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Sharp threshold of blow-up and scattering for the fractional Hartree equation

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Abstract

We consider the fractional Hartree equation in the L^2-supercritical case, and find a sharp threshold of the scattering versus blow-up dichotomy for radial data: If $M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < M[Q]^{\frac{s-s_c}{s_c}} E[Q]$ and $M[u_0]^{\frac{s-s_c}{s_c}} \|u_0\|^2_{\dot{H}^s} < M[Q]^{\frac{s-s_c}{s_c}} \|Q\|^2_{\dot{H}^s}$, then the solution $u(t)$ is globally well-posed and scatters; if $M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < M[Q]^{\frac{s-s_c}{s_c}} E[Q]$ and $M[u_0]^{\frac{s-s_c}{s_c}} \|u_0\|^2_{\dot{H}^s} > M[Q]^{\frac{s-s_c}{s_c}} \|Q\|^2_{\dot{H}^s}$, the solution $u(t)$ blows up in finite time. This condition is sharp in the sense that the solitary wave solution $e^{it}Q(x)$ is global but not scattering, which satisfies the equality in the above conditions. Here, Q is the ground-state solution for the fractional Hartree equation.

MSC: 35Q40; 35Q55; 47J30

Keywords: Fractional Schrödinger equation; L^2-supercritical; Scattering; Blow-up

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1. Introduction

In this paper, we study the fractional Hartree equation, which is the L^2-supercritical, nonlinear, fractional Schrödinger equation.

$$iu_t - (-\triangle)^s u + (\frac{1}{|x|^\gamma} * |u|^2)u = 0, \tag{1.1}$$

with $0 < s < 1$ and $2s < \gamma < \min\{N, 4s\}$, where i is the imaginary unit and $u = u(t,x)$: $\mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. The operator $(-\triangle)^s$ is defined by

$$(-\triangle)^s u = \frac{1}{(2\pi)^{\frac{N}{2}}} \int e^{ix\cdot\xi} |\xi|^{2s} \widehat{u}(\xi) d\xi = \mathcal{F}^{-1}[|\xi|^{2s} \mathcal{F}[u](\xi)],$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the Fourier inverse transform in $\mathbb{R}^N$, respectively. The fractional Schrödinger equations were first proposed by Laskin in [28,29] using the theory of functionals over functional measures generated from the Lévy stochastic process and by expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. Here, s is the Lévy index. If $s = \frac{1}{2}$ and $\gamma = 1$, then (1.1) models the dynamics of (pseudo-relativistic) boson stars, where $\frac{1}{|x|}$ is the Newtonian gravitational potential in the appropriate physical units, which is also called the pseudo-relativistic Hartree equation (see [10,30]). The global existence and blow-up have been widely studied in [13,31]. For the classical Hartree equation, a large amount of work has been devoted to the theory of scattering and blow-up, see for example [34–37].

Eq. (1.1) is the L^2-supercritical, nonlinear, fractional Schrödinger equation. Indeed, we remark on the scaling invariance of Eq. (1.1). If $u(t,x)$ is a solution of Eq. (1.1), then $u^\lambda(t,x) = \lambda^{\frac{N-\gamma+2s}{2}} u(\lambda^{2s}t, \lambda x)$ is also a solution of Eq. (1.1). This implies that

(1) $\|u^\lambda\|_{L^{p_c}} = \|u\|_{L^{p_c}}$, where $p_c = \frac{2N}{N-\gamma+2s}$. When $\gamma > 2s$, we see that $p_c > 2$, and Eq. (1.1) is called the L^2-supercritical, nonlinear, fractional Schrödinger equation.
(2) $\dot{H}^{s_c}$-norm is invariant for Eq. (1.1), i.e., $\|u^\lambda\|_{\dot{H}^{s_c}} = \|u\|_{\dot{H}^{s_c}}$, where $s_c = \frac{\gamma-2s}{2}$.

Now, we impose the initial data,

$$u(0,x) = u_0 \in H^s, \tag{1.2}$$

onto (1.1) and consider the Cauchy problem (1.1)–(1.2). Cho et al. in [7,8] established the local well-posedness in H^s as follows: Let $N \geq 2$, $\frac{1}{2} \leq s < 1$ and $0 < \gamma < \min\{N, 4s\}$. If the initial data $u_0 \in H^s$, then there exists a unique solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) on the maximal time interval $I = [0,T)$ such that $u(t,x) \in C(I; H^s) \bigcap C^1(I; H^{-s})$ and either $T = +\infty$ (global existence) or both $0 < T < +\infty$ and $\lim_{t\to T} \|u(t,x)\|_{H^s} = +\infty$ (blow-up). Moreover, for all $t \in I$, $u(t,x)$ satisfies the following conservation laws.

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(i) Conservation of energy:

$$E[u(t)] = \frac{1}{2}\int\limits_{\mathbb{R}^N} \overline{u}(-\Delta)^s u dx - \frac{1}{4}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dxdy = E[u_0]. \tag{1.3}$$

(ii) Conservation of mass:

$$M[u(t)] = \int\limits_{\mathbb{R}^N} |u(t,x)|^2 dx = M[u_0]. \tag{1.4}$$

Now, even less is known about the global well-posedness and scattering results. To the authors' knowledge, Cho et al. in [8] gave some small data results. First, they addressed the energy-supercritical case, i.e., $4s \le \gamma < N$, and set some $\alpha > \frac{\gamma-2s}{2}$. Assume that the initial data $\|u_0\|_{H^\alpha}$ are sufficiently small; then, there exists a unique solution $u \in C_b([0,\infty); H^\alpha) \cap L^2(0,\infty; H^{\alpha+s-1}_{\frac{2N}{N-2}})$, where $H^\alpha_q = (-\Delta)^{-\frac{\alpha}{2}} L^q$. Moreover, there is $\phi^+ \in H^\alpha$ such that

$$\|u(t) - e^{-i(-\Delta)^s}\phi^+\|_{H^\alpha} \to 0, \quad \text{as } t \to +\infty.$$

Moreover, for the energy-subcritical case and for sufficiently small radial data $u_0 \in H^s_r$ (the radial functions in H^s), they presented some global well-posedness results: for $\frac{N}{2N-1} \le s < 1$, $2s < \gamma < \min\{4s, N\}$, there exists a unique solution

$$u(t,x) \in C_b([0,\infty); H^s_{rad}) \cap L^{\frac{6s}{\gamma-2s}}_{loc}(0,\infty; H^s_{\frac{2N}{N-\frac{2\gamma-4s}{3}}}).$$

However, they did not consider the scattering results in this case. On the other hand, as a typical dispersive wave equation, under certain conditions, the solution of the nonlinear fractional Schrödinger equation (1.1) may blow-up in finite time. In light of the above phenomena, a natural question would be how small of initial data will induce the global existence of the solution. Furthermore, does this global solution scatter at either side of time?

Motivated by this problem, we study the scattering versus blow-up dichotomy of the solutions for the focusing L^2-supercritical, nonlinear, fractional Schrödinger equation (1.1). Similar to studies on the classical semi-linear Schrödinger equation (see [5,33,38]), we attempt to use the variational method to find a sharp threshold of blow-up and global existence of the solutions to (1.1). The first topic is the ground-state solution of the equation

$$(-\Delta)^s Q + Q - (\frac{1}{|x|^\gamma} * |Q|^2)Q = 0, \qquad Q \in H^s(\mathbb{R}^N). \tag{1.5}$$

The existence of a non-trivial solution of Eq. (1.5) has been studied in [19,40], and the stability of related standing waves has been obtained in [9,14,39]. In [40], the second author of this paper obtained a sharp Gagliardo–Nirenberg inequality, which reveals the variational characteristic of the ground-state solutions for Eq. (1.5): Let $N \ge 2$, $0 < s < 1$ and $0 < \gamma < \min\{N, 4s\}$. Then, for all $v \in H^s$,

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$$\int\int \frac{|v(x)|^2|v(y)|^2}{|x-y|^\gamma}dxdy \leq C_{GN}\,\|v\|_2^{\frac{4s-\gamma}{s}}\,\|v\|_{\dot{H}^s}^{\frac{\gamma}{s}}, \tag{1.6}$$

where Q is a solution of (1.5),

$$C_{GN} = \frac{4s}{\gamma}\frac{1}{\|Q\|_2^{\frac{4s-\gamma}{s}}\,\|Q\|_{\dot{H}^s}^{\frac{\gamma-2s}{s}}} = \left(\frac{4s-\gamma}{\gamma}\right)^{\frac{\gamma}{2s}}\frac{4s}{(4s-\gamma)\|Q\|_2^2}. \tag{1.7}$$

Given the fractional operator $(-\triangle)^s$, the classical Virial identity argument fails, and the existence of blow-up solutions for (1.1) presents a particular difficulty. The numerical observations of blow-up solutions have been studied in [1,2], when $s=\frac{1}{2}$, $\gamma=1$. The theoretical proof of the existence of the blow-up solutions of (1.1) has been presented by Cho et al. in [7]. They proved that if $\gamma=2s\geq 1$ and the initial energy is negative, then the life span $[0,T)$ of the corresponding solutions must be finite (i.e., $T<+\infty$). In [40], by establishing some new estimates, Zhu proved the existence of a finite-time blow-up solution for (1.1) with $\gamma=2s$ and the dynamics of blow-up solutions. We note that the sharp threshold of blow-up and global existence for (1.1) with $\gamma>2s$ remains unknown.

In the present paper, we first construct two invariant flows by injecting the sharp Gagliardo–Nirenberg inequality proposed by Zhu in [40], which strongly depend on the scaling index $s_c=\frac{\gamma-2s}{2}$ and conservation laws. Then, we obtain the sharp criteria of blow-up and scattering for the L^2-supercritical, nonlinear, fractional Schrödinger Eq. (1.1) in terms of the arguments in [15,20,21,26]. The main theorem is as follows.

Theorem 1.1. *Let $N\geq 2$ and $2s<\gamma<\min\{N,4s\}$. Assume that $u_0\in H^s$ is radial and $M[u_0]^{\frac{s-s_c}{s_c}}E[u_0]<M[Q]^{\frac{s-s_c}{s_c}}E[Q]$, where Q is the ground-state solution of (1.5).*

(i) *If $\frac{N}{2N-1}\leq s<1$ and*

$$M[u_0]^{\frac{s-s_c}{s_c}}\|u_0\|_{\dot{H}^s}^2 < M[Q]^{\frac{s-s_c}{s_c}}\|Q\|_{\dot{H}^s}^2,$$

then the corresponding solution $u(t)$ of (1.1)–(1.2) exists globally in H^s. Moreover, $u(t)$ scatters in H^s. Specifically, there exists $\phi_\pm\in H^s$ such that $\lim\limits_{t\to\pm\infty}\|u(t)-e^{-it(-\Delta)^s}\phi_\pm\|_{H^s}=0$.

(ii) *Further, if the initial data $u_0\in H^{s_0}$ with $s_0=\max\{2s,\frac{\gamma+1}{2}\}$ and*

$$M[u_0]^{\frac{s-s_c}{s_c}}\|u_0\|_{\dot{H}^s}^2 > M[Q]^{\frac{s-s_c}{s_c}}\|Q\|_{\dot{H}^s}^2$$

satisfies $|x|u_0\in L^2$ and $x\cdot\nabla u_0\in L^2$, then the solution $u(t)$ of (1.1)–(1.2) must blow up in finite time $0<T<+\infty$.

This paper is organized as follows. In Section 2, using the Strichartz estimates, we establish the small data theory and the long-time perturbation theory. We review properties of the ground state Q in Section 3 in connection with the sharp Gagliardo–Nirenberg estimate. We can construct the invariant flows generated by the Cauchy problem of (1.1) and (1.2) and prove Theorem 1.1 for the blow-up part (ii). In Section 4, we introduce the local virial identity and prove

Theorem 1.1, except for the scattering claim in part (i). By assuming that the threshold for scattering is strictly below the threshold claimed, we construct a "critical element", u_c, that stands exactly at the boundary between scattering and non-scattering. This is done through a profile decomposition lemma in H^s. We then show that time slices of $u_c(t)$, as a collection of functions in H^s, form a precompact set in H^s (and thus, u_c has something in common with the soliton $Q(x)$). This enables us to prove that u_c remains localized uniformly in time. In Section 5, by using the localization in Section 4, we deduce a contradiction with the conservation of mass at large times.

