ARTICLE IN PRESS YJDEQ:7878

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Global existence of solutions for a weakly coupled system of semilinear damped wave equations

Kenji Nishihara [a], Yuta Wakasugi [b,*]

[a] *Faculty of Political Science and Economics, Waseda University, Tokyo, 169-8050, Japan*
[b] *Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan*

Received 8 March 2015

Abstract

In this paper, we consider the Cauchy problem for a weakly coupled system of semilinear damped wave equations. We prove the global existence of solutions for small data in the supercritical case for any space dimension. We also give estimates of the weighted energy of solutions and in a special case, we prove an almost optimal estimate. Moreover, in the subcritical case, we give an almost optimal estimate of the lifespan from both above and below.

Keywords: Semilinear damped wave equation; Weakly coupled system; Global existence; Critical exponent; Lifespan

1. Introduction

In this paper, we consider the Cauchy problem for a weakly coupled system of semilinear damped wave equations

$$\begin{cases} \left(\partial_t^2 - \Delta + \partial_t\right) u = F(u), & t > 0,\ x \in \mathbf{R}^N, \\ u(0,x) = \varepsilon u^0(x),\ \partial_t u(0,x) = \varepsilon u^1(x), & x \in \mathbf{R}^N. \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: kenji@waseda.jp (K. Nishihara), yuta.wakasugi@math.nagoya-u.ac.jp (Y. Wakasugi).

http://dx.doi.org/10.1016/j.jde.2015.05.014
0022-0396/

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

Here $u = u(t,x) = {}^t(u_1, u_2, \dots, u_k) : [0,\infty) \times \mathbf{R}^N \to \mathbf{R}^k$ is a real-valued unknown function. The nonlinear term is denoted by $F(u)$, which is defined by

$$F(u) = {}^t\left(|u_k|^{p_1}, |u_1|^{p_2}, \dots, |u_{k-1}|^{p_k}\right) \tag{1.2}$$

with $p_j > 1$ $(1 \le j \le k)$. The initial data $u^0 = {}^t(u_1^0, u_2^0, \dots, u_k^0)$, $u^1 = {}^t(u_1^1, u_2^1, \dots, u_k^1)$ belong to $\left[H^1(\mathbf{R}^N)\right]^k \times \left[L^2(\mathbf{R}^N)\right]^k$. The parameter $\varepsilon > 0$ denotes the amplitude of the initial data.

It is known that there exists the critical exponent for the system (1.1). First, we describe the meaning of the critical exponent. We define the matrix P as

$$P = \begin{pmatrix} 0 & 0 & \cdots & p_1 \\ p_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & p_k & 0 \end{pmatrix} \tag{1.3}$$

and consider $P - I$, where I is the identity matrix. Then, it is clear that

$$|P - I| = (-1)^k + (-1)^{k-1}\prod_{j=1}^k p_j = (-1)^{k-1}\left(\prod_{j=1}^k p_j - 1\right).$$

Therefore, it follows that $|P - I| \ne 0$ and hence, the inverse of $P - I$ exists. Thus, we can define

$$\alpha = {}^t(\alpha_1, \dots, \alpha_k) := (P - I)^{-1} \cdot {}^t(1, \dots, 1). \tag{1.4}$$

We also put $\alpha_{\max} := \max_{1 \le j \le k} \alpha_j$. For the system (1.1), it is expected that the critical exponent is given by

$$\alpha_{\max} = \frac{N}{2}, \tag{1.5}$$

that is, if $\alpha_{\max} < N/2$ (supercritical case), then there exists a unique global solution for small data; if $\alpha_{\max} \ge N/2$ (subcritical or critical case), then the local-in-time solution blows up in finite time.

We note that the exponent α appears in the scaling property of the corresponding parabolic system

$$\begin{cases} (\partial_t - \Delta)v = F(v), \\ v(0,x) = \varepsilon v^0(x). \end{cases} \tag{1.6}$$

Indeed, if $v = {}^t(v_1, \dots, v_k)$ is a solution of the above system, then

$$v^\lambda := {}^t(v_1^\lambda, \dots, v_k^\lambda), \quad v_j^\lambda(t,x) := \lambda^{2\alpha_j} v_j(\lambda^2 t, \lambda x)$$

is also a solution for any $\lambda > 0$, and the L^1-norm of the initial data

$$\|v_j^\lambda(0,x)\|_{L^1} = \lambda^{2\alpha_j - N}\|v_j^0\|_{L^1}$$

is invariant in the critical case $\alpha_j = N/2$.

The asymptotic behavior of solutions and the critical exponent problem for semilinear damped wave equations have been widely investigated after a pioneering work by Matsumura [21]. In particular, it is well known that the asymptotic profile of the solution of the linear damped wave equation

$$(\partial_t^2 - \Delta + \partial_t)u = 0$$

is given by a solution of the corresponding heat equation

$$(\partial_t - \Delta)v = 0$$

with suitable data. This is called the diffusion phenomenon and investigated by many mathematicians (see [2,8,11,13,16,20,23,26,37]). From the viewpoint of the diffusion phenomenon, the semilinear damped wave equation was also studied for a long time (see [7,12,14,15,17–19,35,38]). In particular, for the semilinear damped wave equation

$$\begin{cases} (\partial_t^2 - \Delta + \partial_t)u = |u|^p, \\ (u, \partial_t u)(0,x) = \varepsilon(u_0, u_1)(x), \end{cases} \tag{1.7}$$

where $(u_0, u_1) \in H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ have compact support, it is well known that the critical exponent of (1.7) is given by $p = \rho_F(N) = 1 + 2/N$, that is, if $p > \rho_F(N)$, then there exists a unique global solution for small data; if $p \le \rho_F(N)$, then the local-in-time solution blows up in finite time. This exponent $\rho_F(N)$ is a so-called Fujita's critical exponent and well known as the critical exponent of the corresponding semilinear heat equation

$$(\partial_t - \Delta)v = |v|^p$$

(see [6]).

Turning back to our problem (1.1), we can expect that the structure of the system (1.1) is similar to the corresponding parabolic system (1.6). Escobedo and Herrero [4] determined that when $k = 2$, the critical exponent of (1.6) is given by (1.5). This result has been extended for several direction and we refer the reader to [1,9,22,32,36] and a survey paper by Deng and Levine [3]. In particular, Umeda [36] showed that (1.5) is critical for any $k \ge 1$ and $N \ge 1$.

For our problem (1.1), Sun and Wang [33], Narazaki [24] proved that (1.5) is critical for $k = 2$ and $N \le 3$. After that, the first author [28] studied the asymptotic behavior of solutions, including optimal estimates and asymptotic profile. Recently, the authors [29] extended the result of [24,33] to any $N \ge 1$ and gave the almost optimal estimates of solutions.

For the case $k \ge 3$, Takeda [34] proved that the critical exponent of the system (1.1) is given by (1.5) for $N \le 3$. He also obtained a blow-up result in the case $\alpha_{\max} \ge N/2$ for any $N \ge 1$. Then Ogawa and Takeda [30,31] considered more general nonlinearities like

$$F(u) = {}^t(F_1(u), \dots, F_k(u)), \quad F_i(u) = \prod_{j=1}^k |u_j|^{p_{ij}} \quad (p_{ij} \ge 1 \text{ or } p_{ij} = 0, \ \sum_{j=1}^k p_{ij} > 1)$$

and proved that small data global existence holds if $N \le 3$ and $\alpha_{\max} < N/2$, where $\alpha_{\max} = \max_{1\le j\le k} \alpha_j$ and α is defined by the same as (1.4) with $P = (p_{ij})_{1\le i,j\le k}$. Narazaki [25] further extended these results to $N \ge 4$ by using weighted Sobolev spaces. He proved that if $\alpha_{\max} < N/2$ and the initial data $\varepsilon(u^0, u^1)$ satisfies $(1+|x|^2)^m(u^0,u^1) \in (L^1 \cap H^2) \times (L^1 \cap H^1)$ with suitable integer m, then there exists a unique global solution for sufficiently small ε and the solution satisfies some decay estimates. He also obtained global existence results for slowly decaying data not belonging to L^1 by using modulation spaces.

However, the precise asymptotic behavior of global solutions in the supercritical case and the estimate of the lifespan of solutions in the critical or subcritical case remain open. In this paper we shall give the global existence result in the supercritical case for any $N \ge 1$. Our approach is based on a weighted energy method and we treat initial data belonging to weighted $H^1 \times L^2$ spaces. Also, we prove better estimates of global solutions and in the special case

$$p_1, p_2, \dots, p_{k-1} \le 1 + \frac{2}{N} < p_k,$$

we give an almost optimal estimate. Moreover, we shall give an almost optimal estimate of the lifespan of solutions from both above and below in the subcritical case.

This paper is organized as follows. In the next section, we will state our global existence result in the supercritical case and estimates of global solutions (Theorems 2.2 and 2.3), and the estimate of the lifespan of solutions in the subcritical case (Propositions 2.4 and 2.5). In Section 3, we give proofs of our theorems. The outline of the proof is similar to our previous paper [29]. However, it seems to be difficult to apply directly the same approach as in [29]. The crucial point is the estimate of the nonlinear term. To do this, we appropriately determine the decay rate of the weighted energy by solving a certain linear equation. This part is new and different from [29] (Section 3.2). Finally, in Section 4, we give the proof of Propositions 2.4 and 2.5.

We finish up this section by introducing some notations. Throughout this paper, the letter C indicates the generic constant, which may change from line to line. We use the index $j = 1, \dots, k$ and note that the index $j-1$ is interpreted as k if $j = 1$. The symbol L^p denotes the usual Lebesgue space equipped with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}^N} |f(x)|^p dx \right)^{1/p} \quad (1 \le p < \infty).$$

Moreover, $H^s(\mathbf{R}^N)$ is the usual Sobolev space. For an interval I and a Banach space X, we define $C^r(I; X)$ as the space of r-times continuously differentiable mapping from I to X with respect to the topology in X (if I is a semi-open or closed interval, the differential at the endpoint is interpreted as the one-sided derivative).

