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Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains ☆

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Abstract

We are interested in the existence of least energy sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. Because the so-called nonlocal term $b(\int_\Omega |\nabla u|^2 dx)\Delta u$ is involving in the equation, the variational functional of the equation has totally different properties from the case of $b = 0$. Combining constraint variational method and quantitative deformation lemma, we prove that the problem possesses one least energy sign-changing solution u_b. Moreover, we show that the energy of u_b is strictly larger than the ground state energy. Finally, we regard b as a parameter and give a convergence property of u_b as $b \searrow 0$.

Keywords: Kirchhoff-type equations; Sign-changing solutions; Nonlocal term

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N = 1, 2, 3$, with a smooth boundary $\partial\Omega$. We investigate the existence of least energy sign-changing solutions of the following Kirchhoff type problem

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$$\begin{cases} -\Big(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\Big)\Delta u = f(u), & x \in \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where a, b are positive constants. We assume $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and satisfy the following hypotheses:

(f_1) $f(s) = o(|s|)$ as $s \to 0$;
(f_2) For some constant $p \in (4, 2^*)$, $\lim\limits_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$, where $2^* = +\infty$ for $N = 1, 2$ and $2^* = 6$ for $N = 3$;
(f_3) $\lim\limits_{s\to\infty} \frac{F(s)}{s^4} = +\infty$, where $F(s) = \int_0^s f(t)dt$;
(f_4) $\frac{f(s)}{|s|^3}$ is an increasing function of $s \in \mathbb{R} \setminus \{0\}$.

In recent years, the following elliptic problem

$$\begin{cases} -\Big(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\Big)\Delta u + V(x)u = f(x,u), & x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

has been studied extensively by many researchers, here Ω is a domain in $\mathbb{R}^N$, possibly unbounded, with empty or smooth boundary, $V : \Omega \to \mathbb{R}$, $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$, and $a, b > 0$ are constants. (1.2) is a nonlocal problem as the appearance of the term $(\int_\Omega |\nabla u|^2dx)\Delta u$ implies that (1.2) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.2) particularly interesting. Problem (1.2) arises in an interesting physical context. Indeed, if we set $V(x) = 0$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain in (1.2), then we get the following Kirchhoff Dirichlet problem

$$\begin{cases} -\Big(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\Big)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

proposed by Kirchhoff in [14] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. After the pioneer work of J.L. Lions [15], where a functional analysis approach was proposed to the equation

$$\begin{cases} u_{tt} - \Big(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\Big)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

problem (1.4) began to call attention of several researchers, see [1,2,8,10] and the references therein.

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.3), u denotes the displacement, $f(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. We have to point out that such nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself, for example, population density. For more mathematical and physical background of the problem (1.3), we refer the readers to the papers [1,2,8,12–14,16] and the references therein.

Recently, there has been increasing interest in studying problem (1.2), especially on the existence of positive solutions, multiple solutions, ground states and semiclassical states, see for example, [1,9,11–13,16,18,20] and the references therein. We must point out that there are very few results on the existence of sign-changing solutions for problem (1.2). Recently, only Zhang et al. [17,23] studied the existence of sign-changing solutions of (1.3) via invariant sets of descent flow. To the authors' knowledge, there is no result on the existence of least energy sign-changing solutions for problems (1.2) and (1.3).

Throughout this paper, we denote $H := H_0^1(\Omega)$ the usual Sobolev space equipped with the inner product and norm

$$(u,v) = \int_\Omega \nabla u \nabla v dx, \quad \|u\| = (u,u)^{1/2}.$$

Define the energy functional $I_b : H \to \mathbb{R}$ by

$$I_b(u) := \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2 dx\Big)^2 - \int_\Omega F(u)dx. \tag{1.5}$$

The functional I_b is well-defined for every $u \in H$ and belongs to $\mathcal{C}^1(H,\mathbb{R})$. Moreover, for any $u, \varphi \in H$, we have

$$\langle I_b'(u), \varphi\rangle = a\int_\Omega \nabla u \nabla \varphi dx + b\int_\Omega |\nabla u|^2 dx \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega f(u)\varphi dx. \tag{1.6}$$

Clearly, critical points of I_b are the weak solutions for nonlocal problem (1.1). Furthermore, if $u \in H$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

When $b = 0$, Eq. (1.2) does not depend on the nonlocal term $(\int_\Omega |\nabla u|^2 dx)\Delta u$ any more, i.e., it becomes to the following semilinear equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.7}$$

where we set $a = 1$ for simplicity. In the literature, there are different ways to get the sign-changing solutions of Eq. (1.7). For example, by constructive arguments, Bartsch and Willem [4] proved that, for every integer $k \geq 0$, there is a pair of solutions $u_k^{\pm}$ of (1.7), which have precisely k nodes. Via a variational argument and a version of deformation lemma, Castro, Cossio and Neuberger [7] proved that (1.7), on a bounded domain, possesses a sign-changing solution which changes sign only once. By constructing invariant sets and descending flow, Bartsch, Liu and Weth [3] got a sign-changing solution with precisely two nodal domains for (1.7) when $V(x)$ has a positive lower bound and f satisfies the Ambrosetti–Rabinowitz superlinear condition. By variational method together with the Brouwer degree theory, Bartsch and Weth in [5] obtained three nodal solutions for a singularly perturbed problem of (1.7) on a bounded domain with $V(x)$ being a constant. For more discussions on the existence of sign-changing solution of (1.7) under various conditions on $V(x)$ and f, we refer the reader to the book [24] and the references therein. However, these methods of finding sign-changing solutions for (1.7) heavily rely on the following decompositions, for $u \in H$,

$$\langle I_0'(u), u^+\rangle = \langle I_0'(u^+), u^+\rangle, \ \langle I_0'(u), u^-\rangle = \langle I_0'(u^-), u^-\rangle, \tag{1.8}$$

$$I_0(u) = I_0(u^+) + I_0(u^-), \tag{1.9}$$

where I_0 is the energy functional of (1.7) given by

$$I_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + V(x)u^2 dx - \int_\Omega F(x,u)dx, \quad F(x,u) = \int_0^u f(x,s)ds.$$

