1}$ *in* $\mathrm{Cl}(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u}))$, *such that*

$$\big\| \Phi_{\mathfrak{u}'}\big(t'_j\big) \cdot x_0 \big\|_2 = \|x_0\|_2 \quad \forall x_0 \in \mathbb{E}^c\big(\mathfrak{u}'\big)$$

and

$$\big\| \Phi_{\mathfrak{u}'}\big(t'_j\big) \cdot y_0 \big\|_2 \leqslant C\xi^j \|y_0\|_2 \quad \forall y_0 \in \mathbb{E}^s\big(\mathfrak{u}'\big)$$

for all $j \in \mathbb{N}$ *and any* $\mathfrak{u}' = (\lambda'_j, t'_j)^{+\infty}_{j=1} \in \mathrm{Cl}(\mathrm{Orb}^+_{\vartheta_+}(\mathfrak{u}))$.

Note. According to the classical Lyapunov exponent theory,[5] the stable manifold $\mathbb{E}^s(\mathfrak{u}')$ is invariant with respect to a vector norm $\|\cdot\|$ of $\mathbb{C}^d$ used here in the following sense:

$$\big\| \Phi_{\vartheta^k_+(\mathfrak{u})}(t_{j+k} - t_k) \cdot y_0 \big\| \leqslant C_{\|\cdot\|} \xi^j_{\|\cdot\|} \|y_0\| \quad \forall j \geqslant 1 \text{ for } y_0 \in \mathbb{E}^s\big(\vartheta^k_+(\mathfrak{u})\big)$$

where $C_{\|\cdot\|} > 0$, $\xi_{\|\cdot\|} > 0$ are two constants associated to the norm $\|\cdot\|$. In addition, we do not know if there holds the classical partial stability, i.e.,

$$\big\| \Phi_{\mathfrak{u}'}\big(t'_j\big) \cdot y_0 \big\|_2 \leqslant C\xi^{t'_j} \|y_0\|_2 \quad \forall y_0 \in \mathbb{E}^s\big(\mathfrak{u}'\big)$$

instead of j by t'_j in the statement (3) of Theorem D.

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[5] In fact, here we need only the following simple facts: let $\varphi(j)$, $\psi(j)$ be any given $\mathbb{C}^d$-valued functions defined on $\mathbb{Z}_+$ such that

$$\lambda_\varphi := \limsup_{j \to +\infty} \frac{1}{j} \log \|\varphi(j)\| \quad \text{and} \quad \lambda_\psi := \limsup_{j \to +\infty} \frac{1}{j} \log \|\psi(j)\|,$$

then for any $\alpha, \beta \neq 0$,

$$\limsup_{j \to +\infty} \frac{1}{j} \log \|\alpha\varphi(j) + \beta\psi(j)\| = \max\{\lambda_\varphi, \lambda_\psi\} \quad \text{when } \lambda_\varphi \neq \lambda_\psi$$

and the exponents λ_φ, λ_ψ both are independent of the choice of the vector norm $\|\cdot\|$ on $\mathbb{C}^d$.

As $\sigma_{\mathfrak{u}}$, the discretization of $\mathfrak{u}$, is also Birkhoff recurrent under θ_+ whenever $\mathfrak{u}$ is Birkhoff recurrent under ϑ_+, from Birkhoff's theorems [43, Theorems V.7.07 and V.7.07] which claim: If a dynamics $T : \Omega \to \Omega$ is situated in a *complete* metric space Ω, then $\omega \in \Omega$ is (Birkhoff) recurrent if and only if $\mathrm{Cl}_\Omega(\mathrm{Orb}_T^+(\omega))$ is a compact, *minimal set*; here "minimal set" means that it is nonempty, closed and invariant, and has no proper subset possessing these three properties, it follows that $\mathrm{Cl}_{\Sigma_{\mathcal{I}}^+}(\mathrm{Orb}_{\theta_+}^+(\sigma_{\mathfrak{u}}))$ is a compact minimal set of the compact metric space $\Sigma_{\mathcal{I}}^+$ under θ_+, where $\mathcal{I} = \mathbb{K} \times [0, \Delta]$ is as in the discretization of (**A**). Meanwhile from Lemma 5.2 below, it follows that $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ is Birkhoff recurrent under ϑ_+ if and only if $\mathrm{Cl}(\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u}))$ is a compact, minimal set in $\mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$. This shows that Birkhoff's theorems are still valid in our situation without the completeness by Lemma 1.1.

So, the discrete-time version of Theorem D can be formulated as follows:

Theorem D′. *Let* $\mathbf{S} : \mathcal{I} \ni i \mapsto S_i \in \mathrm{GL}(d, \mathbb{C})$ *be continuous such that* $\|S_i\|_2 \leqslant 1$ *for all* $i \in \mathcal{I}$. *If* Λ *is a compact minimal subset of* $\Sigma_{\mathcal{I}}^+$ *under the dynamics* θ_+, *then there holds one of the following three statements.*

(1) (**S**) *is* Λ*-absolutely exponentially stable,*[6] *i.e.,*

$$\limsup_{j \to +\infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma)\|_2 < 0 \quad \forall \sigma \in \Lambda.$$

(2) (**S**) *is of* Λ*-rigid motion, that is to say,*

$$\|\mathbf{S}(j, \sigma) \cdot x\|_2 = \|x\|_2 \quad \forall j \geqslant 1 \text{ and } x \in \mathbb{C}^d, \text{ for all } \sigma \in \Lambda.$$

(3) (**S**) *is* Λ*-partially stable, that is to say, for some integer* $1 \leqslant \mathbf{i} < d$, *some constants* $0 < \xi < 1$, $C > 0$, *and a continuous splitting*

$$\Lambda \ni \sigma \mapsto \mathbb{C}^d = \mathbb{E}^c(\sigma) \oplus \mathbb{E}^s(\sigma), \quad \dim \mathbb{E}^s(\sigma) = \mathbf{i}$$

with the invariance property $\mathbf{S}(j, \sigma) \cdot \mathbb{E}^{c/s}(\sigma) = \mathbb{E}^{c/s}(\theta_+^j(\sigma))$ *for all* $j \geqslant 1$, *such that*

$$\|\mathbf{S}(j, \sigma) \cdot x_0\|_2 = \|x_0\|_2 \quad \forall j \geqslant 1 \text{ for } x_0 \in \mathbb{E}^c(\sigma)$$

and

$$\|\mathbf{S}(j, \sigma) \cdot y_0\|_2 \leqslant C\xi^j \|y_0\|_2 \quad \forall j \geqslant 1 \text{ for } y_0 \in \mathbb{E}^s(\sigma)$$

for all $\sigma \in \Lambda$.

Here $\mathbf{S}(n, \sigma) = S_{i_n} \cdots S_{i_1}$ *for any* $\sigma = (i_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$ *and* $n \geqslant 1$.

It should be noted here that the above Theorem D′ has been proved by Ian D. Morris in the case where Λ is a compact minimal subset of the *two-sided* Markov shift transformation

$$\theta : \Sigma_{\mathcal{I}} \to \Sigma_{\mathcal{I}}; \quad (i_j)_{j \in \mathbb{Z}} \mapsto (i_{j+1})_{j \in \mathbb{Z}}.$$

---

[6] Since here Λ is a compact θ_+-invariant set, the Λ-absolute exponential stability is equivalent to the Λ-uniform exponential stability.

See [40, Theorems 2.1 and 2.2]. Since for any two-sided switching signal $(i_j)_{j \in \mathbb{Z}} \in \Sigma_{\mathcal{I}}$ or $\mathfrak{u} \colon \mathbb{R} \to \mathbb{K}$ the associated switched dynamics is, in fact, completely determined by the one-sided subsequence $(i_j)_{j=1}^{+\infty}$ or $\mathfrak{u}_{\restriction \mathbb{R}_+}$, it appears more reasonable for us to choose the one-sided Markov shift on $\Sigma_{\mathcal{I}}^+$ as the driving system here; see Footnote 1 in Section 1.1.1.

We will prove Theorem D in Section 5.1 based on Theorem D'. And we will prove Theorem D' in Section 5.2 using an approach completely different from [40] to overcome the difficulty caused by the noninvertibility of the one-sided Markov shift θ_+. Moreover, following our framework, one could easily obtain a more general continuous-time version than Theorem D, see Theorem D'' in Section 5.3.

5.1. Proof of Theorem D

Let

$$\pi \colon \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K}) \ni \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \mapsto \sigma_{\mathfrak{u}} = (\lambda_j, \Delta_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+,$$

where $\Delta_j = t_j - t_{j-1}$ for all $j \geqslant 1$ and $\mathcal{I} = \mathbb{K} \times [0, \Delta]$, as in Section 1.1. According to Theorem D', to prove Theorem D it is sufficient to prove the following basic fact.

Lemma 5.2. *If $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$ is Birkhoff recurrent in $\mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ under ϑ_+, then*

$$\pi\big(\mathrm{Cl}\big(\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u})\big)\big) = \mathrm{Cl}_{\Sigma_{\mathcal{I}}^+}\big(\mathrm{Orb}_{\theta_+}^+(\sigma_{\mathfrak{u}})\big).$$

So, $\mathrm{Cl}(\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u}))$ is a compact, minimal subset of $\mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ under ϑ_+.