2. Main results

In order to describe our results, we define the weak solution of (1.1). Let $T > 0$ and let us define

$$X(T) := C\left([0,T); H^1(\mathbf{R}^N)\right) \cap C^1\left([0,T); L^2(\mathbf{R}^N)\right).$$

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

We say that a function $u = {}^t(u_1, \dots, u_k) \in [X(T)]^k$ is a weak solution of the Cauchy problem (1.1) on the interval $[0, T)$ if it holds that

$$\int\limits_{[0,T)\times\mathbf{R}^N} u_j(t,x)\left(\partial_t^2\phi(t,x) - \Delta\phi(t,x) - \partial_t\phi(t,x)\right)dxdt$$
$$= \varepsilon\int\limits_{\mathbf{R}^N}\left\{(u_j^0(x) + u_j^1(x))\phi(0,x) - u_j^0(x)\partial_t\phi(0,x)\right\}dx$$
$$+ \int\limits_{[0,T)\times\mathbf{R}^N} |u_{j-1}|^{p_j}\phi(t,x)dxdt$$

for any $\phi \in C_0^\infty([0, T) \times \mathbf{R}^N)$ and $j = 1, \dots, k$, where we use the notation $j - 1 = k$ when $j = 1$.

Next, we define

$$\psi = \psi(t,x) = \frac{|x|^2}{4(2+\lambda)(1+t)}, \tag{2.1}$$

where $\lambda > 0$ is a constant, which is associated with the loss of decay of the global solution and is determined later. We also put

$$I_j(u^0, u^1) := \int\limits_{\mathbf{R}^N} e^{2\psi(0,x)}\left(\left|u_j^0(x)\right|^2 + \left|\nabla u_j^0(x)\right|^2 + \left|u_j^1(x)\right|^2\right)dx,$$
$$I_j(t; u) := \int\limits_{\mathbf{R}^N} e^{2\psi(t,x)}\left(\left|u_j(t,x)\right|^2 + \left|\nabla u_j(t,x)\right|^2 + \left|\partial_t u_j(t,x)\right|^2\right)dx$$

and

$$I_0 := \sum_{j=1}^k I_j(u^0, u^1), \quad I(t; u) := \sum_{j=1}^k I_j(t; u). \tag{2.2}$$

First, we describe a local existence result.

Proposition 2.1. *Let $\lambda > 0$ and let p_j $(j = 1, \dots, k)$ satisfy*

$$1 < p_j < \infty \ (N = 1, 2), \quad 1 < p_j \le \frac{N}{N-2} \ (N \ge 3), \tag{2.3}$$

and assume that $(u^0, u^1) \in [H^1(\mathbf{R}^N)]^k \times [L^2(\mathbf{R}^N)]^k$ satisfies $I_0 < +\infty$. Then, there exist $\tilde{T}_\varepsilon \in (0, \infty]$ and a unique weak solution $u = {}^t(u_1, \dots, u_k)$ of (1.1) satisfying $u \in [X(\tilde{T}_\varepsilon)]^k$. Moreover, if $\tilde{T}_\varepsilon < +\infty$, then we have

$$\lim_{t \uparrow \tilde{T}_\varepsilon} I(t; u) = +\infty.$$

The above proposition can be proven by a standard argument (see for example, [15]).

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

Our main result is the following:

Theorem 2.2. *We assume that p_j $(j = 1, \dots, k)$ satisfy (2.3) and the supercritical condition*

$$\alpha_{\max} < \frac{N}{2}. \tag{2.4}$$

Then, for any $\delta > 0$, there exists a constant $\lambda > 0$ (see (2.1)) such that the following holds: If the initial data $(u^0, u^1) \in [H^1(\mathbf{R}^N)]^k \times [L^2(\mathbf{R}^N)]^k$ satisfies $I_0 < +\infty$, then, there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem (1.1) admits a unique global weak solution $u = {}^t(u_1, \dots, u_k)$ satisfying $u \in [X(\infty)]^k$ and the estimates

$$\int_{\mathbf{R}^N} e^{2\psi(t,x)} \left(|\partial_t u_j(t,x)|^2 + |\nabla u_j(t,x)|^2 \right) dx \le C\varepsilon^2 I_0 (1+t)^{-N\alpha_j/\alpha_{\max} + N/2 - 1 + \delta}, \tag{2.5}$$

$$\int_{\mathbf{R}^N} e^{2\psi(t,x)} u_j(t,x)^2 dx \le C\varepsilon^2 I_0 (1+t)^{-N\alpha_j/\alpha_{\max} + N/2 + \delta}, \tag{2.6}$$

where $C = C(N, \delta, p_1, \dots, p_k) > 0$ is a constant.

Remark 2.1. The explicit form of the nonlinearity (1.2) is not essential. Indeed, our result is available for the nonlinear term $F(u) = {}^t(F_1(u_k), F_2(u_1), \dots, F_k(u_{k-1}))$ satisfying $F_j \in C^1(\mathbf{R}; \mathbf{R})$, $F_j(0) = 0$ and

$$|F_j(a) - F_j(b)| \le C(|a| + |b|)^{p_j - 1}|a - b|,$$

$$|F_j'(a) - F_j'(b)| \le C \begin{cases} (|a| + |b|)^{p_j - 2}|a - b|, & p_j > 2, \\ |a - b|^{p_j - 1} & 1 < p_j \le 2 \end{cases}$$

for $1 \le j \le k$.

The estimates (2.5) and (2.6) are not optimal in general. Indeed, in the special case

$$p_1, p_2, \dots, p_{k-1} \le 1 + \frac{2}{N} < p_k, \tag{2.7}$$

then we have the following estimate:

Theorem 2.3. *In addition to the assumption of Theorem 2.2, we further assume the condition (2.7). Then, the global solution u satisfies*

$$\int_{\mathbf{R}^N} e^{2\psi(t,x)} \left(|\partial_t u_j(t,x)|^2 + |\nabla u_j(t,x)|^2 \right) dx \le C\varepsilon^2 I_0 (1+t)^{-l_j - 1 + \delta}, \tag{2.8}$$

$$\int_{\mathbf{R}^N} e^{2\psi(t,x)} u_j(t,x)^2 dx \le C\varepsilon^2 I_0 (1+t)^{-l_j + \delta}, \tag{2.9}$$

where the decay rates l_j $(j = 1, \dots, k)$ *are given by*

$$\begin{aligned}
l_1 &= N\left(p_1 - \frac{2}{N} - \frac{1}{2}\right) - \delta_1, \\
l_2 &= N\left(p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N} - \frac{1}{2}\right) - \delta_2, \\
l_3 &= N\left(p_3\left(p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N}\right) - \frac{2}{N} - \frac{1}{2}\right) - \delta_3, \\
&\vdots \\
l_{k-1} &= N\left(p_{k-1}\left(p_{k-2}\left(\cdots\left(p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N}\right)\cdots\right) - \frac{2}{N}\right) - \frac{2}{N} - \frac{1}{2}\right) - \delta_{k-1}, \\
l_k &= \frac{N}{2},
\end{aligned} \tag{2.10}$$

$\delta_1, \dots, \delta_{k-1} > 0$ *are arbitrary small numbers and* $C = C(N, \delta, \delta_1, \dots, \delta_{k-1}, p_1, \dots, p_k) > 0$ *is a constant.*

Remark 2.2. The above decay rates l_j $(j = 1, \dots, k)$ are better than those of (2.5) and (2.6). We can expect the above decay rates are almost optimal under the condition (2.7). This means that if we can take $\delta = \delta_1 = \dots = \delta_{k-1} = 0$, then the estimates (2.8) and (2.9) become optimal in view of the decay rates (see [28] for the case $k = 2$). However, in general cases, it remains open whether we can find the optimal decay rate of solutions.

Next, we give an estimate of the lifespan of solutions in subcritical cases. The lifespan of the local solution is defined by

$$T_\varepsilon := \sup\left\{T \in (0, \infty) \,|\, \text{There exists a unique weak solution}\, u \in [X(T)]^k \text{ for (1.1)}\right\}.$$

For the estimate of T_ε from above, we have the following.

Proposition 2.4. *Let* p_j $(j = 1, \dots, k)$ *satisfy the condition* (1.3) *and we assume the subcritical condition*

$$\alpha_{\max} > \frac{N}{2}.$$

Moreover, we assume that the initial data satisfy $(u^0, u^1) \in [H^1(\mathbf{R}^N)]^k \times [L^2(\mathbf{R}^N)]^k$ *and*

$$\liminf_{R \to \infty} \int_{|x| < R} (u_j^0(x) + u_j^1(x))dx > 0 \quad (j = 1, \dots, k). \tag{2.11}$$

Then there exists a constant $C > 0$ *such that the lifespan of the solution is estimated as*

$$T_\varepsilon \le C\varepsilon^{-1/\kappa}$$

with

$$\kappa = \alpha_{\max} - \frac{N}{2}. \tag{2.12}$$

For the estimate of T_ε from below, we have the following.

Proposition 2.5. *Let p_j $(j = 1, \dots, k)$ satisfy (2.3) and assume the subcritical or critical condition $\alpha_{\max} \ge N/2$. Let the initial data (u^0, u^1) be in $[H^1(\mathbf{R}^N)]^k \times [L^2(\mathbf{R}^N)]^k$ and $I_0 < +\infty$. Then for any $\delta > 0$, there exists a constant $C > 0$ such that the lifespan of the solution of the system (1.1) is estimated as*

$$C\varepsilon^{-1/\kappa+\delta} \le T_\varepsilon \tag{2.13}$$

for any $\varepsilon \in (0, 1]$, where κ is given by (2.12).

Remark 2.3. From the above two propositions, we have an almost optimal estimate of T_ε from both above and below in the subcritical case $\alpha_{\max} > N/2$. However, in the critical case $\alpha_{\max} = N/2$, the estimate (2.13) seems to be far from the optimal estimate. Moreover, we do not have any estimate of T_ε from above.