When $b > 0$, the nonlocal term $(\int_\Omega |\nabla u|^2 dx)\Delta u$ is involved in the equation, for the variational functional I_b given by (1.5), it is easy to see that

$$I_b(u) = I_b(u^+) + I_b(u^-) + \frac{b}{2}\int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx, \tag{1.10}$$

$$\langle I_b'(u), u^+\rangle = \langle I_b'(u^+), u^+\rangle + b\int_\Omega |\nabla u^-|^2 dx \int_\Omega |\nabla u^+|^2 dx, \tag{1.11}$$

$$\langle I_b'(u), u^-\rangle = \langle I_b'(u^-), u^-\rangle + b\int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx. \tag{1.12}$$

Clearly, the functional I_b does no longer satisfy the decompositions (1.8) and (1.9). Hence, the methods of getting sign-changing solutions of (1.7) seems not applicable to problem (1.1). In fact, there are some essential differences in investigating the sign-changing solutions of the problem (1.1) between $b = 0$ and $b > 0$, because of the so called nonlocal term $(\int_\Omega |\nabla u|^2 dx)\Delta u$. Motivated by [5], in order to get a sign-changing solution for the problem (1.1), we first try to seek a minimizer of the energy functional I_b over the following constraint:

$$\mathcal{M}_b = \Big\{u \in H,\ u^{\pm} \neq 0 \quad \text{and} \quad \langle I_b'(u), u^+\rangle = \langle I_b'(u), u^-\rangle = 0\Big\}, \tag{1.13}$$

and then we show that the minimizer is a sign-changing solution of (1.1). Note that the paper [5] is concerned with Eq. (1.7), but in our problem (1.1) the nonlocal term $(\int_\Omega |\nabla u|^2 dx)\Delta u$

is involved, as we mentioned above the functional I_b has no more the properties (1.8), (1.9), and it is rather difficult to show that $\mathcal{M}_b \neq \varnothing$. Thus, we must introduce some new ideas to apply variational method as in [5] to get a sign-changing solution for the problem (1.1). Roughly speaking, we prove $\mathcal{M}_b \neq \varnothing$ by using the *parametric method* and *implicit function theorem*. We do it successfully by proving that, if $u \in H$ with $u^\pm \neq 0$, there is a unique pair $(s,t) \in (\mathbb{R}_+ \times \mathbb{R}_+)$, such that $su^+ + tu^- \in \mathcal{M}_b$, see Lemma 2.1. To show that the minimizer of the constrained problem is a sign-changing solution, we take advantage of *quantitative deformation lemma* and *degree theory*.

Our first main result can be stated as follows.

Theorem 1.1. *If the assumptions* (f_1)*–*(f_4) *hold, then the problem* (1.1) *possesses one least energy sign-changing solution* u_b*, which has precisely two nodal domains.*

Another aim of the paper is to show that the energy of any sign-changing solution of (1.1) is strictly larger than the ground state energy. This is trivial for the typical equation (1.7), i.e., the problem (1.2) with $b = 0$, $a = 1$. In fact, if we denote the Nehari manifold associated to (1.7) by

$$\mathcal{N}_0 = \Big\{u \in H \setminus \{0\} : \langle I_0'(u), u\rangle = 0\Big\}, \tag{1.14}$$

and let

$$c_0 := \inf_{u \in \mathcal{N}_0} I_0(u), \tag{1.15}$$

then, for any sign-changing solution $w \in H$ of (1.7), it follows from (1.8), (1.9) that $w^\pm \in \mathcal{N}_0$. Moreover, if the nonlinearity $f(x,s)$ satisfies conditions (see [4], A_2–A_6) analogous to (f_1)–(f_4), then we can deduce that

$$I_0(w) = I_0(w^+) + I_0(w^-) \geq 2c_0. \tag{1.16}$$

It is well-known that the minimizer of (1.15) is indeed a ground state solution of the problem (1.7), and $c_0 > 0$ is the *ground state energy*, i.e., the least energy of all weak solutions of (1.7). Therefore, (1.16) implies that, the energy of any sign-changing solution of Eq. (1.7) is larger than two times the least energy, this property is called *energy doubling* by Weth in [21]. However, if $b > 0$ in (1.1), the property (1.16) is still unknown for the functional I_b. Indeed, let $w_b \in H$ be a sign-changing solution of (1.1), it follows from (1.11) and (1.12) that

$$w_b^\pm \notin \mathcal{N}_b := \Big\{u \in H \setminus \{0\} : \langle I_b'(u), u\rangle = 0\Big\}. \tag{1.17}$$

Then, although (1.10) shows that

$$I_b(w_b) > I_b(w_b^+) + I_b(w_b^-)$$

we still cannot deduce that $I_b(w_b) \geq 2c_b$, where

$$c_b := \inf_{u \in \mathcal{N}_b} I_b(u). \tag{1.18}$$

From these observations, it is even not easy to compare $I_b(u_b)$ with c_b. However, taking advantage of the auxiliary function ϕ which is given in Lemma 2.3, we have the following theorem.

Theorem 1.2. *If the assumptions of Theorem 1.1 hold, then $c_b > 0$ is achieved and*

$$I_b(u_b) > c_b,$$

where u_b is the least energy sign-changing solution obtained in Theorem 1.1. In particular, c_b is achieved either by a positive or a negative function.

Theorem 1.2 indicates that the energy of any sign-changing solution of (1.1) is strictly larger than the ground state energy.

It is obvious that the energy of the sign-changing solution u_b obtained in Theorem 1.1 depends on b. As a by-product of this paper, we give a convergence property of u_b as $b \searrow 0$, which reflects some relationship between $b > 0$ and $b = 0$ in problem (1.1). Our main results in this direction can be stated as the following theorem.

Theorem 1.3. *If the assumptions of Theorem 1.1 hold, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, there exists a subsequence, still denoted by $\{b_n\}$, such that u_{b_n} convergent to u_0 strongly in H as $n \to \infty$, where u_0 is a least energy sign-changing solution of the problem*

$$\begin{cases} -a\Delta u = f(u), & x \in \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1.19}$$

which changes sign only once.

The proof of Theorem 1.3 includes three steps: we first prove $\{u_{b_n}\}$ is bounded in H, then we prove $u_{b_n} \to u_0$ strongly in H, and we finally prove that u_0 is just a least energy sign-changing solution of (1.19).

