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Harnack inequality for nonlinear elliptic equations with strong absorption

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Abstract

We develop Harnack inequalities for two different types of equations. First we consider a fully nonlinear uniformly elliptic equation related to the Pucci's maximal and the minimal operators. Next we consider a quasilinear equation related to the p-Laplacian. In both cases we consider lower order terms of Keller–Osserman type. Although the equations considered are quite different, we employ a unified method to approach both problems and the results we find are similar.

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Keywords: Pucci extremal operators; Viscosity solutions; p-Laplace equations; Blow-up solutions; Harnack inequality

1. Introduction

In a recent paper [10], Martin Dindoš studied the Harnack inequality for non-negative classical solutions of $\Delta u = f(u)$ in domains in $\mathbb{R}^n$. In [10], Dindoš used a strict convexity condition and the Keller–Osserman condition on the nonlinear term f to obtain a global L^∞ estimate of all non-negative solutions to the aforementioned equation. The estimate was achieved by comparing nonnegative solutions to boundary blow-up solutions, the existence of which is assured by the

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Keller–Osserman condition. As an important ingredient Dindoš introduced a growth condition on f at infinity which, in conjunction with the global estimate, led to Harnack inequality for nonnegative solutions. The results in [10] extend the work of Finn and McOwen in [11].

A weakened version of the conditions used in [10] have been used in [16] to establish Harnack inequality for general second order uniformly elliptic equations in non-divergence form. In this paper we wish to extend Dindoš' work by not only replacing the principal operator with more general elliptic operators of two types, but also by significantly weakening some of the other conditions used in Dindoš' work. We use these conditions to develop Harnack inequality for non-negative viscosity solutions of fully nonlinear equations. The generalized Dindoš' condition with $p > 1$ allows to extend further the result to operators with different homogeneity degree in the gradient such as the p-Laplace operator. More specifically we first investigate the Harnack inequality for non-negative (viscosity) solutions of a fully nonlinear equation $H(x, u, Du, D^2u) = g(x, u)$, where $H(x, t, \xi, X)$ satisfies appropriate structural condition related to the Pucci extremal operators. Assuming that $g(x, t)$ satisfies $f(t) \le g(x, t) \le Tf(t)$ for some constant $T \ge 1$, and some non-negative function f that satisfies Dindoš' condition, we shall prove a Harnack inequality for non-negative viscosity solutions of the equations described above. For basic results on fully nonlinear equations we refer the reader to [3,7] and the references therein.

The second part of our investigation will focus on developing the Harnack inequality for non-negative weak solutions of the quasilinear equation $\operatorname{div}(|Du|^{p-2}Du) + b(x)u^{p-1} = g(x, u)$, where $p > 1$, and $g(x, t)$ satisfies the same condition as before but with f now satisfying a generalized Dindoš' condition (depending on $p > 1$). As pointed out earlier, the existence of weak boundary blow-up solutions of $\operatorname{div}(|Du|^{p-2}Du) = f(u)$ plays an important role in our approach. This infinite boundary value problem for such equations has been investigated by many authors. For the case $p = 2$ see [2,12,15] and references therein. For general $p > 1$ we refer to [9,13].

In recent years there has been considerable interest in absorption equations with nonlinear principal parts. The reader is referred to the interesting papers [4,5,8,14] and the references therein.

The paper is organized as follows. In Section 2, after introducing some basic facts on fully nonlinear uniformly elliptic equations, we derive Harnack inequality for non-negative viscosity solutions of differential inequalities involving the Pucci extremal operators with lower order terms. This, which is of independent interest in itself, would serve as the basic tool for proving our main Harnack inequality. We then establish the existence of viscosity supersolutions to Pucci maximal operators with lower order terms with nonlinear terms satisfying the Keller–Osserman condition. These supersolutions are used to develop a uniform global L^∞ estimate for all non-negative solutions to such equations. The Dindoš' condition, together with the above mentioned results, provide the necessary tools to derive the desired Harnack Inequality, Theorem 2.8.

In Section 3, we look at a class of quasilinear equations and recall some basic results about them that will aid in our study of the Harnack inequality. Next, we introduce a general version of the Dindoš' condition that is suited to the study of Harnack inequality of quasilinear equations. This section follows the same general approach of Section 2 to develop the Harnack inequality, Theorem 3.8, for non-negative weak solutions of the quasilinear equations under consideration.

In both Sections 2 and 3, we employ a useful estimate involving the nonlinear term f to derive the Harnack inequality. This estimate is proved in Appendix A as a consequence of the generalized Dindoš condition.

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2. The fully nonlinear case in the viscosity setting

Let Ω be a bounded domain in $\mathbb{R}^n$ and $\mathcal{S}_n$ be the space of $n \times n$ real symmetric matrices with the partial ordering $X \leq Y$, which stands for $Y - X$ positive semi-definite.

A mapping $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ is said to be uniformly elliptic in Ω if

$$\lambda \mathrm{Tr}(Y - X) \leq H(x, t, \xi, Y) - H(x, t, \xi, X) \leq \Lambda \mathrm{Tr}(Y - X),$$

for all $(x, t, \xi, X, Y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \times \mathcal{S}_n$, $Y \geq X$, where Tr denotes the trace and λ, Λ are positive real numbers such that $\lambda \leq \Lambda$.

If $u \in C^2(\Omega)$ we denote by Du and D^2u the gradient and the Hessian matrix of u. The operator $H[\cdot]$ acting on $u \in C^2(\Omega)$ as

$$H[u](x) = H(x, u(x), Du(x), D^2u(x))$$

will be called in turn uniformly elliptic (with ellipticity constants λ and Λ).

Special uniformly elliptic operators are the extremal Pucci operators, the maximal and the minimal one, respectively defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \sup_{\lambda I_n \leq A \leq \Lambda I_n} \mathrm{Tr}(AX)$$
$$\mathcal{M}^-_{\lambda,\Lambda}(X) = \inf_{\lambda I_n \leq A \leq \Lambda I_n} \mathrm{Tr}(AX),$$

I_n being the $n \times n$ identity matrix. These operators satisfy the following properties for all $X, Y \in S_n$:

(i) $\mathcal{M}^-_{\lambda,\Lambda}(X) = -\mathcal{M}^+_{\lambda,\Lambda}(-X)$;
(ii) $\mathcal{M}^-_{\lambda,\Lambda}(X) + \mathcal{M}^-_{\lambda,\Lambda}(Y) \leq \mathcal{M}^-_{\lambda,\Lambda}(X + Y) \leq \mathcal{M}^+_{\lambda,\Lambda}(X + Y) \leq \mathcal{M}^+_{\lambda,\Lambda}(X) + \mathcal{M}^+_{\lambda,\Lambda}(Y)$.

All linear uniformly elliptic operators with ellipticity constants λ and Λ

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}u + \sum_{i=1}^{n} b_i(x) D_i u + c(x)u$$

are uniformly elliptic in the sense introduced above with the same ellipticity constants. In this case $H[u] = Lu$ with

$$H(x, t, \xi, X) = \mathrm{Tr}(A(x)X) + B^T(x)\xi + c(x)t,$$

where $A(x) = [a_{ij}(x)] \in \mathcal{S}_n$ and $B(x) = (b_i(x)) \in \mathbb{R}^n$. The extremal Pucci operators are fully nonlinear elliptic operators with ellipticity constants λ and Λ.