We conclude this section by introducing some notations. $L^q := L^q(\mathbb{R}^N)$, $\|\cdot\|_q := \|\cdot\|_{L^q(\mathbb{R}^N)}$, the time-space mixed norm $\|u\|_{L^qX} := \left(\int_{\mathbb{R}} \|u(t,\cdot)\|_X^q\right)^{\frac{1}{q}}$, $H^s := H^s(\mathbb{R}^N)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^N)$, and $\int \cdot dx := \int_{\mathbb{R}^N} \cdot dx$. $\mathcal{F}v = \widehat{v}$ denotes the Fourier transform of v, which for $v \in L^1(\mathbb{R}^N)$ is given by $\mathcal{F}v = \widehat{v}(\xi) := \int e^{-ix\cdot\xi} v(x)dx$ for all $\xi \in \mathbb{R}^N$, and $\mathcal{F}^{-1}v$ is the inverse Fourier transform of $v(\xi)$. $\Re z$ and $\Im z$ are the real and imaginary parts of the complex number z, respectively. $\overline{z}$ denotes the complex conjugate of the complex number z. The various positive constants will be denoted by C or c.

2. Local theory and Strichartz estimate

In this paper, we study the Cauchy problem (1.1)–(1.2) in the form of the following integral equation:

$$u(t) = U(t)u_0 + i\int_0^t U(t-t^1)\left(\frac{1}{|x|^\gamma} * |u|^2\right)u(t^1)dt^1$$

where

$$U(t)\phi(x) = e^{-i(-\Delta)^s t}\phi(x) = \frac{1}{(2\pi)^{\frac{N}{2}}}\int e^{i(x\cdot\xi - |\xi|^{2s})}\widehat{\phi}(\xi)d\xi.$$

In this section, we first recall the local theory for Eq. (1.1) by the radial Strichartz estimate (see [17,25]).

Definition 2.1. For the given $\theta \in [0, s)$, we state that the pair (q, r) is θ-level admissible, denoted by $(q, r) \in \Lambda_\theta$, if

$$q, r \geq 2, \quad \frac{2s}{q} + \frac{N}{r} = \frac{N}{2} - \theta \tag{2.1}$$

and

$$\frac{4N+2}{2N-1} \leq q \leq \infty, \quad \frac{1}{q} \leq \frac{2N-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{or } 2 \leq q < \frac{4N+2}{2N-1}, \quad \frac{1}{q} < \frac{2N-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right). \tag{2.2}$$

Correspondingly, we denote the dual θ-level admissible pair by $(q', r') \in \Lambda'_\theta$ if $(q, r) \in \Lambda_{-\theta}$ with (q', r') is the Hölder dual to (q, r).

Proposition 2.2 *(See [17]). Assume that $N \geq 2$ and that ϕ, f are radial; then for $q_j, r_j \geq 2$, $j = 1, 2$,*

$$\|U(t)\phi\|_{L^{q_1}L^{r_1}} \leq C\|D^{\theta}\phi\|_2, \tag{2.3}$$

where $D^{\theta} = (-\triangle)^{\frac{\theta}{2}}$,

$$\|\int_0^t U(t-t^1)f(t^1)dt^1\|_{L^{q_1}L^{r_1}} \leq C\|f\|_{L^{q_2'}L^{r_2'}}, \tag{2.4}$$

in which $\theta \in \mathbb{R}$, the pairs (q_j, r_j) satisfy the range conditions (2.2) and the gap condition

$$\frac{2s}{q_1} + \frac{N}{r_1} = \frac{N}{2} - \theta, \quad \frac{2s}{q_2} + \frac{N}{r_2} = \frac{N}{2} + \theta.$$

Definition 2.3. We define the following Strichartz norm

$$\|u\|_{S(\Lambda_{s_c})} = \sup_{(q,r)\in\Lambda_{s_c}} \|u\|_{L^qL^r}.$$

Let (q', r') be the Hölder dual to (q, r), and define the dual Strichartz norm

$$\|u\|_{S'(\Lambda_{-s_c})} = \inf_{(q',r')\in\Lambda'_{s_c}} \|u\|_{L^{q'}L^{r'}} = \inf_{(q,r)\in\Lambda_{-s_c}} \|u\|_{L^{q'}L^{r'}}.$$

Remark 2.4. Notice that if

$$s \in [\frac{N}{2N-1}, 1) \subset (\frac{1}{2}, 1),$$

the gap condition (2.1) with $\theta = 0$ right implies the range condition (2.2), which further means that Λ_0 is nonempty. That is we have a full set of 0-level admissible Strichartz estimates without loss of derivatives in radial case. Moreover, denoting

$$q_c = r_c = \frac{2N+4s}{N+2s-\gamma}, \tag{2.5}$$

we check that $(q_c, r_c) \in \Lambda_{s_c} \neq \emptyset$ is an s_c-level admissible pair.

By Proposition 2.2, for u_0, $f = (\frac{1}{|x|^{\gamma}} * |u|^2)u$ radial, we then have that

$$\|U(t)u_0\|_{S(\Lambda_0)} \leq C\|u_0\|_2$$

and

$$\left\|\int_0^t U(t-t^1)(\frac{1}{|x|^{\gamma}} * |u|^2)u(t^1)dt^1\right\|_{S(\Lambda_0)} \leq C\|(\frac{1}{|x|^{\gamma}} * |u|^2)u\|_{S'(\Lambda_0)}.$$

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Together with Sobolev embedding, we obtain

$$\|U(t)\phi\|_{S(\Lambda_{s_c})} \le c\|\phi\|_{\dot{H}^{s_c}},$$

$$\left\| \int_0^t U(t-t^1)(\frac{1}{|x|^\gamma} * |u|^2)u(t^1)dt^1 \right\|_{S(\Lambda_{s_c})} \le C\|D^{s_c}((\frac{1}{|x|^\gamma} * |u|^2)u)\|_{S'(\Lambda_0)}$$

and

$$\left\| \int_0^t U(t-t^1)(\frac{1}{|x|^\gamma} * |u|^2)u(t^1)dt^1 \right\|_{S(\Lambda_{s_c})} \le C\|(\frac{1}{|x|^\gamma} * |u|^2)u\|_{S'(\Lambda_{-s_c})}.$$

Next, we write $S(\Lambda_\theta; I)$ to indicate its restriction to a time subinterval $I \subset (-\infty, +\infty)$.

Proposition 2.5 *(Small data). Let $\|u_0\|_{\dot{H}^{s_c}} \le A$ be radial. Then, there exists $\delta_{sd} = \delta_{sd}(A) > 0$ such that if $\|U(t)u_0\|_{S(\Lambda_{s_c})} \le \delta_{sd}$, then $u = u(t)$ solving (1.1) is global, and*

$$\|u\|_{S(\Lambda_{s_c})} \le 2\|U(t)u_0\|_{S(\Lambda_{s_c})}, \tag{2.6}$$

$$\|D^{s_c}u\|_{S(\Lambda_0)} \le 2c\|u_0\|_{\dot{H}^{s_c}}. \tag{2.7}$$

(Note that by the Strichartz estimates, the hypotheses are satisfied if $\|u_0\|_{\dot{H}^{s_c}} \le C\delta_{sd}$.)

Proof. Set

$$\Phi_{u_0}(v) = U(t)u_0 + i\int_0^t U(t-t^1)(\frac{1}{|\cdot|^\gamma} * |v|^2)v(t^1)dt^1.$$

By the Strichartz estimates, we have

$$\|D^{s_c}\Phi_{u_0}(v)\|_{S(\Lambda_0)} \le c\|u_0\|_{\dot{H}^{s_c}} + c\|D^{s_c}[(\frac{1}{|\cdot|^\gamma} * |v|^2)v]\|_{L^{q'}L^{r'}}$$

and

$$\|\Phi_{u_0}(v)\|_{S(\Lambda_{s_c})} \le \|U(t)u_0\|_{S(\Lambda_{s_c})} + c\|D^{s_c}[(\frac{1}{|\cdot|^\gamma} * |v|^2)v]\|_{L^{q'}L^{r'}},$$

with $(q', r') \in \Lambda_0'$. Applying the fractional Leibnitz [8,23,24], the Hölder inequalities and the Hardy–Littlewood–Sobolev inequalities, we have

$$\begin{aligned}\|D^{s_c}[(\frac{1}{|\cdot|^\gamma} * |v|^2)v]\|_{L^{q'}L^{r'}} &\le c\|D^{s_c}v(\frac{1}{|\cdot|^\gamma} * |v|^2)\|_{L^{q'}L^{r'}} + c\|[\frac{1}{|\cdot|^\gamma} * Re(\bar{v}D^{s_c}v)]v\|_{L^{q'}L^{r'}} \\ &\le c\|D^{s_c}v\|_{L^{q_1}L^{r_1}}\|v\|^2_{L^{q_c}L^{r_c}} + c\|v\|_{L^{q_c}L^{r_c}}\|\bar{v}D^{s_c}v\|_{L^aL^b}\end{aligned}$$

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$$\leq c\|D^{s_c}v\|_{L^{q_1}L^{r_1}}\|v\|^2_{L^{q_c}L^{r_c}} + c\|v\|_{L^{q_c}L^{r_c}}\|v\|_{L^{q_c}L^{r_c}}\|D^{s_c}v\|_{L^{\gamma_3}L^{\rho_3}}$$
$$\leq c\|v\|^2_{S(\Lambda_{s_c})}\|D^{s_c}v\|_{S(\Lambda_0)},$$

where the pairs (q,r), (q_1,r_1), $(\gamma_3,\rho_3) \in \Lambda_0$, satisfying that

$$\frac{1}{q_1} = 1 - \frac{1}{q} - \frac{2}{q_c}, \quad \frac{1}{r_1} = 2 - \frac{\gamma}{N} - \frac{1}{r} - \frac{2}{r_c},$$
$$\frac{1}{a} = \frac{1}{q_c} + \frac{1}{\gamma_3} = 1 - \frac{1}{q} - \frac{1}{q_c}, \quad \frac{1}{b} = \frac{1}{r_c} + \frac{1}{\rho_3} = 1 - \frac{\gamma}{N} + 1 - \frac{1}{r} - \frac{1}{r_c},$$
$$\frac{1}{\gamma_3} = 1 - \frac{1}{q} - \frac{2}{q_c}, \quad \frac{1}{\rho_3} = 2 - \frac{\gamma}{N} + 1 - \frac{2}{r_c}.$$

Let $\delta_{sd} \leq \min\left(\frac{1}{\sqrt{8}c}, \frac{1}{8c^3A}\right)$, and

$$B = \left\{v \mid \|v\|_{S(\Lambda_{s_c})} \leq 2\|U(t)u_0\|_{S(\Lambda_{s_c})}, \|D^{s_c}v\|_{S(\Lambda_0)} \leq 2c\|u_0\|_{\dot{H}^{s_c}}\right\}.$$

Then, $\Phi_{u_0} : B \to B$ and is a contraction on B; thus, the fixed point principle gives the result. □

Proposition 2.6. *If $u_0 \in H^s$ is radial and $u = u(t)$ is global with both bounded s_c-level Strichartz norm $\|u\|_{S(\Lambda_{s_c})} < \infty$ and uniformly bounded H^s norm $\sup_{t\in[0,+\infty)} \|u\|_{H^s} \leq B$, then $u(t)$ scatters in H^s as $t \to +\infty$. Specifically, there exists $\phi^+ \in H^s$ such that*

$$\lim_{t\to+\infty} \|u(t) - U(t)\phi^+\|_{H^s} = 0.$$

Proof. We can obtain from the integral equation

$$u(t) = U(t)u_0 + i\int_0^t U(t-t^1)(\frac{1}{|\cdot|^\gamma} * |u|^2)u(t^1)dt^1 \tag{2.8}$$

that

$$u(t) - U(t)\phi^+ = -i\int_t^\infty U(t-t^1)(\frac{1}{|\cdot|^\gamma} * |u|^2)u(t^1)dt^1, \tag{2.9}$$

where $\phi^+ = u_0 + i\int_0^\infty U(-t^1)(\frac{1}{|\cdot|^\gamma} * |u|^2)u(t^1)dt^1$. By the Hardy–Littlewood–Sobolev inequality and the Strichartz estimates, for $0 \leq \alpha \leq s$, there exist some $(q,r) \in \Lambda_0$, $(q_1,r_1) \in \Lambda_0'$ such that

$$\left\|D^\alpha\left(\int_I U(t-s)\left((\frac{1}{|\cdot|^\gamma} * |u|^2)u(s,x)\right)ds\right)\right\|_{L_I^qL^r} \leq C\left\|D^\alpha\left((\frac{1}{|\cdot|^\gamma} * |u|^2)u\right)\right\|_{L_I^{q_1}L^{r_1}}$$
$$\leq C\|D^\alpha u\|_{L_I^qL^r}\|\frac{1}{|\cdot|^\gamma} * |u|^2\|_{L_I^{q_2}L^{r_2}}$$
$$\leq C\|D^\alpha u\|_{L_I^qL^r}\|u\|^2_{L_I^{q_c}L^{r_c}}, \tag{2.10}$$

where $I \subset [0, +\infty)$,

$$\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q} = \frac{2}{q_c} + \frac{1}{q}, \quad \frac{1}{r_1} = \frac{1}{r} + \frac{1}{r_2} = \frac{1}{r} + \frac{\gamma}{N} + \frac{2}{r_c} - 1.$$