3. Proof of Theorems 2.2 and 2.3

In order to prove Theorem 2.2, by Proposition 2.1, it suffices to prove that $I(t; u)$ defined by (2.2) of the local solution u does not diverge in a finite time, provided that ε is sufficiently small. To prove this, we employ a weighted energy method, which is originally developed by Todorova and Yordanov [35] and refined by several mathematicians (see [15,27]). We define the weighted energy of u_j $(j = 1, \dots, k)$ as

$$W_j(t) := (1+t)^{l_j+1-\delta} \int_{\mathbf{R}^N} e^{2\psi}(|\partial_t u_j|^2 + |\nabla u_j|^2)dx + (1+t)^{l_j-\delta} \int_{\mathbf{R}^N} e^{2\psi}|u_j|^2 dx$$

with the function ψ defined by (2.1), where $l_j \in \mathbf{R}$, $\delta > 0$ are determined later. We also define

$$M(t) := \sup_{0 \le s \le t} \left(\sum_{j=1}^{k} W_j(s) \right).$$

Then we can prove the following a priori estimate:

Proposition 3.1. *If p_j $(j = 1, \dots, k)$ satisfy (2.3) and the supercritical condition (2.4) holds, then there exist $l = {}^t(l_1, \dots, l_k) \in \mathbf{R}^k$ and $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, there is $\lambda = \lambda(N, \delta) > 0$ such that for any local solution $u = {}^t(u_1, \dots, u_k)$ of (1.1) as in Proposition 2.1, we have*

$$M(t) \le C\varepsilon^2 I_0 + C \sum_{j=1}^{k} \left(M(t)^{p_j} + M(t)^{(p_j+1)/2} \right)$$

with some constant $C > 0$.

If we obtain the above estimate, we can immediately prove

$$M(t) \le C\varepsilon^2 I_0 \tag{3.1}$$

for sufficiently small ε and this guarantees the existence of the global solution.

Remark 3.1. As we will see in the proof of the above proposition, $l = {}^t(l_1, \dots, l_k)$ is determined by (3.41) and $\delta_0 = \delta_0(N, p_1, \dots, p_k) > 0$ is taken sufficiently small so that the numbers γ_{j1}, γ_{j2} $(j = 1, \dots, k)$ defined by (3.36) satisfy $\gamma_{ji} < -1$. The number λ depends on $\delta \in (0, \delta_0]$ and is given by

$$\lambda = \frac{8\delta}{3N - 4\delta}. \tag{3.2}$$

We decompose the proof of Proposition 3.1 into two parts. The first part is the estimate of the linear part and the second one is the estimate of the nonlinear part.

3.1. Estimates for the linear part

Now we assume that

$$l_j \le N/2 \quad (j = 1, \dots, k). \tag{3.3}$$

Under this condition, we can apply the weighted energy method and obtain an estimate of $W_j(t)$:

$$W_j(t) \le C\varepsilon^2 I_0 + CN_j(t), \tag{3.4}$$

where $N_j(t)$ is the nonlinear term

$$N_j(t) := \int_0^t \left[(1+s)^{l_j+1-\delta} \int_{\mathbf{R}^N} e^{2\psi} |u_{j-1}|^{2p_j} dx + (1+s)^{l_j-\delta} \int_{\mathbf{R}^N} e^{2\psi} |u_{j-1}|^{p_j} |u_j| dx \right] ds. \tag{3.5}$$

More precisely, we prove the following lemma.

Lemma 3.2. *We assume the condition (3.3). Then it follows that*

$$W_j(t) + L_j(t) \le C\varepsilon^2 I_0 + CN_j(t), \tag{3.6}$$

where $L_j(t)$ is given by

$$L_j(t) = \int_0^t \Bigg[(1+s)^{l_j+1-\delta} \int_{\mathbf{R}^N} e^{2\psi} \left(|\partial_t u_j|^2 + (-\psi_t)(|\partial_t u_j|^2 + |\nabla u_j|^2) \right) dx$$
$$+ (1+s)^{l_j-\delta} \int_{\mathbf{R}^N} e^{2\psi} (1 + (-\psi_t))(|\partial_t u_j|^2 + |\nabla u_j|^2) dx$$
$$+ (1+s)^{l_j-\delta} \int_{\mathbf{R}^N} e^{2\psi} \left(|\nabla \psi|^2 + \frac{1}{1+s} \right) |u_j|^2 dx \Bigg] ds. \tag{3.7}$$

Proof. First, by the definition of ψ (see (2.1)), it is easy to see that

$$-\psi_t(t,x) = \frac{|x|^2}{4(2+\lambda)(1+t)^2}, \tag{3.8}$$
$$\nabla \psi(t,x) = \frac{x}{2(2+\lambda)(1+t)}, \tag{3.9}$$
$$\Delta \psi(t,x) = \left(\frac{N}{4} - \lambda_1 \right) \frac{1}{1+t}. \tag{3.10}$$

Hereafter, λ_i $(i = 1, 2, \dots)$ denote positive numbers depending on λ such that $\lim_{\lambda \to 0} \lambda_i = 0$. From (3.8) and (3.9), we can also easily have

$$-\psi_t(t,x) = (2+\lambda)|\nabla \psi(t,x)|^2. \tag{3.11}$$

Multiplying the j-th equation in (1.1) by $e^{2\psi} \partial_t u_j$, we have

$$\frac{1}{2} \partial_t \left[e^{2\psi} (|\partial_t u_j|^2 + |\nabla u_j|^2) \right] - \nabla \cdot \left(e^{2\psi} \partial_t u_j \nabla u_j \right)$$
$$+ e^{2\psi} \left(1 + (-\psi_t) - \frac{|\nabla \psi|^2}{-\psi_t} \right) |\partial_t u_j|^2 + \frac{e^{2\psi}}{-\psi_t} \left| \psi_t \nabla u_j - \partial_t u_j \nabla \psi \right|^2$$
$$= e^{2\psi} |u_{j-1}|^{p_j} \partial_t u_j. \tag{3.12}$$

Noting (3.11), we have

$$1 - \frac{|\nabla \psi|^2}{-\psi_t} = 1 - \frac{1}{2+\lambda} \ge \frac{1}{2}. \tag{3.13}$$

The Schwarz inequality implies

$$2 \left| (-\psi_t) \partial_t u_j \nabla u_j \cdot \nabla \psi \right| \le \frac{4}{5} (-\psi_t)^2 |\nabla u_j|^2 + \frac{5}{4} |\partial_t u_j|^2 |\nabla \psi|^2$$

and hence,

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

$$\frac{e^{2\psi}}{-\psi_t}\left|\psi_t\nabla u_j-\partial_t u_j\nabla\psi\right|^2\geq e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u_j|^2-\frac{1}{4}|\partial_t u_j|^2\frac{|\nabla\psi|^2}{(-\psi_t)}\right)$$
$$=e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u_j|^2-\frac{1}{4(2+\lambda)}|\partial_t u_j|^2\right). \quad (3.14)$$

Here we have used (3.11). On the other hand, the right-hand side of (3.12) is estimated as

$$e^{2\psi}|u_{j-1}|^{p_j}|\partial_t u_j|\leq e^{2\psi}\left(\frac{1}{8}|\partial_t u_j|^2+2|u_{j-1}|^{2p_j}\right). \quad (3.15)$$

Applying (3.13), (3.14) and (3.15) to (3.12), we obtain

$$\frac{1}{2}\partial_t\left[e^{2\psi}(|\partial_t u_j|^2+|\nabla u_j|^2)\right]-\nabla\cdot\left(e^{2\psi}\partial_t u_j\nabla u_j\right)$$
$$+e^{2\psi}\left(\frac{1}{4}+(-\psi_t)\right)|\partial_t u_j|^2+e^{2\psi}\frac{-\psi_t}{5}|\nabla u_j|^2$$
$$\leq Ce^{2\psi}|u_{j-1}|^{2p_j}. \quad (3.16)$$

Also, by multiplying the j-th equation in (1.1) by $e^{2\psi}u_j$, it follows that

$$\partial_t\left[e^{2\psi}\left(u_j\partial_t u_j+\frac{1}{2}u_j^2\right)\right]-\nabla\cdot\left(e^{2\psi}\left(u_j\nabla u_j+(\nabla\psi)u_j^2\right)\right)$$
$$+e^{2\psi}\left((-\psi_t)-2|\nabla\psi|^2+(\Delta\psi)\right)u_j^2+|(\nabla+\nabla\psi)e^{\psi}u_j|^2$$
$$+e^{2\psi}\left(2(-\psi_t)u_j\partial_t u_j-|\partial_t u_j|^2\right)$$
$$=e^{2\psi}|u_{j-1}|^{p_j}u_j. \quad (3.17)$$

We note that (3.11) and (3.10) lead to

$$(-\psi_t)-2|\nabla\psi|^2+(\Delta\psi)=\lambda|\nabla\psi|^2+\left(\frac{N}{4}-\lambda_1\right)\frac{1}{1+t}. \quad (3.18)$$

The Schwarz inequality implies

$$|(\nabla+\nabla\psi)e^{\psi}u_j|^2=e^{2\psi}|\nabla u_j+2u_j(\nabla\psi)|^2\geq e^{2\psi}\left(\lambda_2|\nabla u_j|^2-\lambda_2|\nabla\psi|^2u_j^2\right), \quad (3.19)$$
$$\left|2(-\psi_t)u_j\partial_t u_j\right|\leq\lambda_2(-\psi_t)u_j^2+\frac{1}{\lambda_2}(-\psi_t)|\partial_t u_j|^2. \quad (3.20)$$

Combining (3.17) with (3.18), (3.19), (3.20), we have

$$
\begin{aligned}
&\partial_t \left[e^{2\psi} \left(u_j \partial_t u_j + \frac{1}{2} u_j^2 \right) \right] - \nabla \cdot \left(e^{2\psi} \left(u_j \nabla u_j + (\nabla \psi) u_j^2 \right) \right) \\
&\quad + e^{2\psi} \left\{ \lambda_2 |\nabla u_j|^2 + \lambda_3 |\nabla \psi|^2 u_j^2 + \left(\frac{N}{4} - \lambda_1 \right) \frac{1}{1+t} u_j^2 - \left(\frac{-\psi_t}{\lambda_2} + 1 \right) |\partial_t u_j|^2 \right\} \\
&\le e^{2\psi} |u_{j-1}|^{p_j} |u_j|.
\end{aligned} \tag{3.21}
$$

