

Time periodic and almost time periodic solutions to rotating stratified fluids subject to large forces

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Abstract

Consider the 3D incompressible Boussinesq equations for rotating stably stratified fluids. It is shown that this set of equations possesses a unique time periodic or almost time periodic solutions for external forces satisfying these properties, which, however, do *not necessarily need to be small*. An explicit bound on the size of the external force, depending on the buoyancy frequency N , is given, which then allows for the unique existence of time periodic or almost periodic solutions. In particular, the size of the external forces can be taken large with respect to the buoyancy frequency. The approach depends crucially on the *dispersive effect* of the rotation and the stable stratification.

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1. Introduction

Consider the time periodic problem for the 3D Boussinesq equations, describing the motion of viscous, incompressible fluids under the effects of rotation and stable stratification on the halfline \mathbb{R}_+

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$$\begin{cases} \partial_t v + (v \cdot \nabla)v = \nu \Delta v - \Omega e_3 \times v - \nabla p + \theta e_3 + g & t > 0, x \in \mathbb{R}^3, \\ \partial_t \theta + (v \cdot \nabla)\theta = \kappa \Delta \theta - N^2 v_3 + h & t > 0, x \in \mathbb{R}^3, \\ \nabla \cdot v = 0 & t > 0, x \in \mathbb{R}^3, \\ v(0, x) = v_0(x), \quad \theta(0, x) = \theta_0(x) & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

as well as the periodic and almost periodic problem on the whole time line \mathbb{R}

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = \nu \Delta v - \Omega e_3 \times v - \nabla p + \theta e_3 + g & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ \partial_t \theta + (v \cdot \nabla)\theta = \kappa \Delta \theta - N^2 v_3 + h & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ \nabla \cdot v = 0 & t \in \mathbb{R}, x \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

where the rotation and the mean stratification gradient are aligned parallel to the vertical axis $e_3 = (0, 0, 1)^T$.

Here, $v = (v_1(t, x), v_2(t, x), v_3(t, x))^T$, $p = p(t, x)$ and $\theta = \theta(t, x)$ are the unknown functions, representing the velocity field, the scalar pressure and the thermal disturbance about a mean state in hydrostatic balance, respectively, while $g = (g_1(t, x), g_2(t, x), g_3(t, x))^T$ and $h = h(t, x)$ are given time periodic external forces. The initial velocity and initial thermal disturbance are denoted by $v_0 = (v_{0,1}(x), v_{0,2}(x), v_{0,3}(x))^T$ and $\theta_0 = \theta_0(x)$. The coefficients $\nu > 0$ and $\kappa > 0$ denote the kinetic viscosity constant and the heat conductivity constant. $N > 0$ is the Brunt–Väisälä (buoyancy) frequency for the constant stratification and $\Omega \in \mathbb{R} \setminus \{0\}$ is the angular frequency of the background rotation. For the derivation and the physical background of the above systems (1.1) and (1.2), we refer e.g. to Majda [27], Babin, Mahalov, Nicolaenko [2–4], and Chemin, Desjardin, Gallagher and Grenier [9].

Observe that in the special case $N = \Omega = 0$ and $\theta = h = 0$, the system (1.2) corresponds to the classical Navier–Stokes equations in the whole time line:

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = \nu \Delta v - \nabla p + g & t \in \mathbb{R}, x \in \mathbb{R}^3, \\ \nabla \cdot v = 0 & t \in \mathbb{R}, x \in \mathbb{R}^3. \end{cases} \quad (1.3)$$

The problem of existence of time periodic or almost time periodic solutions to viscous, incompressible fluids with corresponding periodic forces has a long history. For a recent survey on periodic solutions for the Navier–Stokes equations, see the article by Galdi and Kyed [12]. Before commenting on several key results in this direction below, let us note that typical results on the existence of unique mild or strong time periodic solutions to (1.3) always require suitable *smallness* conditions on the outer forces.

The main objective of this manuscript is to prove the unique existence of time periodic and almost time periodic solutions to (1.1) and (1.2) for external forces, which are *not necessarily small*. To be more precise, we develop an explicit bound on the size of the external force by the *buoyancy frequency* N , which then allows for the unique existence of time periodic solutions to (1.1) and (1.2). Our approach is crucially based on the *dispersive effect* of the rotation and the stable stratification.