Since $\|u\|_{L^{q_c}_{[0,\infty)} L^{r_c}} < \infty$, we can partition $[0, +\infty)$ into a union of $I_j = [t_j, t_{j+1}], 1 \le j \le N$, such that for every $1 \le j \le N$, $\|u\|_{L^{q_c}_{I_j} L^{r_c}} < \delta$($\delta$ is sufficiently small). Thus, by (2.8) and (2.10), for $0 \le \alpha \le s, \forall 1 \le j \le N$,

$$\begin{aligned}
\|D^\alpha u\|_{L^q_{I_j} L^r} &\le \|U(t)u(t_j)\|_{L^q_{I_j} L^r} + \left\| D^\alpha \left(\int_{I_j} U(t-s) \left((\frac{1}{|\cdot|^\gamma} * |u|^2) u(s,x) \right) ds \right) \right\|_{L^q_{I_j} L^r} \\
&\le \|U(t)u(t_j)\|_{L^q_{I_j} L^r} + C \|D^\alpha u\|_{L^q_{I_j} L^r} \|u\|^2_{L^{q_c}_{I_j} L^{r_c}} \\
&\le CB + C\delta^2 \|D^\alpha u\|_{L^q_{I_j} L^r}.
\end{aligned}$$

By choosing δ such that $C\delta^2 < \frac{1}{2}$, we see that $\|D^\alpha u\|_{L^q_{I_j} L^r} < \infty$, $1 \le j \le N$. So we have $\|D^\alpha u\|_{L^q L^r} < \infty$. By (2.9), we have for $0 \le \alpha \le s$,

$$\|D^\alpha (u(t) - U(t)\phi^+)\|_2 \le \|u\|^2_{L^{q_c}_{[t,\infty)} L^{r_c}} \|D^\alpha u\|_{L^q_{[t,\infty)} L^r}.$$

Taking $\alpha = 0$, $\alpha = s$ in the above inequality and sending $t \to +\infty$, we obtain the claim. □

Proposition 2.7 *(Long-time perturbation theory). For any given $A \gg 1$, there exist $\epsilon_0 = \epsilon_0(A) \ll 1$ and $c = c(A)$ such that the following holds: Let $u = u(t,x) \in H^s$ be radial and solve (1.1) for all t. Let $\tilde{u} = \tilde{u}(t,x) \in H^s$ for all t, and set*

$$e \equiv i\tilde{u}_t - (-\Delta)^s \tilde{u} + (\frac{1}{|\cdot|^\gamma} * |\tilde{u}|^2)\tilde{u}.$$

If

$$\|\tilde{u}\|_{S(\Lambda_{s_c})} \le A, \quad \|e\|_{S'(\Lambda_{-s_c})} \le \epsilon_0 \ \text{ and } \ \|U(t-t_0)(u(t_0) - \tilde{u}(t_0))\|_{S(\Lambda_{s_c})} \le \epsilon_0,$$

then

$$\|u\|_{S(\Lambda_{s_c})} \le c = c(A) < \infty.$$

Proof. Define $w = u - \tilde{u}$. Then, w solves the equation

$$iw_t - (-\Delta)^s w + (\frac{1}{|\cdot|^\gamma} * |w + \tilde{u}|^2)w + (\frac{1}{|\cdot|^\gamma} * |w + \tilde{u}|^2)\tilde{u} - (\frac{1}{|\cdot|^\gamma} * |\tilde{u}|^2)\tilde{u} + e = 0.$$

Specifically,

$$
\begin{aligned}
&i w_t - (-\Delta)^s w + (\tfrac{1}{|\cdot|^\gamma} * |w|^2)w + (\tfrac{1}{|\cdot|^\gamma} * (\bar{w}\tilde{u}))w + (\tfrac{1}{|\cdot|^\gamma} * (w\bar{\tilde{u}}))w \\
&\quad +(\tfrac{1}{|\cdot|^\gamma} * |w|^2)\tilde{u} + (\tfrac{1}{|\cdot|^\gamma} * |\tilde{u}|^2)w + (\tfrac{1}{|\cdot|^\gamma} * (\bar{w}\tilde{u}))\tilde{u} + (\tfrac{1}{|\cdot|^\gamma} * (w\bar{\tilde{u}}))\tilde{u} + e = 0.
\end{aligned} \tag{2.11}
$$

Because $\|\tilde{u}\|_{S(\Lambda_{s_c})} \le A$, we can partition $[t_0, \infty)$ into $N = N(A)$ intervals $I_j = [t_j, t_{j+1})$ such that for each $0 \le j \le N-1$, $\|\tilde{u}\|_{S(\Lambda_{s_c}; I_j)} < \delta$ with the sufficiently small δ to be specified later. The integral equation of (2.11) with initial time t_j is

$$
w(t) = U(t-t_j)w(t_j) + i\int_{t_j}^{t} U(t-s)W(\cdot, s)ds, \tag{2.12}
$$

where

$$
\begin{aligned}
W = &(\frac{1}{|\cdot|^\gamma} * |w|^2)w + (\frac{1}{|\cdot|^\gamma} * (\bar{w}\tilde{u}))w + (\frac{1}{|\cdot|^\gamma} * (w\bar{\tilde{u}}))w \\
&+ (\frac{1}{|\cdot|^\gamma} * |w|^2)\tilde{u} + (\frac{1}{|\cdot|^\gamma} * |\tilde{u}|^2)w + (\frac{1}{|\cdot|^\gamma} * (\bar{w}\tilde{u}))\tilde{u} + (\frac{1}{|\cdot|^\gamma} * (w\bar{\tilde{u}}))\tilde{u} + e.
\end{aligned}
$$

Applying the inhomogeneous Strichartz estimate (2.4) on I_j, we have for $(q_1, r_1) \in \Lambda_{-s_c}$

$$
\begin{aligned}
\|w\|_{S(\Lambda_{s_c}; I_j)} \le{}& \|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\Lambda_{s_c}; I_j)} + c\|(\frac{1}{|\cdot|^\gamma} * |w|^2)w\|_{L^{q_1'}_{I_j} L^{r_1'}} \\
&+ c\|(\frac{1}{|\cdot|^\gamma} * (\bar{w}\tilde{u}))w\|_{L^{q_1'}_{I_j} L^{r_1'}} + c\|(\frac{1}{|\cdot|^\gamma} * (w\bar{\tilde{u}}))w\|_{L^{q_1'}_{I_j} L^{r_1'}} + c\|(\frac{1}{|\cdot|^\gamma} * |w|^2)\tilde{u}\|_{L^{q_1'}_{I_j} L^{r_1'}} \\
&+ c\|(\frac{1}{|\cdot|^\gamma} * |\tilde{u}|^2)w\|_{L^{q_1'}_{I_j} L^{r_1'}} + c\|(\frac{1}{|\cdot|^\gamma} * (\bar{w}\tilde{u}))\tilde{u}\|_{L^{q_1'}_{I_j} L^{r_1'}} + c\|(\frac{1}{|\cdot|^\gamma} * (w\bar{\tilde{u}}))\tilde{u}\|_{L^{q_1'}_{I_j} L^{r_1'}} \\
&+ \|e\|_{S'(\Lambda_{-s_c})}.
\end{aligned} \tag{2.13}
$$

Under the condition $\frac{N}{2N-1} \le s < 1$, we easily obtain that any $(q_i, r_i), i = 1, 2$ solving

$$
\begin{cases}
\frac{1}{q_1'} = \frac{2}{q_c} + \frac{1}{q_2} = 1 - \frac{\gamma}{N+2s} + \frac{1}{q_2}, \\
\frac{1}{r_1'} = \frac{\gamma}{N} + \frac{2}{r_c} + \frac{1}{r_2} - 1 = \frac{\gamma}{N} - \frac{\gamma}{N+2s} + \frac{1}{r_2}
\end{cases} \tag{2.14}
$$

should satisfy the range condition (2.2). Hence, for the above pair $(q_1, r_1) \in \Lambda_{-s_c}$, we can find $(q_2, r_2) \in \Lambda_{s_c}$ and apply the Hardy–Littlewood–Sobolev inequality and Hölder inequalities to find that

$$
\begin{aligned}
&\|(\frac{1}{|\cdot|^\gamma} * |\tilde{u}|^2)w\|_{L^{q_1'}_{I_j} L^{r_1'}} \le \|\tilde{u}\|^2_{L^{q_c}_{I_j} L^{r_c}} \|w\|_{L^{q_2}_{I_j} L^{r_2}} \le \|\tilde{u}\|^2_{S(\Lambda_{s_c}; I_j)} \|w\|_{S(\Lambda_{s_c}; I_j)} \le \delta^2 \|w\|_{S(\Lambda_{s_c}; I_j)}, \\
&\|(\frac{1}{|\cdot|^\gamma} * |w|^2)\tilde{u}\|_{L^{q_1'}_{I_j} L^{r_1'}} \le \|\tilde{u}\|_{L^{q_2}_{I_j} L^{r_2}} \|w\|^2_{L^{q_c}_{I_j} L^{r_c}} \le \|\tilde{u}\|_{S(\Lambda_{s_c}; I_j)} \|w\|^2_{S(\Lambda_{s_c}; I_j)} \le \delta \|w\|^2_{S(\Lambda_{s_c}; I_j)}.
\end{aligned} \tag{2.15}
$$

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Similarly, we have other terms estimated in the same way, and we substitute all the estimates in (2.13) to obtain

$$\begin{aligned}\|w\|_{S(\Lambda_{s_c};I_j)} &\le \|U(t-t_j)w(t_j)\|_{S(\Lambda_{s_c};I_j)} + c\delta^2\|w\|_{S(\Lambda_{s_c};I_j)} \\ &\quad + c\delta\|w\|_{S(\Lambda_{s_c};I_j)} + c\|w\|^3_{S(\Lambda_{s_c};I_j)} + c\|e\|_{S'(\dot{H}^{-s_c};I_j)} \\ &\le \|U(t-t_j)w(t_j)\|_{S(\Lambda_{s_c};I_j)} + c\delta^2\|w\|_{S(\Lambda_{s_c};I_j)} \\ &\quad + c\delta\|w\|^2_{S(\Lambda_{s_c};I_j)} + c\|w\|^3_{S(\Lambda_{s_c};I_j)} + c\epsilon_0.\end{aligned} \tag{2.16}$$

Now, if $\delta \le \min(1, \frac{1}{2\sqrt{c}})$ and

$$\|U(t-t_j)w(t_j)\|_{S(\Lambda_{s_c};I_j)} + c\epsilon_0 \le \min(1, \frac{1}{8\sqrt{c}}), \tag{2.17}$$

we obtain

$$\|w\|_{S(\Lambda_{s_c};I_j)} \le 2\|U(t-t_j)w(t_j)\|_{S(\Lambda_{s_c};I_j)} + 2c\epsilon_0. \tag{2.18}$$

Next, we take $t = t_{j+1}$ in (2.12) and apply $U(t-t_{j+1})$ to both sides. We then obtain

$$U(t-t_{j+1})w(t_{j+1}) = U(t-t_j)w(t_j) + i\int_{t_j}^{t_{j+1}} U(t-s)W(\cdot,s)ds. \tag{2.19}$$

Note that the Duhamel integral is confined to I_j. Similar to (2.16), we have the estimate

$$\begin{aligned}\|U(t-t_{j+1})w(t_{j+1})\|_{S(\Lambda_{s_c})} &\le \|e^{i(t-t_j)\Delta}w(t_j)\|_{S(\Lambda_{s_c})} + c\delta^2\|w\|_{S(\Lambda_{s_c};I_j)} \\ &\quad + c\delta\|w\|_{S(\Lambda_{s_c};I_j)} + c\|w\|^3_{S(\Lambda_{s_c};I_j)} + c\epsilon_0.\end{aligned}$$

Then, (2.17) and (2.18) imply

$$\|U(t-t_{j+1})w(t_{j+1})\|_{S(\Lambda_{s_c})} \le 2\|U(t-t_j)w(t_j)\|_{S(\Lambda_{s_c})} + 2c\epsilon_0.$$

Now, iterate the beginning with $j = 0$, and we obtain

$$\|U(t-t_j)w(t_j)\|_{S(\Lambda_{s_c})} \le 2^j\|U(t-t_0)w(t_0)\|_{S(\Lambda_{s_c})} + (2^j-1)2c\epsilon_0 \le 2^{j+2}c\epsilon_0.$$

Because the second part of (2.17) is needed for each I_j, $0 \le j \le N-1$, we require that

$$2^{N+2}c\epsilon_0 \le \min(1, \frac{1}{2\sqrt{6c}}). \tag{2.20}$$

Recall that δ is an absolute constant to satisfy (2.17); the given A determines the number of time intervals N. Then, by (2.20), ϵ_0 is determined by $N = N(A)$. Thus, the iteration completes our proof. □

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3. Variational characteristic and invariant sets

In this section, we first recall some variational characteristic of the ground state for Eq. (1.1) given in [40]. Then, we can construct the invariant flows generated by the Cauchy problem of (1.1) and (1.2). Finally, we give some refined estimates of the invariant set of the global solutions, which are crucial for proving that the global solutions will be scattering.