The paper is organized as follows. In Section 2, we prove several lemmas, which are crucial to prove our main results. In Section 3, by quantitative deformation lemma and degree theory, we first show that the minimizer of the constrained problem is a sign-changing solution. We then prove Theorems 1.2 and 1.3 by some energy estimations and comparisons.

2. Some preliminary lemmas

We use constraint minimization on $\mathcal{M}_b$ to seek a critical point of I_b. We begin this section by checking that the set $\mathcal{M}_b$ is nonempty in H.

Lemma 2.1. *Assume that (f_1)–(f_4) hold, if $u \in H$ with $u^\pm \neq 0$, then there is a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_b$.*

Proof. Fixed $u \in H$ with $u^\pm \neq 0$, we denote $B := \int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx$ for convenience. Then, $su^+ + tu^-$ is contained in $\mathcal{M}_b$ if and only if

$$\begin{cases} as^2\|u^+\|^2 + bs^4\Big(\int_\Omega |\nabla u^+|^2dx\Big)^2 + bBs^2t^2 - \int_\Omega f(su^+)su^+dx = 0, \\ at^2\|u^-\|^2 + bt^4\Big(\int_\Omega |\nabla u^-|^2dx\Big)^2 + bBt^2s^2 - \int_\Omega f(tu^-)tu^-dx = 0. \end{cases} \tag{2.1}$$

Hence, the problem is reduced to verify that there is only one solution $(s, t) \in (\mathbb{R}_+ \times \mathbb{R}_+)$ of system (2.1).

We consider the solvability of the following system with a parameter $\mu \in [0, 1]$,

$$\begin{cases} as^2\|u^+\|^2 + bs^4\Big(\int_\Omega |\nabla u^+|^2dx\Big)^2 + \mu bBs^2t^2 - \int_\Omega f(su^+)su^+dx = 0, \\ at^2\|u^-\|^2 + bt^4\Big(\int_\Omega |\nabla u^-|^2dx\Big)^2 + \mu bBs^2t^2 - \int_\Omega f(tu^-)tu^-dx = 0. \end{cases} \tag{2.2}$$

Define

$$\mathcal{Z} := \Big\{\mu \;\Big|\; 0 \le \mu \le 1 \text{ such that (2.2) is uniquely solvable in } \mathbb{R}_+ \times \mathbb{R}_+\Big\}, \tag{2.3}$$

and set

$$\begin{aligned} g_\mu(s,t) &:= as^2\|u^+\|^2 + bs^4\Big(\int_\Omega |\nabla u^+|^2dx\Big)^2 + \mu bBs^2t^2 - \int_\Omega f(su^+)su^+dx, \\ h_\mu(s,t) &:= at^2\|u^-\|^2 + bt^4\Big(\int_\Omega |\nabla u^-|^2dx\Big)^2 + \mu bBs^2t^2 - \int_\Omega f(tu^-)tu^-dx. \end{aligned} \tag{2.4}$$

Claim 1. The set $\mathcal{Z}$ contains 0, i.e., $0 \in \mathcal{Z}$.

Since $g_0(s,t)$ is independent of t and $h_0(s,t)$ is independent of s, without loss of generality, we need only to prove that there is a unique $t > 0$ such that $h_0(s,t) = 0$. Since $u^- \neq 0$, from (f_1)–(f_4) that $h_0(s, 0) = 0$, $h_0(s,t) > 0$ for $t > 0$ small and $h_0(s,t) < 0$ for t large. Suppose that there exist t_1, t_2, such that $0 < t_1 < t_2$ and $h_0(s,t_1) = h_0(s,t_2) = 0$, then

$$\frac{a}{t_1^2}\|u^-\|^2 + b\Big(\int_\Omega |\nabla u^-|^2dx\Big)^2 = \int_\Omega \frac{f(t_1u^-)}{t_1^3}u^-dx$$

and this identity is also true if t_1 is replaced by t_2. Therefore,

$$a\Big(\frac{1}{t_1^2} - \frac{1}{t_2^2}\Big)\|u^-\|^2 = \int_\Omega \Big(\frac{f(t_1u^-)}{(t_1u^-)^3} - \frac{f(t_2u^-)}{(t_2u^-)^3}\Big)|u^-|^4dx,$$

which is absurd in view of (f_4) and $0 < t_1 < t_2$. Then the proof of Claim 1 is completed.

Claim 2. The set $\mathcal{Z}$ is open and closed in [0, 1].

We first prove that $\mathcal{Z}$ is open in $[0,1]$. Suppose that $\mu_0 \in \mathcal{Z}$ and $(\bar{s}, \bar{t}) \in \mathbb{R}_+ \times \mathbb{R}_+$ is the unique solution of (2.2) with $\mu = \mu_0$. By direct calculation, we have

$$\frac{\partial g_\mu(s,t)}{\partial s}\Big|_{(\bar{s},\bar{t})} = a\bar{s}\|u^+\|^2 + 3b\bar{s}^3\Big(\int_\Omega |\nabla u^+|^2 dx\Big)^2 + \mu_0 bB\bar{s}\bar{t}^2 - \int_\Omega f'(\bar{s}u^+)\bar{s}|u^+|^2 dx, \tag{2.5}$$

$$\frac{\partial g_\mu(s,t)}{\partial t}\Big|_{(\bar{s},\bar{t})} = 2\mu_0 bB\bar{s}^2\bar{t}, \qquad \frac{\partial h_\mu(s,t)}{\partial s}\Big|_{(\bar{s},\bar{t})} = 2\mu_0 bB\bar{s}\bar{t}^2, \tag{2.6}$$

$$\frac{\partial h_\mu(s,t)}{\partial t}\Big|_{(\bar{s},\bar{t})} = a\bar{t}\|u^-\|^2 + 3b\bar{t}^3\Big(\int_\Omega |\nabla u^-|^2 dx\Big)^2 + \mu_0 bB\bar{s}^2\bar{t} - \int_\Omega f'(\bar{t}u^-)\bar{t}|u^-|^2 dx. \tag{2.7}$$

Set the matrix

$$M = \begin{bmatrix} \frac{\partial g_\mu(\bar{s},\bar{t})}{\partial s}, & \frac{\partial g_\mu(\bar{s},\bar{t})}{\partial t} \\ \frac{\partial h_\mu(\bar{s},\bar{t})}{\partial s}, & \frac{\partial h_\mu(\bar{s},\bar{t})}{\partial t} \end{bmatrix}.$$