Note that our systems (1.1) and (1.2) are invariant under the scaling

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad F_\lambda(t, x) = \lambda^2 F(\lambda^2 t, \lambda x), \quad N_\lambda = \lambda^2 N$$

for $\lambda > 0$ (see (2.3) in Section 2). Thus, a natural scaling invariant space for time periodic external forces F is $L^{\frac{3}{2}}(\mathbb{R}^3)$ or $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$ and the works [13,16,36] on the unique existences of time periodic and almost periodic solutions to the Navier–Stokes equations (1.3) and related systems were proved for small external forces F in the scaling critical space $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$.

In the following, we decompose the time periodic external force via the Craya–Herring cyclic basis

$$\mathbb{P}F = P_+F + P_-F + P_0F,$$

and making use of the dispersive estimates for rotation and stable stratification, we are able to show the unique existence of time periodic and almost periodic solutions to (1.1) and (1.2) for large $P_{\pm}F$ in $L^{p,\infty}(\mathbb{R}^3)$ with some $p > 3/2$ and small P_0F in the $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, provided the buoyancy frequency N is sufficiently high.

Some comments on previous works on the system (1.1) and on related time periodic problems are in order: Babin, Mahalov and Nicolaenko [3,4] proved global well-posedness for the initial value problem (1.1) for the case of large N and $|\Omega|$ in the space periodic setting \mathbb{T}^3 . Charve [6] proved a similar result in the whole space \mathbb{R}^3 by making use of the dispersive nature of the rotation and the stable stratification. The asymptotics of solutions, as the buoyancy frequency N and the rotation speed $|\Omega|$ tend to infinity, was studied in [3,4,7,8,11].

Maremonti [28,29] proved the existence of time-periodic solutions for the Navier–Stokes equations (1.3) in the whole space \mathbb{R}^3 as well as in the half space \mathbb{R}_+^3 assuming a smallness condition on the external force. Kozono–Nakao [23] introduced the notion of mild solutions to the time periodic problem for (1.3), and showed the unique existence of time periodic mild solutions in \mathbb{R}^d , \mathbb{R}_+^d with $d \geq 3$ and in exterior domains of \mathbb{R}^d with $d \geq 4$.

Yamazaki [36] developed an approach to (1.3) with time-dependent external force by interpolation techniques of weak L^d spaces, and proved the existence and uniqueness of time periodic solutions on exterior domains in \mathbb{R}^d with $d \geq 3$ for *small* time periodic external forces. For stability as well as for spatial asymptotic assertions for the time periodic solutions to (1.3), we refer to [19,33]. Time periodic solutions to (1.3) in two- or three-dimensional bounded domains were studied in [20,30,32,34].

Geissert–Hieber–Nguyen [13] established a general approach to time periodic problem arising in incompressible viscous fluid flows. Based on that approach, they proved the unique existence of time periodic solutions to the Navier–Stokes–Oseen system on exterior domains, to the Navier–Stokes flow past rotating obstacles, to the rotating Navier–Stokes equations under a suitable *smallness* condition on the force. That approach was extended to the almost periodic setting in the whole time line in [16]. For the time periodic solution to the rotating Navier–Stokes equations and the Boussinesq system, we also refer to [17,21,22,35].

This paper is organized as follows. Section 2 is devoted to the presentation of our main results. In Section 3 we establish linear homogeneous estimates for the semigroup e^{-tL_N} and in Section 4 we give linear and nonlinear inhomogeneous estimates for the Duhamel terms via real interpolation arguments. Then, in Sections 5 and 6, we present the proofs of Theorem 2.1 and Theorems 2.2–2.3, respectively.

Throughout this paper, we shall denote by C the constants which may change from line to line. In particular, $C = C(\cdot, \dots, \cdot)$ will denote the constants, which depend only on the quantities appearing in parentheses.