Moreover, for all pure second order uniformly elliptic operators H we have

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq H[u] \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2u).$$

In particular, $\Delta u = \mathrm{Tr}(D^2u) = \mathcal{M}^{\pm}_{1,1}(D^2u)$.

For computational purposes, we recall the following alternative description of the Pucci operators:

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \Lambda \mathrm{Tr}(X^+) - \lambda \mathrm{Tr}(X^-) = \Lambda \sum_{e_i>0} |e_i(X)| - \lambda \sum_{e_i<0} |e_i(X)|,$$

$$\mathcal{M}^-_{\lambda,\Lambda}(X) = \lambda \mathrm{Tr}(X^+) - \Lambda \mathrm{Tr}(X^-) = \lambda \sum_{e_i>0} |e_i(X)| - \Lambda \sum_{e_i<0} |e_i(X)|,$$

where $X^\pm$ are the positive and negative parts of X such that $X^\pm \geq 0$, $X = X^+ - X^-$, $X^+X^- = 0 = X^-X^+$, while $e_i(X)$ are the eigenvalues of X.

If $w(x) = \varphi(|x|)$ is a radial function, then the Hessian matrix $D^2w(x)$ has eigenvalues $\varphi''(|x|)$ (which is simple) and $\frac{\varphi'(|x|)}{|x|}$ (with multiplicity $n-1$).

Therefore, if φ is a convex non-decreasing function, we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2w(x)) = \Lambda\, \varphi''(|x|) + \Lambda(n-1)\,\frac{\varphi'(|x|)}{|x|} = \Lambda\,\Delta w,$$

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w(x)) = \lambda\, \varphi''(|x|) + \lambda(n-1)\,\frac{\varphi'(|x|)}{|x|} = \lambda\,\Delta w.$$

In terms of Pucci operators the uniform ellipticity of H can be equivalently stated as

$$\begin{aligned}\mathcal{M}^-_{\lambda,\Lambda}(Y-X) &\leq H(x,t,\xi,Y) - H(x,t,\xi,X)\\ &\leq \mathcal{M}^+_{\lambda,\Lambda}(Y-X), \quad \forall X,Y \in \mathcal{S}_n, \;\; Y \geq X.\end{aligned}$$

We will also consider a Lipschitz-continuous dependence on the gradient variable:

$$\begin{aligned}\mathcal{M}^-_{\lambda,\Lambda}(Y-X) - \beta|\eta-\xi| &\leq H(x,t,\eta,Y) - H(x,t,\xi,X)\\ &\leq \mathcal{M}^+_{\lambda,\Lambda}(Y-X) + \beta|\eta-\xi|, \quad \forall X,Y \in \mathcal{S}_n, \;\; Y \geq X,\end{aligned}$$

for a constant $\beta \geq 0$. In passing we point out that both of the above inequalities are equivalent to the corresponding inequalities without the requirement $Y \geq X$.

Now, we briefly report on viscosity solutions of fully nonlinear elliptic equations. We say that a function $u \in USC(\Omega)$ (upper semi-continuous in Ω) is a viscosity subsolution of equation $H[u] = g$, equivalently a solution of $H[u] \geq g(x)$, if for every point $x \in \Omega$ and every test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a maximum at x, we have

$$H(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \geq g(x)\,.$$

Analogously, a function $u \in LSC(\Omega)$ (lower semi-continuous in Ω) is a viscosity supersolution of equation $H[u] = g$, equivalently a solution of $H[u] \leq g(x)$, if for every point $x \in \Omega$ and every test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a minimum at x, we have

$$H(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \leq g(x)\,.$$

A continuous function u which is both a subsolution and a supersolution will be called a viscosity solution of equation $H[u] = g$ in Ω.

The above definition provides a generalization of the concept of solution in the sense that a classical solution $u \in C^2(\Omega)$ is *a fortiori* a viscosity solution whereas conversely a viscosity solution $u \in C^2(\Omega)$ is in turn a classical solution.

A comparison principle for sub and supersolutions is provided by [7, Theorem 3.3].

Here below we also state the Harnack inequality for viscosity solutions of fully nonlinear uniformly elliptic equations. Denote with $B(z, R)$ the ball with center at z and radius R.

Theorem 2.1. *Suppose $u \in C(B(z, R))$ is a non-negative viscosity solution of differential inequalities*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \beta|Du| + \alpha u \geq 0$$
$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \beta|Du| - (\alpha + c(x))u \leq 0$$

in $B(z, R)$, where α and β are non-negative numbers and $c(x)$ is a non-negative continuous function in $B(z, R)$. Then

$$\sup_{B(z,R/3)} u(x) \leq C \inf_{B(z,R/3)} u(x)$$

where C is a positive constant depending only on

$$n,\ \frac{\Lambda}{\lambda},\ \frac{\beta}{\lambda} R,\ \frac{\alpha}{\lambda} R^2,\ \frac{R^2}{\lambda} \sup_{B(z,2R/3)} c(x)$$

and independent of u.

Proof. From [1] we know that Theorem 2.1 holds true with $\alpha + c(x) \equiv 0$. This relies on the following interior estimates for subsolutions and supersolutions, known respectively as the local maximum principle (LMP) and the weak Harnack inequality (WHI). Let $k \in L^n(B(z, 2R/3))$.

(LMP) *Let w be a continuous solution in $B(z, R)$ of the differential inequality*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2w) + \beta|Dw| \geq -k^-.$$

Then for all $p > 0$

$$\sup_{B(z,R/3)} w \leq C \left\{ \left(\fint_{B(z,2R/3)} (w^+)^p \, dx \right)^{\frac{1}{p}} + \frac{R}{\lambda} \|k^-\|_{L^n(B(z,3R/4))} \right\}$$

where C is a positive constant depending only on n, $\frac{\Lambda}{\lambda}$, $\frac{\beta R}{\lambda}$ and p.

(WHI) *Let $v \geq 0$ be a continuous solution in $B(z, R)$ of the differential inequality*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2v) - \beta|Dv| \leq k^+.$$

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Then there exists $p_0 > 0$ *such that*

$$\left(\fint_{B(z,2R/3)} v^{p_0}\, dx \right)^{\frac{1}{p_0}} \le C_0 \left(\inf_{B(z,R/3)} u + \frac{R}{\lambda} \|k^+\|_{L^n(B(z,3R/4))} \right),$$

where p_0 *and* C_0 *are positive constants depending only on* n, $\frac{\Lambda}{\lambda}$ *and* $\frac{\beta R}{\lambda}$.

To show Theorem 2.1 in the general case $\alpha + c(x) \ge 0$, we shall use (LMP) and (WHI) with $k = 0$. We divide the proof into 6 steps.

1. Let $u \ge 0$ be a solution in $B(z, R)$ of the differential inequalities

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \beta|Du| + \gamma u \ge 0, \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \beta|Du| - \gamma u \le 0, \end{aligned} \tag{1}$$

where γ is a positive upper bound for $\alpha + c(x)$ in $B(z, 2R/3)$.