By the condition (f_4), for $s \neq 0$, we have

$$f'(s)s^2 - 3f(s)s > 0. \tag{2.8}$$

Then

$$\frac{\partial g_\mu(s,t)}{\partial s}\Big|_{(\bar{s},\bar{t})} < -2a\bar{s}\|u^+\|^2 - 2\mu_0 bB\bar{s}\bar{t}^2,$$

and

$$\frac{\partial h_\mu(s,t)}{\partial t}\Big|_{(\bar{s},\bar{t})} < -2a\bar{t}\|u^-\|^2 - 2\mu_0 bB\bar{s}^2\bar{t}.$$

Thus, we conclude that

$$\det M > \Big(2a\bar{s}\|u^+\|^2 + 2\mu_0 bB\bar{s}\bar{t}^2\Big)\Big(2a\bar{t}\|u^-\|^2 + 2\mu_0 bB\bar{s}^2\bar{t}\Big) - (2\mu_0 bB\bar{s}^2\bar{t})(2\mu_0 bB\bar{s}\bar{t}^2) > 0.$$

Then, the implicit function theorem implies that we can find open neighborhoods U_0 of μ_0 and $A_0 \subset \mathbb{R}_+ \times \mathbb{R}_+$ of $(\bar{s}, \bar{t})$ such that the system (2.2) is uniquely solvable in $U_0 \times A_0$.

Suppose that there is $\mu_1 \in U_0$ such that the second solution $(\widetilde{s}, \widetilde{t})$ of (2.2) exists in $(\mathbb{R}_+ \times \mathbb{R}_+) \setminus A_0$, then by the implicit function theorem again, we can find a solution curve $\big(\mu, (\widetilde{s}(\mu), \widetilde{t}(\mu))\big)$ in $(\mu_1 - \varepsilon, \mu_1 + \varepsilon) \times (\mathbb{R}_+ \times \mathbb{R}_+)$ which satisfies (2.2) and goes through $(\mu_1, (\widetilde{s}, \widetilde{t}))$. Assume $\mu_0 <$

μ_1 for a while and extend this curve as much as possible. Since it cannot be defined at μ_0 and enter into $U_0 \times A_0$, there should be a point $\mu_2 \in [\mu_0, \mu_1)$ such that $(\widetilde{s}(\mu), \widetilde{t}(\mu))$ exists in $(\mu_2, \mu_1]$ and blows up as $\mu \to \mu_2^+$. However, this is impossible. In fact, if (s, t) having sufficiently large norm, by (f_3), the left-hand side of (2.2) is strictly negative for at least one of them. This gives a contradiction. Thus, $U_0 \subset \mathcal{Z}$. The case $\mu_0 > \mu_1$ is similar.

We next prove that $\mathcal{Z}$ is closed in $[0, 1]$. Let $\{\mu_n\}$ be a sequence in $\mathcal{Z}$ converging to $\mu_0 \in [0, 1]$ and $(s_n, t_n) \in (\mathbb{R}_+ \times \mathbb{R}_+)$ be the solution of (2.2) for μ_n. By the preceding argument the sequence (s_n, t_n) is bounded above. Thus we may assume that (s_n, t_n) converges to a solution $(s_0, t_0) \in (\mathbb{R}_+ \times \mathbb{R}_+)$ of (2.2) for μ_0. Combine (2.2) and (f_1)–(f_2), by Sobolev embedding theorem, we can get

$$a(s_n)^2\|u^+\|^2 \le \int_\Omega f(s_n u^+) s_n u^+ dx \le \frac{a}{2}(s_n)^2\|u^+\|^2 + C(s_n)^p\|u^+\|^p. \tag{2.9}$$

Since $p > 4$, we then conclude that $0 < C_1 \le s_n$ is uniformly in n, thus $s_0 \ge C_1 > 0$. Similarly, we conclude $t_0 \ge C_2 > 0$, where C, C_1 and C_2 are constants. So $(s_0, t_0) \in (\mathbb{R}_+ \times \mathbb{R}_+)$. Also, the fact that (s_0, t_0) is the unique solution in $\mathbb{R}_+ \times \mathbb{R}_+$ again follows from the implicit function theorem. Claim 2 is therefore proved.

From the above two claims, we can easily get the conclusion of Lemma 2.1. □

Lemma 2.2. *Assume that (f_1)–(f_4) hold, suppose that $u \in H$ such that, $g_1(1,1) \le 0$ and $h_1(1,1) \le 0$, where $g_1(s,t)$, $h_1(s,t)$ are given as (2.4) with $\mu = 1$. Then the unique pair (s_u, t_u) of positive numbers obtained in Lemma 2.1 satisfies $0 < s_u, t_u \le 1$.*

Proof. Suppose that $s_u \ge t_u > 0$, since $s_u u^+ + t_u u^- \in \mathcal{M}_b$, then we have

$$\begin{aligned}
&a s_u^2\|u^+\|^2 + b s_u^4\Big(\int_\Omega |\nabla u^+|^2 dx\Big)^2 + b s_u^4 \int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx \\
&\ge a s_u^2\|u^+\|^2 + b s_u^4\Big(\int_\Omega |\nabla u^+|^2 dx\Big)^2 + b s_u^2 t_u^2 \int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx \\
&= \int_\Omega f(s_u u^+) s_u u^+ dx.
\end{aligned} \tag{2.10}$$

The assumption $g_1(1,1) \le 0$ gives that

$$a\|u^+\|^2 + b\Big(\int_\Omega |\nabla u^+|^2 dx\Big)^2 + b\int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx \le \int_\Omega f(u^+)u^+ dx. \tag{2.11}$$

Combine (2.10) and (2.11), we then get

$$\Big(\frac{1}{s_u^2} - 1\Big) a\|u^+\|^2 \ge \int_\Omega \Big[\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}\Big](u^+)^4 dx.$$

If $s_u > 1$, the left side of this inequality is negative. But from (f_4), the right side is positive, thus we must have $s_u \le 1$. Then the proof is completed. □