2. Preliminaries and main results

We start by rewriting the systems (1.1) and (1.2) within the Craya–Herring cyclic basis and refer to [18, Section 2] for the details on this basis. Combining the velocity field with the rescaled thermal disturbance, we introduce the function $u := (v, \theta/N)^T = (v_1, v_2, v_3, \theta/N)^T$ and put

$$L_{v,\kappa} := \begin{pmatrix} -v\Delta & 0 & 0 & 0 \\ 0 & -v\Delta & 0 & 0 \\ 0 & 0 & -v\Delta & 0 \\ 0 & 0 & 0 & -\kappa\Delta \end{pmatrix}, \quad J_{\Omega,N} := \begin{pmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & 0 \end{pmatrix}$$

and $\tilde{\nabla} := (\nabla, 0)^T = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, 0)^T$. The original systems (1.1) and (1.2) may then be rewritten as

$$\begin{cases} \partial_t u + L_{v,\kappa} u + J_{\Omega,N} u + (u \cdot \tilde{\nabla})u + \tilde{\nabla} p = f, \\ \tilde{\nabla} \cdot u = 0, \\ u(0, x) = u_0(x) \end{cases} \quad (2.1)$$

where $f := (g, h/N)^T$ and $u_0 = (v_0, \theta_0/N)^T$. Next, we denote by \mathbb{P} the extended Helmholtz projection onto the divergence-free vector fields for the velocity v , which is defined by

$$\mathbb{P} := \left(\frac{(\delta_{jk} + R_j R_k)_{1 \leq j, k \leq 3}}{0} \middle| \frac{0}{1} \right),$$

where $\{R_j\}_{1 \leq j \leq 3}$ denote the Riesz transforms on \mathbb{R}^3 . Applying the Helmholtz projection \mathbb{P} to (2.1), we obtain the following evolution equation for u :

$$\begin{cases} \partial_t u + L_{v,\kappa} u + \mathbb{P} J_{\Omega,N} \mathbb{P} u + \mathbb{P}(u \cdot \tilde{\nabla})u = \mathbb{P} f, \\ \tilde{\nabla} \cdot u = 0, \\ u(0) = u_0. \end{cases} \quad (2.2)$$

Here, we used the fact that $\mathbb{P}u = u$ since $\tilde{\nabla} \cdot u = 0$.

We now introduce the three assumptions A1, A2 and A3 on the coefficients involved as well as on the external force in the system (2.2):

A1: $v = \kappa > 0$.

A2: The ratio $\mu := \frac{\Omega}{N} \in \mathbb{R} \setminus \{0\}$ is fixed and $\mu^2 \neq 1$.

A3: f is of the form $f = \tilde{\nabla} \cdot F$ for some $F = F(t, x) \in \mathbb{R}^{4 \times 4}$.

Since the size of the constants $v, \kappa > 0$ plays no role in our analysis, we set $v = \kappa = 1$ for simplicity. Then, under the above assumptions, the original systems (1.1) and (1.2) are transformed into the following form:

$$\begin{cases} \partial_t u - \Delta u + N \mathbb{P} J_\mu \mathbb{P} u + \mathbb{P} \tilde{\nabla} \cdot (u \otimes u) = \mathbb{P} \tilde{\nabla} \cdot F, & \tilde{\nabla} \cdot u = 0, \\ u(0, x) = u_0(x), \end{cases} \quad (2.3)$$

where J_μ is the skew-symmetric constant matrix defined by

$$J_\mu := \begin{pmatrix} 0 & -\mu & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For the frequency variable $\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$, we set

$$\xi_h := (\xi_1, \xi_2)^T, \quad \xi_\mu := (\xi_1, \xi_2, \mu\xi_3)^T$$

so that $|\xi_h| = \sqrt{\xi_1^2 + \xi_2^2}$ and $|\xi_\mu| = \sqrt{\xi_1^2 + \xi_2^2 + \mu^2\xi_3^2}$.

We now define an orthonormal basis $\{a_+(\xi), a_-(\xi), a_0(\xi), b_0(\xi)\}$ in \mathbb{C}^4 by

$$\begin{aligned} a_+(\xi) &:= \frac{1}{\sqrt{2}|\xi_h||\xi||\xi_\mu|} \begin{pmatrix} \mu\xi_2\xi_3|\xi| + i\xi_1\xi_3|\xi_\mu| \\ -\mu\xi_1\xi_3|\xi| + i\xi_2\xi_3|\xi_\mu| \\ -i(\xi_1^2 + \xi_2^2)|\xi_\mu| \\ (\xi_1^2 + \xi_2^2)|\xi| \end{pmatrix}, \\ a_-(\xi) &:= \overline{a_+(\xi)}, \quad a_0(\xi) := \frac{1}{|\xi_\mu|} \begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ \mu\xi_3 \end{pmatrix}, \quad b_0(\xi) := \frac{1}{|\xi|} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{pmatrix}. \end{aligned} \quad (2.4)$$

Note that $\langle \widehat{u}(\xi), b_0(\xi) \rangle_{\mathbb{C}^4} = 0$ for the vector fields u satisfying the divergence-free condition $\widetilde{\nabla} \cdot u = 0$.