Lemma 3.1 *(See [40]). Let $N \geq 2$, $0 < s < 1$ and $0 < \gamma < \min\{N, 4s\}$. Suppose that Q is the ground-state solution of (1.5). Then, we have the following Pohozaev identities:*

$$\int \overline{Q}(-\triangle)^s Q dx + \int |Q|^2 dx - \int\int \frac{|Q(x)|^2|Q(y)|^2}{|x-y|^\gamma} dxdy = 0, \tag{3.1}$$

$$\frac{N-2s}{2}\int \overline{Q}(-\triangle)^s Q dx + \frac{N}{2}\int |Q|^2 dx - \frac{2N-\gamma}{4}\int\int \frac{|Q(x)|^2|Q(y)|^2}{|x-y|^\gamma} dxdy = 0. \tag{3.2}$$

Remark 3.2. Let Q be the ground-state solution of (1.5). In terms of the Pohozaev identities (3.1) and (3.2), we can obtain the following properties.

(i)

$$\int\int \frac{|Q(x)|^2|Q(y)|^2}{|x-y|^\gamma} dxdy = \frac{4s}{\gamma}\|Q\|^2_{\dot{H}^s} = \frac{4s}{4s-\gamma}\|Q\|_2^2.$$

(ii)

$$E[Q] = \frac{1}{2}\int \overline{Q}(-\triangle)^s Q dx - \frac{1}{4}\int\int \frac{|Q(x)|^2|Q(y)|^2}{|x-y|^\gamma} dxdy = \frac{\gamma-2s}{2(4s-\gamma)}\|Q\|_2^2.$$

(iii)

$$E[Q]M[Q]^{\frac{s-s_c}{s_c}} = \frac{\gamma-2s}{2(4s-\gamma)}\|Q\|_2^{\frac{2s}{s_c}}.$$

(iv)

$$\|Q\|^2_{\dot{H}^s} M[Q]^{\frac{s-s_c}{s_c}} = \frac{\gamma}{4s-\gamma}\|Q\|_2^{\frac{2s}{s_c}}.$$

The general fractional Laplacian was first proposed by Caffarelli and Silvestre in [4], and many researchers have studied the related time-independent Schrödinger equations with the fractional Laplacian (see [6,11,12,16,22,32]).

For the Cauchy problem (1.1)–(1.2), we can construct the following two invariant evolution flows by the sharp G–N inequality (1.6) and the conservation laws. Let $u \in H^s \setminus \{0\}$, and define

$$K_1 = \{\|u\|^2_{\dot{H}^s} M[u]^{\frac{s-s_c}{s_c}} < \|Q\|^2_{\dot{H}^s} M[Q]^{\frac{s-s_c}{s_c}},\ E[u]M[u]^{\frac{s-s_c}{s_c}} < E[Q]M[Q]^{\frac{s-s_c}{s_c}}\}$$

and

$$K_2 = \{\|u\|_{\dot{H}^s}^2 M[u]^{\frac{s-s_c}{s_c}} > \|Q\|_{\dot{H}^s}^2 M[Q]^{\frac{s-s_c}{s_c}},\ E[u]M[u]^{\frac{s-s_c}{s_c}} < E[Q]M[Q]^{\frac{s-s_c}{s_c}}\}.$$

Proposition 3.3. *Let $N \geq 2$ and Q be the ground-state solution of (1.5). If $0 < s < 1$ and $2s < \gamma < \min\{N, 4s\}$, then K_1 and K_2 are invariant manifolds of (1.1).*

Proof. Denote

$$V(u) := \int\int \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^\gamma} dxdy.$$

Multiplying the definition of energy by $M[u]^{\frac{s-s_c}{s_c}}$ and using (1.6), we have

$$\begin{aligned} M[u]^{\frac{s-s_c}{s_c}} E[u] &= \frac{1}{2}\|u(t)\|_2^{\frac{2(s-s_c)}{s_c}} \|D^s u(t)\|_2^2 - \frac{1}{4}V(u)\|u\|_2^{\frac{2(s-s_c)}{s_c}} \\ &\geq \frac{1}{2}(\|u(t)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t)\|_2)^2 - \frac{C_{GN}}{4}(\|u(t)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t)\|_2)^{\frac{\gamma}{s}}. \end{aligned}$$

Define $f(y) = \frac{1}{2}y^2 - \frac{1}{4}C_{GN}y^{\frac{\gamma}{s}}$. Then, $f'(y) = y\left(1 - C_{GN}\frac{\gamma}{4s}y^{\frac{\gamma-2s}{s}}\right)$, and thus, $f'(y) = 0$ when $y_0 = 0$ and $y_1 = \|Q\|_2^{\frac{s-s_c}{s_c}} \|D^s Q\|_2$. The graph of f has a local minimum at y_0 and a local maximum at y_1. Remark 3.2 implies that $f_{max} = f(y_1) = M[Q]^{\frac{s-s_c}{s_c}} E[Q]$. This combined with energy conservation gives

$$f(\|u(t)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t)\|_2) \leq M[u(t)]^{\frac{s-s_c}{s_c}} E[u(t)] = M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < f(y_1). \tag{3.3}$$

Next, we shall prove Proposition 3.3 in the following two cases:

Case I: If the initial data $u_0 \in K_1$, i.e., $\|u_0\|_2^{\frac{s-s_c}{s_c}} \|D^s u_0\|_2 < y_1$, then by (3.3) and the continuity of $\|D^s u(t)\|_2$ in t, we have for all time $t \in \mathbb{R}$,

$$\|u(t)\|_{\dot{H}^s}^2 M[u(t)]^{\frac{s-s_c}{s_c}} < \|Q\|_{\dot{H}^s}^2 M[Q]^{\frac{s-s_c}{s_c}}. \tag{3.4}$$

Indeed, if (3.4) is not true, then there exists $t_1 \in I$ such that $\|u(t_1)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t_1)\|_2 \geq y_1$. Because the corresponding solution $u(t,x) \in C(I; H^s)$ is continuous with respect to t, there exists $0 < t_0 \leq t_1$ such that $\|u(t_0)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t_0)\|_2 = y_1$. Thus, injecting the conservation of energy $E[u(t_0)] = E[u_0]$ and $\|u(t_0)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t_0)\|_2 = y_1$ into (3.3), we deduce that

$$f(y_1) > M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] = M[u(t_0)]^{\frac{s-s_c}{s_c}} E[u(t_0)] \geq f(\|u(t_0)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t_0)\|_2) = f(y_1).$$

This is a contradiction. Hence, (3.4) is true, which implies that K_1 is an invariant set.

Case II: If the initial data $u_0 \in K_2$, i.e., $\|u_0\|_2^{\frac{s-s_c}{s_c}} \|D^s u_0\|_2 > y_1$, then by (3.3) and the continuity of $\|D^s u(t)\|_2$ in t, we have for all time $t \in I$ that

$$\|u(t)\|_{\dot{H}^s}^2 M[u(t)]^{\frac{s-s_c}{s_c}} > \|Q\|_{\dot{H}^s}^2 M[Q]^{\frac{s-s_c}{s_c}}, \tag{3.5}$$

which implies that K_2 is an invariant set. The proof is similar to **Case I**. □

Remark 3.4. From the argument above, we can refine this analysis to obtain the following. If the condition $\|u_0\|_{\dot{H}^s}^2 M[u_0]^{\frac{s-s_c}{s_c}} < \|Q\|_{\dot{H}^s}^2 M[Q]^{\frac{s-s_c}{s_c}}$ holds, then there exists $\delta > 0$ such that $M[u]^{\frac{s-s_c}{s_c}} E[u] < (1-\delta) M[Q]^{\frac{s-s_c}{s_c}} E[Q]$, and thus, there exists $\delta_0 = \delta_0(\delta)$ such that $\|u(t)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t)\|_2 < (1-\delta_0)\|Q\|_2^{\frac{s-s_c}{s_c}} \|D^s Q\|_2$, where $u = u(t)$ is the corresponding solution to Eq. (1.1).

Theorem 3.5 *(Global versus blow-up dichotomy). Let $u_0 \in H^s$, and let $I = (T_-, T_+)$ be the maximal time interval of existence of $u = u(t)$ solving (1.1).*

(i) *If $u_0 \in K_1$, then $I = (-\infty, +\infty)$, i.e., the solution exists globally in time.*
(ii) *If $u_0 \in K_2 \bigcap H^{s_0}$ is radial, $|x|u_0 \in L^2$ and $x \cdot \nabla u_0 \in L^2$, where $s_0 = \max\{2s, \frac{\gamma+1}{2}\}$, then the corresponding solution $u(t)$ of (1.1) must blow up in a finite time $0 < T < +\infty$.*

Proof. (i) By the invariance of K_1, we see that (3.4) is true. In particular, the H^s-norm of the solution u is bounded, which proves the global existence of the solution in this case.

(ii) Denote $A := \left(\left(\frac{\gamma}{4s-\gamma}\right)^{s_c} \frac{\|Q\|_2^{2s}}{M[u_0]^{s-s_c}}\right)^{\frac{1}{2s_c}}$. Using the invariance of K_2, we have $\|u(t)\|_{\dot{H}^s}^2 > A^2$ for all $t \in I$. It follows from [7,40] that $|x|u(t) \in L^2$ and $x \cdot \nabla u(t) \in L^2$, and for all $t \in I$ (the maximal time interval), $\int \overline{u} x(-\Delta)^{1-s} x u dx$ is non-negative and

$$\int \overline{u} x(-\Delta)^{1-s} x u dx \le \int_0^t \int_0^t \left(2\gamma E[u(\tau)] - (\gamma - 2s)\|u(\tau)\|_{\dot{H}^s}^2\right) d\tau dt + Ct + C. \tag{3.6}$$

Applying the fact that for all $t \in I$, $E[u(t)] = E[u_0] < \frac{\gamma-2s}{2\gamma} A^2$ and $\|u(t)\|_{\dot{H}^s}^2 > A^2$ to (3.6), we deduce that for all $t \in I$

$$\int \overline{u} x(-\Delta)^{1-s} x u dx < \int_0^t \int_0^t \left(2\gamma \frac{\gamma - 2s}{2\gamma} A^2 - (\gamma - 2s) A^2\right) d\tau dt + Ct + C.$$

Hence, there exists a constant $C_0 > 0$ such that for all $t \in I$

$$\int \overline{u} x(-\Delta)^{1-s} x u dx \le -C_0 t^2 + Ct + C.$$

For sufficiently large $|t|$, the left-hand side is negative, while $\int \overline{u} x(-\Delta)^{1-s} x u dx$ is non-negative, which means that both T_- and T_+ are finite. Specifically, the solution $u(t, x)$ of the Cauchy problem (1.1)–(1.2) blows up in finite time. □

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Lemma 3.6. *Let $u_0 \in K_1$. Furthermore, take $\delta > 0$ such that $M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < (1-\delta)M[Q]^{\frac{s-s_c}{s_c}} E[Q]$. If $u = u(t)$ is a solution to problem (1.1) with initial data u_0, then there exists $C_\delta > 0$ such that for all $t \in \mathbb{R}$,*

$$\|D^s u\|_2^2 - \frac{\gamma}{4s} V(u) \geq C_\delta \|D^s u\|_2^2. \tag{3.7}$$

Proof. By Remark 3.4, there exists $\delta_0 = \delta_0(\delta) > 0$ such that for all $t \in \mathbb{R}$,

$$\|u(t)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t)\|_2 < (1-\delta_0)\|Q\|_2^{\frac{s-s_c}{s_c}} \|D^s Q\|_2. \tag{3.8}$$

Let

$$h(t) = \frac{1}{\|Q\|_2^{\frac{2(s-s_c)}{s_c}} \|D^s Q\|_2^2} (\|u(t)\|_2^{\frac{2(s-s_c)}{s_c}} \|D^s u(t)\|_2^2 - \frac{\gamma}{4s} V(u)\|u(t)\|_2^{\frac{2(s-s_c)}{s_c}})$$

and $g(y) = y^2 - y^{\frac{\gamma}{s}}$. By the Gagliardo–Nirenberg estimate (1.6) with the sharp constant C_{GN} (1.7), we can obtain $h(t) \geq g\left(\frac{\|u(t)\|_2^{\frac{s-s_c}{s_c}} \|D^s u(t)\|_2}{\|Q\|_2^{\frac{s-s_c}{s_c}} \|D^s Q\|_2}\right)$. By (3.8), we restrict our attention to $0 \leq y \leq 1-\delta_0$. The elementary argument gives a constant C_δ such that $g(y) \geq C_\delta y^2$ if $0 \leq y \leq 1-\delta_0$. This indeed implies (3.7). □

Lemma 3.7 *(Comparability of gradient and energy). Let $u_0 \in K_1$. Then,*

$$\frac{\gamma - 2s}{2\gamma} \|D^s u(t)\|_2^2 \leq E[u(t)] \leq \frac{1}{2}\|D^s u(t)\|_2^2.$$

Proof. The expression of $E[u(t)]$ gives the second inequality immediately. The first inequality is obtained from

$$\frac{1}{2}\|D^s u\|_2^2 - \frac{1}{4}V(u) \geq \frac{1}{2}\|D^s u\|_2^2 \left(1 - \frac{2s}{\gamma}\left(\frac{\|D^s u\|_2 \|u\|_2^{\frac{s-s_c}{s_c}}}{\|D^s Q\|_2 \|Q\|_2^{\frac{s-s_c}{s_c}}}\right)^{\frac{2s_c}{s}}\right) \geq \frac{\gamma - 2s}{2\gamma}\|D^s u\|_2^2,$$

where we have used (1.6), (1.7) and (3.4). □

To establish the scattering theory, we need the existence result of the wave operator $\Omega^+ : \phi^+ \mapsto v_0$.