Lemma 2.3. *For fixed $u \in H$ with $u^\pm \neq 0$, then, the vector (s_u, t_u) which obtained in Lemma 2.1 is the unique maximum point of the function $\phi : (\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ defined as $\phi(s,t) = I_b(su^+ + tu^-)$.*

Proof. From the proof of Lemma 2.1, (s_u, t_u) is the unique critical point of ϕ in $(\mathbb{R}_+ \times \mathbb{R}_+)$. By the assumption (f_3), we deduce that $\phi(s,t) \to -\infty$ uniformly as $|(s,t)| \to \infty$, so it is sufficient to check that a maximum point cannot be achieved on the boundary of $(\mathbb{R}_+ \times \mathbb{R}_+)$. Without loss of generality, we may assume that $(0, \bar{t})$ is a maximum point of ϕ. Then since

$$\begin{aligned}
\phi(s,\bar{t}) &= I_b(su^+ + \bar{t}u^-) \\
&= \frac{as^2}{2}\int_\Omega |\nabla u^+|^2 dx + \frac{bs^4}{4}\Big(\int_\Omega |\nabla u^+|^2 dx\Big)^2 - \int_\Omega F(su^+)dx \\
&\quad + \frac{bs^2\bar{t}^2}{2}\int_\Omega |\nabla u^+|^2 dx \int_\Omega |\nabla u^-|^2 dx \\
&\quad + \frac{a\bar{t}^2}{2}\int_\Omega |\nabla u^-|^2 dx + \frac{b\bar{t}^4}{4}\Big(\int_\Omega |\nabla u^+|^2 dx\Big)^2 - \int_\Omega F(\bar{t}u^-)dx
\end{aligned}$$

is an increasing function with respect to s if s is small enough, the pair $(0, \bar{t})$ is not a maximum point of ϕ in $(\mathbb{R}_+ \times \mathbb{R}_+)$. □

By Lemma 2.1, we can define the following minimization problem

$$m_b := \inf\Big\{ I_b(u) : u \in \mathcal{M}_b \Big\}. \tag{2.12}$$

Lemma 2.4. *Assume that (f_1)–(f_4) hold, then $m_b > 0$ can be achieved.*

Proof. For every $u \in \mathcal{M}_b$, we have $\langle I_b'(u), u\rangle = 0$. Then by (f_1), (f_2) and Sobolev embedding theorem, we get

$$\begin{aligned}
a\|u\|^2 &\le a\int_\Omega |\nabla u|^2 dx + b\Big(\int_\Omega |\nabla u|^2 dx\Big)^2 = \int_\Omega f(u)u dx \\
&\le \frac{a}{2}\lambda_1 \int_\Omega |u|^2 dx + \widetilde{C}\int_\Omega |u|^p dx \\
&\le \frac{a}{2}\|u\|^2 + C\|u\|^p
\end{aligned} \tag{2.13}$$

where λ_1 is the first eigenvalue of $(-\Delta, H)$. So, there exists a constant $\alpha > 0$ such that $\|u\|^2 \ge \alpha$. And by (2.8), we have

$$f(s)s - 4F(s) \geq 0. \tag{2.14}$$

Then

$$I_b(u) = I_b(u) - \frac{1}{4}\langle I_b'(u), u\rangle \geq \frac{1}{4}\|u\|^2 \geq \frac{1}{4}\alpha.$$

This implies that $m_b \geq \frac{1}{4}\alpha > 0$.

Let $\{u_n\} \subset \mathcal{M}_b$ be such that $I_b(u_n) \to m_b$. Then $\{u_n\}$ is bounded in H, and there exists $u_b \in H$ such that $u_n^{\pm} \rightharpoonup u_b^{\pm}$ weakly in H. Since $u_n \in \mathcal{M}_b$, we have $\langle I_b'(u_n), u_n^{\pm}\rangle = 0$, that is

$$a\int\limits_{\Omega} |\nabla u_n^{\pm}|^2 dx + b\int\limits_{\Omega} |\nabla u_n|^2 dx \int\limits_{\Omega} |\nabla u_n^{\pm}|^2 dx = \int\limits_{\Omega} f(u_n^{\pm})u_n^{\pm} dx. \tag{2.15}$$

Similar as (2.13) there exists a constant $\mu > 0$ such that $\|u_n^{\pm}\|^2 \geq \mu$ for all $n \in \mathbb{N}$. From (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(s)s \leq \varepsilon s^2 + C_\varepsilon |s|^p, \quad \text{for all } s \in \mathbb{R}.$$

Since $u_n \in \mathcal{M}_b$, thus

$$\mu \leq \|u_n^{\pm}\|^2 < \int\limits_{\Omega} f(u_n^{\pm})u_n^{\pm} dx \leq \varepsilon \int\limits_{\Omega} |u_n^{\pm}|^2 dx + C_\varepsilon \int\limits_{\Omega} |u_n^{\pm}|^p dx.$$

Using the boundedness of $\{u_n\}$, there is $C_1 > 0$ such that

$$\mu \leq \varepsilon C_1 + C_\varepsilon \int\limits_{\Omega} |u_n^{\pm}|^p dx.$$

Choosing $\varepsilon = \frac{\mu}{2C_1}$, we get

$$\int\limits_{\Omega} |u_n^{\pm}|^p dx \geq \frac{\mu}{2\bar{C}}.$$

By (2.15) and the compactness of the embedding $H \hookrightarrow L^q(\Omega)$ for $2 \leq q < 2^*$, we get

$$\int\limits_{\Omega} |u_b^{\pm}|^p dx \geq \frac{\mu}{2\bar{C}}, \tag{2.16}$$

thus, $u_b^{\pm} \neq 0$. The conditions (f_1)–(f_2) combined with the compactness lemma of Strauss [19] gives

$$\lim_{n\to\infty} \int\limits_{\Omega} f(u_n^{\pm})u_n^{\pm} dx = \int\limits_{\Omega} f(u_b^{\pm})u_b^{\pm} dx, \qquad \lim_{n\to\infty} \int\limits_{\Omega} F(u_n^{\pm}) dx = \int\limits_{\Omega} F(u_b^{\pm}) dx. \tag{2.17}$$