Let $P(\xi)$ be the symbol matrix of \mathbb{P} defined as $\widehat{\mathbb{P}u}(\xi) = P(\xi)\widehat{u}(\xi)$, which is given explicitly by

$$P(\xi) := \left(\frac{\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2} \right)_{1 \leq j,k \leq 3}}{0} \middle| \begin{array}{c} 0 \\ 1 \end{array} \right).$$

Then, the eigenfrequencies of the skew-symmetric operator $-\mathbb{P}J_\mu\mathbb{P}$ are given by

$$\sigma[-P(\xi)J_\mu P(\xi)] = \left\{ \pm i \frac{|\xi_\mu|}{|\xi|}, 0, 0 \right\},$$

and the orthonormal basis $\{a_+(\xi), a_-(\xi), a_0(\xi), b_0(\xi)\}$ gives the corresponding eigenvectors in the frequency side. Therefore, the semigroup e^{-tL_N} on $X := \{u \in (L^p(\mathbb{R}^3))^4 \mid \widetilde{\nabla} \cdot u = 0\}$ ($1 < p < \infty$) generated by the linear operator $L_N := -\Delta + N\mathbb{P}J_\mu\mathbb{P}$ is explicitly given by

$$e^{-tL_N}\varphi = e^{t\Delta}e^{iNtp_\mu(D)}P_+\varphi + e^{t\Delta}e^{-iNtp_\mu(D)}P_-\varphi + e^{t\Delta}P_0\varphi, \quad (2.5)$$

where

$$\begin{aligned}
 P_j \varphi &:= \mathcal{F}^{-1} \left[\langle \widehat{\varphi}(\xi), a_j(\xi) \rangle_{\mathbb{C}^4} a_j(\xi) \right] \quad (j = \pm, 0), \\
 p_\mu(\xi) &= \frac{|\xi_\mu|}{|\xi|} = \frac{\sqrt{\xi_1^2 + \xi_2^2 + \mu^2 \xi_3^2}}{|\xi|}, \\
 e^{\pm i N t p_\mu(D)} f(x) &:= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi \pm i N t p_\mu(\xi)} \widehat{f}(\xi) d\xi.
 \end{aligned} \tag{2.6}$$

Following the ideas in [13,23] for the construction of time periodic solutions, we consider the integral equations:

$$u(t) = e^{-tL_N} u_0 + \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot \{F(\tau) - (u \otimes u)(\tau)\} d\tau \tag{2.7}$$

for (1.1) with $t > 0$, and

$$u(t) = \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot \{F(\tau) - (u \otimes u)(\tau)\} d\tau \tag{2.8}$$

for (1.2) with $t \in \mathbb{R}$.

We say that u is a *mild solution to the time periodic problem for (1.1) and (1.2)*, if u satisfies the integral equations (2.7) and (2.8) in some function space.

Our main results on time periodic solutions to (1.1) read as follows:

Theorem 2.1 (Periodic solutions on \mathbb{R}_+). *Suppose that Assumptions A1, A2 and A3 hold. Let $T > 0$, and let the exponents s and p satisfy*

$$0 < s < \frac{1}{2}, \quad \max \left\{ \frac{2-s}{3}, \frac{5+2s}{9} \right\} < \frac{1}{p} < \frac{2}{3}. \tag{2.9}$$

Then, there exist positive constants $\delta_1 = \delta_1(\mu, s, p)$, δ_2 and K such that for every $N > 0$ and for every T -periodic external force $F \in BC(\mathbb{R}; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$ satisfying

$$\begin{aligned}
 \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} &\leq \delta_1 N^{\frac{3}{2} \left(\frac{2}{3} - \frac{1}{p} \right)}, \\
 \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} &\leq \delta_2,
 \end{aligned} \tag{2.10}$$

there exists a unique $u_0 \in L^{3,\infty}(\mathbb{R}^3)$ and a unique T -periodic solution $u \in X_K$ to (2.7), where

$$X_K := \left\{ u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3)) \mid \sup_{t>0} \|u(t)\|_{L^{3,\infty}} \leq K \right\}.$$

The following two theorems concern the time periodic and almost periodic solutions to (1.2) in the whole time line.