Proposition 3.8 *(Existence of wave operators). Suppose that $\phi^+ \in H^s$ and*

$$\frac{1}{2} M[\phi^+]^{\frac{s-s_c}{s_c}} \|D^s \phi^+\|_2^2 < M[Q]^{\frac{s-s_c}{s_c}} E[Q]. \tag{3.9}$$

Then, there exists $v_0 \in H^s$ such that $v = v(t)$ globally solves (1.1) satisfying

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$$\|D^s v(t)\|_2 \|v(t)\|_2^{\frac{s-s_c}{s_c}} \le \|D^s Q\|_2 \|Q\|_2^{\frac{s-s_c}{s_c}}, \quad M[v] = \|\phi^+\|_2^2, \quad E[v] = \frac{1}{2}\|D^s\phi^+\|_2^2,$$

and

$$\lim_{t\to+\infty} \|v(t) - U(t)\phi^+\|_{H^s} = 0.$$

Moreover, if $\|U(t)\phi^+\|_{S(\Lambda_{s_c})} \le \delta_{sd}$*, where* δ_{sd} *is defined in Proposition 2.5, then*

$$\|v\|_{S(\Lambda_{s_c})} \le 2\|U(t)\phi^+\|_{S(\Lambda_{s_c})}, \quad \|D^{s_c} v\|_{S(\Lambda_0)} \le 2c\|\phi^+\|_{\dot H^{s_c}}.$$

Proof. In this paper, we always use $v(t) := FNLS(t)v_0$ to denote the solution $v = v(t)$ of Eq. (1.1) with the initial data $v(0) = v_0$. First, similar to the proof of the small data scattering theory Proposition 2.5, we can solve the integral equation

$$v(t) = U(t)\phi^+ - i\int_t^\infty U(t-t^1)\left(\frac{1}{|\cdot|^\gamma} * |v|^2\right)v(t^1)dt^1 \tag{3.10}$$

for $t \ge T$ with T large. In fact, there exists $T \gg 1$ such that $\|U(t)\phi^+\|_{S(\Lambda_{s_c};[T,+\infty))} \le \delta_{sd}$. Now, from (3.10), we again obtain by the Strichartz estimate and the Hardy–Littlewood–Sobolev inequality that

$$\begin{aligned}
\|D^s v(t)\|_{S(\Lambda_0;[T,\infty))} &\le c\|D^s\phi^+\|_{L^2} + c\|D^s[(\frac{1}{|\cdot|^\gamma} * |v|^2)v]\|_{L^{q'}_{[T,+\infty)}L^{r'}} \\
&\le c\|D^s\phi^+\|_{L^2} + c\|D^s v\|_{L^{q_1}_{[T,+\infty)}L^{r_1}} \|v\|^2_{L^{q_2}_{[T,+\infty)}L^{r_2}} \\
&\quad + c\|v\|_{L^{\gamma_1}_{[T,+\infty)}L^{\rho_1}} \|v\|_{L^{\gamma_2}_{[T,+\infty)}L^{\rho_2}} \|D^s v\|_{L^{\gamma_3}_{[T,+\infty)}L^{\rho_3}} \\
&\le c\|D^s\phi^+\|_{L^2} + c\|v\|^2_{S(\Lambda_{s_c};[T,+\infty))} \|D^s v(t)\|_{S(\Lambda_0;[T,+\infty))},
\end{aligned}$$

where $(q,r), (q_1,r_1), (\gamma_3,\rho_3) \in \Lambda_0$, $(q_2,r_2), (\gamma_1,\rho_1), (\gamma_2,\rho_2) \in \Lambda_{s_c}$, which indeed can be chosen as $(q_2,r_2) = (\gamma_1,\rho_1) = (\gamma_2,\rho_2) = (q_c,r_c) \in \Lambda_{s_c}$, with (q_c,r_c) defined by (2.5). Similarly,

$$\|v(t)\|_{S(\Lambda_0;[T,+\infty))} \le c\|\phi^+\|_{L^2} + c\|v\|^2_{S(\Lambda_{s_c};[T,+\infty))} \|v(t)\|_{S(\Lambda_0;[T,+\infty))}.$$

Following Proposition 2.5, we obtain for sufficiently large T

$$\|v\|_{S(\Lambda_0;[T,+\infty))} + \|D^s v\|_{S(\Lambda_0;[T,+\infty))} < 2c\|\phi^+\|_{H^s}.$$

Using a similar approach with $t > T$, we obtain

$$\|v - U(t)\phi^+\|_{S(\Lambda_0;[T,+\infty))} + \|D^s(v - e^{it\Delta}\phi^+)\|_{S(\Lambda_0;[T,+\infty))} \to 0, \quad \text{as} \quad T \to +\infty,$$

which implies $v(t) - U(t)\phi^+ \to 0$ in H^s, and thus, $M[v] = \|\phi^+\|_2^2$. Because $U(t)\phi^+ \to 0$ in L^p for any $p \in (2, \frac{2N}{N-2s}]$ as $t \to +\infty$, by the Hardy–Littlewood–Sobolev inequality, $V(U(t)\phi^+) \to 0$. This together with the fact that $\|D^s U(t)\phi^+\|_2$ is conserved implies

$$E[v]=\lim_{t\to+\infty}(\frac{1}{2}\|D^sU(t)\phi^+\|_2^2-\frac{1}{4}V(U(t)\phi^+))=\frac{1}{2}\|D^s\phi^+\|_2^2.$$

Considering (3.9), we immediately obtain $M[v]^{\frac{s-s_c}{s_c}}E[v]<E[Q]M[Q]^{\frac{s-s_c}{s_c}}$. Note that

$$\begin{aligned}\lim_{t\to+\infty}\|v(t)\|_2^{\frac{2(s-s_c)}{s_c}}\|D^sv(t)\|_2^2&=\lim_{t\to+\infty}\|U(t)\phi^+\|_2^{\frac{2(s-s_c)}{s_c}}\|D^sU(t)\phi^+\|_2^2\\&=\|\phi^+\|_2^{\frac{2(s-s_c)}{s_c}}\|D^s\phi^+\|_2^2\leq 2E[Q]M[Q]^{\frac{s-s_c}{s_c}}\\&=\frac{\gamma-2s}{\gamma}\|Q\|_2^{\frac{2(s-s_c)}{s_c}}\|D^sQ\|_2^2,\end{aligned}$$

where we used (3.9) and Remark 3.2 in the last two steps. Thus, due to Theorem 3.5, we can evolve $v(t)$ from T back to time 0 and complete our proof. □

4. Critical solution and compactness

From this section, we begin to prove the scattering part of Theorem 1.1. Let $u=u(t)$ be the solution of (1.1) such that the assumption of Theorem 1.1 holds. Then, we know from Theorem 3.5 that $u(t)$ is globally well-posed. Thus, combined with Proposition 2.6, our goal is to show that

$$\|u\|_{S(\Lambda_{s_c})}<\infty, \tag{4.1}$$

which implies that the solution of (1.1) is H^s scattering.

We say that $SC(u_0)$ holds if (4.1) is true for the solution $u=u(t)$ with the initial data u_0.

We first claim that there exists $\delta>0$ such that if $E[u_0]M[u_0]^{\frac{s-s_c}{s_c}}<\delta$ and $\|u_0\|_2^{\frac{s-s_c}{s_c}}\|D^su_0\|_2<\|Q\|_2^{\frac{s-s_c}{s_c}}\|D^sQ\|_2$, then (4.1) holds. Indeed, if

$$E[u_0]M[u_0]^{\frac{s-s_c}{s_c}}<\frac{s_c}{\gamma}\delta_{sd}^{\frac{2s}{s_c}},$$

where δ_{sd} is simply the $C\delta_{sd}$ appearing in Proposition 2.5, and $\|u_0\|_2^{\frac{s-s_c}{s_c}}\|D^su_0\|_2<\|Q\|_2^{\frac{s-s_c}{s_c}}\|D^sQ\|_2$, we obtain from Lemma 3.7 that

$$\|u_0\|_{\dot H^{s_c}}^2\leq\|u_0\|_2^{\frac{2(s-s_c)}{s}}\|D^su_0\|_2^{\frac{2s_c}{s}}\leq\left(\frac{\gamma}{s_c}E[u_0]M[u_0]^{\frac{s-s_c}{s_c}}\right)^{\frac{s_c}{s}}\leq\delta_{sd}^2,$$

which implies that $SC(u_0)$ holds by the small data theory. The claim holds for $\delta=\frac{s_c}{\gamma}\delta_{sd}^{\frac{2s}{s_c}}$. Now, for each δ, we define the set S_δ to be the collection of all such initial data in H^s:

$$S_\delta=\{u_0\in H^s:\ \ E[u_0]M[u_0]^{\frac{s-s_c}{s_c}}<\delta\ \text{ and }\ M[u_0]^{\frac{s-s_c}{s_c}}\|D^su_0\|_2^2<M[Q]^{\frac{s-s_c}{s_c}}\|D^sQ\|_2^2\}.$$

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We also define that $(ME)_c = \sup\{\delta : \ u_0 \in S_\delta \Rightarrow SC(u_0)$ holds$\}$. If $(ME)_c = M[Q]^{\frac{s-s_c}{s_c}} E[Q]$, then we are done. Thus, we assume now that

$$(ME)_c < M[Q]^{\frac{s-s_c}{s_c}} E[Q]. \tag{4.2}$$

Then, there exists a sequence of solutions u_n to (1.1) with H^s initial data $u_{n,0}$ (note from the beginning of the above section that we can rescale them to satisfy $\|u_n\|_2 = 1$) such that $\|D^s u_{n,0}\|_2 < \|Q\|_2^{\frac{s-s_c}{s_c}} \|D^s Q\|_2$ and $E[u_{n,0}] \downarrow (ME)_c$ as $n \to \infty$, and $SC(u_0)$ does not hold for any n.

Our goal in this section is to show the existence of an H^s solution u_c to (1.1) with initial data $u_{c,0}$ such that $\|u_{c,0}\|_2^{\frac{s-s_c}{s_c}} \|D^s u_{c,0}\|_2 < \|Q\|_2^{\frac{s-s_c}{s_c}} \|D^s Q\|_2$ and $M[u_{c,0}]^{\frac{s-s_c}{s_c}} E[u_{c,0}] = (ME)_c$ for which $SC(u_{c,0})$ does not hold. Moreover, we will show that $\{u_c(t,\cdot)|0 \le t < \infty\}$ is precompact in H^s. This will play an important role in the rigidity theorem in the next section, which will ultimately leads to a contradiction.

Prior to fulfilling our main task, we will first introduce a profile decomposition lemma that is highly similar to that in [20], which is for the cubic Schrödinger equation in the spirit of Keraani's arguments in [27].