Next, we adding the estimates (3.16) and (3.21). In order to control the bad term $-(-\psi_t/\lambda_2 + 1)|\partial_t u_j|^2$ in (3.21), we multiply (3.21) by a small parameter $\nu > 0$ and add it to (3.16). Then we obtain

$$
\begin{aligned}
&\partial_t \left[e^{2\psi} \left(\frac{1}{2} (|\partial_t u_j|^2 + |\nabla u_j|^2) + \nu \left(u_j \partial_t u_j + \frac{1}{2} u_j^2 \right) \right) \right] \\
&\quad - \nabla \cdot \left(e^{2\psi} \left(\partial_t u_j \nabla u_j + \nu \left(u_j \nabla u_j + (\nabla \psi) u_j^2 \right) \right) \right) \\
&\quad + e^{2\psi} \left(\left(\frac{1}{4} - \nu \right) + \left(1 - \frac{\nu}{\lambda_2} \right) (-\psi_t) \right) |\partial_t u_j|^2 \\
&\quad + e^{2\psi} \left(\nu \lambda_2 + \frac{-\psi_t}{5} \right) |\nabla u_j|^2 \\
&\quad + \nu e^{2\psi} \left(\lambda_3 |\nabla \psi|^2 u_j^2 + \left(\frac{N}{4} - \lambda_1 \right) \frac{1}{1+t} u_j^2 \right) \\
&\le C e^{2\psi} \left(|u_{j-1}|^{2p_j} + |u_{j-1}|^{p_j} |u_j| \right).
\end{aligned} \tag{3.22}
$$

Therefore, taking the parameter ν so small that

$$
1/4 - \nu > 0 \quad \text{and} \quad 1 - \nu/\lambda_2 > 0 \tag{3.23}
$$

and integrating (3.22) over $\mathbf{R}^N$, we can deduce that

$$
\begin{aligned}
&\frac{d}{dt} \left[\int_{\mathbf{R}^N} e^{2\psi} \left(\frac{1}{2} (|\partial_t u_j|^2 + |\nabla u_j|^2) + \nu \left(u_j \partial_t u_j + \frac{1}{2} u_j^2 \right) \right) dx \right] \\
&\quad + c_0 \int_{\mathbf{R}^N} e^{2\psi} (1 + (-\psi_t))(|\partial_t u_j|^2 + |\nabla u_j|^2) dx \\
&\quad + \nu \int_{\mathbf{R}^N} e^{2\psi} \left(\lambda_3 |\nabla \psi|^2 u_j^2 + \left(\frac{N}{4} - \lambda_1 \right) \frac{1}{1+t} u_j^2 \right) dx \\
&\le C \int_{\mathbf{R}^N} e^{2\psi} \left(|u_{j-1}|^{2p_j} + |u_{j-1}|^{p_j} |u_j| \right) dx
\end{aligned} \tag{3.24}
$$

with some constant $c_0 > 0$.

We further multiply (3.24) by $(t_0+t)^{l_j-\delta}$ with a large parameter $t_0 \ge 1$ depending on λ, ν and have

$$
\begin{aligned}
&\frac{d}{dt}\left[(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2+|\nabla u_j|^2)+\nu\left(u_j\partial_t u_j+\frac{1}{2}u_j^2\right)\right)dx\right]\\
&\quad-(l_j-\delta)(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2+|\nabla u_j|^2)+\nu\left(u_j\partial_t u_j+\frac{1}{2}u_j^2\right)\right)dx\\
&\quad+c_0(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}(1+(-\psi_t))(|\partial_t u_j|^2+|\nabla u_j|^2)dx\\
&\quad+\nu(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\lambda_3|\nabla\psi|^2u_j^2+\left(\frac{N}{4}-\lambda_1\right)\frac{1}{1+t}u_j^2\right)dx\\
&\le C(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(|u_{j-1}|^{2p_j}+|u_{j-1}|^{p_j}|u_j|\right)dx. \qquad (3.25)
\end{aligned}
$$

We must control the second term in the left-hand side. Here we recall the assumption $l_j \le N/2$ (see (3.3)). As we will see in the next subsection, l_j may take the negative value. Thus, we divide the following proof into two parts depending on the sign of l_j.

Case I: $l_j > 0$. In this case we must estimate the second term in (3.25). First, we remark that $l_j-\delta>0$ for sufficiently small $\delta>0$. We also note that

$$
\nu\left|u_j\partial_t u_j\right| \le \nu\lambda_4 u_j^2+\frac{\nu}{4\lambda_4}|\partial_t u_j|^2. \qquad (3.26)
$$

Then the second term in (3.25) is estimated as

$$
\begin{aligned}
&\left|(l_j-\delta)(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2+|\nabla u_j|^2)+\nu\left(u_j\partial_t u_j+\frac{1}{2}u_j^2\right)\right)dx\right|\\
&\quad\le\left|(l_j-\delta)(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{1}{2}+\frac{\nu}{4\lambda_4}\right)|\partial_t u_j|^2+\frac{1}{2}|\nabla u_j|^2\right)dx\right|\\
&\qquad+\left|(l_j-\delta)(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{\nu}{2}+\nu\lambda_4\right)u_j^2\right)dx\right|. \qquad (3.27)
\end{aligned}
$$

Here we note that the first term in the right-hand side of (3.27) is controlled by the third term of (3.25), provided that the parameter t_0 is sufficiently large. Indeed, if t_0 is sufficiently large depending on c_0, λ, ν, δ, then it follows that

$$
\begin{aligned}
&c_0(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}(1+(-\psi_t))(|\partial_t u_j|^2+|\nabla u_j|^2)dx \\
&\quad - \left|(l_j-\delta)(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{1}{2}+\frac{\nu}{4\lambda_4}\right)|\partial_t u_j|^2+\frac{1}{2}|\nabla u_j|^2\right)dx\right| \\
&\ge c_1(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}(1+(-\psi_t))(|\partial_t u_j|^2+|\nabla u_j|^2)dx \qquad (3.28)
\end{aligned}
$$

with some constant $c_1>0$. Moreover, the second term of the right-had side of (3.27) can be controlled by the forth term of (3.25). In fact, for a given $\delta>0$, we determine the parameter λ so that $\delta=3\lambda_1$. Then, as in Remark 3.1, we see that

$$
\lambda=\frac{8\delta}{3N-4\delta}
$$

and have

$$
\begin{aligned}
&\nu(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{N}{4}-\lambda_1\right)\frac{1}{1+t}u_j^2\right)dx \\
&\quad - \left|(l_j-\delta)(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{\nu}{2}+\nu\lambda_4\right)u_j^2\right)dx\right| \\
&\ge \nu(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{N}{4}-\frac{\delta}{3}\right)-\left(\frac{1}{2}+\lambda_4\right)(l_j-\delta)\right)u_j^2dx \\
&\ge c_2(t_0+t)^{l_j-1-\delta}\int_{\mathbf{R}^N} e^{2\psi}u_j^2dx \qquad (3.29)
\end{aligned}
$$

with some constant $c_2>0$, provided that λ_4 is sufficiently small. Here we have used the assumption $l_j\le N/2$ (see (3.3)). Consequently, from (3.26), (3.27), (3.28) and (3.29) we have

$$
\begin{aligned}
&\frac{d}{dt}\left[(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2+|\nabla u_j|^2)+\nu\left(u_j\partial_t u_j+\frac{1}{2}u_j^2\right)\right)dx\right] \\
&\quad + c_1(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}(1+(-\psi_t))(|\partial_t u_j|^2+|\nabla u_j|^2)dx \\
&\quad + c_3(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(|\nabla\psi|^2+\frac{1}{t_0+t}\right)u_j^2
\end{aligned}
$$

$$\leq C(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(|u_{j-1}|^{2p_j}+|u_{j-1}|^{p_j}|u_j|\right)dx \tag{3.30}$$

with some constant $c_3 > 0$.

Case II: $l_j \leq 0$. In this case the second term of (3.25) is positive and so we can omit this term (see (3.33) for the positivity of the integrand). Then we can immediately obtain the same estimate as (3.30).

To reach the conclusion, we turn back to the inequality (3.16). Integrating (3.16) over $\mathbf{R}^N$ and multiplying it by $(t_0+t)^{l_j+1-\delta}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}(|\partial_t u_j|^2+|\nabla u_j|^2)dx\right]\\
&\quad-\frac{1}{2}(l_j+1-\delta)(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}(|\partial_t u_j|^2+|\nabla u_j|^2)dx\\
&\quad+(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{1}{4}+(-\psi_t)\right)|\partial_t u_j|^2+\frac{-\psi_t}{5}|\nabla u_j|^2\right)dx\\
&\leq C(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{2p_j}dx.
\end{aligned} \tag{3.31}$$

To control the second term of the above inequality, we use the second term of (3.30). Calculating (3.30) $+\,\mu\times$ (3.31) with a small parameter $\mu > 0$, we can deduce that

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\left[(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}(|\partial_t u_j|^2+|\nabla u_j|^2)dx\right]\\
&\quad+\frac{d}{dt}\left[(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2+|\nabla u_j|^2)+\nu\left(u_j\partial_t u_j+\frac{1}{2}u_j^2\right)\right)dx\right]\\
&\quad+\left(c_1-\frac{\mu}{2}(l_j+1-\delta)\right)(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}(1+(-\psi_t))(|\partial_t u_j|^2+|\nabla u_j|^2)dx\\
&\quad+\mu(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\left(\frac{1}{4}+(-\psi_t)\right)|\partial_t u_j|^2+\frac{-\psi_t}{5}|\nabla u_j|^2\right)dx\\
&\quad+c_3(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(|\nabla\psi|^2+\frac{1}{t_0+t}\right)u_j^2\\
&\leq C(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{2p_j}dx+C(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{p_j}|u_j|dx.
\end{aligned}$$