By the weak semicontinuity of norm, we have

$$\begin{aligned} &a\|u_b^\pm\|^2 + b\int_\Omega |\nabla u_b|^2 dx \int_\Omega |\nabla u_b^\pm|^2 dx \\ &\le \liminf_{n\to\infty}\left\{a\|u_n^\pm\|^2 + b\int_\Omega |\nabla u_n|^2 dx \int_\Omega |\nabla u_n^\pm|^2 dx\right\}. \end{aligned} \tag{2.18}$$

Then from (2.17) we get

$$a\|u_b^\pm\|^2 + b\int_\Omega |\nabla u_b|^2 dx \int_\Omega |\nabla u_b^\pm|^2 dx \le \int_\Omega f(u_b^\pm)u_b^\pm dx. \tag{2.19}$$

From (2.19) and Lemma 2.2, there exists $(s_{u_b}, t_{u_b}) \in (0, 1] \times (0, 1]$ such that

$$\bar{u}_b := s_{u_b}u_b^+ + t_{u_b}u_b^- \in \mathcal{M}_b.$$

Since condition (f_4) implies that $H(s) := sf(s) - 4F(s)$ is a non-negative function, increasing in $|s|$, we then have

$$\begin{aligned} m_b &\le I_b(\bar{u}_b) = I_b(\bar{u}_b) - \frac{1}{4}\langle I_b'(\bar{u}_b), \bar{u}_b\rangle \\ &= \frac{1}{4}\|\bar{u}_b\|^2 + \frac{1}{4}\int_\Omega \Big(f(\bar{u}_b)\bar{u}_b - 4F(\bar{u}_b)\Big)dx \\ &= \frac{1}{4}\|s_{u_b}u_b^+\|^2 + \frac{1}{4}\|t_{u_b}u_b^-\|^2 + \frac{1}{4}\int_\Omega \Big(f(s_{u_b}u_b^+)s_{u_b}u_b^+ - 4F(s_{u_b}u_b^+)\Big)dx \\ &\quad + \frac{1}{4}\int_\Omega \Big(f(t_{u_b}u_b^-)t_{u_b}u_b^- - 4F(t_{u_b}u_b^-)\Big)dx \\ &\le \frac{1}{4}\|u_b\|^2 + \frac{1}{4}\int_\Omega \Big(f(u_b)u_b - 4F(u_b)\Big)dx \\ &\le \liminf_{n\to\infty}\Big[I_b(u_n) - \frac{1}{4}\langle I_b'(u_n), u_n\rangle\Big] = m_b. \end{aligned} \tag{2.20}$$

We then deduce that $s_{u_b} = t_{u_b} = 1$. Thus, $\bar{u}_b = u_b$ and $I_b(u_b) = m_b$. □

3. Proof of main results

The main aim of this section is to prove our main results. We first prove that the minimizer u_b for the minimization problem (2.12) is indeed a sign-changing solution of (1.1), that is, $I_b'(u_b) = 0$.

Proof of Theorem 1.1. Using the quantitative deformation lemma, we prove that $I_b'(u_b)=0$.

It is clear that $I_b'(u_b)u_b^+ = 0 = I_b'(u_b)u_b^-$. It follows from Lemma 2.3 that, for $(s,t)\in(\mathbb{R}_+\times\mathbb{R}_+)$ and $(s,t)\neq(1,1)$,

$$I_b(su_b^+ + tu_b^-) < I_b(u_b^+ + u_b^-) = m_b. \tag{3.1}$$

If $I_b'(u_b)\neq 0$, then there exist $\delta>0$ and $\lambda>0$ such that

$$\|I_b'(v)\| \geq \lambda, \quad \text{for all } \|v-u_b\|\leq 3\delta.$$

Let $D := (\frac{1}{2},\frac{3}{2})\times(\frac{1}{2},\frac{3}{2})$ and $g(s,t) := su_b^+ + tu_b^-$. It follows from Lemma 2.3 again that

$$\bar{m}_b := \max_{\partial D} I_b\circ g < m_b \tag{3.2}$$

For $\varepsilon := \min\{(m_b-\bar{m}_b)/2, \lambda\delta/8\}$ and $S := B(u_b,\delta)$, [see [22], Lemma 2.3] yields a deformation η such that

(a) $\eta(1,u)=u$ if $u\notin I_b^{-1}([m_b-2\varepsilon, m_b+2\varepsilon])\cap S_{2\delta}$;
(b) $\eta(1, I_b^{m_b+\varepsilon}\cap S)\subset I_b^{m_b-\varepsilon}$;
(c) $I_b\big(\eta(1,u)\big)\leq I_b(u)$ for all $u\in H$.

It is clear that

$$\max_{(s,t)\in\bar{D}} I_b\big(\eta(1,g(s,t))\big) < m_b. \tag{3.3}$$

We prove that $\eta(1,g(D))\cap\mathcal{M}_b\neq\varnothing$, contradicting to the definition of m_b. Let us define $h(s,t) := \eta(1,g(s,t))$ and

$$\Psi_0(s,t) := \Big(I_b'\big(g(s,t)\big)u_b^+, I_b'\big(g(s,t)\big)u_b^-\Big) = \Big(I_b'(su_b^+ + tu_b^-)u_b^+, I_b'(su_b^+ + tu_b^-)u_b^-\Big),$$

$$\Psi_1(s,t) := \Big(\frac{1}{s}I_b'\big(h(s,t)\big)h^+(s,t), \frac{1}{t}I_b'\big(h(s,t)\big)h^-(s,t)\Big).$$

Lemma 2.1 and the degree theory now yields $\deg(\Psi_0, D, 0)=1$. It follows from (3.2) that $g=h$ on ∂D. Consequently, we obtain $\deg(\Psi_1,D,0)=\deg(\Psi_0,D,0)=1$. Therefore, $\Psi_1(s_0,t_0)=0$ for some $(s_0,t_0)\in D$, so that $\eta(1,g(s_0,t_0))=h(s_0,t_0)\in\mathcal{M}_b$, which is a contradiction. From this, u_b is a critical point of I_b, and so, a sign-changing solution for problem (1.1).