2. Setting $u = \psi w = \varphi v$ for positive smooth functions ψ and φ, from (1) we get

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}\left(D^2w + \frac{D\psi}{\psi} \otimes Dw + Dw \otimes \frac{D\psi}{\psi}\right) + \beta|Dw| \\ &\quad + \left(\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi) + \beta|D\psi| + \gamma\psi\right)\frac{w}{\psi} \ge 0, \\ &\mathcal{M}^-_{\lambda,\Lambda}\left(D^2v + \frac{D\varphi}{\varphi} \otimes Dv + Dv \otimes \frac{D\varphi}{\varphi}\right) - \beta|Dv| \\ &\quad + \left(\mathcal{M}^-_{\lambda,\Lambda}(D^2\varphi) - \beta|D\varphi| - \gamma\varphi\right)\frac{v}{\varphi} \le 0. \end{aligned}$$

We are proceeding as v and w would be smooth functions, but this is allowed in the viscosity sense since ψ and φ are smooth (see Lemma 1 of [6]).

Moreover, we have used here the sub-additivity of $\mathcal{M}^+_{\lambda,\Lambda}$ and the super-additivity of $\mathcal{M}^-_{\lambda,\Lambda}$. Using these properties again and noticing that

$$\begin{aligned} Tr\left(A\left(\frac{D\psi}{\psi} \otimes Dw + Dw \otimes \frac{D\psi}{\psi}\right)\right) &\le \frac{2}{\psi}|AD\psi|\,|Dw| \\ Tr\left(A\left(\frac{D\varphi}{\varphi} \otimes Dv + Dv \otimes \frac{D\varphi}{\varphi}\right)\right) &\ge -\frac{2}{\varphi}|AD\varphi|\,|Dv|, \end{aligned}$$

taking sup and inf over $\lambda I_n \le A \le \Lambda I_n$, we get

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2w) + \left(2\Lambda\tfrac{|D\psi|}{\psi} + \beta\right)|Dw| + \left(\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi) + \beta|D\psi| + \gamma\psi\right)\frac{w}{\psi} \ge 0, \\ &\mathcal{M}^-_{\lambda,\Lambda}(D^2v) - \left(2\Lambda\tfrac{|D\varphi|}{\varphi} + \beta\right)|Dv| + \left(\mathcal{M}^-_{\lambda,\Lambda}(D^2\varphi) - \beta|D\varphi| - \gamma\varphi\right)\frac{v}{\varphi} \le 0. \end{aligned}$$

3. We choose as φ a positive and convex smooth function such that

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2\varphi) - \beta|D\varphi| - \gamma\varphi \geq 0.$$

To do this, we set $\varphi(x) = h(x_1) = e^{ax_1}$ with

$$a = \frac{\beta}{\lambda} + \sqrt{\frac{\gamma}{\lambda}} \tag{2}$$

in order to have

$$\begin{aligned}(\mathcal{M}^{-}_{\lambda,\Lambda}(D^2\varphi) - \beta|D\varphi| - \gamma\varphi) &= \left(\lambda h'' - \beta h' - \gamma h\right)\\ &= e^{ax_1}\left(\lambda a^2 - \beta a - \gamma\right) \geq 0.\end{aligned}$$

4. Concerning ψ, we need a positive and concave smooth function such that

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2\psi) + \beta|D\psi| + \gamma\psi \leq 0$$

in a suitable slab $S = \{x \in \mathbb{R}^n : 0 < x_1 < d\}$.

To do this we take $\psi(x) = 2 - h(x_1) = 2 - e^{ax_1}$ with a as in (2). Using the previous computation we have

$$\begin{aligned}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2\psi) + \beta|D\psi| + \gamma\psi &= -\lambda h'' + \beta h' + \gamma(2 - h)\\ &= -(\lambda h'' - \beta h' - \gamma h) + 2\gamma(1 - h)\\ &\leq 2\gamma\,(1 - e^{ax_1}) \leq 0.\end{aligned}$$

In order to have $\psi > 0$ we choose $d = \frac{\delta}{a}$ with a positive number $\delta < \log 2$ so that $\psi(x) \geq 2 - e^{\delta} > 0$.

5. From the above, we deduce that if $B(z, r)$ is a ball of radius $r \leq d/2$, which we may suppose contained in the slab $S = \{x \in \mathbb{R}^n : 0 < x_1 < d\}$, then in $B(z, r)$ we have

$$\begin{aligned}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 w) + \left(2\Lambda \tfrac{|D\psi|}{\psi} + \beta\right)|Dw| \geq 0,\\ \mathcal{M}^{-}_{\lambda,\Lambda}(D^2 v) - \left(2\Lambda \tfrac{|D\varphi|}{\varphi} + \beta\right)|Dv| \leq 0.\end{aligned}$$

Choosing $\delta = \log(3/2)$, it turns out that $1 \leq \varphi \leq 3/2$, $1/2 \leq \psi \leq 1$, $\psi \leq \varphi \leq 3\psi$ and $|D\varphi|/\varphi \leq 3a$, $|D\psi|/\psi \leq 3a$ where a is as in (2). Therefore we have

$$\begin{aligned}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2 w) + (6\Lambda a + \beta)\,|Dw| \geq 0,\\ \mathcal{M}^{-}_{\lambda,\Lambda}(D^2 v) - (6\Lambda a + \beta)\,|Dv| \leq 0.\end{aligned}$$

Note that $u \le w \le 3v \le 3u$. Now we apply the local maximum principle (LMP) to $w = \frac{u}{\psi}$ with $p = p_0$ and then the weak Harnack inequality (WHI) to $v = \frac{u}{\varphi}$ on $B(z, r)$ obtaining

$$\begin{aligned}
\sup_{B(z,r/3)} u \le \sup_{B(z,r/3)} w &\le C \left(\fint_{B(z,2r/3)} \quad dx \right)^{\frac{1}{p_0}} \\
&\le 3C \left(\fint_{B(z,2r/3)} v^{p_0} \, dx \right)^{\frac{1}{p_0}} \le 3CC_0 \inf_{B(z,r/3)} v \\
&\le 3CC_0 \inf_{B(z,r/3)} u.
\end{aligned}$$

Note that C and C_0 depend only on n, $\frac{\Lambda}{\lambda}$, $\frac{\beta}{\lambda} r$ and $\frac{\Lambda}{\lambda} ar = \frac{\Lambda}{\lambda}\left(\frac{\beta}{\lambda} r + \sqrt{\frac{\gamma}{\lambda}} r\right)$.

Resuming we have

$$\sup_{B(z,r/3)} u \le C \inf_{B(z,r/3)} u,$$

provided $0 < r \le d/2$, with $C = C(n, \frac{\Lambda}{\lambda}, \frac{\beta r}{\lambda}, \frac{\gamma r^2}{\lambda})$.

6. Assume now that u satisfies the differential inequalities (1) in a ball $B(z, R)$ with an arbitrary radius $R > 0$. Suppose also, as we may, that $z = 0$. Let us consider $r = d/2$ and set $\rho = \frac{r}{R}$.