Taking the parameter $\mu > 0$ so that $c_1 - \frac{\mu}{2}(l_j + 1 - \delta) > 0$, we can conclude that

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\left[(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}(|\partial_t u_j|^2 + |\nabla u_j|^2)dx\right] \\
&\quad + \frac{d}{dt}\left[(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2 + |\nabla u_j|^2) + \nu\left(u_j\partial_t u_j + \frac{1}{2}u_j^2\right)\right)dx\right] \\
&\quad + c_4(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(|\partial_t u_j|^2 + (-\psi_t)(|\partial_t u_j|^2 + |\nabla u_j|^2)\right)dx \\
&\quad + c_4(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left((1+(-\psi_t))((|\partial_t u_j|^2 + |\nabla u_j|^2)\right)dx \\
&\quad + c_4(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(|\nabla\psi|^2 + \frac{1}{t_0+t}\right)|u_j|^2 dx \\
&\le C(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{2p_j}dx + C(t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{p_j}|u_j|dx
\end{aligned}$$

with some constant $c_4 > 0$. Finally, integrating over $[0,t]$, we obtain

$$\begin{aligned}
&\frac{\mu}{2}(t_0+t)^{l_j+1-\delta}\int_{\mathbf{R}^N} e^{2\psi}(|\partial_t u_j|^2 + |\nabla u_j|^2)dx \\
&\quad + (t_0+t)^{l_j-\delta}\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2 + |\nabla u_j|^2) + \nu\left(u_j\partial_t u_j + \frac{1}{2}u_j^2\right)\right)dx \\
&\quad + c_4 L_j(t) \\
&\le C\varepsilon^2 I_0 + CN_j(t),
\end{aligned} \tag{3.32}$$

where $L_j(t)$ and $N_j(t)$ are defined by (3.7) and (3.5), respectively. Noting that

$$\nu|u_j\partial_t u_j| \le \frac{\nu}{4}u_j^2 + \nu|\partial_t u_j|^2$$

and recalling $\nu < 1/4$ (see (3.23)), we see that

$$\begin{aligned}
&\int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{2}(|\partial_t u_j|^2 + |\nabla u_j|^2) + \nu\left(u_j\partial_t u_j + \frac{1}{2}u_j^2\right)\right)dx \\
&\quad \ge \int_{\mathbf{R}^N} e^{2\psi}\left(\frac{1}{4}(|\partial_t u_j|^2 + |\nabla u_j|^2) + \frac{\nu}{4}u_j^2\right)dx.
\end{aligned} \tag{3.33}$$

By noting that $(t_0+t) \sim (1+t)$, (3.32) and (3.33) imply

$$W_j(t) + L_j(t) \le C\varepsilon^2 I_0 + CN_j(t),$$

which completes the proof of (3.6). $\square$

3.2. Estimates for the nonlinear part

To prove Proposition 3.1, it suffices to control the nonlinear term $N_j(t)$ defined by (3.5). In order to estimate $N_j(t)$, we use the following lemma:

Lemma 3.3 *(Gagliardo–Nirenberg inequality). (See [5].) If* $1 < p < \infty$ $(N = 1, 2)$, $1 < p \le N/(N-2)$ $(N \ge 3)$, *then we have*

$$\|f\|_{L^{2p}} \le C\|\nabla f\|_{L^2}^{\sigma_{2p}} \|f\|_{L^2}^{1-\sigma_{2p}}, \quad \sigma_{2p} = \frac{N(p-1)}{2p},$$
$$\|f\|_{L^{p+1}} \le C\|\nabla f\|_{L^2}^{\sigma_{p+1}} \|f\|_{L^2}^{1-\sigma_{p+1}}, \quad \sigma_{p+1} = \frac{N(p-1)}{2(p+1)}.$$

From the above lemma, we obtain

$$\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{2p_j}dx$$
$$\le C\left((1+t)^{-1/2}\|e^{\psi}u_{j-1}\|_{L^2} + \|e^{\psi}\nabla u_{j-1}\|_{L^2}\right)^{2p_j\sigma_{2p_j}} \|e^{\psi}u_{j-1}\|_{L^2}^{2p_j(1-\sigma_{2p_j})}$$

and

$$\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{p_j}|u_j|dx$$
$$\le C\left((1+t)^{-1/2}\|e^{\psi}u_{j-1}\|_{L^2} + \|e^{\psi}\nabla u_{j-1}\|_{L^2}\right)^{p_j\sigma_{p_j+1}} \|e^{\psi}u_{j-1}\|_{L^2}^{p_j(1-\sigma_{p_j+1})}$$
$$\times\left((1+t)^{-1/2}\|e^{\psi}u_j\|_{L^2} + \|e^{\psi}\nabla u_j\|_{L^2}\right)^{\sigma_{p_j+1}} \|e^{\psi}u_j\|_{L^2}^{1-\sigma_{p_j+1}}.$$

Indeed, we note that

$$\int_{\mathbf{R}^N} e^{2\psi}|u_{j-1}|^{2p_j}dx = \|e^{\psi/p_j}u_{j-1}\|_{L^{2p_j}}^{2p_j}$$
$$\le \|\nabla(e^{\psi/p_j}u_{j-1})\|_{L^2}^{2p_j\sigma_{2p_j}} \|e^{\psi/p_j}u_{j-1}\|_{L^2}^{2p_j(1-\sigma_{2p_j})}$$

and (3.9) implies

$$|\nabla\psi| e^{\psi/p_j} = C(1+t)^{-1/2} \left(\frac{|x|^2}{1+t} \right)^{1/2} e^{\psi/p_j} \le C(1+t)^{-1/2} e^{\psi}.$$

The second inequality can be proven in the same way. These estimates give a bound for $N_j(t)$ as

$$N_j(t) \le C \int_0^t \left[(1+s)^{\gamma_{j1}} M(t)^{p_j} + (1+s)^{\gamma_{j2}} M(t)^{(p_j+1)/2} \right] ds,$$

and hence, we have

$$M(t) \le C\varepsilon^2 I_0 + C \sum_{j=1}^k \int_0^t \left[(1+s)^{\gamma_{j1}} M(t)^{p_j} + (1+s)^{\gamma_{j2}} M(t)^{(p_j+1)/2} \right] ds, \tag{3.34}$$

where

$$\begin{aligned} \gamma_{j1} &= l_j + 1 - \delta + \frac{1}{2}\left(-l_{j-1} - 1 + \delta\right) 2p_j\sigma_{2p_j} + \frac{1}{2}\left(-l_{j-1} + \delta\right) 2p_j(1-\sigma_{2p_j}), \\ \gamma_{j2} &= l_j - \delta + \frac{1}{2}\left(-l_{j-1} - 1 + \delta\right) p_j\sigma_{p_j+1} + \frac{1}{2}\left(-l_{j-1} + \delta\right) p_j(1-\sigma_{p_j+1}) \\ &\quad + \frac{1}{2}\left(-l_j - 1 + \delta\right) \sigma_{p_j+1} + \frac{1}{2}\left(-l_j + \delta\right)(1-\sigma_{p_j+1}). \end{aligned} \tag{3.35}$$

Therefore, it suffices to show that both γ_{j1} and γ_{j2} are strictly less than -1.

To prove this, we note that

$$2p\sigma_{2p} = N(p-1), \quad (p+1)\sigma_{p+1} = \frac{N(p-1)}{2}$$

and calculate γ_{j1}, γ_{j2} as

$$\begin{aligned} \gamma_{j1} &= l_j + 1 - p_j l_{j-1} - \frac{1}{2} \cdot 2p_j\sigma_{2p_j} + (p_j - 1)\delta \\ &= \left\{ l_j - p_j l_{j-1} - \frac{N}{2}(p_j - 1) \right\} + 1 + (p_j - 1)\delta, \\ \gamma_{j2} &= \frac{l_j}{2} - \frac{p_j l_{j-1}}{2} - \frac{1}{2} \cdot (p_j + 1)\sigma_{p_j+1} + \frac{1}{2}(p_j - 1)\delta \\ &= \frac{1}{2}\left\{ l_j - p_j l_{j-1} - \frac{N}{2}(p_j - 1) \right\} + \frac{1}{2}(p_j - 1)\delta. \end{aligned} \tag{3.36}$$

Therefore, if

$$\left\{ l_j - p_j l_{j-1} - \frac{N}{2}(p_j - 1) \right\} < -2 \quad (j = 1, \ldots, k) \tag{3.37}$$

holds, then taking δ sufficiently small, we can obtain $\gamma_{ji} < -1$ $(j = 1, \ldots, k,\ i = 1, 2)$.

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

Now we prove that there exist some $l_1, \dots, l_k$ such that (3.37) holds under the supercritical condition. Let $\eta > 0$ be a small number determined later. Instead of the inequality (3.37), we consider the following linear equation of l_j:

$$\left\{ l_j - p_j l_{j-1} - \frac{N}{2}(p_j - 1) \right\} = -(2+\eta) \quad (j = 1, \dots, k). \tag{3.38}$$

Using the vector notation $l = {}^t(l_1, \dots, l_k)$, we can rewrite the equation (3.38) as

$$-(P-I)l - \frac{N}{2}(P-I)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = -(2+\eta)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where P is defined by (1.3) and I is the identity matrix. Multiplying the both-side of the above equation by $(P-I)^{-1}$, we have

$$l = (2+\eta)\alpha - \frac{N}{2}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{3.39}$$

(recall that α is defined by (1.4)). Finally, we choose the number $\eta > 0$ so that l satisfies the condition (3.3) to apply Lemma 3.2. To obtain better estimates, we have to take larger η. Hence, let us choose η so that

$$(2+\eta)\alpha_{\max} - \frac{N}{2} = \frac{N}{2},$$

that is,

$$\eta = \frac{N - 2\alpha_{\max}}{\alpha_{\max}}. \tag{3.40}$$

We note that the supercritical condition (2.4) guarantees that $\eta > 0$. From this choice and (3.39), we have

$$l_j = (2+\eta)\alpha_j - \frac{N}{2} = N\frac{\alpha_j}{\alpha_{\max}} - \frac{N}{2}. \tag{3.41}$$

It is obvious that the above l_j $(j = 1, \dots, k)$ satisfy the condition (3.3). Thus, we can apply the weighted energy method (Lemma 3.2) and it holds from (3.38) that

$$\begin{aligned}
\gamma_{j1} &= -(2+\eta) + 1 + (p_j - 1)\delta = -1 - \eta + (p_j - 1)\delta, \\
\gamma_{j2} &= -\frac{1}{2}(2+\eta) + \frac{1}{2}(p_j - 1)\delta = -1 - \frac{\eta}{2} + \frac{1}{2}(p_j - 1)\delta.
\end{aligned}$$

Thus, choosing δ sufficiently small (it depends on p_j and η), we have $\gamma_{j1}, \gamma_{j2} < -1$ $(j = 1, \dots, k)$ and this completes the proof of Proposition 3.1. Moreover, from (3.41) and (3.1), we can

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

immediately obtain the estimates (2.5) and (2.6). Finally, we note that the parameters in the above proof are determined in the following order:

$$\eta \to \delta \to \lambda \to \nu \to t_0 \to \mu.$$

The number η defined by (3.40) depends only on N and p_j $(j = 1, \dots, k)$ and so do the other parameters.