Theorem 2.2 (Periodic solutions on \mathbb{R}). Suppose that Assumptions A1, A2 and A3 hold. Let $T > 0$ and let the exponents s and p satisfy (2.9) in Theorem 2.1. Then, there exist positive constants $\delta_1 = \delta_1(\mu, s, p)$, δ_2 and K such that for every $N > 0$ and for every T -periodic external force $F \in BC(\mathbb{R}; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$ satisfying

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t \in \mathbb{R}} \|P_- F(t)\|_{L^{p,\infty}} &\leq \delta_1 N^{\frac{3}{2}(\frac{2}{3} - \frac{1}{p})}, \\ \sup_{t \in \mathbb{R}} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} &\leq \delta_2, \end{aligned} \quad (2.11)$$

(2.8) admits a unique T -periodic solution $u \in Y_K$, where

$$Y_K := \left\{ u \in BC(\mathbb{R}; L^{3,\infty}(\mathbb{R}^3)) \mid \sup_{t \in \mathbb{R}} \|u(t)\|_{L^{3,\infty}} \leq K \right\}.$$

Our last theorem concerns the unique existence of almost periodic solutions to (1.2). Given a Banach space X , a function $f \in C(\mathbb{R}; X)$ is called *almost periodic* if for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that for each $a \in \mathbb{R}$ one can take $T \in [a, a + L_\varepsilon)$ such that

$$\sup_{t \in \mathbb{R}} \|f(t + T) - f(t)\|_X < \varepsilon.$$

We denote the space of all almost time periodic functions $f : \mathbb{R} \rightarrow X$ by $AP(\mathbb{R}; X)$. It is well known that $AP(\mathbb{R}; X)$ is a Banach space equipped with the norm $\|f\|_{AP} = \sup_{t \in \mathbb{R}} \|f(t)\|_X$. Moreover, for Banach spaces X, Y and Z , if $f \in AP(\mathbb{R}; X)$, $g \in AP(\mathbb{R}; Y)$ and the product $\cdot : X \times Y \rightarrow Z$ is continuous, then $g \cdot f \in AP(\mathbb{R}; Z)$.

For further information on almost periodic functions, we refer to the works of Corduneanu [10], Levitan–Zhikov [25], [1] and [16].

Theorem 2.3 (Almost periodic solutions on \mathbb{R}). Suppose that Assumptions A1, A2 and A3 hold. Let the exponents s and p satisfy (2.9) in Theorem 2.1. Then, there exist positive constants $\delta_1 = \delta_1(\mu, s, p)$, δ_2 and K such that for every $N > 0$ and for every almost periodic external force $F \in BC(\mathbb{R}; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$ satisfying

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t \in \mathbb{R}} \|P_- F(t)\|_{L^{p,\infty}} &\leq \delta_1 N^{\frac{3}{2}(\frac{2}{3} - \frac{1}{p})}, \\ \sup_{t \in \mathbb{R}} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} &\leq \delta_2, \end{aligned} \quad (2.12)$$

(2.8) admits a unique almost periodic solution $u \in Y_K$, where

$$Y_K := \left\{ u \in BC(\mathbb{R}; L^{3,\infty}(\mathbb{R}^3)) \mid \sup_{t \in \mathbb{R}} \|u(t)\|_{L^{3,\infty}} \leq K \right\}.$$

Remark 2.4. The size conditions (2.10)–(2.12) imply that for large $P_\pm F$ in the scaling sub-critical space $L^{p,\infty}(\mathbb{R}^3)$ and small $P_0 F$ in the scaling critical space $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, our systems

(1.1) and (1.2) possess unique time periodic and almost periodic mild solutions for a sufficiently high buoyancy frequency $N \geq N_0$ and a large Coriolis parameter $|\Omega| \geq |\mu|N_0$, where

$$N_0 \geq \left(\frac{1}{\delta_1} \left(\sup_t \|P_+ F(t)\|_{L^{p,\infty}} + \sup_t \|P_- F(t)\|_{L^{p,\infty}} \right) \right)^{\frac{2}{3(\frac{2}{3}-\frac{1}{p})}}.$$

It further follows from (2.10)–(2.12) that we need to assume the smallness condition in the scaling critical space only on $P_0 F$. In particular, the size of $P_{\pm} F$ can be taken proportionally to the strength of the stable stratification.