The re-scaled function $u_\rho(y) = u(y/\rho)$, $|y| \le r$, satisfies the following differential inequalities

$$\begin{aligned}
\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_\rho) + \frac{\beta}{\rho}|Du_\rho| + \frac{\gamma}{\rho^2} u_\rho \ge 0, \\
\mathcal{M}^-_{\lambda,\Lambda}(D^2 u_\rho) - \frac{\beta}{\rho}|Du_\rho| - \frac{\gamma}{\rho^2} u_\rho \le 0
\end{aligned}$$

in $B(0, r)$. Then we may apply the result obtained in the previous step with $u = u_\rho$ to get

$$\sup_{B(0,r/3)} u_\rho \le C \inf_{B(0,r/3)} u_\rho.$$

Turning back to $x = y/\rho$, we get

$$\sup_{B(0,R/3)} u \le C \inf_{B(0,R/3)} u.$$

We conclude observing that the positive constant C, according to the coefficients of the differential inequalities satisfied by u_ρ, depends on n, $\frac{\Lambda}{\lambda}$, $\frac{\beta r/\rho}{\lambda} = \frac{\beta R}{\lambda}$ and $\frac{\gamma r^2/\rho^2}{\lambda} = \frac{\gamma R^2}{\lambda}$. This ends the proof. □

In the sequel we will make use of the following assumptions:

$$f \text{ positive continuous increasing in } (0,\infty), \tag{3}$$

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad \text{where } F(s) = \int_0^s f(t)dt. \tag{4}$$

The latter is usually called Keller–Osserman condition.

2.1. A boundary blow-up super-solution

Here we assume (3) and (4), and define (see Remark 2.3 of [16]) the non-increasing continuous function

$$\Psi(t) = \int_t^\infty \frac{ds}{\sqrt{F(s)-F(t)}}.$$

We have $\Psi(t) \to 0$ as $t \to \infty$, and the inverse Φ, the non-increasing continuous function such that

$$\int_{\Phi(r)}^\infty \frac{ds}{\sqrt{F(s)-F(\Phi(r))}} = r,$$

satisfies $\Phi(r) \to \infty$ as $r \to 0^+$.

For the convenience of the reader, we recall the following result from Lemma 2.5 (Keller, Osserman) of [16].

Lemma 2.2. *Assume (3) and (4). Then for all $\kappa > 0$, $z \in \mathbb{R}^n$ and $R > 0$, there exists a radial positive solution $w \in C^2(B(z,R))$, radially increasing and strictly convex, of the boundary blow-up problem*

$$\begin{cases} \Delta w = \kappa f(w) & \text{in } B(z,R) \\ w = \infty & \text{on } \partial B(z,R). \end{cases} \tag{5}$$

Moreover

$$\Phi(\sqrt{2\kappa}R) \le w(z) \le \Phi\left(\sqrt{\tfrac{2\kappa}{n}}R\right) \tag{6}$$

and

$$|Dw(x)| \le \frac{\kappa R}{n} f(w(x)). \tag{7}$$

The above lemma will provide boundary blow-up supersolutions when the Laplace operator is replaced by the fully nonlinear maximal operator occurring in the right-hand side of the structure condition (19), defined later on. To show this, we notice the following simple consequence of assumptions (3) and (4).

Lemma 2.3. *Assume (3) and (4). Then*

$$\lim_{t\to\infty}\frac{f(t)}{t}=\infty.$$

Proof. We refer to [13] (see the remark to Lemma 2.1 therein). □

Combining Lemma 2.2 and Lemma 2.3, we obtain the following existence result of boundary blow-up supersolutions.

Lemma 2.4. *Assume (3) and (4) and let $\alpha,\beta\geq 0$. Then there exists $R_0>0$ such that for each $z\in\mathbb{R}^n$ and $R\in(0,R_0)$ there is a positive supersolution $w\in C^2(B(z,R))$ of the boundary blow-up problem*

$$\begin{cases}\mathcal{M}^+_{\lambda,\Lambda}(D^2w)+\beta|Dw|+\alpha w=f(w) & \text{in } B(z,R)\\ w=\infty & \text{on } \partial B(z,R).\end{cases}\tag{8}$$

Moreover

$$\Phi\left(\frac{R}{\sqrt{2\Lambda}}\right)\leq w(z)\leq\Phi\left(\frac{R}{\sqrt{2\Lambda n}}\right).\tag{9}$$

Proof. The proof proceeds along the lines of Lemma 3.1 of [16]. R_0 will be chosen in the sequel.

We solve the blow-up problem (5) as in Lemma 2.2 with $\kappa=\frac{1}{4\Lambda}$, noting that the solution w is convex and hence $\mathcal{M}^+_{\lambda,\Lambda}(D^2w)=\Lambda\Delta w$. Taking also into account the gradient estimate (7), we get

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2w)+\beta|Dw|+\alpha w&=\Lambda\Delta w+\beta|Dw|+\alpha w\\ &\leq\left(\frac{1}{4}+\frac{\beta R}{4\Lambda n}+\frac{\alpha w}{f(w)}\right)f(w).\end{aligned}\tag{10}$$

By virtue of Lemma 2.3 there exists $t_*>0$ such that

$$\frac{\alpha t}{f(t)}\leq\frac{1}{2}\quad\text{as}\quad t\geq t_*.\tag{11}$$

Since $\Phi(r)\to\infty$ as $r\to 0^+$, we choose R_0 small enough to have $\Phi\left(\frac{R_0}{\sqrt{2\Lambda}}\right)\geq t_*$. Using the radial monotonicity of w and recalling (6) with $\kappa=1/(4\Lambda)$ we find

$$w(x)\geq w(z)\geq\Phi\left(\frac{R}{\sqrt{2\Lambda}}\right)\geq\Phi\left(\frac{R_0}{\sqrt{2\Lambda}}\right)\geq t_*\quad\text{in } B(z,R)$$

for $R < R_0$. Hence, by (11) we have

$$\frac{\alpha w}{f(w)} \le \frac{1}{2} \quad \forall x \in B(z, R).$$

Eventually passing to a smaller R_0, we can also make $\frac{\beta R_0}{\Lambda n} \le 1$ so that we can conclude from (10) that in $B(z, R)$ we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + \beta|Dw| + \alpha w \le f(w).$$

Finally, (9) follows from (6) with $\kappa = 1/(4\Lambda)$. □

2.2. Global estimate of solutions

Here we assume the following growth condition (stronger than (3))

$$f \text{ positive continuous and } t \to \frac{f(t)}{t} \text{ is non-decreasing in } (0, \infty). \tag{12}$$

Assumption (12) provides an estimate for non-negative subsolutions.

Theorem 2.5. *Assume (4) and (12), and let* $\alpha, \beta \ge 0$. *Then there exists a non-increasing function* $\eta : (0, \infty) \to (0, \infty)$ *such that, if u is any non-negative subsolution of the equation*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + \beta|Du| + \alpha u = f(u)$$

in a (bounded) domain Ω *of* $\mathbb{R}^n$*, we have*

$$u(z) \le \eta(d(z)) \quad \forall z \in \Omega, \tag{13}$$

where $d(z) = dist(z, \partial\Omega)$.

Proof. Let κ and R_0 be the positive constants of Lemma 2.4. Following the proof of Theorem 3.3 of [16], let $z \in \Omega$ and $0 < R < \min(d(z), R_0)$.