Lemma 4.1 *(Profile expansion). Let $\phi_n(x)$ be a radial and uniformly bounded sequence in H^s. Then, for each M, there exists a subsequence of ϕ_n, also denoted by ϕ_n, and*

(1) *for each $1 \le j \le M$, there exists a (fixed in n) profile $\psi^j(x)$ in H^s,*
(2) *for each $1 \le j \le M$, there exists a sequence (in n) of time shifts t_n^j,*
(3) *there exists a sequence (in n) of remainders $W_n^M(x)$ in H^s such that*

$$\phi_n(x) = \sum_{j=1}^{M} U(-t_n^j)\psi^j(x) + W_n^M(x).$$

The time and space sequences have a pairwise divergence property, i.e., for $1 \le j \ne k \le M$, we have

$$\lim_{n\to+\infty} |t_n^j - t_n^k| = +\infty. \tag{4.3}$$

The remainder sequence has the following asymptotic smallness property:

$$\lim_{M\to+\infty} [\lim_{n\to+\infty} \|U(t)W_n^M\|_{S(\Lambda_{s_c})}] = 0. \tag{4.4}$$

For fixed M and any $0 \le \alpha \le s$, we have the asymptotic Pythagorean expansion:

$$\|\phi_n\|_{\dot H^\alpha}^2 = \sum_{j=1}^{M} \|\psi^j\|_{\dot H^\alpha}^2 + \|W_n^M\|_{\dot H^\alpha}^2 + o_n(1). \tag{4.5}$$

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Remark 4.2. The proof of the linear profile decomposition could simply follow the proof in [15] without any significant changes. Furthermore, from the proof, the vanishing property (4.4) could be improved to

$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|U(t)W_n^M\|_{L^qL^r}]=0,\quad \forall(q,r) \text{ satisfies (2.1) with } \theta=s_c, \tag{4.6}$$

especially,

$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|U(t)W_n^M\|_{L^\infty L^{\frac{2N}{N-2s_c}}}]=0. \tag{4.7}$$

Lemma 4.3 *(Energy Pythagorean expansion). In the situation of Lemma 4.1, we have*

$$E[\phi_n]=\sum_{j=1}^{M}E[U(-t_n^j)\psi^j]+E[W_n^M]+o_n(1). \tag{4.8}$$

Proof. According to (4.5), it suffices to establish that for all $M\ge 1$,

$$V(\phi_n)=\sum_{j=1}^{M}V(U(-t_n^j)\psi^j)+V(W_n^M). \tag{4.9}$$

There are only two cases to consider. Case 1. There exists some j for which t_n^j converges to a finite number, which without loss of generality, we assume is 0. In this case, we will show that $\lim\limits_{n\to+\infty}V(W_n^M)=0$ for $M>j$, $\lim\limits_{n\to+\infty}V(U(-t_n^k)\psi^k)=0$ for all $k\ne j$, and $\lim\limits_{n\to+\infty}V(\phi_n)=V(\psi^j)$, which gives (4.9). Case 2. For all j, $|t_n^j|\to+\infty$. In this case, we will show that $\lim\limits_{n\to+\infty}V(U(-t_n^k)\psi^k)=0$ for all k and that $\lim\limits_{n\to+\infty}V(\phi_n)=\lim\limits_{n\to+\infty}V(W_n^M)$, which gives (4.9) again.

For Case 1, we infer from the proof of Lemma 4.1 that $W_n^{j-1}\rightharpoonup\psi^j$, as $n\to+\infty$. By the compactness of the embedding $H_r^s\hookrightarrow L^p, \forall p\in(2,\frac{2N}{N-2s})$, it follows from that Hardy–Littlewood–Sobolev inequalities that $V(W_n^{j-1})\to V(\psi^j)$ as $n\to+\infty$. Let $k\ne j$. Then, we obtain from (4.3) that $|t_n^k|\to+\infty$. As argued in the proof of Lemma 4.1, from the Sobolev embedding and the L^p spacetime decay estimates (or the dispersive estimates; see [18]) of the linear flow, we find that $V(U(-t_n^k)\psi^k)\to 0$, as $n\to+\infty$. Recalling that

$$W_n^{j-1}=\phi_n-U(-t_n^1)\psi^1-\cdots-U(-t_n^{j-1})\psi^{j-1},$$

we conclude that $V(\phi_n)\to V(\psi^j)$. Because

$$W_n^M=W_n^{j-1}-\psi^j-U(-t_n^{j+1})\psi^{j+1}-\cdots-U(-t_n^M)\psi^M,$$

we also conclude that $\lim\limits_{n\to+\infty}V(W_n^M)=0$ for $M>j$.

Case 2 follows similarly from the proof of Case 1. □

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Proposition 4.4 *(Existence of a critical solution). There exists a global solution $u_c = u_c(t)$ in H^s with initial data $u_{c,0}$ such that $\|u_{c,0}\|_2 = 1$,*

$$E[u_c] = (ME)_c < M[Q]^{\frac{s-s_c}{s_c}} E[Q], \quad \|D^s u_c\|_2^2 < M[Q]^{\frac{s-s_c}{s_c}} \|D^s Q\|_2^2, \ \text{for all } \ 0 \le t < \infty,$$

and

$$\|u_c\|_{S(\Lambda_{s_c})} = \infty.$$

Proof. Recall that we have obtained the sequence $\|u_n\|_2 = 1$ described at the beginning of this section satisfying $\|D^s u_{n,0}\|_2^2 < M[Q]^{\frac{s-s_c}{s_c}} \|D^s Q\|_2^2$ and $E[u_{n,0}] \downarrow (ME)_c$ as $n \to +\infty$. Each u_n is global and non-scattering $\|u_n\|_{S(\Lambda_{s_c})} = \infty$. We apply Lemma 4.1 to $u_{n,0}$, which is uniformly bounded in H^s, to obtain

$$u_{n,0}(x) = \sum_{j=1}^{M} U(-t_n^j)\psi^j(x) + W_n^M(x). \tag{4.10}$$

Then, by Lemma 4.3 (Energy Pythagorean expansion), we further have

$$\sum_{j=1}^{M} \lim_{n\to+\infty} E[U(-t_n^j)\psi^j] + \lim_{n\to+\infty} E[W_n^M] = \lim_{n\to+\infty} E[u_{n,0}] = (ME)_c.$$

Also by the profile expansion, we have

$$\|D^s u_{n,0}\|_2^2 = \sum_{j=1}^{M} \|D^s U(-t_n^j)\psi^j\|_2^2 + \|D^s W_n^M\|_2^2 + o_n(1),$$

and

$$1 = \|u_{n,0}\|_2^2 = \sum_{j=1}^{M} \|\psi^j\|_2^2 + \|W_n^M\|_2^2 + o_n(1). \tag{4.11}$$

We know from the proof of Lemma 3.7 that each energy is nonnegative, and thus,

$$\lim_{n\to+\infty} E[U(-t_n^j)\psi^j] \le (ME)_c. \tag{4.12}$$

Claim A: only one $\psi^j \neq 0$.

If more than one $\psi^j \neq 0$, we will show a contradiction in the following, and thus, the profile expansion will be reduced to the case in which only one profile is non-trivial.

For this, by (4.11), we must have $M[\psi^j] < 1$ for each j, which together with (4.12), implies that for sufficiently large n,

$$M[U(-t_n^j)\psi^j]^{\frac{s-s_c}{s_c}} E[U(-t_n^j)\psi^j] < (ME)_c.$$

For a given j, if $|t_n^j| \to +\infty$, we assume $t_n^j \to +\infty$ or $t_n^j \to -\infty$ up to a subsequence. In this case, by the proof of Lemma 4.3, we have $\lim\limits_{n\to+\infty} V(U(-t_n^k)\psi^k) = 0$, and thus, $\frac{1}{2}\|\psi^j\|_2^{\frac{2(s-s_c)}{s_c}}\|D^s\psi^j\|_2^2 = \frac{1}{2}\|U(-t_n^j)\psi^j\|_2^{\frac{2(s-s_c)}{s_c}}\|D^sU(-t_n^j)\psi^j\|_2^2 < (ME)_c$. Then, we obtain from the existence of wave operators (Proposition 3.8) that there exists $\tilde{\psi}^j$ such that

$$\|FNLS(-t_n^j)\tilde{\psi}^j - U(-t_n^j)\psi^j\|_{H^s} \to 0, \ \text{ as } \ n \to +\infty$$

with

$$\|\tilde{\psi}^j\|_2^{\frac{s-s_c}{s_c}}\|D^sFNLS(t)\tilde{\psi}^j\|_2 < \|Q\|_2^{\frac{s-s_c}{s_c}}\|D^sQ\|_2$$
$$\|\tilde{\psi}^j\|_2 = \|\psi^j\|_2, \quad E[\tilde{\psi}^j] = \frac{1}{2}\|D^s\psi^j\|_2^2,$$

and thus,

$$M[\tilde{\psi}^j]^{\frac{s-s_c}{s_c}}E[\tilde{\psi}^j] < (ME)_c, \quad \|FNLS(t)\tilde{\psi}^j\|_{S(\Lambda_{s_c})} < \infty.$$

If, on the other hand, for the given j, $t_n^j \to t'$ finite, then by the continuity of the linear flow in H^s, we have

$$U(-t_n^j)\psi^j \to U(-t')\psi^j \quad \text{strongly in} \ \ H^s.$$

In this case, we set $\tilde{\psi}^j = FNLS(t')[U(-t')\psi^j]$ so that $FNLS(-t')\tilde{\psi}^j = U(-t')\psi^j$.

Above all, in either case, we have a new profile $\tilde{\psi}^j$ for the given ψ^j such that

$$\|FNLS(-t_n^j)\tilde{\psi}^j - U(-t_n^j)\psi^j\|_{H^s} \to 0, \ \text{ as } \ n \to +\infty.$$

As a result, we can replace $U(-t_n^j)\psi^j$ by $FNLS(-t_n^j)\tilde{\psi}^j$ in (4.10) and obtain

$$u_{n,0}(x) = \sum_{j=1}^{M} FNLS(-t_n^j)\tilde{\psi}^j(x) + \tilde{W}_n^M(x),$$

where

$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|U(t)\tilde{W}_n^M\|_{S(\Lambda_{s_c})}] = 0.$$

To use the perturbation theory to obtain a contradiction, we set $v^j(t) = FNLS(t)\tilde{\psi}^j$, $u_n(t) = FNLS(t)u_{n,0}$ and

$$\tilde{u}_n(t) = \sum_{j=1}^{M} v^j(t - t_n^j).$$

Then, we have

$$i\partial_t\tilde{u}_n - (-\Delta)^s\tilde{u}_n + (\frac{1}{|\cdot|^\gamma} * |\tilde{u}_n|^2)\tilde{u}_n = e_n,$$

where

$$e_n = (\frac{1}{|\cdot|^\gamma} * |\tilde{u}_n|^2)\tilde{u}_n - \sum_{j=1}^{M}(\frac{1}{|\cdot|^\gamma} * |v^j(t-t_n^j)|^2)v^j(t-t_n^j).$$

In the near future, we will prove the following two claims to obtain the contradiction:

- Claim 1 – There exists a large constant A independent of M such that the following holds: For any M, there exists $n_0 = n_0(M)$ such that for $n > n_0$, $\|\tilde{u}_n\|_{S(\Lambda_{s_c})} \le A$.
- Claim 2 – For each M and $\epsilon > 0$, there exist $n_1 = n_1(M,\epsilon)$ such that for $n > n_1$, $\|e_n\|_{L^{q_1'}L^{r_1'}} \le \epsilon$ for some pair $(q_1, r_1) \in \Lambda_{-s_c}$.

Note that if the two claims hold true, because $\tilde{u}_n(0) - u_n(0) = \tilde{W}_n^M$, there exists $M_1 = M_1(\epsilon)$ such that for each $M > M_1$, there exists $n_2 = n_2(M)$ satisfying $\|U(t)(\tilde{u}_n(0) - u_n(0))\|_{S(\Lambda_{s_c})} \le \epsilon$. Thus, now by the long-time perturbation theory Proposition 2.7, we have for sufficiently large n and M that $\|u_n\|_{S(\Lambda_{s_c})} < \infty$, which is a contradiction, giving Claim A. Thus, it suffices to show the above claims.