3.3. *Proof of Theorem 2.3*

In the inequality (3.37), we first put $l_k = N/2$. Then we have the condition for l_1:

$$l_1 < N\left(p_1 - \frac{2}{N} - \frac{1}{2}\right). \tag{3.42}$$

Therefore, we choose $l_1 = N\left(p_1 - \frac{2}{N} - \frac{1}{2}\right) - \delta_1$ with sufficiently small $\delta_1 > 0$. Using (3.42) and (3.37) again, we have

$$l_2 < N\left[p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N} - \frac{1}{2}\right] - p_2\delta_1$$

and hence, we choose

$$l_2 = N\left[p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N} - \frac{1}{2}\right] - \delta_2$$

with $\delta_2 = (p_2 + 1)\delta_1$. By this procedure, we define $l_1, \dots, l_{k-1}$ and in particular, l_{k-1} is given by

$$l_{k-1} = N\left(p_{k-1}\left(p_{k-2}\left(\cdots\left(p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N}\right)\cdots\right) - \frac{2}{N}\right) - \frac{2}{N} - \frac{1}{2}\right) - \delta_{k-1} \tag{3.43}$$

with $\delta_{k-1} = (p_{k-1} + 1)\delta_{k-2}$. We note that the assumption (2.7) implies the condition (3.3). Therefore, it suffices to check

$$\left\{l_k - p_k l_{k-1} - \frac{N}{2}(p_k - 1)\right\} < -2. \tag{3.44}$$

In order to check the condition (3.44), we prove the following lemma:

Lemma 3.4. *If (2.7) holds, then $\alpha_k < N/2$ is equivalent with*

$$1 + \frac{2}{N} < p_k\left(p_{k-1}\left(p_{k-2}\left(\cdots\left(p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N}\right)\cdots\right) - \frac{2}{N}\right) - \frac{2}{N}\right). \tag{3.45}$$

Proof. By using the cofactor matrix, we can calculate α as

$$\alpha_1 = \frac{1}{\prod_{j=1}^k p_j - 1}\left\{1 + \prod_{j\neq 2} p_j + \prod_{j\neq 2,3} p_j + \cdots + p_k p_1 + p_1\right\},$$

$$\alpha_2 = \frac{1}{\prod_{j=1}^k p_j - 1}\left\{1 + \prod_{j\neq 3} p_j + \prod_{j\neq 3,4} p_j + \cdots + p_1 p_2 + p_2\right\},$$

$$\vdots$$

$$\alpha_k = \frac{1}{\prod_{j=1}^k p_j - 1}\left\{1 + \prod_{j\neq 1} p_j + \prod_{j\neq 1,2} p_j + \cdots + p_{k-1} p_k + p_k\right\}.$$

From the above expression, we can rewrite the condition $\alpha_k < N/2$ as

$$1 + \frac{2}{N} < \prod_{j=1}^k p_j - \frac{2}{N}\left\{\prod_{j\neq 1} p_j + \prod_{j\neq 1,2} p_j + \cdots + p_{k-1}p_k + p_k\right\}.$$

We can compute the right-hand side as

$$\begin{aligned}
&\prod_{j=1}^k p_j - \frac{2}{N}\left\{\prod_{j\neq 1} p_j + \prod_{j\neq 1,2} p_j + \cdots + p_{k-1}p_k + p_k\right\}\\
&= p_k\left[\left(\prod_{j\neq k} p_j - \frac{2}{N}\left\{\prod_{j\neq 1,k} p_j + \prod_{j\neq 1,2,k} p_j + \cdots + p_{k-1}\right\}\right) - \frac{2}{N}\right]\\
&= p_k\left[p_{k-1}\left[\left(\prod_{j\neq k,k-1} p_j - \frac{2}{N}\left\{\prod_{j\neq 1,k,k-1} p_j + \prod_{j\neq 1,2,k,k-1} p_j + \cdots + p_{k-2}\right\}\right)\right.\right.\\
&\qquad \left.\left. - \frac{2}{N}\right] - \frac{2}{N}\right]\\
&= \cdots\\
&= p_k\left(p_{k-1}\left(p_{k-2}\left(\cdots\left(p_2\left(p_1 - \frac{2}{N}\right) - \frac{2}{N}\right)\cdots\right) - \frac{2}{N}\right) - \frac{2}{N}\right).
\end{aligned}$$

This proves Lemma 3.4. $\square$

Let us complete the proof of Theorem 2.3. By (3.43) and (3.45), we can deduce that

$$\begin{aligned}
&\left\{l_j - p_j l_{j-1} - \frac{N}{2}(p_j - 1)\right\}\\
&\quad = \frac{N}{2} - Np_k\left(p_{k-1}\left(\cdots\left(p_1 - \frac{2}{N}\right)\cdots\right) - \frac{2}{N} - \frac{1}{2}\right) - \frac{N}{2}(p_k - 1) + p_k\delta_{k-1}
\end{aligned}$$

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

$$
\begin{aligned}
&= N - Np_k\left(p_{k-1}\left(\cdots\left(p_1 - \frac{2}{N}\right)\cdots\right) - \frac{2}{N}\right) + p_k\delta_{k-1} \\
&< N - N\left(1 + \frac{2}{N}\right) \\
&= -2,
\end{aligned}
$$

provided that δ_{k-1} is sufficiently small. Thus, the condition (3.44) is checked and this completes the proof of Theorem 2.3.

4. Estimates of lifespan

In this section, we give a proof of Propositions 2.4 and 2.5.

4.1. Estimates of the lifespan from above

First, we prove Proposition 2.4. The proof is similar to that of our previous result [29] in which we followed [10] and proved in the case $k = 2$. For our problem, we combine the argument in [29] and that in [34].

Proof of Proposition 2.4. Let $\eta(t)$ be a test function defined by

$$
\eta(t) = \begin{cases} 1 & 0 \le t \le 1/2, \\ \dfrac{\exp(-1/(1-t^2))}{\exp(-1/(t^2 - 1/4)) + \exp(-1/(1-t^2))} & 1/2 < t < 1, \\ 0 & t \ge 1 \end{cases}
$$

and let $\phi(x) = \eta(|x|)$. Then we can easily see that $\eta \in C_0^\infty([0, \infty))$, $\phi \in C_0^\infty(\mathbf{R}^N)$ and

$$
\left|\left(\frac{d}{dt}\right)^i \eta(t)\right| \le C\eta(t)^{1/p}, \ (i = 1, 2) \quad |\Delta\phi(x)| \le C\phi(x)^{1/p} \tag{4.1}
$$

for $p > 1$. Indeed, putting $\mu(t) = \eta(t)^{1/r}$ with $1/p + 2/r = 1$, we have $\mu(t) \in C_0^\infty([0, \infty))$, $0 \le \mu(t) \le 1$ and hence,

$$
\begin{aligned}
\left|\eta''(t)\right| = \left|\left(\frac{d}{dt}\right)^2 \mu(t)^r\right| &= \left|r(r-1)\left(\mu'(t)\right)^2 \mu(t)^{r-2} + r\mu''(t)\mu(t)^{r-1}\right| \\
&\le C\mu(t)^{r-2} \le C\eta(t)^{1-2/r} = C\eta(t)^{1/p}.
\end{aligned}
$$

We can prove the other estimates of (4.1) in the same way. Let τ_0, R_0 be constants depending only on N, u^0, u^1 determined later. First, we note that if $T_\varepsilon \le \tau_0$, then this yields that $T_\varepsilon \le \tau_0\varepsilon^{-1/\kappa}$ for any $0 < \varepsilon \le 1$, $\kappa > 0$. Therefore, hereafter we assume that $T_\varepsilon \ge \tau_0$. Let u be a weak local solution of the system (1.1) on $[0, T_\varepsilon)$. We also assume that $\alpha_{\max} > N/2$. We note that without loss of generality, we may assume that

$$
\alpha_{\max} = \alpha_k.
$$

Let $\tau \in (\tau_0, T_\varepsilon)$ and $R \in [R_0, \infty)$ be parameters and let

$$\psi_{\tau,R}(t,x) = \eta_\tau(t)\phi_R(x) = \eta(t/\tau)\phi(x/R).$$

We note that

$$\operatorname{supp} \psi_{\tau,R} = [0,\tau] \times \left\{ x \in \mathbf{R}^N \,|\, |x| \le R \right\} \tag{4.2}$$

and

$$\psi_{\tau,R}(t,x) \equiv 1 \text{ on } [0,\tau/2] \times \left\{ x \in \mathbf{R}^N \,|\, |x| \le R/2 \right\}.$$