Comparing a subsolution u in Ω with the supersolution w of the boundary blow-up problem (8) (see Lemma 2.4), we will show later on that $u \le w$ in $B(z, R)$. Assuming this, since $w \ge t_*$ by the choice of R_0 (see the proof of Lemma 2.4), we may use the right hand side of (9) to find

$$u(z) \le w(z) \le \Phi\left(\frac{R}{\sqrt{2\Lambda n}}\right).$$

Finally, as $R \to \min(d(z), R_0)$ we obtain estimate (13) with

$$\eta(r) = \Phi\Big(\frac{\min(r, R_0)}{\sqrt{2\Lambda n}}\Big). \tag{14}$$

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We are left with showing that $u \le w$ in $B(z, R)$. To this end, suppose by contradiction $A_z = \{x \in B(z, R) : u(x) > w(x)\} \neq \emptyset$. Note that $A_z \subset\subset B(z, R)$, so $u = w$ on ∂A_z. Then by using (12) we find

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \beta|Du| + \left(\alpha - \frac{f(w(x))}{w(x)}\right)u \\ &\geq \left(\frac{f(u(x))}{u(x)} - \frac{f(w(x))}{w(x)}\right)u \geq 0, \qquad x \in A_z. \end{aligned} \tag{15}$$

On the other hand, by Lemma 2.4 we have

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2w) + \beta|Dw| + \left(\alpha - \frac{f(w(x))}{w(x)}\right)w \leq 0. \tag{16}$$

Observing again that $w \ge t_*$, from (11) we get $c(x) \equiv \alpha - \frac{f(w(x))}{w(x)} \le 0$ on A_z. Therefore, by (15) and (16) we obtain

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \mathcal{M}^+_{\lambda,\Lambda}(D^2w) + \beta(|Du| - |Dw|) \\ &\geq -c(x)(u-w) \geq 0 \quad \forall x \in A_z. \end{aligned}$$

Recalling that

$$\mathcal{M}^+_{\lambda,\Lambda}(X - Y) \geq \mathcal{M}^+_{\lambda,\Lambda}(X) - \mathcal{M}^+_{\lambda,\Lambda}(Y)$$

for all $X, Y \in S_n$ we conclude that the following holds in A_z:

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2(u-w)) + \beta|D(u-w)| \\ &\geq \mathcal{M}^+_{\lambda,\Lambda}(D^2u) - \mathcal{M}^+_{\lambda,\Lambda}(D^2w) + \beta(|Du| - |Dw|) \geq 0. \end{aligned}$$

This, together with the condition $u = w$ on ∂A_z allows us to invoke the maximum principle (see [1, Theorem 1.2] for instance) to conclude that $u = w$ in A_z. However, this is in contradiction with our earlier assumption that $A_z \neq \emptyset$. Therefore $u \le w$ in $B(z, R)$, as claimed. □

2.3. The Harnack inequality

We make the following assumption introduced by M. Dindoš in [10].

$$\exists\, \theta > 1 \ \text{ such that } \ \liminf_{t\to\infty} \frac{f(\theta t)}{\theta f(t)} > 1. \tag{17}$$

As noted in [16], (17) implies (4). We also recall a result proved in [16].

Lemma 2.6. *If f satisfies (3) and (17) then*

$$\lim_{r\to 0^+} \frac{r^2 f(\Phi(r))}{\Phi(r)} < \infty.$$

Proof. See Appendix A for $p = 2$ or Lemma 2.4 of [16]. □

As a consequence of Lemma 2.6 and Theorem 2.5 we have

Corollary 2.7. *If η is the function defined in (14) then*

$$\frac{r^2 f(\eta(r))}{\eta(r)} \le C, \quad 0 < r < R_1,$$

where C is a constant and R_1 is the radius of the largest ball contained in Ω.

We are now in a position to establish the Harnack inequality for the equation

$$H(x,u,Du,D^2u) = g(x,u), \quad x \in \Omega. \tag{18}$$

On $H(x,t,\xi,X)$ we assume the structural condition

$$\mathcal{M}^-_{\lambda,\Lambda}(X) - \beta|\xi| - \alpha t \le H(x,t,\xi,X) \le \mathcal{M}^+_{\lambda,\Lambda}(X) + \beta|\xi| + \alpha t, \tag{19}$$

where α and β are non-negative constants, $t \ge 0$, $\xi \in \mathbb{R}^n$ and $X \in S_n$.

On $g(x,t)$ we suppose that there exists a real number $T \ge 1$ such that for $x \in \Omega$ and $t \ge 0$ we have

$$f(t) \le g(x,t) \le Tf(t). \tag{20}$$

For example we may have $g(x,t) = t^q(2 + \sin(xt))$ with $q > 1$. Then one may take $f(t) = t^q$ and $T = 3$.

Theorem 2.8 *(Harnack inequality). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that H satisfies the structural condition (19). Suppose g satisfies condition (20) with f satisfying conditions (12) and (17). If u is a non-negative viscosity solution of the equation (18) then, if $z \in \Omega$ there is a positive constant C, independent of u and z, such that*

$$\sup_{B(z,d(z)/3)} u \le C \inf_{B(z,d(z)/3)} u.$$

Proof. Thanks to the conditions (19) and (20), the function u satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \beta|Du| + \alpha u \ge f(u)$$

as well as

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \beta|Du| - \alpha u \le Tf(u).$$

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The function $u_\varepsilon = u + \varepsilon$, with $\varepsilon > 0$, is in turn a positive solution of the differential inequalities

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_\varepsilon) + \beta|Du_\varepsilon| + \alpha u_\varepsilon \geq 0,$$
$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 u_\varepsilon) - \beta|Du_\varepsilon| - \left(\alpha + \frac{Tf(u(x))}{u_\varepsilon(x)}\right) u_\varepsilon \leq 0.$$

Now we wish to invoke Theorem 2.1 with u_ε instead of u and

$$c(x) = \frac{Tf(u(x))}{u_\varepsilon(x)}$$

to obtain

$$\sup_{B(z,d(z)/3)} (u+\varepsilon) \leq C \inf_{B(z,d(z)/3)} (u+\varepsilon), \tag{21}$$

with C depending only on

$$n,\ \frac{\Lambda}{\lambda},\ \frac{\beta}{\lambda} d(z),\ \frac{\alpha}{\lambda} d^2(z) \ \text{and}\ \frac{T}{\lambda} \max_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon(x)} d^2(z).$$

Since $d(z) \leq \operatorname{diam}(\Omega) < \infty$, to finish the proof it suffices to show that

$$\max_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon(x)} d^2(z)$$

is uniformly bounded, independently of ε and $z \in \Omega$. To this end, we shall use Theorem 2.5 and condition (12). We observe that condition (12) implies that the function $t \to f(t)/(t+\varepsilon)$ is increasing. Hence,

$$\frac{f(u(x))}{u_\varepsilon(x)} \leq \frac{f(\eta(d(x)))}{\eta(d(x)) + \varepsilon} \leq \frac{f(\eta(d(x)))}{\eta(d(x))} \quad \text{in } \Omega.$$

Following [16] we note that if $x \in B(z, 2d(z)/3)$ then $d(x) \geq d(z)/3$. Therefore, for any $x \in B(z, 2d(z)/3)$, since η is non-increasing, we find

$$\frac{f(u(x))}{u_\varepsilon(x)} \leq \frac{f(\eta(d(z)/3))}{\eta(d(z)/3)} \quad \text{in } \Omega.$$

Hence,

$$d^2(z) \max_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon(x)} \leq 9(d(z)/3)^2 \frac{f(\eta(d(z)/3))}{\eta(d(z)/3)}.$$

By Corollary 2.7 we conclude that

$$d^2(z) \max_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon(x)} \leq 9C$$

uniformly in ε and independently of $z \in \Omega$ and u.