Let M_0 be sufficiently large such that $\|U(t)\tilde{W}_n^{M_0}\|_{S(\Lambda_{s_c})} \le \delta_{sd}$. Thus, we know from the definition of $\tilde{W}_n^{M_0}$ that for each $j > M_0$, it holds that $\|U(t)v^j(-t_n^j)\|_{S(\Lambda_{s_c})} \le \delta_{sd}$. Similar to the small data scattering and Proposition 3.8, we obtain

$$\|v^j(t-t_n^j)\|_{S(\Lambda_{s_c})} \le 2\|U(t)v^j(-t_n^j)\|_{S(\Lambda_{s_c})} \le 2\delta_{sd}, \tag{4.13}$$

and

$$\|D^{s_c}v^j(t-t_n^j)\|_{S(\Lambda_0)} \le c\|v^j(-t_n^j)\|_{\dot{H}^{s_c}} \quad \text{for } j > M_0. \tag{4.14}$$

Recall that $\|v^j(-t_n^j) - U(-t_n^j)\psi^j\|_{\dot{H}^{s_c}} \to 0$ as $n \to +\infty$. Then, (4.14) implies for n large and $j > M_0$ that

$$\|D^{s_c}v^j(t-t_n^j)\|_{S(\Lambda_0)} \le c\|U(-t_n^j)\psi^j\|_{\dot{H}^{s_c}} = c\|\psi^j\|_{\dot{H}^{s_c}}. \tag{4.15}$$

Thus, by elementary calculation, we have that

$$\begin{aligned}\|\tilde{u}_n\|_{L^{q_c}L^{q_c}}^{q_c} &= \sum_{j=1}^{M_0}\|v^j\|_{L^{q_c}L^{q_c}}^{q_c} + \sum_{j=M_0+1}^{M}\|v^j\|_{L^{q_c}L^{q_c}}^{q_c} + crossterms \\ &\le \sum_{j=1}^{M_0}\|v^j\|_{L^{q_c}L^{q_c}}^{q_c} + c\sum_{j=M_0+1}^{M}\|\psi^j\|_{\dot{H}^{s_c}}^{q_c} + crossterms.\end{aligned} \tag{4.16}$$

Note first that by (4.3), the *crossterm* can be made bounded by taking n_0 as sufficiently large. On the other hand, by (4.10) and Lemma 4.1,

$$\|u_{n,0}\|^2_{\dot H^{s_c}} = \sum_{j=1}^{M_0} \|\psi^j\|^2_{\dot H^{s_c}} + \sum_{j=M_0+1}^{M} \|\psi^j\|^2_{\dot H^{s_c}} + \|W_n^M\|^2_{\dot H^{s_c}} + o_n(1), \tag{4.17}$$

which shows that the quantity $\sum_{j=M_0+1}^{M} \|\psi^j\|^{\frac{2(N+2s)}{N+2s-\gamma}}_{\dot H^{s_c}}$ is bounded independently of M. Hence, (4.16) gives that $\|\tilde u_n\|_{L^{q_c}L^{q_c}}$ is bounded independently of M for $n > n_0$. A similar argument will show that $\|\tilde u_n\|_{L^\infty L^{\frac{2N}{N-2s_c}}}$ is also bounded independently of M provided that $n > n_0$ is sufficiently large. According to the definition of the Strichartz norm introduced in Section 2, the boundness of $\|\tilde u_n\|_{S(\Lambda_{s_c})}$ can be obtained by interpolation between the two exponents. Then, finally, we have obtained that Claim 1 holds true.

Now, we turn to prove the second claim. We easily have the following expansion of e_n:

$$\begin{aligned}
e_n &= \left(\frac{1}{|\cdot|^\gamma} * |\sum_{j=1}^{M} v^j(t-t_n^j)|^2\right) \sum_{j=1}^{M} v^j(t-t_n^j) - \sum_{j=1}^{M} \left(\frac{1}{|\cdot|^\gamma} * |v^j(t-t_n^j)|^2\right) v^j(t-t_n^j) \\
&= \left(\frac{1}{|\cdot|^\gamma} * \left(|\sum_{j=1}^{M} v^j(t-t_n^j)|^2 - \sum_{j=1}^{M} |v^j(t-t_n^j)|^2\right)\right) \sum_{j=1}^{M} v^j(t-t_n^j) \\
&\quad + \left(\frac{1}{|\cdot|^\gamma} * \sum_{j=1}^{M} |v^j(t-t_n^j)|^2\right) \sum_{j=1}^{M} v^j(t-t_n^j) - \sum_{j=1}^{M} \left(\frac{1}{|\cdot|^\gamma} * |v^j(t-t_n^j)|^2\right) v^j(t-t_n^j) \\
&= \left(\frac{1}{|\cdot|^\gamma} * \left(|\sum_{j=1}^{M} v^j(t-t_n^j)|^2 - \sum_{j=1}^{M} |v^j(t-t_n^j)|^2\right)\right) \sum_{j=1}^{M} v^j(t-t_n^j) \\
&\quad + \sum_{j=1}^{M} \left(\frac{1}{|\cdot|^\gamma} * |v^j(t-t_n^j)|^2\right) \sum_{k\neq j} v^k(t-t_n^k).
\end{aligned}$$

The focus now is on how to estimate the cross terms. Assume first that $j \neq k$ and $|t_n^j - t_n^k| \to +\infty$; then, taking one of the cross terms for example, we have

$$\left\|(\frac{1}{|\cdot|^\gamma} * |v^j|^2)(t-t_n^j) v^k(t-t_n^k)\right\|_{L^{q_1'}L^{r_1'}} = \left\|(\frac{1}{|\cdot|^\gamma} * |v^j|^2)(t) v^k(t+t_n^j-t_n^k)\right\|_{L^{q_1'}L^{r_1'}}. \tag{4.18}$$

Using a similar argument as in (2.15), for the above pair $(q_1, r_1) \in \Lambda_{-s_c}$, we can find $(q_2, r_2) \in \Lambda_{s_c}$ and apply the Hardy–Littlewood–Sobolev inequality and Hölder inequalities to obtain

$$\begin{aligned}
\left\|(\frac{1}{|\cdot|^\gamma} * |v^j|^2)(t) v^k(t+t_n^j-t_n^k)\right\|_{L^{q_1'}L^{r_1'}} &\le \|v^j\|^2_{L^{q_c}L^{r_c}} \|v^k\|_{L^{q_2}L^{r_2}} \\
&\le \|v^j\|^2_{S(\Lambda_{s_c};I_j)} \|v^k\|_{S(\Lambda_{s_c};I_j)}.
\end{aligned}$$

If $j \neq k$, by (4.3), $|t_n^j - t_n^k| \to +\infty$, and then, we find that (4.18) goes to zero as $n \to \infty$. Observe that all other cross terms will have the same property through similar estimates, and we have proved Claim 2.

Claim 1 and Claim 2 imply **Claim A**. We have reduced the profile expansion to the case in which $\psi^1 \neq 0$, and $\psi^j = 0$ for all $j \geq 2$. We now begin to show the existence of a critical solution.

By (4.11), we have $M[\psi^1] \leq 1$, and by (4.12), we have $\lim_{n\to+\infty} E[U(-t_n^1)\psi^1] \leq (ME)_c$. If t_n^1 converges and, without loss of generality, $t_n^1 \to 0$ as $n \to +\infty$, we take $\tilde{\psi}^1 = \psi^1$, and then, we have $\|FNLS(-t_n^1)\tilde{\psi}^1 - U(-t_n^1)\psi^1\|_{H^s} \to 0$ as $n \to +\infty$. If, on the other hand, $t_n^1 \to +\infty$, then by the proof of Lemma 4.3, we have again $\lim_{n\to+\infty} V(U(-t_n^1)\psi^1) = 0$, and thus,

$$\frac{1}{2}\|D^s\psi^1\|_2^2 = \lim_{n\to+\infty} E[U(-t_n^1)\psi^1] \leq (ME)_c.$$

Therefore, by Proposition 3.8, there exist $\tilde{\psi}^1$ such that $M[\tilde{\psi}^1] = M[\psi^1] \leq 1$, $E[\tilde{\psi}^1] = \frac{1}{2}\|D^s\psi^1\|_2^2 \leq (ME)_c$, and $\|FNLS(-t_n^1)\tilde{\psi}^1 - U(-t_n^1)\psi^1\|_{H^s} \to 0$ as $n \to +\infty$.

In either case, if we set $\tilde{W}_n^M = W_n^M + (U(-t_n^1)\psi^1 - FNLS(-t_n^1)\tilde{\psi}^1)$, then by the Strichartz estimates, we have

$$\|U(t)\tilde{W}_n^M\|_{S(\Lambda_{s_c})} \leq \|U(t)W_n^M\|_{S(\Lambda_{s_c})} + c\|U(-t_n^1)\psi^1 - FNLS(-t_n^1)\tilde{\psi}^1\|_{S(\Lambda_{s_c})},$$

and thus,

$$\lim_{n\to+\infty} \|U(t)\tilde{W}_n^M\|_{S(\Lambda_{s_c})} = \lim_{n\to+\infty} \|U(t)W_n^M\|_{S(\Lambda_{s_c})}.$$

Therefore, we have

$$u_{n,0} = FNLS(-t_n^1)\tilde{\psi}^1) + \tilde{W}_n^M$$

with $M(\tilde{\psi}^1) \leq 1$, $E(\tilde{\psi}^1) \leq (ME)_c$ and $\lim_{M\to+\infty} [\lim_{n\to+\infty} \|U(t)\tilde{W}_n^M\|_{S(\Lambda_{s_c})}] = 0$. Let $u_c = u_c(t)$ be the solution to (1.1) with initial data $u_{c,0} = \psi^1$. Now, if we claim that $\|u_c\|_{S(\Lambda_{s_c})} = \infty$, then it must hold that $M[u_c] = 1$ and $E[u_c] = (ME)_c$, which will complete the proof. Thus, it suffices to establish this claim. We argue by contradiction to suppose otherwise that

$$A \equiv \|FNLS(t - t_n^1)\tilde{\psi}^1\|_{S(\Lambda_{s_c})} = \|FNLS(t)\tilde{\psi}^1\|_{S(\Lambda_{s_c})} = \|u_c\|_{S(\Lambda_{s_c})} < \infty.$$

By the long-time perturbation theory Proposition 2.7, we obtain $\epsilon_0 = \epsilon_0(A)$. Taking M as sufficiently large and $n_2(M)$ as large enough that for $n > n_2$, it holds that $\|W_n^M\|_{S(\Lambda_{s_c})} \leq \epsilon_0$. Similar to the proof in the first case, Proposition 2.7 implies that there exists a large n such that $\|u_c\|_{S(\Lambda_{s_c})} < \infty$, which is a contradiction. □

Proposition 4.5 *(Precompactness of the flow of the critical solution). Let $u_c = u_c(t)$ be as in Proposition 4.4; then, if $\|u_c\|_{S([0,+\infty);\Lambda_{s_c})} = \infty$,*

$$\{u_c(t, \cdot)|\ t \in [0, +\infty)\} \subset H^s$$

is precompact in H^s. A corresponding conclusion is reached if $\|u_c\|_{S((-\infty,0];\Lambda_{s_c})} = \infty$.

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Proof. We will argue by contradiction and write $u = u_c$ for short. Otherwise, we will obtain an $\eta > 0$ and a sequence $t_n \to +\infty$ such that for all $n \neq n'$,

$$\|u(t_n, \cdot) - u(t_{n'}, \cdot)\|_{H^s} \geq \eta. \tag{4.19}$$

We take $\phi_n = u(t_n)$ in the profile expansion Lemma 4.1 to obtain the profiles ψ^j and a remainder W_n^M such that $u(t_n) = \sum_{j=1}^M U(-t_n^j)\psi^j + W_n^M$ with $|t_n^j - t_n^k| \to +\infty$ as $n \to +\infty$ for any $j \neq k$. Then, Lemma 4.3 gives

$$\sum_{j=1}^{M} \lim_{n\to+\infty} E[U(-t_n^j)\psi^j] + \lim_{n\to+\infty} E[W_n^M] = E[u(t_n)] = (ME)_c.$$

Similar to the proof of Lemma 3.7, we know that each energy is non-negative, and thus, for any j,

$$\lim_{n\to+\infty} E[U(-t_n^j)\psi^j] \leq (ME)_c.$$

Moreover, by (4.5), we have

$$\sum_{j=1}^{M} M[\psi^j] + \lim_{n\to+\infty} M[W_n^M] = \lim_{n\to+\infty} M[u(t_n)] = 1.$$

If more than one $\psi^j \neq 0$, following the proof in Proposition 4.4, we can show that this case will contradict the definition of the critical solution $u = u_c$. Thus, we will address the case in which only $\psi^1 \neq 0$ and $\psi^j = 0$ for all $j > 1$, and thus,

$$u(t_n) = U(-t_n^1)\psi^1 + W_n^M. \tag{4.20}$$

In addition, as in the proof of Proposition 4.4, we find that $M[\psi^1] = 1$, $\lim_{n\to+\infty} E[U(-t_n^1)\psi^1] = (ME)_c$, $\lim_{n\to+\infty} M[W_n^M] = 0$ and $\lim_{n\to+\infty} E[W_n^M] = 0$. Thus, by Lemma 3.7, we obtain

$$\lim_{n\to+\infty} \|W_n^M\|_{H^s} = 0. \tag{4.21}$$

We claim now that t_n^1 converges to some finite t^1 up to a subsequence. Note that if this holds, because $U(-t_n^1)\psi^1 \to e^{-it^1\Delta}\psi^1$ in H^s and by (4.20), (4.21) implies that $u(t_n)$ converges in H^s, which contradicts (4.19); we thus conclude our proof.