Now, we show that u_b has exactly two nodal domains, to this end, we assume by contradiction that

$$u_b = u_1+u_2+u_3$$

with

$$u_i\neq 0,\ u_1\geq 0,\ u_2\leq 0 \quad \text{and} \quad \text{suppt}(u_i)\cap\text{suppt}(u_j)=\varnothing, \quad \text{for } i\neq j,\ i,j=1,2,3$$

and

$$\langle I_b'(u_b), u_i \rangle = 0, \quad \text{for } i = 1, 2, 3.$$

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Then, by Lemma 2.1, there is a unique pair (s_v, t_v) of positive numbers such that

$$s_v v^+ + t_v v^- \in \mathcal{M}_b,$$

or equivalently,

$$s_v u_1 + t_v u_2 \in \mathcal{M}_b.$$

And so,

$$I_b(s_v u_1 + t_v u_2) \geq m_b. \tag{3.4}$$

Moreover, using the fact that $I_b'(u_b)u_i = 0$, it follows that

$$\langle I_b'(v), v^\pm \rangle < 0.$$

From Lemma 2.2, we have that

$$(s_v, t_v) \in (0, 1] \times (0, 1].$$

On the other hand,

$$\begin{aligned} 0 = \frac{1}{4}\langle I_b'(u_b), u_3 \rangle &= \frac{a}{4}\int_\Omega |\nabla u_3|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u_3|^2 dx\Big)^2 \\ &\quad + \frac{b}{4}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_3|^2 dx + \frac{b}{4}\int_\Omega |\nabla u_2|^2 dx \int_\Omega |\nabla u_3|^2 dx - \frac{1}{4}\int_\Omega f(u_3)u_3 dx \\ &< I_b(u_3) + \frac{b}{4}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_3|^2 dx + \frac{b}{4}\int_\Omega |\nabla u_2|^2 dx \int_\Omega |\nabla u_3|^2 dx. \end{aligned} \tag{3.5}$$

Then, similar as (2.20), we can calculate that

$$\begin{aligned} I_b(s_v u_1 + t_v u_2) &= I_b(s_v u_1) + I_b(t_v u_2) + \frac{b s_v^2 t_v^2}{2}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_2|^2 dx \\ &= \frac{a s_v^2}{4}\|u_1\|^2 + \frac{1}{4}\int_\Omega \Big(f(s_v u_1)s_v u_1 - 4F(s_v u_1)\Big)dx + \frac{a t_v^2}{4}\|u_2\|^2 \\ &\quad + \frac{1}{4}\int_\Omega \Big(f(t_v u_2)t_v u_2 - 4F(t_v u_2)\Big)dx \end{aligned}$$

$$\begin{aligned}
&\le \frac{a}{4}\|u_1\|^2 + \frac{1}{4}\int_\Omega \Big(f(u_1)u_1 - 4F(u_1)\Big)dx + \frac{a}{4}\|u_2\|^2 \\
&\quad + \frac{1}{4}\int_\Omega \Big(f(u_2)u_2 - 4F(u_2)\Big)dx \\
&= I_b(u_1) + I_b(u_2) + \frac{b}{2}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_2|^2 dx \\
&\quad + \frac{b}{4}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_3|^2 dx + \frac{b}{4}\int_\Omega |\nabla u_2|^2 dx \int_\Omega |\nabla u_3|^2 dx. \qquad (3.6)
\end{aligned}$$

Then, from (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
m_b &\le I_b(s_v u_1 + t_v u_2) < I_b(u_1) + I_b(u_2) + I_b(u_3) + \frac{b}{2}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_2|^2 dx \\
&\quad + \frac{b}{2}\int_\Omega |\nabla u_1|^2 dx \int_\Omega |\nabla u_3|^2 dx + \frac{b}{2}\int_\Omega |\nabla u_2|^2 dx \int_\Omega |\nabla u_3|^2 dx \\
&= I_b(u_b) = m_b,
\end{aligned}$$

which is a contradiction. This way, $u_3 = 0$, and u_b has exactly two nodal domains. □

By Theorem 1.1, we know that the problem (1.1) has a least energy sign-changing solution u_b which changes sign only once. We now prove that the energy of u_b is strictly larger than the ground state energy.

Proof of Theorem 1.2. Let $\mathcal{N}_b$ and c_b be given by (1.17) and (1.18), respectively. Then, similar as the proof of Lemma 2.4, for each $b > 0$, we can deduce that there exists $v_b \in \mathcal{N}_b$ such that $I_b(v_b) = c_b > 0$. By Corollary 2.9 in [12], the critical points of the functional I_b on $\mathcal{N}_b$ are critical points of I_b in H, we can conclude that $I_b'(v_b) = 0$. Thus, v_b is a ground state solution of (1.1).

From Theorem 1.1, we know that the problem (1.1) has a least energy sign-changing solution u_b which changes sign only once. Suppose that $u_b = u_b^+ + u_b^-$. As the proof of Claim 1 in Lemma 2.1, there is unique $t_{u_b^+} > 0$ such that

$$t_{u_b^+} u_b^+ \in \mathcal{N}_b.$$

Then, by Lemma 2.3, we get

$$c_b \le I_b(t_{u_b^+} u_b^+) = I_b(t_{u_b^+} u_b^+ + 0) < I_b(u_b^+ + u_b^-) = m_b,$$

that is $I_b(u_b) > c_b$, which implies that $c_b > 0$ cannot be achieved by a sign-changing function. This completes the proof. □

Now, we are in a situation to prove Theorem 1.3. In the following, we regard $b > 0$ as a parameter in problem (1.1). We shall analyze the convergence property of u_b as $b \searrow 0$.

Proof of Theorem 1.3. For any $b > 0$, let $u_b \in H$ be the least energy sign-changing solution of (1.1) obtained in Theorem 1.1, which changes sign only once.