Proof. We need to prove only that $\mathrm{Cl}_{\Sigma_{\mathcal{I}}^+}(\mathrm{Orb}_{\theta_+}^+(\sigma_{\mathfrak{u}})) \subseteq \pi(\mathrm{Cl}(\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u})))$, where $\sigma_{\mathfrak{u}} = (\lambda_j, \Delta_j)_{j=1}^{+\infty}$ belongs to $\Sigma_{\mathcal{I}}^+$ with $\Delta_j = t_j - t_{j-1}$ for all $j \geqslant 1$. Let $\sigma' = (\lambda_j', \Delta_j')_{j=1}^{+\infty}$ be an ω-limit point of $\sigma_{\mathfrak{u}}$ under θ_+. Then there is a sequence $k_j \to +\infty$ such that $\theta_+^{k_j}(\sigma_{\mathfrak{u}})$ converges to σ' as $j \to +\infty$. We claim that σ' belongs to $\pi(\mathrm{Cl}(\mathrm{Orb}_{\vartheta_+}^+(\mathfrak{u})))$. For that, we only need to prove that $\sum_{j=1}^{+\infty} \Delta_j' = +\infty$.

By contradiction, assume that $\sum_{j=1}^{+\infty} \Delta_j' < +\infty$. Then for an arbitrary $\gamma > 0$ with $\gamma \ll \frac{1}{4}$, there exists an integer $N_\gamma \geqslant 1$ such that $\sum_{j=1}^{+\infty} \Delta_{j+N_\gamma}' < \gamma$.

Since θ_+ is continuous, we have get that $\theta_+^{N_\gamma + k_j}(\sigma_{\mathfrak{u}}) \to \theta_+^{N_\gamma}(\sigma')$ as $j \to +\infty$.

On the other hand, by $\theta_+^{N_\gamma}(\sigma_{\mathfrak{u}}) = (\lambda_{j+N_\gamma}, \Delta_{j+N_\gamma})_{j=1}^{+\infty}$ we see $\sum_{j=1}^{+\infty} \Delta_{j+N_\gamma} = +\infty$. So, one can find an integer $L \geqslant 1$ so that $\sum_{j=1}^{L} \Delta_{j+N_\gamma} \geqslant 1$. Moreover, there is an $\varepsilon > 0$ sufficiently small such that for any $\hat{\sigma} = (\hat{\lambda}_j, \hat{\Delta}_j)_{j=1}^{+\infty} \in \Sigma_{\mathcal{I}}^+$,

$$\sum_{j=1}^{L} \hat{\Delta}_j \geqslant \frac{1}{2} \quad \text{whenever } \mathrm{d}_\varrho\big(\theta_+^{N_\gamma}(\sigma_{\mathfrak{u}}), \hat{\sigma}\big) < \varepsilon.$$

Since $\theta_+^{N_\gamma}(\sigma_{\mathfrak{u}})$ is also Birkhoff recurrent under θ_+, there is some integer $\ell(\varepsilon) \geqslant 1$ such that for any $j \geqslant 1$, one can find some $n_j \in [N_\gamma + k_j, N_\gamma + k_j + \ell(\varepsilon) - 1]$ satisfying $\mathrm{d}_\varrho(\theta_+^{N_\gamma}(\sigma_{\mathfrak{u}}), \theta_+^{n_j}(\sigma_{\mathfrak{u}})) < \varepsilon$.

Finally, one can choose some $J > 1$ sufficiently large such that

$$\sum_{i=1}^{L+\ell(\varepsilon)} \Delta_{i+N_\gamma+k_j} \leqslant 2\gamma \quad \text{whenever } j \geqslant J.$$

However,

$$\frac{1}{2} \leqslant \sum_{i=1}^{L} \Delta_{i+n_j} \leqslant \sum_{i=1}^{L+\ell(\varepsilon)} \Delta_{i+N_\gamma+k_j} \leqslant 2\gamma$$

for all $j \geqslant J$; it is a contradiction.

This shows the lemma. $\square$

This fact is nontrivial, because $(\mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K}), d_\varrho)$ is not complete from Lemma 1.1.

5.2. Proof of Theorem D′

In this subsection, we will devote our attention to proving Theorem D′ using some known result cited from [14,15] and Theorem A′.

5.2.1. A semi hyperbolicity theorem

For simply proving Theorem D′, we will need a semi hyperbolicity theorem cited from [14]. Here, for the convenience we will introduce it in details.

We now adopt the terminology of cocycle. Let $\mathbf{S} \colon \mathcal{I} \to \mathrm{GL}(d, \mathbb{C})$; $i \mapsto S_i$ be a continuous, nonsingular, $d \times d$, complex matrix-valued function on a separable metric space $\mathcal{I}$, where $\mathcal{I}$ need not be compact. Let W be a compact θ_+-invariant closed subset of the discrete-time switching-signal space $\Sigma_\mathcal{I}^+$ that is a separable metric space with the induced metric $d_\varrho(\cdot, \cdot)$ as in Section 1.1. We consider the induced cocycle by $\mathbf{S}$, also write as $\mathbf{S}$,

$$\mathbf{S} \colon \mathbb{Z}_+ \times W \to \mathrm{GL}(d, \mathbb{C}); \quad (j, \sigma) \mapsto \mathbf{S}(j, \sigma) = \begin{cases} \mathrm{Id}_{\mathbb{C}^d} & \text{for } j = 0; \\ S_{\sigma(j)} \cdots S_{\sigma(1)} & \text{for } j \geqslant 1, \end{cases}$$

which is driven by $\theta_+ \colon \sigma = (i_j)_{j=1}^{+\infty} \mapsto \theta_+(\sigma) = (i_{j+1})_{j=1}^{+\infty}$ restricted to the compact invariant subspace W.

Let $\hat{\chi} \in \mathbb{R}$ be arbitrarily given and assume that to 'a.s.' $\sigma \in W$, there exists a measurable direct decomposition of $\mathbb{C}^d$ into subspaces[7]

$$\sigma \mapsto \mathbb{E}^s(\sigma, \hat{\chi}) \oplus \mathbb{E}^u(\sigma, \hat{\chi}) \quad \text{with } \mathbf{S}(j, \sigma) \cdot \mathbb{E}^{s/u}(\sigma, \hat{\chi}) = \mathbb{E}^{s/u}\big(\theta_+^j(\sigma), \hat{\chi}\big) \, \forall j \geqslant 1$$

such that

$$\lim_{j \to +\infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma) \cdot x\|_2 < \hat{\chi} \quad \forall x \in \mathbb{E}^s(\sigma, \hat{\chi}) \setminus \{\mathbf{0}\}$$

and

$$\lim_{j \to +\infty} \frac{1}{j} \log \|\mathbf{S}(j, \sigma) \cdot x\|_2 > \hat{\chi} \quad \forall x \in \mathbb{E}^u(\sigma, \hat{\chi}) \setminus \{\mathbf{0}\}.$$

$\mathbb{E}^s(\sigma, \hat{\chi})$ and $\mathbb{E}^u(\sigma, \hat{\chi})$ are called the $\hat{\chi}$-stable and $\hat{\chi}$-unstable manifolds of $\mathbf{S}$ at the regular base point $\sigma \in W$, respectively. Moreover, we call

$$\mathrm{Index}(\sigma, \hat{\chi}) = \dim \mathbb{E}^s(\sigma, \hat{\chi})$$

the $\hat{\chi}$-index of $\mathbf{S}$ at the base point $\sigma \in W$.

---

[7] We notice that since θ_+ is not invertible, such a decomposition need not exist in general.

We should notice here that $\sigma \mapsto \mathbb{E}^{s/u}(\sigma, \hat{\chi})$ is only measurable, not necessarily continuous, with respect to σ in W.

Next, we introduce the notation "$\hat{\chi}$-semi hyperbolicity".

Definition 5.3. (See [14].) Given $\hat{\chi} \in \mathbb{R}$, **S** is called *$\hat{\chi}$-semi hyperbolic* on W, provided that **S** is almost uniformly $\hat{\chi}$-expanding along $\mathbb{E}^u(\sigma, \hat{\chi})$; that is to say, there are constants $\varrho' > 0$ and $C' > 0$, which both are independent of σ, such that

$$\left\| \mathbf{S}(j, \sigma) \cdot x \right\|_2 \geqslant C' \|x\|_2 \exp\left(j(\hat{\chi} + \varrho')\right) \quad \forall x \in \mathbb{E}^u(\sigma, \hat{\chi}) \text{ and } j \geqslant 1$$

for 'a.s.' $\sigma \in W$.

Here and in the following, 'a.s.' means relative to all θ_+-ergodic Borel probabilities supported on W unless an explicit measure μ is given and write 'μ-a.s.' in this case.

Then, the semi hyperbolicity theorem proved in [14] could be stated as follows:

Theorem 5.4. *(See [14, Theorem 1].) Let* **S** *be $\hat{\chi}$-semi hyperbolic on W. If* Index$(\sigma, \hat{\chi})$ *is constant for 'a.s.' $\sigma \in W$, then* **S** *is almost $\hat{\chi}$-hyperbolic on W; that is to say, there exists a continuous invariant splitting of $\mathbb{C}^d$ into subspaces*

$$\sigma \mapsto \mathbb{E}^u(\sigma, \hat{\chi}) \oplus \mathbb{E}^s(\sigma, \hat{\chi})$$

and constants $\varrho > 0$, $C > 0$ such that

$$\left\| \mathbf{S}(j, \sigma) \cdot x \right\|_2 \leqslant C \|x\|_2 \exp\left(j(\hat{\chi} - \varrho)\right) \quad \forall x \in \mathbb{E}^s(\sigma, \hat{\chi}) \text{ and } j \geqslant 1$$

and

$$\left\| \mathbf{S}(j, \sigma) \cdot y \right\|_2 \geqslant C^{-1} \|y\|_2 \exp\left(j(\hat{\chi} + \varrho)\right) \quad \forall y \in \mathbb{E}^u(\sigma, \hat{\chi}) \text{ and } j \geqslant 1$$

for 'a.s.' $\sigma \in W$.