Now, we show the above claim by contradiction. Suppose that $t_n^1 \to -\infty$. Then,

$$\|U(t)u(t_n)\|_{S(\Lambda_{s_c};[0,+\infty))} \leq \|U(t-t_n^1)\psi^1\|_{S(\Lambda_{s_c};[0,+\infty))} + \|U(t)W_n^M\|_{S(\Lambda_{s_c};[0,+\infty))}.$$

Because

$$\lim_{n\to+\infty} \|U(t-t_n^1)\psi^1\|_{S(\Lambda_{s_c};[0,+\infty))} = \lim_{n\to+\infty} \|U(t)\psi^1\|_{S(\Lambda_{s_c};[-t_n^1,+\infty))} = 0$$

and $\|U(t)W_n^M\|_{S(\Lambda_{s_c})} \leq \frac{1}{2}\delta_{sd}$, by taking n as sufficiently large, we obtain a contradiction to the small data scattering theory. If other $t_n^1 \to +\infty$, we similarly obtain

$$\|U(t)u(t_n)\|_{S(\Lambda_{s_c};(-\infty,0])} \leq \frac{1}{2}\delta_{sd}.$$

Thus, the small data scattering theory (Proposition 2.5) shows that

$$\|u\|_{S(\Lambda_{s_c};(-\infty,t_n])} \leq \delta_{sd}.$$

Because $t_n \to +\infty$ by the assumption in the beginning of our proof, sending $n \to +\infty$, we obtain $\|u\|_{S(\Lambda_{s_c};(-\infty,+\infty))} \leq \delta_{sd}$, which is a contradiction. □

Corollary 4.6. *Let $u = u(t)$ be a solution to (1.1) such that $\mathcal{K}^+ = \{u(t,\cdot)|\ t \in [0,+\infty)\}$ is precompact in H_r^s. Then, for each $\epsilon > 0$, there exists $R > 0$ such that*

$$\int\limits_{|x|>R} |D^s u(t,x)|^2 + |u(t,x)|^2 + (\frac{1}{|\cdot|^\gamma} * |u|^2)|u|^2(t,x)dx \leq \epsilon.$$

Proof. If not, for any $R > 0$, there exists $\epsilon_0 > 0$ and a sequence t_n such that

$$\int\limits_{|x|>R} |D^s u(t_n,x)|^2 + |u(t_n,x)|^2 + (\frac{1}{|\cdot|^\gamma} * |u|^2)|u|^2(t_n,x)dx \geq \epsilon_0.$$

By the precompactness of $\mathcal{K}^+$, there exists $\phi \in H^s$ such that, up to a subsequence of t_n, we have $u(t_n,\cdot) \to \phi$ in H^s. Thus, for any $R > 0$, we obtain

$$\int\limits_{|x|>R} |D^s \phi(x)|^2 + |\phi(x)|^2 + (\frac{1}{|\cdot|^\gamma} * |\phi|^2)|\phi|^2(x)dx \geq \epsilon_0,$$

from which we can easily obtain a contradiction because $\phi \in H^s$ and $V(\phi) \leq c\|\phi\|_{H^s}^4$ by the Hardy–Littlewood–Sobolev inequality. □

5. Rigidity theorem

In this section, we will prove the following Liouville-type theorem.

Theorem 5.1. *Let $N \geq 2$ and $2s < \gamma < \min\{N, 4s\}$. Suppose that $u_0 \in H^s$ is radial and that $u_0 \in K_1$, i.e.,*

$$M[u_0]^{\frac{s-s_c}{s_c}} E[u_0] < M[Q]^{\frac{s-s_c}{s_c}} E[Q], \tag{5.1}$$

and

$$M[u_0]^{\frac{s-s_c}{s_c}} \|u_0\|_{\dot{H}^s}^2 < M[Q]^{\frac{s-s_c}{s_c}} \|Q\|_{\dot{H}^s}^2. \tag{5.2}$$

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Let $u = u(t)$ be the global solution of (1.1) with initial data u_0, and it holds that $\mathcal{K}^+ = \{u(t,\cdot)|\ t \in [0,+\infty)\}$ is precompact in H^s. Then, $u_0 = 0$. The same conclusion holds if $\mathcal{K}^- = \{u(t,\cdot) : t \in (-\infty, 0]\}$ is precompact in H^s.

Before proving the rigidity theorem, we follow the same idea of [3] to introduce the localized virial estimate for the radial solutions of (1.1).

For $u \in H^s$ with $s \geq \frac{1}{2}$, we need the auxiliary function $u_m = u_m(t,x)$, defined as

$$u_m := c_s \frac{1}{-\Delta + m} u(t) = c_s \mathcal{F}^{-1} \frac{\widehat{u}(t,\xi)}{|\xi|^2 + m} \tag{5.3}$$

with $c_s = \sqrt{\frac{sin\pi s}{\pi}}$, turns out to be a convenient normalization factor. By Balakrishnan's formula in semi-group theory used in [3], for any $u \in H^s$, we have the identity

$$\int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 dx dm = s\|(-\Delta)^{\frac{s}{2}} u\|_2^2. \tag{5.4}$$

We obtain a counterpart of Corollary 4.6.

Corollary 5.2. *Let $u = u(t,x)$ be a solution to (1.1) such that $\mathcal{K}^+ = \{u(t,\cdot)|\ t \in [0,+\infty)\}$ is precompact in H_r^s. Then, for each $\epsilon > 0$, there exists $R > 0$ such that*

$$\int_0^\infty m^s \int_{|x|>R} |\nabla u_m|^2 dx dm + \int_{|x|>R} |u(t,x)|^2 + (\frac{1}{|\cdot|^\gamma} * |u|^2)|u|^2(t,x) dx \leq \epsilon.$$

Proof of Theorem 5.1. It suffices to address the $\mathcal{K}^+$ case, since the $\mathcal{K}^-$ case follows similarly. For some given real-valued function $\varphi \in C_c^\infty$, which is radial, with

$$\varphi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

For $R > 0$, define the localized virial of $u = u(t,x) \in H^s$ to be the quantity given by

$$\mathcal{M}_R(t) := 2Im \int_{\mathbb{R}^N} \bar{u}(t,x) R \nabla\varphi(\frac{x}{R}) \cdot \nabla u(t,x) dx.$$

Following the method used in [3], we have the identity

$$\mathcal{M}_R'(t) = \int_0^\infty m^s \int_{\mathbb{R}^N} \left(4\overline{\partial_k u_m}(\partial_{kl}^2 \varphi(\frac{x}{R}))\partial_l u_m - (\frac{1}{R^2}\Delta^2\varphi(\frac{x}{R}))|u_m|^2 \right) dx dm + I,$$

where

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$$I = 2R \int_{\mathbb{R}^N} \nabla\phi(\frac{x}{R})(\nabla(\frac{1}{|\cdot|^\gamma}) * |u|^2)|u|^2 dx$$
$$= -\gamma R \int\int (\nabla\phi(\frac{x}{R}) - \nabla\phi(\frac{x}{R})) \cdot \frac{x-y}{|x-y|^{\gamma+2}} |u(x)|^2 |u(y)|^2$$

By the definition of φ, we have

$$\mathcal{M}'_R(t) = 8\int_0^\infty m^s \int_{|x|\le R} |\nabla u_m|^2 dx + 4\int_0^\infty m^s \int_{R<|x|<2R} \partial_r^2 \varphi\left(\frac{x}{R}\right) |\nabla u_m|^2 dx dm \tag{5.5}$$
$$- \frac{1}{R^2}\int_0^\infty m^s \int_{|x|>R} \Delta^2\varphi\left(\frac{x}{R}\right)|u_m|^2 dx dm + I.$$

We rewrite I as

$$I = -\gamma R \int\int \left(\nabla\varphi\left(\frac{x}{R}\right) - \nabla\varphi\left(\frac{y}{R}\right)\right) \cdot \frac{x-y}{|x-y|^{\gamma+2}} |u(x)|^2|u(y)|^2 dx dy$$
$$= -2\gamma \int\int_{\{|x|\le R, |y|\le R\}} \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dx dy$$
$$- \gamma R \left[\int\int_\Omega + \int\int_\Lambda\right] \left(\nabla\varphi\left(\frac{x}{R}\right) - \nabla\varphi\left(\frac{y}{R}\right)\right) \frac{x-y}{|x-y|^{\gamma+2}} |u(x)|^2|u(y)|^2 dx dy,$$

where

$$\Omega = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : R < |x| < 2R\} \bigcup \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : R < |y| < 2R\}$$

and

$$\Lambda = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x| > 2R, |y| < R\} \bigcup \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x| < R, |y| > 2R\}.$$

Then, by the properties of φ, we estimate I in the following form.

$$I = -2\gamma \int\int \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dx dy$$
$$+ O\left(\int\int_{\{|x|\ge R\}} \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dx dy + \int\int_{\{|y|\ge R\}} \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dx dy\right)$$
$$+ O\left(R \int\int_{\{|x|>R, |x-y|>\frac{R}{2}\}} \left(\nabla\varphi\left(\frac{x}{R}\right) - \nabla\varphi\left(\frac{y}{R}\right)\right) \frac{x-y}{|x-y|^{\gamma+2}} |u(x)|^2|u(y)|^2 dx dy\right)$$

$$+ O\left(R \int \int_{\{|x|>R, |x-y|<\frac{R}{2}\}} \left(\nabla\varphi\left(\frac{x}{R}\right) - \nabla\varphi\left(\frac{y}{R}\right)\right) \frac{x-y}{|x-y|^{\gamma+2}} |u(x)|^2 |u(y)|^2 dxdy \right)$$

$$= -2\gamma \int\int \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dxdy + O\left(\int_{|x|>R} (\frac{1}{|\cdot|^\gamma} * |u|^2)|u|^2 dx \right).$$

From (5.5), we obtain

$$\mathcal{M}'_R(t) = 8\int_0^\infty m^s \int_{|x|\le R} |\nabla u_m|^2 dx + 4\int_0^\infty m^s \int_{R<|x|<2R} \partial_r^2 \varphi\left(\frac{x}{R}\right) |\nabla u_m|^2 dxdm$$

$$- \frac{1}{R^2} \int_0^\infty m^s \int_{|x|>R} \Delta^2 \varphi\left(\frac{x}{R}\right) |u_m|^2 dxdm + I$$

$$\ge \left(8\int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 dx - 2\gamma V(u) \right) + A_R(u)$$

$$= 2\gamma \left(\frac{4s}{\gamma} \|D^s u\|_2^2 - V(u) \right) + A_R(u),$$

where by Corollary 5.2,

$$A_R(u(t)) \le c \left(\|D^s u\|^2_{L^2(|x|>R)} + \frac{1}{R^2}\|u\|^2_{L^2(|x|>R)} + \int_{|x|>R} (\frac{1}{|\cdot|^\gamma} * |u|^2)|u|^2 dx \right) \tag{5.6}$$

$$\to 0, \quad \text{as} \quad R \to +\infty.$$

Let $\delta \in (0, 1)$ be a positive constant satisfying $E[u_0] < (1-\delta)E[Q]M[Q]^{\frac{s-s_c}{s_c}}$. It follows from Lemma 3.6 and Lemma 3.7 that $\frac{4s}{\gamma}\|D^s u\|_2^2 - V(u) \ge C_\delta \|D^s u_0\|_2^2$, and for large R,

$$\mathcal{M}'_R(t) \ge C_\delta \|D^s u_0\|_2^2. \tag{5.7}$$

Integrating (5.7) over $[0, t]$, we obtain

$$|\mathcal{M}_R(t) - \mathcal{M}_R(0)| \ge C_\delta t \|D^s u_0\|_2^2$$

On the other hand, by [3], we should have

$$|\mathcal{M}_R(t) - \mathcal{M}_R(0)| \le C_R(\|u\|^2_{H^{\frac{1}{2}}} + \|u_0\|^2_{H^{\frac{1}{2}}}) \le C_R(\|u\|^2_{H^s} + \|u_0\|^2_{H^s}) \le C_R\|Q\|^2_{H^s},$$

which is a contradiction for large t unless $u_0 = 0$. □

Now, we can finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that by Proposition 4.5, the critical solution u_c constructed in Section 4 satisfies the hypotheses in Theorem 5.1. Therefore, to complete the proof of Theorem 1.1, we should apply Theorem 5.1 to u_c and find that $u_{c,0} = 0$, which contradicts the fact that $\|u_c\|_{S(\Lambda_{s_c})} = +\infty$. This contradiction shows that $SC(u_0)$ holds. Thus, by Proposition 2.6, we have shown that H^s scattering holds. □

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