Passing to the limit as ε tends to zero in (22) we get the desired result. □

3. On a class of quasi-linear elliptic equations

Consider the quasi-linear elliptic operator

$$\mathcal{Q}u := \Delta_p u + \mathcal{B}(x,u), \tag{22}$$

where $1 < p < \infty$ and $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$. We assume that $\mathcal{B}(x,t)$ satisfies the following structural condition:

$$\exists\, \mu \geq 0 \;\text{ such that }\; |\mathcal{B}(x,t)| \leq \mu t^{p-1} \quad \forall (x,t) \in \Omega \times \mathbb{R}^+. \tag{23}$$

Remark 3.1. The lack of satisfactory comparison principle for quasilinear operators $\mathcal{Q}$ including a gradient term is the main reason for taking $\mathcal{B}$ in (22) to be independent of the gradient term.

We say $u \in W^{1,p}_{loc}(\Omega)$ is a solution (resp., subsolution or supersolution) of $\mathcal{Q}u = g(x,u)$ if and only if $g(x,u(x)) \in L^1_{loc}(\Omega)$ and, for all $\tau \in C^1_0(\Omega)$, $\tau \geq 0$ we have

$$\int_\Omega |Du|^{p-2}Du \cdot D\tau - \int_\Omega \mathcal{B}(x,u)\tau = (\leq,\, \geq) - \int_\Omega g(x,u)\tau.$$

We shall indicate this by writing $\mathcal{Q}u = (\geq,\, \leq)\, g(x,u)$, respectively.

We will make use of the following Harnack inequality which follows from [18].

Theorem 3.2. *Let $1 < p < n$, and suppose $\mathcal{B}(x,t)$ satisfies the structure condition (23). Let $B(z,R) \subset \Omega$, and let $u \in W^{1,p}_{loc}(\Omega)$ be a non-negative weak solution of the inequalities*

$$\Delta_p u + \mu u^{p-1} \geq 0$$
$$\Delta_p u - (\mu + c(x))u^{p-1} \leq 0,$$

where μ is a non-negative constant and $c(x)$ is a non-negative continuous function. Then

$$\sup_{B(z,R/3)} u(x) \leq C \inf_{B(z,R/3)} u(x),$$

where C depends on n, p and $R^p\|\mu + c(x)\|_{L^\infty(B(z,2R/3))}$.

Proof. In [18, Theorem 7.2.1] the following inequality is proved:

$$\sup_{B(z,R/4)} u(x) \leq C \inf_{B(z,R/4)} u(x),$$

where C depends on n, p and $R^p\|\mu + c(x)\|_{L^\infty(B(z,R))}$. Replacing R with $2R/3$ we find

$$\sup_{B(z,R/6)} u(x) \leq C \inf_{B(z,R/6)} u(x),$$

with C depending on n, p and $R^p\|\mu + c(x)\|_{L^\infty(B(z,2R/3))}$. If we apply twice the latter inequality in a ball with radius $R/3$ then we get

$$\sup_{B(z,R/3)} u(x) \le C^2 \inf_{B(z,R/3)} u(x),$$

that is, the inequality stated in our theorem with C^2 in place of C. □

We wish to use the above theorem to derive a Harnack inequality for non-negative solutions of

$$\mathcal{Q}u = g(x,u), \tag{24}$$

where g satisfies (20) with f satisfying (3) as well as the following generalized Dindoš condition.

$$\exists\, \theta > 1 \text{ such that } \liminf_{t\to\infty} \frac{f(\theta t)}{\theta^{p-1} f(t)} > 1. \tag{25}$$

Remark 3.3. Condition (25) with $p = 2$ returns condition (17), which has already been used in Section 2 and goes back to Dindoš [10]. Also note that if f satisfies (25) then $f^{\frac{1}{p-1}}$ satisfies the standard Dindoš condition (17). Therefore, by Lemma 2.2 of [16] there are $\sigma > 0$, $t^* > 0$ and $\rho > 1$ such that

$$f^{\frac{1}{p-1}}(t) > \sigma t^{\rho} \quad \forall t > t^*.$$

It follows that the generalized Dindoš condition (25) implies the following generalized Keller–Osserman condition:

$$\int_1^\infty \frac{dt}{(F(t))^{1/p}} < \infty, \quad F(t) = \int_0^t f(s)\,ds, \quad \forall\, t > 0. \tag{26}$$

We introduce the following function

$$\Psi(t) = \int_t^\infty \frac{ds}{(q(F(s)-F(t)))^{1/p}}, \qquad t > 0, \qquad q = \frac{p}{p-1}.$$

It is well-known that Ψ is a continuous and decreasing function such that

$$\lim_{t\to\infty} \Psi(t) = 0.$$

Let Φ be the inverse of Ψ, that is

$$\int_{\Phi(t)}^\infty \frac{ds}{\big(q\big(F(s) - F(\Phi(t))\big)\big)^{1/p}} = t, \quad 0 < t < \Psi(0+).$$

We observe that $\Phi(0+) = \infty$.

Remark 3.4. If f satisfies (3) and (26) then

$$\lim_{t\to\infty}\frac{f(t)}{t^{p-1}}=\infty.$$

For a proof, see [13] (remark to Lemma 2.1 therein).

Suppose f satisfies (3) and (26). Given $R>0$, $\kappa>0$ and a ball $B:=B(z,R)\subset\mathbb{R}^n$, $n\geq 2$, it is well known that the following boundary blow-up problem

$$\begin{cases} \operatorname{div}(|Dw|^{p-2}Dw) = \kappa f(w) & \text{in } B \\ w = \infty & \text{on } \partial B, \end{cases}$$

admits a radial solution $w(x)=\varphi(|x-z|)$ that belongs to $C^1(B(z,R))$. Moreover φ satisfies the following, in the sense of distributions:

$$\begin{cases} \left(r^{n-1}|\varphi'(r)|^{p-2}\varphi'(r)\right)' = r^{n-1}\kappa f(\varphi(r)), \quad r\in(0,R), \\ \varphi(0)>0, \ \ \varphi'(0)=0, \ \ \varphi(R)=\infty. \end{cases}$$

Let us make note of the following observation (see [17, Theorem 2.1]):

$$\Phi\left(\kappa^{1/p}R\right)\leq\varphi(0)\leq\Phi\left((\kappa/n)^{1/p}R\right).$$

We summarize the above discussion as follows.

Lemma 3.5. *Let $1<p<\infty$, $R>0$, $\kappa>0$ and $n\geq 2$. Suppose f satisfies (3) and (26). Given $z\in\mathbb{R}^n$ there is a radially increasing function $w\in C^1(B(z,R))$ such that*

$$\begin{cases} \Delta_p w = \kappa f(w) & \text{in } B(z,R) \\ w=\infty & \text{on } \partial B(z,R). \end{cases} \tag{27}$$

Moreover we have

$$\Phi\left(\kappa^{1/p}R\right)\leq w(z)\leq\Phi\left((\kappa/n)^{1/p}R\right).$$

Furthermore, since w is radially increasing we have

$$w(x)\geq\Phi\left(\kappa^{1/p}R\right)\quad\forall x\in B(z,R).$$

Lemma 3.6. *Let $B(z,R)\subset\Omega$ and let $\mu\geq 0$. Suppose f satisfies (3) and (26). Then there is $R_0>0$ sufficiently small such that for each $0<R<R_0$ the problem*

$$\begin{cases} \Delta_p w+\mu w^{p-1} = f(w) & \text{in } B(z,R) \\ w=\infty & \text{on } \partial B(z,R) \end{cases} \tag{28}$$

admits a supersolution $w\in C^1(B(z,R))$. Moreover,

$$\Phi\big((1/2)^{\frac{1}{p}} R\big) \le w(z) \le \Phi\big((1/(2n))^{\frac{1}{p}} R\big).$$