Note. This result implies immediately that **S** is contracting uniformly restricted to $\mathbb{E}^s$ on a θ_+-invariant closed subset of W of total probability 1 if $\hat{\chi} < 0$.

This theorem has been proven in [14] using Liao theory.

5.2.2. Proof of Theorem D$'$

For convenience, for any $\sigma \in \Sigma_{\mathcal{I}}^+$ write

$$\mathbb{E}^c(\sigma) = \left\{ x \in \mathbb{C}^d \colon \left\| S(n, \sigma) \cdot x \right\|_2 = \|x\|_2 \, \forall n \geqslant 1 \right\}.$$

Since the Euclidean norm $\| \cdot \|_2$ is pre-extremal for **S** by the hypothesis of the theorem, $\mathbb{E}^c(\sigma)$ is a linear subspace of $\mathbb{C}^d$. It is easy to check the invariance $S_{i_1} \cdot \mathbb{E}^c(\sigma) \subseteq \mathbb{E}^c(\theta_+(\sigma))$ for $\sigma = (i_j)_{j=1}^{+\infty}$.

Proof of Theorem D$'$. If, for any θ_+-ergodic Borel probability μ with supp$(\mu) \subseteq \Lambda$, (**S**) is exponentially stable μ-almost surely, then (**S**) is Λ-absolutely exponentially stable from [19] and so the property (1) of Theorem D$'$ holds. And we then could stop our proof here. So, we next assume that (**S**) is not exponentially stable $\mathbb{P}$-almost surely for some θ_+-ergodic Borel probability $\mathbb{P}$ with supp$(\mathbb{P}) \subseteq \Lambda$.

Then from Theorem A′, it follows that $\dim \mathbb{E}^c(\sigma) \geqslant 1$ for $\mathbb{P}$-a.s. $\sigma \in \Lambda$. As Λ is θ_+-minimal, from the Birkhoff recurrence of every $\sigma \in \Lambda$ and the continuity of **S**, it follows that there exists a continuous, non-zero distribution $\sigma \mapsto \mathbb{E}^c(\sigma)$ over Λ satisfying that for any $\sigma \in \Lambda$ and any $j \geqslant 1$

$$\mathbf{S}(j,\sigma) \cdot \mathbb{E}^c(\sigma) = \mathbb{E}^c\big(\theta_+^j(\sigma)\big) \quad \text{and} \quad \big\|\mathbf{S}(j,\sigma) \cdot x_0\big\|_2 = \|x_0\|_2 \quad \forall x_0 \in \mathbb{E}^c(\sigma),$$

and that to any $v \in \mathbb{C}^d - \mathbb{E}^c(\sigma)$ there are some $\mathbf{n}_v \geqslant 1$ and $0 \leqslant \delta_v < 1$ so that

$$\big\|\mathbf{S}(j,\sigma) \cdot v\big\|_2 \leqslant \delta_v \|v\|_2 \quad \forall j \geqslant \mathbf{n}_v.$$

It is easily seen that $\dim \mathbb{E}^c(\sigma) \equiv d - \mathbf{i} \ \forall \sigma \in \Lambda$ for some integer $0 \leqslant \mathbf{i} \leqslant d - 1$.

If $\mathbf{i} = 0$ then, the property (2) of Theorem D′ holds and we could stop the proof here. Therefore, we next assume $1 \leqslant \mathbf{i} \leqslant d - 1$.

Write $\mathbb{E}^c(\sigma)^\perp = \{x \in \mathbb{C}^d \mid \langle x, \mathbb{E}^c(\sigma) \rangle = 0\}$, i.e., the orthogonal complement of $\mathbb{E}^c(\sigma)$ in $\mathbb{C}^d$, for all $\sigma \in \Lambda$, and define a natural (Grassmannian) topological vector bundle

$$\mathbb{E}^c(\Lambda)^\perp = \bigsqcup_{\sigma \in \Lambda} \mathbb{E}^c(\sigma)^\perp.$$

Further, we define, based on $\theta_{+ \restriction \Lambda} : \Lambda \to \Lambda$, a linear skew-product dynamical system as follows:

$$\mathbf{S}^\perp : \mathbb{Z}_+ \times \mathbb{E}^c(\Lambda)^\perp \to \mathbb{E}^c(\Lambda)^\perp; \quad \big(j, (\sigma, y)\big) \mapsto \big(\theta_+^j(\sigma), \mathbf{S}^\perp(j,\sigma) \cdot y\big)$$

where

$$\mathbf{S}^\perp(j,\sigma) : \mathbb{E}^c(\sigma)^\perp \to \mathbb{E}^c\big(\theta_+^j(\sigma)\big)^\perp$$

is defined by the projection of $\mathbf{S}(j,\sigma) \cdot y$ onto $\mathbb{E}^c(\theta_+^j(\sigma))^\perp$ for any $y \in \mathbb{E}^c(\sigma)^\perp$, such that

$$\big\|\mathbf{S}^\perp(j,\sigma) \cdot y\big\|_2 \leqslant \big\|\mathbf{S}(j,\sigma) \cdot y\big\|_2 \leqslant \|y\|_2 \quad \forall j \geqslant 1.$$

Then, it follows from Theorem A′ that for any θ_+-ergodic Borel probability μ supported on Λ,

$$\chi^\perp(\mu) := \lim_{j \to +\infty} \frac{1}{j} \log \big\|\mathbf{S}^\perp(j,\sigma)\big\|_2 < 0 \quad \mu\text{-a.s. } \sigma \in \Lambda.$$

Moreover, from the upper-semi continuity of $\chi^\perp(\mu)$ with respect to μ ([15, Proposition 5], also see [19]), one could find some $\gamma < 0$ such that $\chi^\perp(\mu) \leqslant \gamma$ for all θ_+-ergodic Borel probability μ supported on Λ.

Thus, from the Liao spectrum theorem, for example see [15, Main Theorem 1], it follows that for any θ_+-ergodic Borel probability μ supported on Λ, the linear switched system (**S**) (or the cocycle **S**) has $\mathbf{i}$ negative Lyapunov exponents which are less than or equal to γ, counting with multiplicity, for μ-a.s. $\sigma \in \Lambda$. Then, from the multiplicative ergodic theorem [27,44], one could find a θ_+-invariant Borel subset $\Gamma \subset \Lambda$ with $\mu(\Gamma) = 1$ for any θ_+-ergodic Borel probability μ on Λ, such that there is a *measurable*, invariant splitting of $\mathbb{C}^d$ into subspaces

$$\Gamma \ni \sigma \mapsto \mathbb{C}^d = \mathbb{E}^c(\sigma) \oplus \mathbb{E}^s(\sigma)$$

satisfying

$$\lim_{j \to +\infty} \frac{1}{j} \log \big\|\mathbf{S}(j,\sigma) \cdot y\big\|_2 \leqslant \gamma \quad \forall y \in \mathbb{E}^s(\sigma) \setminus \{\mathbf{0}\}.$$

From the semi hyperbolicity theorem (Theorem 5.4) with $\hat{\chi} = \gamma/2$ and $W = \Lambda$, it follows that $\mathbf{S}_{\restriction \mathbb{E}^s(\Gamma)}$ is contracting uniformly. So, the above splitting over Γ could be extended onto the closure $\mathrm{Cl}(\Gamma)$. Since Λ is minimal, there follows $\mathrm{Cl}(\Gamma) = \Lambda$. So, the property (3) of Theorem D′ holds.

This thus proves Theorem D′. $\quad\square$

5.3. A more general continuous-time version

Let Ω be a Polish space. A more general continuous-time version of Theorem D′ can be stated as follows:

Theorem D″. *Let $\Phi_{\cdot}(\cdot) : \Omega \times \mathbb{R}_+ \to \mathrm{GL}(d, \mathbb{C})$ be a continuous linear cocycle based on a semiflow (Ω, φ) such that*

$$\left\| \Phi_\omega(t) \right\|_2 \leqslant \beta^t \quad \forall \omega \in \Omega \text{ and } t \geqslant 0$$

for some constant $\beta \geqslant 1$. If Λ is a φ-invariant compact minimal subset of Ω, then

(1) *either*

$$\limsup_{t \to +\infty} \frac{1}{t} \log \left\| \Phi_\omega(t) \right\|_2 < \log \beta \quad \forall \omega \in \Lambda$$

(2) *or,*

$$\left\| \Phi_\omega(t) \right\|_2 = \beta^t \|x\|_2 \quad \forall t \geqslant 0, \ x \in \mathbb{C}^d \text{ for all } \omega \in \Lambda;$$

(3) *or $\Phi_\omega(t)$ is Λ-partially stable, i.e., for an integer $1 \leqslant \mathbf{i} < d$, two constants $0 < \xi < 1$, $C > 0$, and a continuous splitting*

$$\Lambda \ni \omega \mapsto \mathbb{C}^d = \mathbb{E}^c(\omega) \oplus \mathbb{E}^s(\omega), \quad \dim \mathbb{E}^s(\omega) = \mathbf{i}$$

with the invariance property $\Phi_\omega(t) \mathbb{E}^{c/s}(\omega) = \mathbb{E}^{c/s}(\varphi(t, \omega))$ for all $t > 0$, such that

$$\left\| \Phi_\omega(t) \cdot x_0 \right\|_2 = \beta^t \|x_0\|_2 \quad \forall t \geqslant 0 \text{ for } x_0 \in \mathbb{E}^c(\omega)$$

and

$$\left\| \Phi_\omega(t) \cdot y_0 \right\|_2 \leqslant C \beta^t \xi^t \|y_0\|_2 \quad \forall t \geqslant 0 \text{ for } y_0 \in \mathbb{E}^s(\omega)$$

for all $\omega \in \Lambda$.