Step 1. We claim that, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, $\{u_{b_n}\}$ is bounded in H. Choose a nonzero function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with $\varphi^\pm \neq 0$. From (2.14), for $s \neq 0$, we have

$$f(s)s > 4F(s).$$

Then, (f_3) implies that, for any $b \in [0, 1]$, there exists a pair (λ_1, λ_2) of positive numbers, which does not depend on b, such that

$$\begin{cases} a\lambda_1^2\|\varphi^+\|^2 + b\lambda_1^4\Big(\int_\Omega |\nabla\varphi^+|^2 dx\Big)^2 + bB_\varphi\lambda_1^2\lambda_2^2 - \int_\Omega f(\lambda_1\varphi^+)\lambda_1\varphi^+ dx < 0, \\ a\lambda_2^2\|\varphi^-\|^2 + b\lambda_2^4\Big(\int_\Omega |\nabla\varphi^-|^2 dx\Big)^2 + bB_\varphi\lambda_1^2\lambda_2^2 - \int_\Omega f(\lambda_2\varphi^-)\lambda_2\varphi^- dx < 0, \end{cases}$$

where $B_\varphi = \int_\Omega |\nabla\varphi^+|^2 dx \int_\Omega |\nabla\varphi^-|^2 dx$. In view of Lemmas 2.1 and 2.2, for any $b \in [0, 1]$, there is a unique pair $\big(s_\varphi(b), t_\varphi(b)\big) \in (0, 1] \times (0, 1]$ such that

$$\bar{\varphi} := s_\varphi(b)\lambda_1\varphi^+ + t_\varphi(b)\lambda_2\varphi^- \in \mathcal{M}_b. \tag{3.7}$$

Thus, for any $b \in [0, 1]$, we have

$$\begin{aligned} I_b(u_b) &\le I_b(\bar{\varphi}) = I_b(\bar{\varphi}) - \frac{1}{4}\langle I_b'(\bar{\varphi}), \bar{\varphi}\rangle \\ &= \frac{a}{4}\|\bar{\varphi}\|^2 + \frac{1}{4}\int_\Omega \Big(f(\bar{\varphi})\bar{\varphi} - 4F(\bar{\varphi})\Big)dx \\ &\le \frac{a}{4}\|\bar{\varphi}\|^2 + \frac{1}{4}\int_\Omega \Big(C_1\bar{\varphi}^2 + C_2\bar{\varphi}^p\Big)dx \\ &\le \Big\{\frac{a}{4}\|\lambda_1\varphi^+\|^2 + \frac{a}{4}\|\lambda_2\varphi^-\|^2 + \frac{1}{4}\int_\Omega \Big(C_1\lambda_1^2|\varphi^+|^2 + C_1\lambda_2^2|\varphi^-|^2\Big)dx \\ &\quad + \frac{1}{4}\int_\Omega \Big(C_2\lambda_1^p|\varphi^+|^p + C_2\lambda_2^p|\varphi^-|^p\Big)dx\Big\} := C_0, \end{aligned} \tag{3.8}$$

where C_0 does not depend on b. For n large enough, it follows that

$$C_0 + 1 \ge I_{b_n}(u_{b_n}) = I_{b_n}(u_{b_n}) - \frac{1}{4}\langle I_{b_n}'(u_{b_n}), u_{b_n}\rangle \ge \frac{a}{4}\|u_{b_n}\|^2. \tag{3.9}$$

Then, $\{u_{b_n}\}$ is bounded in H.

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Step 2. There exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $u_{b_n} \rightharpoonup u_0$ weakly in H. Then, u_0 is a weak solution of (1.19). Since u_{b_n} is the least energy sign-changing solution of (1.1) with $b = b_n$, then by the compactness of the embedding $H \hookrightarrow L^q(\Omega)$ for $2 \le q < 2^*$, we deduce that $u_{b_n} \to u_0$ strongly in H as $n \to \infty$. In fact,

$$\begin{aligned}
&\|u_{b_n} - u_0\|^2 \\
&= \langle I'_{b_n}(u_{b_n}) - I'_0(u_0), u_{b_n} - u_0\rangle - b_n \int_\Omega |\nabla u_{b_n}|^2 dx \int_\Omega \nabla u_{b_n}\big(\nabla u_{b_n} - \nabla u_0\big) dx \\
&\quad + \int_\Omega f(u_{b_n})\big(u_{b_n} - u_0\big) dx - \int_\Omega f(u_0)\big(u_{b_n} - u_0\big) dx \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Then, $u_0 \neq 0$ and u_0 changes sign only once.

Step 3. Suppose that v_0 is a least energy sign-changing solution of (1.19), the existence of v_0 was proved by Bartsch, Weth and Willem in [6], Proposition 3.1. By Lemma 2.1, for each $b_n > 0$, there is a unique pair $\big(s_{b_n}, t_{b_n}\big)$ of positive numbers such that

$$s_{b_n} v_0^+ + t_{b_n} v_0^- \in \mathcal{M}_{b_n}.$$

Then, we have

$$\begin{aligned}
&a(s_{b_n})^2\|v_0^+\|^2 + b_n(s_{b_n})^4\Big(\int_\Omega |\nabla v_0^+|^2 dx\Big)^2 + b_n(s_{b_n} t_{b_n})^2 \int_\Omega |\nabla v_0^+|^2 dx \int_\Omega |\nabla v_0^-|^2 dx \\
&= \int_\Omega f(s_{b_n} v_0^+) s_{b_n} v_0^+ dx,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&a(t_{b_n})^2\|v_0^-\|^2 + b_n(t_{b_n})^4\Big(\int_\Omega |\nabla v_0^-|^2 dx\Big)^2 + b_n(s_{b_n} t_{b_n})^2 \int_\Omega |\nabla v_0^+|^2 dx \int_\Omega |\nabla v_0^-|^2 dx \\
&= \int_\Omega f(t_{b_n} v_0^-) t_{b_n} v_0^- dx.
\end{aligned} \tag{3.11}$$

Recall that $v_0^\pm$ satisfies

$$a\|v_0^\pm\|^2 = \int_\Omega f(v_0^\pm) v_0^\pm dx. \tag{3.12}$$

Up to a subsequence, one can easily check that

$$(s_{b_n}, t_{b_n}) \to (1, 1), \quad \text{as } n \to \infty. \tag{3.13}$$

Now, we can prove u_0 is a least energy sign-changing solution of (1.19) which changes sign only once. From (3.13) and Lemma 2.3, we have

$$\begin{aligned} I_0(v_0) \le I_0(u_0) &= \lim_{n\to\infty} I_{b_n}(u_{b_n}) = \lim_{n\to\infty} I_{b_n}(u_{b_n}^+ + u_{b_n}^-) \\ &\le \lim_{n\to\infty} I_{b_n}\left(s_{b_n} v_0^+ + t_{b_n} v_0^-\right) = I_0(v_0^+ + v_0^-) = I_0(v_0). \end{aligned} \tag{3.14}$$

This completes the proof of Theorem 1.3. □

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