Proof. By Remark 3.4, we fix $t_0 > 0$ such that

$$f(t) \ge 2\mu t^{p-1} \quad \forall\, t \ge t_0.$$

Since $\Phi(0+) = \infty$, choose $R_0 > 0$ such that

$$\Phi\left(2^{-1/p} R_0\right) \ge t_0.$$

Given $0 < R < R_0$ and $z \in \Omega$ such that $B(z, R) \subset \Omega$, let $w \in C^1(B(z, R))$ be the radial solution of (27) given in Lemma 3.5 with $\kappa = 1/2$. Let us recall that

$$w(x) \ge \Phi(2^{-1/p} R) \ge t_0 \quad \forall\, x \in B(z, R).$$

Then for $0 < R < R_0$ we note that

$$-\mu w^{p-1} + f(w) = \frac{1}{2} f(w) + \left(\frac{f(w)}{2w^{p-1}} - \mu\right) w^{p-1} \ge \frac{1}{2} f(w).$$

It follows that, if $\tau \in C_0^1(B(z, R))$ with $\tau \ge 0$ then

$$\begin{aligned}&\int\limits_{B(z,R)} |Dw|^{p-2} Dw \cdot D\tau - \int\limits_{B(z,R)} \mu w^{p-1}\tau + \int\limits_{B(z,R)} f(w)\tau \\ &\ge \int\limits_{B(z,R)} |Dw|^{p-2} Dw \cdot D\tau + \frac{1}{2} \int\limits_{B(z,R)} f(w)\tau.\end{aligned}$$

Since $\Delta_p w = \frac{1}{2} f(w)$ (in the sense of distributions) in $B(z, R)$ we conclude that w is a supersolution of (28) in $B(z, R)$, as claimed. □

To proceed further, we need the following additional condition on f:

$$f \text{ positive continuous and } t \to \frac{f(t)}{t^{p-1}} \text{ is non-decreasing in } (0, \infty). \tag{29}$$

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $B(z, R) \subset \Omega$. Suppose $\mathcal{B}(x, t)$ satisfies the structure condition (23) and that f satisfies conditions (26) and (29). Then there is a non-increasing function $\eta : (0, \infty) \to (0, \infty)$ such that for any non-negative subsolution u of $\mathcal{Q}u = f(u)$ we have*

$$u(x) \le \eta(d(x)) \qquad \textit{for a.e } x \in \Omega.$$

Proof. Since $u \geq 0$, using the structural condition (23) on $\mathcal{B}$ we have

$$\Delta_p u + \mu u^{p-1} \geq \mathcal{Q}u \geq f(u),$$

that is

$$\Delta_p u + \left(\mu - \frac{f(u)}{u^{p-1}}\right) u^{p-1} \geq 0. \tag{30}$$

We recall from Lemma 3.6 above that

$$\Delta_p w + \left(\mu - \frac{f(w)}{w^{p-1}}\right) w^{p-1} \leq 0. \tag{31}$$

Let $z \in \Omega$, and take $0 < R < \min(d(z), R_0)$ where R_0 is as in Lemma 3.6. By the proof of Lemma 3.6 we note that

$$f(t) \geq 2\mu t^{p-1} \quad \forall\, t \geq t_0$$

and that $w \geq t_0$ in $B(z, R)$. In particular we have

$$\mu - \frac{f(w)}{w^{p-1}} \leq 0 \;\text{ in }\; B(z, R).$$

Let us show that $u \leq w$ in $B(z, R)$. Suppose, by way of contradiction, the open set $A_z := \{x \in B(z, R) : u(x) > w(x)\}$ is non-empty. Note that $A_z \subset\subset \Omega$ and so $u = w$ on the boundary ∂A_z. Then from (30) and condition (29) we see that

$$\Delta_p u + \left(\mu - \frac{f(w)}{w^{p-1}}\right) u^{p-1} \geq 0 \quad \text{in } B(z, R). \tag{32}$$

Hence, by (31), (32) and the comparison principle (see [9, Theorem 2.2] for instance) we get $u \leq w$ in A_z, which is an obvious contradiction. Therefore $u \leq w$ in $B(z, R)$ as claimed. In particular

$$u(z) \leq w(z) \leq \Phi\left((2n)^{-1/p} R\right).$$

Letting $R \to \min(d(z), R_0)$ we find that

$$u(z) \leq \eta(d(z)), \quad \text{where} \quad \eta(t) := \Phi\left((2n)^{-1/p} \min(t, R_0)\right). \quad \square$$

Theorem 3.8 *(Harnack inequality). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $\mathcal{B}(x, t)$ satisfies the structure condition (23). Suppose g satisfies condition (20) with f satisfying conditions (25) and (29). If $1 < p < n$ there is a positive constant C, independent of $z \in \Omega$ and any non-negative solution $u \in W^{1,p}_{loc}(\Omega)$ of equation (24) in Ω, such that*

$$\sup_{B(z,d(z)/3)} u \leq C \inf_{B(z,d(z)/3)} u.$$

Proof. By (20) and (23), from (24) we have

$$\begin{cases} \Delta_p u + \mu u^{p-1} \geq f(u), \\ \Delta_p u - \mu u^{p-1} \leq T f(u). \end{cases}$$

With $u_\epsilon = u + \epsilon$ we find

$$\begin{cases} \Delta_p u_\epsilon + \mu u_\epsilon^{p-1} \geq 0, \\ \Delta_p u_\epsilon - \left(\mu + \frac{Tf(u)}{u_\epsilon^{p-1}}\right) u_\epsilon^{p-1} \leq 0. \end{cases}$$

Now we wish to invoke Theorem 3.2 with u_ε instead of u and $R = d(z)$ to obtain

$$\sup_{B(z,d(z)/3)} (u + \varepsilon) \leq C \inf_{B(z,d(z)/3)} (u + \varepsilon), \tag{33}$$

with C depending only on n, p and

$$d^p(z) \left\| \mu + \frac{Tf(u)}{u_\epsilon^{p-1}} \right\|_{L^\infty(B(z,2d(z)/3))}.$$

We have to show that

$$d^p(z) \sup_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon^{p-1}(x)}$$

is bounded in Ω uniformly with respect to ε and $z \in \Omega$. The proof is similar to the fully nonlinear equation case. Using condition (29) and Theorem 3.7 we find

$$\frac{f(u(x))}{u_\varepsilon^{p-1}(x)} \leq \frac{f(\eta(d(x)))}{(\eta(d(x)) + \varepsilon)^{p-1}} \leq \frac{f(\eta(d(x)))}{\eta^{p-1}(d(x))} \quad \text{in } \Omega.$$

On noting that if $x \in B(z, 2d(z)/3)$ then $d(x) \geq d(z)/3$, for any $x \in B(z, 2d(z)/3)$ (since η is non-increasing), we find

$$\frac{f(u(x))}{u_\varepsilon^{p-1}(x)} \leq \frac{f(\eta(d(z)/3))}{\eta^{p-1}(d(z)/3)} \quad \text{in } \Omega.$$

Hence,

$$d^p(z) \sup_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon^{p-1}(x)} \leq 3^p (d(z)/3)^p \frac{f(\eta(d(z)/3))}{\eta^{p-1}(d(z)/3)}.$$

Recalling that

$$\eta(t) := \Phi\left((2n)^{-1/p} \min(t, R_0)\right),$$

by Lemma A.1 (see Appendix A below) we conclude that

$$d^p(z) \sup_{B(z,2d(z)/3)} \frac{f(u(x))}{u_\varepsilon(x)} \le 3^p C$$

uniformly with respect to ε and $z \in \Omega$.