By a modification of the proof of Theorem D′, one could obtain the statement. So, we omit the details here.

A very interesting case for Theorem D″ is that Λ is the hull of an almost periodic control function $\mathfrak{u} : \mathbb{R}_+ \to \mathbb{C}^{d \times d}$ and φ is the one-sided translation flow; see Footnote 1 in Section 1.1.1.

5.4. Remarks

For a general topological dynamics $T : \Omega \to \Omega$, the set $W(T)$ of weakly Birkhoff recurrent points is of total measure 1 from Theorem 2.3, but need not be true for the set of all the Birkhoff recurrent points.

Clearly a Birkhoff recurrent point must be weakly Birkhoff recurrent, but the converse is not the case as is shown by a counterexample in Z. Zhou and W. He [62] for the one-sided Markov shift θ_+ of finitely many letters. Here, we could construct a simple example as follows:

Example 5.5. Let $\mathcal{I} = \{0, 1\}$ endowed with the discrete-topology metric. Let $\mathbf{p} = (1/2, 1/2)$ be a 2-dimensional probability vector. Then the corresponding Markovian probability measure $\mu_{\mathbf{p}}$, defined in the means

$$\mu_{\mathbf{p}}\big([i_1, \ldots, i_\ell]\big) = \left(\frac{1}{2}\right)^\ell \quad \forall (i_1, \ldots, i_\ell) \in \mathcal{I}^\ell,$$

is ergodic for the one-sided Markov shift $\theta_+ : \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$ such that $\mathrm{supp}(\mu_{\mathbf{p}}) = \Sigma_{\mathcal{I}}^+$. Since the periodic points of θ_+ are dense in $\Sigma_{\mathcal{I}}^+$, $\Sigma_{\mathcal{I}}^+$ is not minimal. Thus, for $\mu_{\mathbf{p}}$-a.s. $\sigma \in \Sigma_{\mathcal{I}}^+$, it is weakly Birkhoff recurrent from Theorem 2.3, but not Birkhoff recurrent.

This example shows that Theorem A/A′ is essentially different from Theorem D/D′.

Parallel to the continuous-time case considered in Section 3, the discrete-time linear switched dynamics (**S**) is called to be

(1) "switched convergent", if to each $x_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ there corresponds some $\sigma_{x_0} \in \Sigma_{\mathcal{I}}^+$ satisfying

$$\big\| \mathbf{S}(j, \sigma_{x_0}) \cdot x_0 \big\|_2 \to 0 \quad \text{as } j \to +\infty;$$

(2) "exponentially stabilizable", if there exist two constants $\alpha < 0$ and $\beta > 0$ such that to each $x_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ there corresponds some $\sigma'_{x_0} \in \Sigma_{\mathcal{I}}^+$ satisfying

$$\big\| \mathbf{S}(j, \sigma'_{x_0}) \cdot x_0 \big\|_2 \leqslant \beta \|x_0\|_2 \exp(j\alpha) \quad \forall j \geqslant 1.$$

Next, using Theorem D′ we are going to prove that if **S** is product bounded and switched convergent, then there must be a subset restricted to which (**S**) is either absolutely exponentially stable or partially stable. For this, we need first the following lemma due to Z. Sun:

Lemma 5.6. *(See [53, Theorem 1].) The following two statements are equivalent*:

(i) **S** *is switched convergent.*
(ii) **S** *is exponentially stabilizable.*

Now, the following is a result of the statements of Theorem D′ and Lemma 5.6.

Corollary 5.7. *Let* $\mathbf{S} : \mathcal{I} \to \mathrm{GL}(d, \mathbb{C})$ *satisfy* $\|S_i\|_2 \leqslant 1$ *for all* $i \in \mathcal{I}$, *where* $\mathcal{I}$ *is compact. If* (**S**) *is switched convergent, then there must be at least one* θ_+-*minimal subset* $\Lambda \subset \Sigma_{\mathcal{I}}^+$ *such that* (**S**) *is either* Λ-*absolutely exponentially stable or* Λ-*partially stable.*

Proof. By Lemma 5.6, (**S**) is exponentially stabilizable. Let $\alpha < 0$, $\beta > 0$ be defined as in the item (ii) of Lemma 5.6 above. We define the set

$$\Sigma_{\mathrm{s.c.}} = \big\{ \sigma \in \Sigma_{\mathcal{I}}^+ \,\big|\, \exists x_\sigma \in \mathbb{C}^d \setminus \{\mathbf{0}\} \text{ so that } \big\| \mathbf{S}(j, \sigma) \cdot x_\sigma \big\|_2 \leqslant \beta \|x_\sigma\|_2 \exp(j\alpha) \; \forall j \geqslant 1 \big\}.$$

Clearly, $\Sigma_{\text{s.c.}}$ is θ_+-invariant, and nonempty by the exponential stabilizability. Moreover, $\Sigma_{\text{s.c.}}$ is a closed subset of $\Sigma_{\mathcal{I}}^+$; in fact, let $\{\sigma_{(j)}\}_{j=1}^{\infty}$ be a sequence in $\Sigma_{\text{s.c.}}$ satisfying $\sigma_{(j)} \to \sigma_{(0)}$ as $j \to +\infty$; one can take a sequence of unit vectors $\{x_{(j)}\}$ in $\mathbb{C}^d$ with

$$\left\| \mathbf{S}(\ell, \sigma_{(j)}) \cdot x_{(j)} \right\|_2 \leqslant \beta \exp(\ell\alpha) \quad \forall \ell \geqslant 1;$$

by the compactness of the unit sphere in $\mathbb{C}^d$, there is no loss of generality in assuming that $x_{(j)} \to x_{(0)}$ with $\|x_{(0)}\|_2 = 1$; so, $\|\mathbf{S}(\ell, \sigma_{(0)}) \cdot x_{(0)}\|_2 \leqslant \beta \exp(\ell\alpha) \ \forall \ell \geqslant 1$; this implies that $\sigma_{(0)} \in \Sigma_{\text{s.c.}}$. Thus, from the compactness of $\Sigma_{\mathcal{I}}^+$ and so $\Sigma_{\text{s.c.}}$, one can find a θ_+-minimal subset $\Lambda \subseteq \Sigma_{\text{s.c.}}$. Then, the statement follows immediately from Theorem D$'$.

This proves the corollary. $\square$

Remark 5.8. From Theorem E and Lemma 6.1 to be proven in Section 6.1, we see that Λ given by Corollary 5.7 might be required to consist of periodically switched signals.

6. Approximation and stabilizability of linear systems driven by minimal dynamics

How to design, for an initial state $x_0 \in \mathbb{C}^d$, a stabilizing switching signal $\mathfrak{u}(t)$ for a linear switched system (**A**) is a primary synthesis issue in the theory of control; see [57,9,50–52,28,29] for example. However, we need first to establish a theoretic foundation before one begins to design such a stabilizing switching signal; that is the existence theorem of stabilizing switching signal/law for any initial state. This section will be devoted to this topic. It is just the aspects (3) described in Section 1.2. The main results are Theorem E and Theorem F proved in this section. Theorem 6.10 below is an ergodic result that seems important for further study of rotation numbers.

6.1. Approximation of stable manifold by periodically switched signals

Corresponding to "partial hyperbolicity" in the differentiable dynamical systems, as is shown by Corollary 5.7 before, here the "partial stability" defined as in the statement of Theorem D$'$ could be an interesting, typical phenomenon in the linear control theory. However, the stable manifold $\mathbb{E}^s(\sigma)$ depends completely upon the infinite switching sequence $\sigma = (i_j)_{j=1}^{+\infty}$, not upon any finite-length sub-word $(i_1, \ldots, i_\ell)$ of σ.

So, for applications in engineering, in the partially stable case we often need to find suitable ways to approach arbitrarily the stable manifold bundle $\mathbb{E}^s(\Lambda)$, since for any input $(\sigma, x_0) \in \mathbb{E}^s(\Lambda)$ the output/solution $\{x_n\}_{n \geqslant 0}$ of (**S**) is exponentially stable. For this, we can obtain the following theorem.