We also define

$$U_{j-1}(\tau,R) = \int_0^\tau \int_{\mathbf{R}^N} |u_{j-1}|^{p_j} \psi_{\tau,R} dx dt$$

and

$$J_j(R) = \int_{\mathbf{R}^N} \left(u_0^j(x) + u_1^j(x) \right) \phi_R(x) dx.$$

Then, by the definition of the weak solution, we have

$$\begin{aligned} U_{j-1}(\tau,R) + \varepsilon J_j(R) &= \int_0^\tau \int_{\mathbf{R}^N} u_j \left(\partial_t^2 - \Delta - \partial_t \right) \psi_{\tau,R} dx dt \\ &=: K_1 + K_2 + K_3. \end{aligned} \tag{4.3}$$

Let us estimate K_i $(i = 1,2,3)$. By (4.1), we deduce that

$$\begin{aligned} K_1 &= \int_0^\tau \int_{\mathbf{R}^N} u_j \tau^{-2} \eta''(t/\tau) \phi_R(x) dx dt \\ &\le \tau^{-2} \int_0^\tau \int_{\mathbf{R}^N} |u_j| \eta(t/\tau)^{1/p_{j+1}} \phi_R(x) dx dt \\ &\le \tau^{-2} \int_0^\tau \int_{\mathbf{R}^N} |u_j| \psi_{\tau,R}(t,x)^{1/p_{j+1}} dx dt. \end{aligned}$$

Here we have also used that $0 \le \phi_R(x) \le 1$. Applying the Hölder inequality, we obtain

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

$$K_1 \le \tau^{-2} \left(\int_0^\tau \int_{\mathbf{R}^N} |u_j|^{p_{j+1}} \psi_{\tau,R} dx dt \right)^{1/p_{j+1}} \left(\int_0^\tau \int_{|x|\le R} dx dt \right)^{1/p'_{j+1}}$$
$$\le C\tau^{-2+1/p'_{j+1}} R^{N/p'_{j+1}} U_j(\tau, R)^{1/p_{j+1}}. \tag{4.4}$$

Here we note that the support of $\psi_{\tau,R}$ is given by (4.2) and hereafter, p' denotes the Hölder conjugate of p. In the same manner, we can deduce that

$$K_2 \le C\tau^{1/p'_{j+1}} R^{-2+N/p'_{j+1}} U_j(\tau, R)^{1/p_{j+1}}, \tag{4.5}$$
$$K_3 \le C\tau^{-1+1/p'_{j+1}} R^{N/p'_{j+1}} U_j(\tau, R)^{1/p_{j+1}}. \tag{4.6}$$

By (4.3), (4.4), (4.5), (4.6), we can conclude

$$U_{j-1}(\tau, R) + \varepsilon J_j(R) \le CD(p'_{j+1}, \tau, R) U_j(\tau, R)^{1/p_{j+1}}, \tag{4.7}$$

where D is defined by

$$D(q, \tau, R) = \tau^{-2+1/q} R^{N/q} + \tau^{1/q} R^{-2+N/q} + \tau^{-1+1/q} R^{N/q} \tag{4.8}$$

for $q > 1$, $\tau \in (\tau_0, T_\varepsilon)$, $R \in (R_0, \infty)$.

Next, by the assumption (2.11), there exist constants $R_0, c_0 > 0$ such that for any $R \ge R_0$, we have

$$J_j(R) \ge c_0 \quad (j = 1, \dots, k). \tag{4.9}$$

From this and (4.7), we can compute

$$\begin{aligned}
&U_{k-1} + \varepsilon J_k(R) \\
&\quad \le CD(p'_1, \tau, R) U_k(\tau, R)^{1/p_1} \\
&\quad \le CD(p'_1, \tau, R) D(p'_2, \tau, R)^{1/p_1} U_1(\tau, R)^{1/(p_1 p_2)} \\
&\quad \le \cdots \\
&\quad \le CD(p'_1, \tau, R) D(p'_2, \tau, R)^{1/p_1} \cdots D(p'_k, \tau, R)^{1/(p_1 \cdots p_{k-1})} U_{k-1}(\tau, R)^{1/(p_1 \cdots p_k)}.
\end{aligned}$$

By using (4.9) again, we can rewrite this inequality as

$$\begin{aligned}
\varepsilon \le{}& CD(p'_1, \tau, R) D(p'_2, \tau, R)^{1/p_1} \cdots D(p'_k, \tau, R)^{1/(p_1 \cdots p_{k-1})} U_{k-1}(\tau, R)^{1/(p_1 \cdots p_k)} \\
&- U_{k-1}(\tau, R).
\end{aligned}$$

Now we use an elementary inequality

$$ac^b - c \le (1-b) b^{b/(1-b)} a^{1/(1-b)}$$

for $a > 0$, $0 < b < 1$, $c \geq 0$. We can immediately prove this by considering the maximal value of the function $f(c) = ac^b - c$. From this, we can conclude

$$\varepsilon \leq C\left[D(p_1', \tau, R)D(p_2', \tau, R)^{1/p_1} \cdots D(p_k', \tau, R)^{1/(p_1 \cdots p_{k-1})}\right]^{(p_1 \cdots p_k)'}.$$

Now we put $\tau_0 = \max\{1, R_0^2\}$ and $R = \tau^{1/2}$. Then we have

$$\varepsilon \leq C\left[D(p_1', \tau, \tau^{1/2})D(p_2', \tau, \tau^{1/2})^{1/p_1} \cdots D(p_k', \tau, \tau^{1/2})^{1/(p_1 \cdots p_{k-1})}\right]^{(p_1 \cdots p_k)'}. \tag{4.10}$$

Noting that

$$D(q, \tau, \tau^{1/2}) \leq C\tau^{-1+(N+2)/(2q)}$$

for $\tau \geq 1$, we use the following lemma, which was proven by Takeda [34, Lemma 14].

Lemma 4.1. *Let $p_j > 1$ $(j = 1, \ldots, k)$ and let α_j $(j = 1, \ldots, k)$ be defined as (1.4). Then it is true that*

$$\begin{aligned}
&\left(-1 + \frac{N+2}{p_1'}\right) + \frac{1}{p_1}\left(-1 + \frac{N+2}{p_2'}\right) + \frac{1}{p_1 p_2}\left(-1 + \frac{N+2}{p_3'}\right) \\
&\quad + \cdots + \frac{1}{p_1 p_2 \cdots p_{k-1}}\left(-1 + \frac{N+2}{p_k'}\right) \\
&= \frac{p_1 p_2 \cdots p_k - 1}{p_1 p_2 \cdots p_k}\left(\frac{N}{2} - \alpha_k\right).
\end{aligned}$$

Using this lemma, we further estimate the right-hand side of (4.10) as

$$\varepsilon \leq C\tau^{N/2-\alpha_k}.$$

Noting $\alpha_k > N/2$, we conclude that

$$\tau \leq C\varepsilon^{-1/\kappa}$$

with $\kappa = \alpha_k - N/2$. Since τ is arbitrarily in $[\tau_0, T_\varepsilon)$, we have

$$T_\varepsilon \leq C\varepsilon^{-1/\kappa},$$

which completes the proof. □

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

4.2. *Estimates of the lifespan from below*

Next, we give a proof of Proposition 2.5.

Proof of Proposition 2.5. We use the same notation as in the previous section. Let $\varepsilon \in (0, 1]$. First, we note that if $T_\varepsilon = +\infty$, then the estimate (2.13) is obvious. Therefore, we assume that $T_\varepsilon < +\infty$ in the following argument. In the following, we assume that $l = {}^t(l_1, \dots, l_k)$ satisfies (3.3) and

$$\gamma_{j1} + 1 > 0, \quad \gamma_{j2} + 1 > 0, \tag{4.11}$$

where γ_{j1}, γ_{j2} are defined by (3.35). Then, we can prove the same estimate as (3.34) and hence,

$$\begin{aligned} M(t) &\le C_0\varepsilon^2 + C\sum_{j=1}^{k}\int_0^t \Big[(1+s)^{\gamma_{j1}} M(t)^{p_j} + (1+s)^{\gamma_{j2}} M(t)^{(p_j+1)/2}\Big] ds \\ &\le C_0\varepsilon^2 + C\sum_{j=1}^{k}\Big[(1+t)^{\gamma_{j1}+1} M(t)^{p_j} + (1+t)^{\gamma_{j2}+1} M(t)^{(p_j+1)/2}\Big] \end{aligned} \tag{4.12}$$

with some constant $C_0 > 0$, which depends on I_0. We note that the first inequality of (4.12) implies

$$M(0) \le C_0\varepsilon^2 I_0.$$

Therefore, we can take $T'_\varepsilon > 0$ as the smallest time such that

$$M(T'_\varepsilon) = 2C_0\varepsilon^2 I_0. \tag{4.13}$$

Indeed, if such a time T'_ε does not exist, then it implies that $M(t) < 2C_0\varepsilon^2 I_0$ for all $t \ge 0$ and hence, we have the existence of the global solution. Substituting (4.13) into (4.12), we deduce that

$$2C_0\varepsilon^2 \le C_0\varepsilon^2 + C\sum_{j=1}^{k}\Big[(1+T'_\varepsilon)^{\gamma_{j1}+1}(2C_0\varepsilon^2)^{p_j} + (1+T'_\varepsilon)^{\gamma_{j2}+1}(2C_0\varepsilon^2)^{(p_j+1)/2}\Big].$$

We rewrite this as

$$C_0\varepsilon^2 \le C\sum_{j=1}^{k}\Big[(1+T'_\varepsilon)^{\gamma_{j1}+1}(2C_0\varepsilon^2)^{p_j} + (1+T'_\varepsilon)^{\gamma_{j2}+1}(2C_0\varepsilon^2)^{(p_j+1)/2}\Big].$$

Next, we replace the right-hand side by the maximal term:

$$C_0\varepsilon^2 \le 2kC \max_{1\le j\le k}\Big\{(1+T'_\varepsilon)^{\gamma_{j1}+1}(2C_0\varepsilon^2)^{p_j},\ (1+T'_\varepsilon)^{\gamma_{j2}+1}(2C_0\varepsilon^2)^{(p_j+1)/2}\Big\}. \tag{4.14}$$