Passing to the limit as ε tends to zero in (33) we get the desired result. □

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Appendix A

Lemma A.1. *If f satisfies (3) and (25) then*

$$\limsup_{t\to 0^+} \frac{t^p f(\Phi(t))}{(\Phi(t))^{p-1}} < \infty.$$

Proof. By (25) we fix ϱ such that

$$1 < \varrho < \liminf_{t\to\infty} \frac{f(\theta t)}{\theta^{p-1} f(t)}.$$

There is M_ϱ such that

$$f(\theta t) \ge (\varrho\theta^{p-1}) f(t) \quad \forall t \ge M_\varrho.$$

Iterating this, for any positive integer k we obtain

$$f(\theta^k t) \ge (\varrho\theta^{p-1})^k f(t) \qquad \forall t \ge M_\varrho.$$

Since $F(2t) \ge 2F(t)$ we note that for $s \ge 2t$ we have

$$(F(s) - F(t))^{1/p} = \left[F(s)\left(1 - \frac{F(t)}{F(s)}\right)\right]^{1/p} \ge \left(\frac{F(s)}{2}\right)^{1/p}.$$

Therefore

$$\Psi(t) = \int_t^{2t} \frac{ds}{(q(F(s) - F(t)))^{1/p}} + \int_{2t}^{\infty} \frac{ds}{(q(F(s) - F(t)))^{1/p}}$$

$$\leq \frac{1}{q^{1/p}} \int_t^{2t} \frac{ds}{(f(t)(s-t))^{1/p}} + \left(\frac{2}{q}\right)^{1/p} \int_{2t}^{\infty} \frac{ds}{(F(s))^{1/p}}$$

$$\leq q^{\frac{1}{q}} \left(\frac{t^{p-1}}{f(t)}\right)^{1/p} + \left(\frac{2}{q}\right)^{1/p} \int_{2t}^{\infty} \frac{ds}{(F(s))^{1/p}}.$$

Now we observe that

$$\int_{2t}^{\infty} \frac{ds}{(F(s))^{1/p}} = 2\int_t^{\infty} \frac{ds}{(F(2s))^{1/p}} \leq 2\int_t^{\infty} \frac{ds}{(sf(s))^{1/p}}$$

$$= 2\sum_{k=0}^{\infty} \int_{\theta^k t}^{\theta^{k+1}t} \frac{ds}{(sf(s))^{1/p}}$$

$$= 2\sum_{k=0}^{\infty} \int_{\theta^k t}^{\theta^{k+1}t} \frac{ds}{(s^p(f(s)/s^{p-1}))^{1/p}}.$$

For $t \geq M_\varrho$ and $\theta^k t \leq s \leq \theta^{k+1} t$ we find that

$$\frac{f(s)}{s^{p-1}} \geq \frac{f(\theta^k t)}{(\theta^{k+1}t)^{p-1}} \geq \frac{(\varrho\theta^{p-1})^k f(t)}{(\theta^{k+1}t)^{p-1}} = \frac{\varrho^k f(t)}{(\theta t)^{p-1}}.$$

Therefore we have

$$\int_{\theta^k t}^{\theta^{k+1}t} \frac{ds}{(s^p(f(s)/s^{p-1}))^{1/p}} \leq \left(\frac{\theta^{p-1}t^{p-1}}{\varrho^k f(t)}\right)^{1/p} \int_{\theta^k t}^{\theta^{k+1}t} \frac{ds}{s}$$

$$= \theta^{(p-1)/p} \ln(\theta) \left(\frac{1}{\varrho^{1/p}}\right)^k \left(\frac{t^{p-1}}{f(t)}\right)^{1/p}.$$

Thus we find

$$\Psi(t) \leq C\left(\frac{t^{p-1}}{f(t)}\right)^{1/p},$$

where $C := C(\theta, p, \varrho)$ is a positive constant. The stated inequality follows from this. □

References

[1] M.E. Amendola, L. Rossi, A. Vitolo, Harnack inequalities and ABP estimates for nonlinear second order elliptic equations in unbounded domains, Abstr. Appl. Anal. (2008), Art. ID 178534, 19 pp.

[2] C. Bandle, M. Marcus, Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (2) (1995) 155–171.

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[3] L.A. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, AMS Colloquium Publications, Providence, Rhode Island, 1995.
[4] I. Capuzzo Dolcetta, F. Leoni, A. Vitolo, Entire subsolutions of fully nonlinear degenerate elliptic equations, Bull. Inst. Math. Acad. Sin. (N.S.) 9 (2) (2014) 147–161.
[5] I. Capuzzo Dolcetta, F. Leoni, A. Vitolo, On the inequality $F(x, D^2u) \geq f(u) + g(u)|Du|^q$, Math. Ann. 365 (1–2) (2016) 423–448.
[6] I. Capuzzo Dolcetta, A. Vitolo, A qualitative Phragmèn–Lindelöf theorem for fully nonlinear elliptic equations, J. Differential Equations 243 (2007) 578–592.
[7] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second-order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[8] G. Díaz, A note on the Liouville method applied to elliptic eventually degenerate fully nonlinear equations governed by the Pucci operators and the Keller–Osserman condition, Math. Ann. 353 (1) (2012) 145–159.
[9] G. Díaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, Nonlinear Anal. 20 (1993) 97–125.
[10] M. Dindoš, Large solutions for Yamabe and similar problems on domains in Riemannian manifolds, Trans. Amer. Math. Soc. 363 (10) (2011) 5131–5178.
[11] D.L. Finn, R.C. McOwen, Singularities and asymptotics for the equation $\Delta_g u - u^q = Su$, Indiana Univ. Math. J. 42 (4) (1993) 1487–1523.
[12] J. García-Melián, Boundary behavior of large solutions to elliptic equations with singular weights, Nonlinear Anal. 67 (2007) 530–536.
[13] F. Gladiali, G. Porru, Estimates for explosive solutions to p-Laplace equations, in: Progress in Partial Differential Equations, Pont–Mousson, 1997, vol. 1, in: Pitman Res. Notes Math. Ser., vol. 383, Longman, Harlow, 1998, pp. 117–127.
[14] V. Julin, Generalized Harnack inequality for nonhomogeneous elliptic equations, Arch. Ration. Mech. Anal. 216 (2) (2015) 673–702.
[15] M. Marcus, L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equations, J. Evol. Equ. 3 (2004) 637–652.
[16] A. Mohammed, G. Porru, Harnack inequality for non-divergence structure semilinear elliptic equations, Adv. Nonlinear Anal., http://doi.org/10.1515/anona-2016-0050.
[17] A. Olofsson, A priori estimates of Osserman–Keller type, Differential Integral Equations 16 (6) (2003) 737–756.
[18] P. Pucci, J. Serrin, Maximum principles for elliptic partial differential equations, in: Handbook of Differential Equations: Stationary Partial Differential Equations, in: Handb. Differ. Equ., vol. IV, Elsevier/North-Holland, Amsterdam, 2007, pp. 355–483.
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