Theorem E. *Let* $\mathbf{S} \colon \mathcal{I} \ni i \mapsto S_i \in \mathrm{GL}(d, \mathbb{C})$ *be continuous, where* $\mathcal{I}$ *is a compact metric space. Assume* (**S**) *is* Λ-*partially stable for a* θ_+-*invariant minimal subset* Λ *of* $\Sigma_{\mathcal{I}}^+$. *If* $\{\sigma^{(k)}\}_{k=1}^{+\infty}$ *is a sequence of periodically switched signals with*

$$\mathrm{Orb}_{\theta_+}^+\left(\sigma^{(k)}\right) \xrightarrow{\text{in the sense of Hausdorff metric}} \Lambda \quad \text{as } k \to +\infty,$$

then for any $\hat{\chi} < 0$ *sufficiently large*

$$\mathbb{E}^s\left(\mathrm{Orb}_{\theta_+}^+\left(\sigma^{(k)}\right), \hat{\chi}\right) \xrightarrow{\text{in the sense of Grassmann}} \mathbb{E}^s(\Lambda) \quad \text{as } k \to +\infty.$$

Here $\mathbb{E}^s(\mathrm{Orb}_{\theta_+}^+(\sigma^{(k)}), \hat{\chi}) = \bigsqcup_{\sigma \in \mathrm{Orb}_{\theta_+}^+(\sigma^{(k)})} \mathbb{E}^s(\sigma, \hat{\chi})$ is the $\hat{\chi}$-stable manifold bundle over the periodical orbit $\mathrm{Orb}_{\theta_+}^+(\sigma^{(k)})$ and $\mathbb{E}^s(\sigma, \hat{\chi})$ is defined in the same manner as in Section 5.2.1. In fact, if $\sigma^{(k)} = (i_j)_{j=1}^{+\infty}$ has period τ_k, then $\mathbb{E}^s(\sigma^{(k)}, \hat{\chi})$ is just the direct sum of the eigenspaces associated to the eigenvalues with absolute value $< \exp(\tau_k \hat{\chi})$ of the product matrix $S_{i_{\tau_k}} \cdots S_{i_1}$.

In addition, since Λ is minimal, such a sequence $\{\sigma^{(k)}\}_{k=1}^{+\infty}$ always could be selected out from the following closing lemma.

Lemma 6.1. *Let $\theta_+ : \Sigma_{\mathcal{I}}^+ \to \Sigma_{\mathcal{I}}^+$ be the one-sided Markov shift based on a compact metric space $\mathcal{I}$. Then, for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that whenever $d_\varrho(\sigma, \theta_+^\tau(\sigma)) < \delta$ for $\sigma \in \Sigma_{\mathcal{I}}^+$ and $\tau \geqslant 1$, one can find a point σ' satisfying the following two conditions:*

(1) $\theta_+^\tau(\sigma') = \sigma'$, i.e., σ' is a periodically switched signal of period τ;
(2) $d_\varrho(\theta_+^k(\sigma), \theta_+^k(\sigma')) < \varepsilon$ for $0 \leqslant k \leqslant \tau$.

Here $d_\varrho(\cdot, \cdot)$ is the metric on $\Sigma_{\mathcal{I}}^+$ defined as before.

Proof. This statement follows easily from the definition of the metric $d_\varrho(\cdot, \cdot)$ and a standard argument similar to that of $\mathcal{I} = \{1, 2\}$. $\quad\square$

In the proof of Theorem E, we will employ a theorem quoted from [16]. Here we will introduce it in details for the convenience of readers.

Let $\sigma \in \Sigma_{\mathcal{I}}^+$ be a periodically switched signal of period $\tau \geqslant 3$. We simply write

$$\mathcal{O} = \{\sigma, \theta_+(\sigma), \dots, \theta_+^{\tau-1}(\sigma)\}.$$

Clearly, θ_+ is 1-to-1 restricted to $\mathcal{O}$. The following is an improved version of the classical Alexseev theorem.

Theorem 6.2. *(See [16, Lemma 3.3].) Let $\lfloor \cdot \rfloor_w$ be a vector norm of $\mathbb{C}^d$, for any $w \in \mathcal{O}$. Suppose that $\mathbb{C}^d$ has a Whitney decomposition*

$$\mathbb{C}^d = \mathbb{F}^1(w) \oplus \mathbb{F}^2(w) \quad \forall w \in \mathcal{O}$$

such that $\dim \mathbb{F}^1(w)$ is constant and that $\mathbf{S}(1, w)$ and $\mathbf{S}^{-1}(1, \theta_+^{-1}(w))$ could be, respectively, represented as

$$\begin{bmatrix} F_{11}(w) & F_{12}(w) \\ F_{21}(w) & F_{22}(w) \end{bmatrix} : \mathbb{F}^1(w) \oplus \mathbb{F}^2(w) \to \mathbb{F}^1(\theta_+(w)) \oplus \mathbb{F}^2(\theta_+(w))$$

and

$$\begin{bmatrix} \widehat{F}_{11}(w) & \widehat{F}_{12}(w) \\ \widehat{F}_{21}(w) & \widehat{F}_{22}(w) \end{bmatrix} : \mathbb{F}^1(w) \oplus \mathbb{F}^2(w) \to \mathbb{F}^1(\theta_+^{-1}(w)) \oplus \mathbb{F}^2(\theta_+^{-1}(w))$$

for all $w \in \mathcal{O}$, where θ_+^{-1} is restricted to $\mathcal{O}$. Assume for any $w \in \mathcal{O}$

$$\max\{\lfloor F_{11}(w) \rfloor, \lfloor \widehat{F}_{11}^{-1}(w) \rfloor\} < e^{\varsigma_1}, \qquad \max\{\lfloor F_{22}^{-1}(w) \rfloor, \lfloor \widehat{F}_{22}(w) \rfloor\} < e^{-\varsigma_2}$$

and

$$\max\{\lfloor F_{12}(w) \rfloor, \lfloor F_{21}(w) \rfloor, \lfloor \widehat{F}_{12}(w) \rfloor, \lfloor \widehat{F}_{21}(w) \rfloor\} < \varepsilon e^{-|\varsigma_1 + \varsigma_2|/2}$$

where

$$-\infty < \varsigma_1 < \varsigma_2 < +\infty \quad \text{and} \quad 0 < \varepsilon < \min\{1 - e^{(\varsigma_1 - \varsigma_2)/2}, e^{(\varsigma_2 - \varsigma_1)/2} - 1\}.$$

Let

$$\chi_- = \frac{\varsigma_1 + \varsigma_2}{2} - \log\big[e^{(\varsigma_2 - \varsigma_1)/2} - \varepsilon\big],$$

$$\chi_+ = \frac{\varsigma_1 + \varsigma_2}{2} + \log\big[e^{(\varsigma_2 - \varsigma_1)/2} - \varepsilon\big].$$

Then there is an invariant decomposition of $\mathbb{C}^d$ *into subspaces*

$$\mathbb{C}^d = \mathbb{E}^1(w) \oplus \mathbb{E}^2(w) \quad \text{with } \dim \mathbb{E}^1(w) = \dim \mathbb{F}^1(w) \; \forall w \in \mathcal{O}$$

such that

$$\big\lfloor \mathbf{S}(\ell, w) \cdot v_1 \big\rfloor_{\theta_+^\ell(w)} \leqslant K e^{\ell \chi_-} \lfloor v_1 \rfloor_w \quad \forall v_1 \in \mathbb{E}^1(w), \; \ell \geqslant 1$$

and

$$\big\lfloor \mathbf{S}(\ell, w) \cdot v_2 \big\rfloor_{\theta_+^\ell(w)} \geqslant K^{-1} e^{\ell \chi_+} \lfloor v_2 \rfloor_w \quad \forall v_2 \in \mathbb{E}^2(w), \; \ell \geqslant 1.$$

Here $K > 0$ *is a constant.*

Remark 6.3. The fact that the constants χ_- and χ_+ are such that

$$\chi_- \searrow \varsigma_1 \quad \text{and} \quad \chi_+ \nearrow \varsigma_2 \quad \text{as } \varepsilon \to 0,$$

will be useful for proving Theorem E.

Proof of Theorem E. Assume the admissible switching-signal set Λ is nontrivial, i.e., Λ is not itself a periodical orbit. Otherwise, the statement holds trivially.

From the hypothesis of the theorem, there is a sequence $\varepsilon_k \searrow 0$ such that the Hausdorff distance

$$d_H(\mathcal{O}_k, \Lambda) < \varepsilon_k \quad \text{where } \mathcal{O}_k := \mathrm{Orb}_{\theta_+}^+\big(\sigma^{(k)}\big) \text{ for all } k \geqslant 1.$$

For any $w \in \mathcal{O}_k$, there is some $\sigma_w \in \Lambda$ which is such that $d_\varrho(w, \sigma_w) < \varepsilon_k$; so by translation we have the Whitney decomposition

$$\mathbb{C}^d = \mathbb{F}^1(w) \oplus \mathbb{F}^2(w) \quad \text{where } \mathbb{F}^1(w) = \mathbb{E}^s(\sigma_w) \text{ and } \mathbb{F}^2(w) = \mathbb{E}^c(\sigma_w).$$

Here the stable and central manifolds $\mathbb{E}^{s/c}(\sigma_w)$ over σ_w are defined by Theorem D′. Let

$$\frac{1}{2} \log \xi < \varsigma_1 < \varsigma_2 < 0,$$

where ξ is defined by Theorem D′. Since the splitting $\mathbb{E}^s(\Lambda) \oplus \mathbb{E}^c(\Lambda)$ is continuous in $\sigma \in \Lambda$ and $\mathbf{S}(\ell, \sigma)$ is continuous in $\sigma \in \Sigma_{\mathcal{I}}^+$, there holds the condition of the Alexseev theorem (Theorem 6.2) as k large sufficiently. Thus, the statement follows from Theorem 6.2.