We consider the case where the maximal term is given by

$$(1+T'_\varepsilon)^{\gamma_{j1}+1}(2C_0\varepsilon^2)^{p_j}$$

with some $j\in\{1,\dots,k\}$. Then, (4.14) implies

$$C_0\varepsilon^2 \le C(1+T'_\varepsilon)^{\gamma_{j1}+1}(2C_0\varepsilon^2)^{p_j}$$

and hence, we conclude

$$\varepsilon^{-2(p_j-1)/(\gamma_{j1}+1)} \le CT'_\varepsilon. \tag{4.15}$$

On the other hand, when the maximal term of (4.14) is given by

$$(1+T'_\varepsilon)^{\gamma_{j2}+1}(2C_0\varepsilon^2)^{(p_j+1)/2}$$

with some $j\in\{1,\dots,k\}$, we obtain

$$C_0\varepsilon^2 \le (1+T'_\varepsilon)^{\gamma_{j2}+1}(2C_0\varepsilon^2)^{(p_j+1)/2}$$

and hence, we have

$$\varepsilon^{-2((p_j+1)/2-1)/(\gamma_{j2}+1)} \le CT'_\varepsilon. \tag{4.16}$$

Combining the estimates (4.15) and (4.16) and consider the worst case, we can obtain

$$\varepsilon^{-1/\kappa_1} \le CT'_\varepsilon, \tag{4.17}$$

with

$$\kappa_1 = \max_{1\le j\le k}\left\{\frac{\gamma_{j1}+1}{2(p_j-1)},\ \frac{\gamma_{j2}+1}{2((p_j+1)/2-1)}\right\}. \tag{4.18}$$

Let us calculate κ_1. Formally, we put $\delta=0$. Then, from (3.36) we can see that

$$\gamma_{j1}+1=\left\{l_j-p_jl_{j-1}-\frac{N}{2}(p_j-1)\right\}+2=2(\gamma_{j2}+1),$$

which implies

$$\frac{\gamma_{j1}+1}{2(p_j-1)}=\frac{\gamma_{j2}+1}{2((p_j+1)/2-1)}.$$

Therefore, κ_1 is written as

$$\kappa_1=\max_{1\le j\le k}\frac{\gamma_{j1}+1}{2(p_j-1)}.$$

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014

Now, we determine the decay rate of the energy l_j such that

$$\frac{\gamma_{j1}+1}{2(p_j-1)} = \alpha_{\max} - \frac{N}{2} = \kappa$$

for all $j = 1, \dots, k$. This identity can be written as

$$\frac{\left\{l_j - p_j l_{j-1} - \frac{N}{2}(p_j-1)\right\}+2}{2(p_j-1)} = \alpha_{\max} - \frac{N}{2}.$$

This is also equivalent with

$$\left\{l_j - p_j l_{j-1} - \frac{N}{2}(p_j-1)\right\} + 2 = 2(p_j-1)\left(\alpha_{\max} - \frac{N}{2}\right).$$

Therefore, it suffices to determine the vector $l = {}^t(l_1, \dots, l_k)$ as

$$-(P-I)l = \left(2\alpha_{\max} - \frac{N}{2}\right)(P-I)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} - 2\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}.$$

Thus, we conclude

$$l = -\left(2\alpha_{\max} - \frac{N}{2}\right)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} + 2\alpha, \tag{4.19}$$

or, equivalently,

$$l_j = -2(\alpha_{\max} - \alpha_j) + \frac{N}{2}.$$

This l_j is obviously satisfies the condition (3.3) and hence, we can obtain the estimate (4.12) for this choice of l_j.

Turning back to the case $\delta \neq 0$, from the above choice of l_j (4.19), we have

$$\gamma_{j1} + 1 = 2(p_j-1)\left(\alpha_{\max} - \frac{N}{2}\right) + c_{j1}\delta,$$
$$\gamma_{j2} + 1 = 2\left(\frac{p_j+1}{2} - 1\right)\left(\alpha_{\max} - \frac{N}{2}\right) + c_{j2}\delta$$

with some $c_{j1}, c_{j2} > 0$ $(j = 1, \dots, k)$. Thus, the condition (4.11) is valid and κ_1 defined by (4.18) satisfies

$$\kappa_1 = \alpha_{\max} - \frac{N}{2} + c_0\delta$$

with some $c_0 > 0$. Therefore, the estimate (4.17) becomes

$$\varepsilon^{-1/\kappa_1} \le CT'_\varepsilon$$

with

$$\frac{1}{\kappa_1} = \frac{1}{\alpha_{\max} - N/2} - c_1\delta,$$

where $c_1 > 0$ is some constant. Finally (with appropriately modification of δ) we have the desired estimate

$$\varepsilon^{-1/\kappa+\delta} \le CT_\varepsilon,$$

where κ is defined by (2.12). □

Acknowledgments

This work was supported in part by Grant-in-Aid for Scientific Research (C) 25400184 (K. Nishihara) and Grant-in-Aid for JSPS Fellows 13J01835 (Y. Wakasugi) of Japan Society for the Promotion of Science.

References

[1] Y. Aoyagi, K. Tsutaya, Y. Yamauchi, Global existence of solutions for a reaction–diffusion system, Differential Integral Equations 20 (2007) 1321–1339.
[2] R. Chill, A. Haraux, An optimal estimate for the difference of solutions of two abstract evolution equations, J. Differential Equations 193 (2003) 385–395.
[3] K. Deng, H.A. Levine, The role of critical exponents in blow-up theorems: the sequel, J. Math. Anal. Appl. 243 (2000) 85–126.
[4] M. Escobedo, M.A. Herrero, Boundedness and blow up for a semilinear reaction–diffusion system, J. Differential Equations 89 (1991) 176–202.
[5] A. Friedman, Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.
[6] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sec. I 13 (1966) 109–124.
[7] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation with super critical nonlinearities, Differential Integral Equations 17 (2004) 637–652.
[8] T. Hosono, T. Ogawa, Large time behavior and L^p–L^q estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (2004) 82–118.
[9] T. Igarashi, N. Umeda, Existence and nonexistence of global solutions in time for a reaction–diffusion system with inhomogeneous term, Funkcial. Ekvac. 51 (2008) 17–37.
[10] M. Ikeda, Y. Wakasugi, A note on the lifespan of solutions to the semilinear damped wave equation, Proc. Amer. Math. Soc. 143 (2015) 163–171.
[11] R. Ikehata, Diffusion phenomenon for linear dissipative wave equations in an exterior domain, J. Differential Equations 186 (2002) 633–651.
[12] R. Ikehata, Y. Miyaoka, T. Nakatake, Decay estimates of solutions for dissipative wave equations in $\mathbf{R}^N$ with lower power nonlinearities, J. Math. Soc. Japan 56 (2004) 365–373.
[13] R. Ikehata, K. Nishihara, Diffusion phenomenon for second order linear evolution equations, Studia Math. 158 (2003) 153–161.
[14] R. Ikehata, M. Ohta, Critical exponents for semilinear dissipative wave equations in $\mathbf{R}^N$, J. Math. Anal. Appl. 269 (2002) 87–97.

[15] R. Ikehata, K. Tanizawa, Global existence of solutions for semilinear damped wave equations in $\mathbf{R}^N$ with noncompactly supported initial data, Nonlinear Anal. 61 (2005) 1189–1208.
[16] G. Karch, Selfsimilar profiles in large time asymptotics of solutions to damped wave equations, Studia Math. 143 (2000) 175–197.
[17] T. Kawakami, Y. Ueda, Asymptotic profiles to the solutions for a nonlinear damped wave equation, Differential Integral Equations 26 (2013) 781–814.
[18] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995) 617–653.
[19] T.T. Li, Y. Zhou, Breakdown of solutions to $\Box u + u_t = |u|^{1+\alpha}$, Discrete Contin. Dyn. Syst. 1 (1995) 503–520.
[20] P. Marcati, K. Nishihara, The L^p–L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations 191 (2003) 445–469.
[21] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. 12 (1976) 169–189.
[22] K. Mochizuki, Q. Huang, Existence and behavior of solutions for a weakly coupled system of reaction–diffusion equations, Methods Appl. Anal. 5 (1998) 109–124.
[23] T. Narazaki, L^p–L^q estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan 56 (2004) 585–626.
[24] T. Narazaki, Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations, Discrete Contin. Dyn. Syst. (Suppl.) (2009) 592–601.
[25] T. Narazaki, Global solutions to the Cauchy problem for a system of damped wave equations, Differential Integral Equations 24 (2011) 569–600.
[26] K. Nishihara, L^p–L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003) 631–649.
[27] K. Nishihara, Decay properties for the damped wave equation with space dependent potential and absorbed semilinear term, Comm. Partial Differential Equations 35 (2010) 1402–1418.
[28] K. Nishihara, Asymptotic behavior of solutions for a system of semilinear heat equations and the corresponding damped wave system, Osaka J. Math. 49 (2012) 331–348.
[29] K. Nishihara, Y. Wakasugi, Critical exponent for the Cauchy problem to the weakly coupled damped wave system, Nonlinear Anal. 108 (2014) 249–259.
[30] T. Ogawa, H. Takeda, Global existence of solutions for a system of nonlinear damped wave equations, Differential Integral Equations 23 (2010) 635–657.
[31] T. Ogawa, H. Takeda, Large time behavior of solutions for a system of nonlinear damped wave equations, J. Differential Equations 251 (2011) 3090–3113.
[32] J. Renclawowicz, Global existence and blow-up for a completely coupled Fujita type system, Appl. Math. 27 (2000) 203–218.
[33] F. Sun, M. Wang, Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping, Nonlinear Anal. 66 (2007) 2889–2910.
[34] H. Takeda, Global existence and nonexistence of solutions for a system of nonlinear damped wave equations, J. Math. Anal. Appl. 360 (2009) 631–650.
[35] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001) 464–489.
[36] N. Umeda, Blow-up and large time behavior of solutions of a weakly coupled system of reaction–diffusion equations, Tsukuba J. Math. 27 (2003) 31–46.
[37] H. Yang, A. Milani, On the diffusion phenomenon of quasilinear hyperbolic waves, Bull. Sci. Math. 124 (2000) 415–433.
[38] Qi S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001) 109–114.

Please cite this article in press as: K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.05.014