This completes the proof of Theorem E. $\quad\square$

6.2. Switching-exponential stabilizability and rotation number

Let $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{C}^{d \times d}$ and $0 < \Delta < +\infty$ both be arbitrarily given. An input $(y_0, \mathfrak{u})$ in $(\mathbb{C}^d \setminus \{\mathbf{0}\}) \times \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ is said to be *switching stable* for $\mathbf{A}$, if the output $\{y_j = \Phi_\mathfrak{u}(t_j) \cdot y_0\}_{j \geqslant 1}$ of $(\mathbf{A})$ is exponentially switching-stable, not necessarily exponentially stable in the traditional sense; that is to say, the switching indicator

$$\zeta(y_0, \mathfrak{u}, \mathbf{A}) := \limsup_{j \to +\infty} \frac{1}{j} \log \| \Phi_\mathfrak{u}(t_j) \cdot y_0 \|_2 < 0.$$

The problem that if one can design some switching-stable signal $\mathfrak{u}$ for an initial data $y_0 \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ given previously, is very important in engineering.

In applications, such a switching signal, found for the above problem, should be restricted to some constrained switching-signal subset $\Lambda \subset \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$. Particularly, if Λ is ϑ_+-dynamics minimal, then the problem is trivial in the case of items (1) and (2) of Theorem D. So, we will devote our attention to the case (3) of partial switching-stability.

On such a basis, we need to use the rotation number of a switching signal defined in Section 6.2.1 below. Our main result of this part can be stated as follows:

Theorem F. *Let $0 < \Delta < +\infty$ and $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{R}^{2 \times 2}$ be such that $\| \Phi_\mathfrak{u}(t) \|_2 \leqslant 1$ for all $t > 0$ and any $\mathfrak{u} \in \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$. Assume $(\mathbf{A})$ is Λ-partially switching-stable for a compact, minimal subset Λ of $\mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ under ϑ_+. If there exists a $\mathfrak{u}' \in \Lambda$ having an irrational rotation number $\boldsymbol{\alpha}(\mathfrak{u}'; \mathbf{A})$, then to any $y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$,*

(1) *there exists some $\mathfrak{u}_{y_0} \in \Lambda$ switching stable for y_0 such that $y_0 \in \mathbb{E}^s(\mathfrak{u}_{y_0})$;*
(2) *there exists some $\mathfrak{u}'_{y_0} \in \Lambda$ such that $y_0 \in \mathbb{E}^c(\mathfrak{u}'_{y_0})$.*

Here the stable manifold $\mathbb{E}^s$ and the central manifold $\mathbb{E}^c$ are defined in the same manner as in Theorem D.

This theorem shows that in the partially switching-stable case, the stabilizability problem is very complicated.

6.2.1. Rotation numbers of continuous-time linear switched systems

Rotation number possibly becomes an important tool for study of the chaotic behavior of switched dynamical systems. So, in this part, we will introduce the theory of rotation numbers for linear switched dynamics using a very simple approach.

Hereafter, we let $0 < \Delta < +\infty$ and $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{R}^{2 \times 2}$ be arbitrarily given. We first consider, for an arbitrary $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty} \in \mathcal{L}_\Delta^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$, the continuous-time linear switched dynamical system

$$\dot{x}(t) = A_{\mathfrak{u}(t)} x(t), \quad x(0) = x_0 \in \mathbb{R}^2 \text{ and } t \in \mathbb{R}_+. \tag{6.1}$$

Write

$$A_{\mathfrak{u}(t)} = \begin{bmatrix} A_{\mathfrak{u}(t)}^{11} & A_{\mathfrak{u}(t)}^{12} \\ A_{\mathfrak{u}(t)}^{21} & A_{\mathfrak{u}(t)}^{22} \end{bmatrix}.$$

Introduce the polar coordinates (r, φ) in the $(x_1, x_2)^{\mathsf{T}}$-state space $\mathbb{R}^2$, i.e.,

$$x_1 = r \cos 2\pi \varphi \quad \text{and} \quad x_2 = r \sin 2\pi \varphi,$$

where $r \geqslant 0$, $\varphi \in \mathbb{R}$. Then from Eq. (6.1), one can see that φ satisfies the equation

$$\dot{\varphi} = f_{\mathfrak{u}(t)}(\varphi) = \frac{1}{2\pi} \left\{ A_{\mathfrak{u}(t)}^{21} \cos^2 2\pi \varphi - A_{\mathfrak{u}(t)}^{12} \sin^2 2\pi \varphi + \frac{A_{\mathfrak{u}(t)}^{22} - A_{\mathfrak{u}(t)}^{11}}{2} \sin 4\pi \varphi \right\}. \tag{6.2}$$

For any initial state $\varphi_0 \in \mathbb{R}$, we let $\varphi(t) = \varphi_{\mathfrak{u}}(t, \varphi_0)$ denote the solution of (6.2) with $\varphi(0) = \varphi_0$. Then, for any $x_0 = (r_0 \cos 2\pi \varphi_0, r_0 \sin 2\pi \varphi_0)^{\mathsf{T}}$ with $r_0 > 0$, the solution $x(t) = \Phi_{\mathfrak{u}(t)} \cdot x_0$ of (6.1) satisfies that

$$x(t) = \|x(t)\|_2 \big(\cos 2\pi \varphi_{\mathfrak{u}}(t, \varphi_0), \sin 2\pi \varphi_{\mathfrak{u}}(t, \varphi_0)\big)^{\mathsf{T}} \quad \forall t > 0. \tag{6.3}$$

This motivates us to introduce the following definition.

Definition 6.4. For any $\mathfrak{u} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ and any $\varphi_0 \in \mathbb{R}$, the limit (if exists)

$$\boldsymbol{\alpha}(\mathfrak{u}; \mathbf{A}) = \lim_{j \to +\infty} \frac{\varphi_{\mathfrak{u}}(t_j, \varphi_0) - \varphi_0}{j} \tag{6.4}$$

is called the *rotation number of* (**A**) *at the switching signal* $\mathfrak{u}$.

The following lemma shows that such a rotation number is well defined, i.e., if $\boldsymbol{\alpha}(\mathfrak{u}; \mathbf{A})$ is defined for some initial state $\varphi_0 \in \mathbb{R}$, then it is independent of the choice of φ_0.

Lemma 6.5. *Let* $\mathfrak{u} \in \mathcal{L}_{\Delta}^{\mathrm{pc}}(\mathbb{R}_+, \mathbb{K})$ *be any given. Then, for any* $\varphi_0', \varphi_0'' \in \mathbb{R}$, *one has*

$$\big| \big(\varphi_{\mathfrak{u}}(t, \varphi_0') - \varphi_0'\big) - \big(\varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0''\big) \big| \leqslant 1 \quad \forall t \in \mathbb{R}_+.$$

Proof. From the fact $f_{\mathfrak{u}(t)}(\varphi + 1) = f_{\mathfrak{u}(t)}(\varphi)$ in (6.2), it follows that

$$\varphi_{\mathfrak{u}}(t, \varphi_0 + k) = \varphi_{\mathfrak{u}}(t, \varphi_0) + k \quad \forall t > 0,$$

for all $k \in \mathbb{Z}$ and $\varphi_0 \in \mathbb{R}$. We may assume, without loss of generality, that $\varphi_0' < \varphi_0'' < \varphi_0' + 1$. From the monotone increasing property of $\varphi_{\mathfrak{u}}(t, \varphi_0)$ with respect to $\varphi_0 \in \mathbb{R}$, it follows that

$$\varphi_{\mathfrak{u}}(t, \varphi_0') - \varphi_0' \leqslant \varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0'' + \varphi_0'' - \varphi_0' < \varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0'' + 1.$$

Hence we have

$$\big(\varphi_{\mathfrak{u}}(t, \varphi_0') - \varphi_0'\big) - \big(\varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0''\big) \leqslant 1.$$

On the other hand, from

$$\begin{aligned}
\varphi_{\mathfrak{u}}(t, \varphi_0') - \varphi_0' &= \varphi_{\mathfrak{u}}(t, \varphi_0' + 1) - \varphi_0' - 1 \\
&\geqslant \varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0'' + (\varphi_0'' - \varphi_0') - 1 \\
&> \varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0'' - 1 \quad \text{by } \varphi_0'' - \varphi_0' > 0
\end{aligned}$$

it follows that

$$\big(\varphi_{\mathfrak{u}}(t, \varphi_0') - \varphi_0'\big) - \big(\varphi_{\mathfrak{u}}(t, \varphi_0'') - \varphi_0''\big) \geqslant -1,$$

which implies the required result. $\quad\square$

In the classical theory of ordinary differential equations, for example [12,33], the traditional definition of rotation number requires instead of the denominator in (6.4) by t_j. Following our definition, however, we can obtain the following result in the switched dynamics situation.

Theorem 6.6. *Let $0 < \Delta < +\infty$ and $\mathbf{A} = \{A_1, \ldots, A_K\} \subset \mathbb{R}^{2 \times 2}$ be arbitrary. Assume Λ is a compact ϑ_+-invariant subset of $\mathcal{L}^{\mathrm{pc}}_\Delta(\mathbb{R}_+, \mathbb{K})$. If $\mathbb{P}$ is a ϑ_+-ergodic probability measure supported on Λ, then there exists a* (rotation) *number $\boldsymbol{\alpha}(\mathbb{P}; \mathbf{A})$ such that*

$$\boldsymbol{\alpha}(\mathfrak{u}; \mathbf{A}) = \boldsymbol{\alpha}(\mathbb{P}; \mathbf{A}) \quad \text{for } \mathbb{P}\text{-a.e. } \mathfrak{u} \in \mathcal{L}^{\mathrm{pc}}_\Delta(\mathbb{R}_+, \mathbb{K}).$$

Here

$$\boldsymbol{\alpha}(\mathbb{P}; \mathbf{A}) = \lim_{k \to +\infty} \frac{1}{k} \int \varphi_{\mathfrak{u}}(t_k, 0)\, \mathrm{d}\mathbb{P}(\mathfrak{u})$$

is a well-defined real number.

Proof. Motivated by Lemma 6.5, we define a continuous transformation

$$\Xi : \mathbb{Z}_+ \times \Lambda \to \mathbb{R}; \quad (k, \mathfrak{u}) \mapsto \varphi_{\mathfrak{u}}(t_k, 0) \quad \text{for } \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}.$$

Since $\varphi_{\mathfrak{u}}(t, \boldsymbol{\varphi}_0)$ is the solution of (6.2) with the initial value $\boldsymbol{\varphi}_0$, we have from Lemma 6.5 the following quasi-additivity property:

$$\Xi(\ell, \mathfrak{u}) + \Xi\big(m, \vartheta_+^\ell(\mathfrak{u})\big) - 1 \leqslant \Xi(\ell + m, \mathfrak{u}) \leqslant \Xi(\ell, \mathfrak{u}) + \Xi\big(m, \vartheta_+^\ell(\mathfrak{u})\big) + 1.$$

In fact, for $\mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}$ and $\ell, m \geqslant 1$, we have

$$\vartheta_+^\ell(\mathfrak{u}) = \big(\lambda'_j, t'_j\big)_{j=1}^{+\infty} = (\lambda_{j+\ell}, t_{j+\ell} - t_\ell)_{j=1}^{+\infty}$$

and

$$\begin{aligned}
\Xi(\ell + m, \mathfrak{u}) &= \varphi_{\mathfrak{u}}(t_{\ell+m}, 0) = \varphi_{\vartheta_+^\ell(\mathfrak{u})}\big(t_{\ell+m} - t_\ell, \varphi_{\mathfrak{u}}(t_\ell, 0)\big) \\
&= \varphi_{\mathfrak{u}}(t_\ell, 0) + \big\{\varphi_{\vartheta_+^\ell(\mathfrak{u})}\big(t'_m, \varphi_{\mathfrak{u}}(t_\ell, 0)\big) - \varphi_{\mathfrak{u}}(t_\ell, 0)\big\};
\end{aligned}$$

in addition, note that

$$\varphi_{\vartheta_+^\ell(\mathfrak{u})}\big(t'_m, 0\big) - 1 \leqslant \varphi_{\vartheta_+^\ell(\mathfrak{u})}\big(t'_m, \varphi_{\mathfrak{u}}(t_\ell, 0)\big) - \varphi_{\mathfrak{u}}(t_\ell, 0) \leqslant \varphi_{\vartheta_+^\ell(\mathfrak{u})}\big(t'_m, 0\big) + 1$$

from Lemma 6.5, and $\Xi(m, \vartheta_+^\ell(\mathfrak{u})) = \varphi_{\vartheta_+^\ell(\mathfrak{u})}(t'_m, 0)$; thus there holds the quasi-additivity.

Since Λ is compact, $\{\Xi(k, \cdot)\}_{k=1}^{+\infty}$ is a sequence of bounded functions on Λ. Now, applying our quasi-additive ergodic theorem (Theorem 6.10 in Section 6.3 below) with $X = \Lambda$, $T = \vartheta_{+\restriction \Lambda}$ and $f_n(\cdot) = \Xi(n, \cdot)$, we can obtain that

$$\lim_{k \to +\infty} \frac{1}{k} \varphi_{\mathfrak{u}}(t_k, 0) = \boldsymbol{\alpha}(\mathbb{P}; \mathbf{A})$$

for $\mathbb{P}$-a.s. $\mathfrak{u} \in \Lambda$.

This ends the proof of Theorem 6.6. ☐

In the classical literature, [33,32] for example, the existence of $\boldsymbol{\alpha}(\mathbb{P}; \mathbf{A})$ is proven by considering the induced skew-product dynamics

$$\Theta : \mathbb{Z}_+ \times \Lambda \times \mathbb{T}^1 \to \Lambda \times \mathbb{T}^1; \quad (k, (\mathfrak{u}, \varphi_0)) \mapsto \big(\vartheta_+^k(\mathfrak{u}), \varphi_{\mathfrak{u}}(t_k, \varphi_0) \bmod 1\big) \quad \text{for } \mathfrak{u} = (\lambda_j, t_j)_{j=1}^{+\infty}.$$

In that way, we will have to construct a Θ-ergodic probability $\mathbb{P}^*$ on $\Lambda \times \mathbb{T}^1$, for which $\mathbb{P}$ is its marginal measure. So, our approach presented here is more simpler and direct than that.

6.2.2. Proof of Theorem F

We first have a simple lemma.

Lemma 6.7. Let $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ be two real sequences in the unit interval $[0,1]$. If $\{b_j\}$ is dense in $[0,1]$ and $\lim_{j\to\infty} a_j/b_j = 1$, then $\{a_j\}$ is also dense in $[0,1]$.

Proof. The statement is obvious and we omit the details. $\square$

Proof of Theorem F. From Theorem D, there is a nontrivial continuous splitting

$$\mathbb{R}^2 = \mathbb{E}^c(\mathfrak{u}) \oplus \mathbb{E}^s(\mathfrak{u}) \quad \forall \mathfrak{u} \in \Lambda.$$

Let $\mathfrak{u}' = (\lambda_j', t_j')_{j=1}^{+\infty} \in \Lambda$ have the rotation number $\boldsymbol{\alpha}(\mathfrak{u}'; \mathbf{A})$ that is irrational. Let $\mathbf{x} \in \mathbb{E}^s(\mathfrak{u}')$ be arbitrarily given with $\|\mathbf{x}\|_2 = 1$ and $\mathbf{x} = (\cos 2\pi \varphi, \sin 2\pi \varphi)^{\mathsf{T}}$ where $0 \leqslant \varphi < 1$. Put

$$b_j = \varphi_{\mathfrak{u}'}(t_j, \varphi) \in \mathbb{R} \quad \forall j \geqslant 1,$$

where $\varphi_{\mathfrak{u}'}(\cdot, \varphi) : \mathbb{R}_+ \to \mathbb{R}$ is the solution of (6.2) with initial value $\varphi(0) = \varphi$ in the case $\mathfrak{u}'$ instead of $\mathfrak{u}$. From Definition 6.4, it follows that $\lim_{j\to\infty} b_j/j \to \boldsymbol{\alpha}(\mathfrak{u}'; \mathbf{A})$. Since $\{j\boldsymbol{\alpha}(\mathfrak{u}'; \mathbf{A}) \bmod 1 \mid j \geqslant 1\}$ is dense in $[0,1]$, from Lemma 6.7 there follows that $\{b_j \bmod 1 \mid j \geqslant 1\}$ is also dense in $[0,1]$. This implies that the sequence

$$\left\{ \mathbf{x}_j := \frac{\Phi_{\mathfrak{u}'}(t_j) \cdot \mathbf{x}}{\|\Phi_{\mathfrak{u}'}(t_j) \cdot \mathbf{x}\|_2} \ \Big| \ j \geqslant 1 \right\}$$

is dense in the unit circle $\mathbb{T}^1$. Notice that $\mathbf{x}_j \in \mathbb{E}^s(\vartheta_+^j(\mathfrak{u}'))$ for all $j \geqslant 1$ from the invariance of the stable manifold bundle $\mathbb{E}^s(\Lambda)$.

Now, consider any initial data $y_0 \in \mathbb{T}^1$. One can find a subsequence $\mathbf{x}_{j_k} \to y_0$. Since Λ is compact, we might assume, without loss of generality, that $\vartheta_+^{j_k}(\mathfrak{u}') \to \mathfrak{u}_{y_0}$ for some $\mathfrak{u}_{y_0} \in \Lambda$. So, it holds that $(\vartheta_+^{j_k}(\mathfrak{u}'), \mathbf{x}_{j_k}) \to (\mathfrak{u}_{y_0}, y_0)$. Clearly, $\mathfrak{u}_{y_0}$ is stable for y_0 from the uniform stability of Φ along $\mathbb{E}^s(\Lambda)$. Then from the classical Lyapunov theory (cf. Footnote 5 in Section 5), there follows easily that $y_0 \in \mathbb{E}^s(\mathfrak{u}_{y_0})$.

This completes the proof of the statement (1) of Theorem F. The statement (2) of Theorem F can be similarly proved.

So, the proof of Theorem F is completed. $\square$

6.2.3. A further question

A nontrivial minimal subset of a dynamical system contains uncountably many orbits. So, it is a hard work to select a stable switching signal for an initial data. In the proof of Theorem F, we have suggested an approach of selection. We further ask the following question:

Question 6.8. Under the situation of Theorem F and let $y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be arbitrarily given. Then, can one find a Borel subset $\Lambda_{y_0} \subset \Lambda$ with $\mathbb{P}(\Lambda_{y_0}) > 0$ or $\dim_{\mathrm{H}}(\Lambda_{y_0}) > 0$ such that every $\mathfrak{u} \in \Lambda_{y_0}$ is switching stable for y_0, for some probability $\mathbb{P}$ supported on Λ?

Here $\dim_{\mathrm{H}}$ means the Hausdorff dimension in the sense of the metric $\mathrm{d}_\varrho(\cdot, \cdot)$ defined in Section 1.1.

An affirmative answer to this question is useful for the fundamental stabilization and optimization problem of linear switched systems.

6.3. The quasi-additive ergodic theorem

In this subsection, we will establish the quasi-additive ergodic theorem, which is a slight improvement of the classical Birkhoff ergodic. This part is of interest independently.

First, we prove a lemma.

Lemma 6.9. *Let* $\{a_n\}_{n=1}^{+\infty}$ *be a sequence of real numbers such that*

$$a_n + a_m - \delta \leqslant a_{n+m} \leqslant a_n + a_m + \delta \quad \forall n, m \geqslant 1,$$

where $\delta \geqslant 0$ *is a constant. Then,*

$$\lim_{n \to +\infty} \frac{a_n}{n} = a^*$$

for some $a^* \in \mathbb{R}$.

Proof. We first fix any $N \geqslant 1$. For $n = mN + k$ where $0 \leqslant k < N$, by

$$a_n = a_{mN+k} \leqslant a_{mN} + a_k + \delta \leqslant (ma_N + m\delta) + (a_k + \delta),$$

one has

$$\limsup_{n \to +\infty} \frac{a_n}{n} \leqslant \frac{a_N}{N} + \frac{\delta}{N}.$$

So, there holds that

$$\liminf_{n \to +\infty} \frac{a_n}{n} \leqslant \limsup_{n \to +\infty} \frac{a_n}{n} \leqslant \frac{a_K}{K} + \frac{\delta}{K} \quad \forall K \geqslant 1.$$

Let $\varepsilon > 0$ be arbitrary. Then there is a $\widehat{K} \geqslant 1$ so large that $\delta/\widehat{K} < \varepsilon$. From

$$\liminf_{n \to +\infty} \frac{a_n}{n} \leqslant \limsup_{n \to +\infty} \frac{a_n}{n} \leqslant \frac{a_{\widehat{K}+k}}{\widehat{K} + k} + \varepsilon \quad \forall k \geqslant 1,$$

it follows that

$$\liminf_{n \to +\infty} \frac{a_n}{n} \leqslant \limsup_{n \to +\infty} \frac{a_n}{n} \leqslant \inf_{n \geqslant \widehat{K}} \frac{a_n}{n} + \varepsilon \leqslant \liminf_{n \to +\infty} \frac{a_n}{n} + \varepsilon.$$

Letting $\varepsilon \searrow 0$, one can find some $a^* < \infty$ satisfying

$$\lim_{n \to +\infty} \frac{a_n}{n} = a^*.$$

Similarly, we can obtain that

$$\liminf_{n \to +\infty} \frac{a_n}{n} \geqslant \frac{a_K}{K} - \frac{\delta}{K} \quad \forall K \geqslant 1,$$

which implies $a^* > -\infty$.

This proves the lemma. □

Now, based on Lemma 6.9 and the classical Birkhoff ergodic theorem [56], we can obtain the following ergodic theorem.

Theorem 6.10 (*Quasi-additive ergodic theorem*). *If T is an ergodic measure-preserving transformation of the probability measure space $(X, \mathcal{B}, \mu)$, and if $\{f_n\}_{n=1}^{+\infty}$ is a sequence of $L^1(\mu)$-functions satisfying the* quasi-additivity *condition*:

$$f_n(x) + f_m(T^n x) - \delta \leqslant f_{m+n}(x) \leqslant f_n(x) + f_m(T^n x) + \delta \quad \forall m, n \geqslant 1$$

for μ-a.e. $x \in X$ where $\delta \geqslant 0$ is a constant, then

$$\frac{1}{n} f_n(x) \to f^* := \lim_{n \to +\infty} \frac{1}{n} \int_X f_n(x) \, d\mu(x)$$

for μ-a.e. $x \in X$. Here f^* is a real number.

We notice that this theorem might be a consequence of the classical Kingman's subadditive ergodic theory [34] by letting $F_n(x) = f_n(x) + \delta$ and then $F_{m+n}(x) \leqslant F_m(x) + F_n(T^m(x))$. However, we would like to present here an elementary proof without using the Kingman subadditive ergodic theorem.

Proof. For simplicity, for any $n \geqslant 1$ we let $f_n^* = \int_X f_n(x) \, d\mu(x)$. Then from Lemma 6.9, it follows that

$$\frac{1}{n} f_n^* \to f^* \quad \text{as } n \to +\infty$$

for some constant $f^* \in \mathbb{R}$. Next, denote

$$\bar{f}(x) = \limsup_{n \to +\infty} \frac{1}{n} f_n(x) \quad \text{and} \quad \underline{f}(x) = \liminf_{n \to +\infty} \frac{1}{n} f_n(x).$$

Observe that the T-invariance: $\bar{f}(x) = \bar{f}(Tx)$ and $\underline{f}(x) = \underline{f}(Tx)$ for all $x \in X$. Now

$$\frac{1}{n} \sum_{j=0}^{n-1} f_1(T^j x) - \delta \leqslant \frac{1}{n} f_n(x) \leqslant \frac{1}{n} \sum_{j=0}^{n-1} f_1(T^j x) + \delta \quad \forall n \geqslant 1$$

and thus by Birkhoff's ergodic theorem [56] we obtain

$$f_1^* - \delta \leqslant \underline{f}(x) \leqslant \bar{f}(x) \leqslant f_1^* + \delta \quad \text{for } \mu\text{-a.e. } x \in X$$

because of the T-ergodicity of μ.

Next, we have a similar, asymptotic, estimate with f_N instead of f_1 in the above inequality, as follows.

Let $f_0(x) \equiv 0$ for all $x \in X$. Fix $N \geqslant 2$ and let $n \gg N$. For each $i = 0, 1, \ldots, N-1$, we write

$$n = i + m_i N + k_i \quad \text{with } 0 \leqslant k_i < N.$$

Then by the quasi-additivity, for μ-a.e. $x \in X$

$$f_n(x) \geqslant f_i(x) + \sum_{\ell=0}^{m_i-1} f_N\big(T^{\ell N+i}x\big) - m_i\delta + f_{k_i}\big(T^{m_iN+i}x\big) - 2\delta$$

and

$$f_n(x) \leqslant f_i(x) + \sum_{\ell=0}^{m_i-1} f_N\big(T^{\ell N+i}x\big) + m_i\delta + f_{k_i}\big(T^{m_iN+i}x\big) + 2\delta$$

and summing over i we have

$$Nf_n(x) \geqslant \sum_{i=0}^{N-1} f_i(x) + \sum_{j=0}^{n-1} f_N\big(T^j x\big) - \sum_{i=0}^{N-1} m_i\delta + \sum_{i=0}^{N-1} f_{n-i-m_iN}\big(T^{m_iN+i}x\big) - 2N\delta$$

and

$$Nf_n(x) \leqslant \sum_{i=0}^{N-1} f_i(x) + \sum_{j=0}^{n-1} f_N\big(T^j x\big) + \sum_{i=0}^{N-1} m_i\delta + \sum_{i=0}^{N-1} f_{n-i-m_iN}\big(T^{m_iN+i}x\big) + 2N\delta.$$

Hence, for μ-a.e. $x \in X$

$$\frac{1}{n}f_n(x) \geqslant \frac{1}{nN}\sum_{j=0}^{n-1} f_N\big(T^j x\big) - \frac{1}{nN}\sum_{i=0}^{N-1} m_i\delta + \frac{1}{nN}\sum_{i=0}^{N-1}\big\{f_i(x) + f_{n-i-m_iN}\big(T^{m_iN+i}x\big)\big\} - \frac{2\delta}{n}$$

and

$$\frac{1}{n}f_n(x) \leqslant \frac{1}{nN}\sum_{j=0}^{n-1} f_N\big(T^j x\big) + \frac{1}{nN}\sum_{i=0}^{N-1} m_i\delta + \frac{1}{nN}\sum_{i=0}^{N-1}\big\{f_i(x) + f_{n-i-m_iN}\big(T^{m_iN+i}x\big)\big\} + \frac{2\delta}{n}.$$

As $n \to +\infty$ the last two terms on the right-hand side converge to zero for μ-a.e. $x \in X$, $\frac{m_i}{n} \to \frac{1}{N}$ and, by the Birkhoff ergodic theorem once again,

$$\frac{1}{N}f_N^* - \frac{\delta}{N} \leqslant \underline{f}(x) \leqslant \bar{f}(x) \leqslant \frac{1}{N}f_N^* + \frac{\delta}{N} \quad \text{for } \mu\text{-a.e. } x \in X$$

which implies

$$f^* \leqslant \underline{f}(x) \leqslant \bar{f}(x) \leqslant f^*$$

for μ-a.e. $x \in X$.

This proves the theorem. $\quad\square$

We notice that in the statement of Theorem 6.10, if $\delta = 0$ then it is just the classical Birkhoff ergodic theorem.

7. Concluding remarks

There are several important new ingredients in the present paper.

First, we have introduced the dynamics concepts "weakly Birkhoff recurrent switching signal" borrowed from Z. Zhou [60], "switching indicator" and "discretization" for continuous-time linear switched dynamics and importantly applied them to the control theory to prove the alternative result Theorem A that is the most important tool of this paper.

Second, we have employed a "semi hyperbolicity" theorem proved by X. Dai in [14] to the study of "partial switching-stability" of continuous-time linear switched systems driven by a minimal dynamics.

Third, to counter the partially stable switched system, we have proved an approximation theorem of stable manifolds using the Alexseev theorem that is an important tool for the hyperbolic theory of differentiable dynamical systems.

And moreover, for a constrained partially switching stable switched system, we have introduced the "rotation number" of a switching signal to choose a stable switching signal for any initial data given previously.

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