3. Linear homogeneous estimates

In this section we establish various smoothing and dispersive estimates for the semigroup e^{-tL_N} acting on the Lorentz spaces $L^{p,1}(\mathbb{R}^3)$ and $L^{p,\infty}(\mathbb{R}^3)$. We start by the following L^p – L^q smoothing type estimates.

Lemma 3.1. *For $1 \leq p \leq 2 \leq q \leq \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$, there exists a positive constant $C = C(p, q, \alpha)$ such that*

$$\left\| \partial_x^\alpha e^{-tL_N} f \right\|_{L^q} \leq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}} \|f\|_{L^p}$$

for all $t > 0$, $N \geq 0$ and all $f \in L^p(\mathbb{R}^3)$ satisfying $\tilde{\nabla} \cdot f = 0$.

Proof. Following the strategy of the proof of [15, Proposition 2.4], we rewrite the semigroup e^{-tL_N} as

$$e^{-tL_N} f = e^{\frac{t}{2}\Delta} [e^{iNtp_\mu(D)} P_+ + e^{-iNtp_\mu(D)} P_- + P_0] e^{\frac{t}{2}\Delta} f.$$

Since $e^{iNtp_\mu(D)} P_+ + e^{-iNtp_\mu(D)} P_- + P_0$ is bounded in $L^2(\mathbb{R}^3)$ by the Plancherel theorem, the desired estimate follows from the classical L^p – L^2 and L^2 – L^q smoothing properties of the heat semigroup $e^{t\Delta}$. \square

Lemma 3.2. (i) *For $1 < p \leq 2 < q < \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$, there exists a positive constant $C = C(p, q, \alpha)$ such that*

$$\left\| \partial_x^\alpha e^{-tL_N} f \right\|_{L^{q,1}} \leq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}} \|f\|_{L^{p,1}}$$

for all $t > 0$, $N \geq 0$ and $f \in L^{p,1}(\mathbb{R}^3)$ with $\tilde{\nabla} \cdot f = 0$.

(ii) *For $1 < p < 2 < q < \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$, there exists a positive constant $C = C(p, q, \alpha)$ such that*

$$\left\| \partial_x^\alpha e^{-tL_N} f \right\|_{L^{q,\infty}} \leq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}} \|f\|_{L^{p,\infty}}$$

for all $t > 0$, $N \geq 0$ and $f \in L^{p,\infty}(\mathbb{R}^3)$ with $\tilde{\nabla} \cdot f = 0$.

Proof. The assertions follow from Lemma 3.1 and real interpolation methods (see [5, Theorems 5.3.1 and 5.3.2] and [26, Theorem 1.6]). Here, we only give a proof for the case (i) with $p = 2$ and $2 < q < \infty$. The remaining cases $1 < p < 2 < q < \infty$ and (ii) can be treated similarly.

Let us choose exponents q_0 and q_1 so that

$$2 < q_0 < q < q_1 < \infty, \quad \frac{1}{q} = \frac{1/2}{q_0} + \frac{1/2}{q_1}.$$

Then it follows from the continuous embeddings $L^{2, \frac{4}{3}}(\mathbb{R}^3) \hookrightarrow L^{2, \frac{4}{3}}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, and the L^2 – L^{q_j} ($j = 0, 1$) estimates given in Lemma 3.1 that

$$\left\| \partial_x^\alpha e^{-tL_N} \right\|_{L^{2, \frac{4}{3}} \rightarrow L^{q_0}} \leq C t^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{q_0}) - \frac{|\alpha|}{2}}, \quad \left\| \partial_x^\alpha e^{-tL_N} \right\|_{L^{2, \frac{4}{3}} \rightarrow L^{q_1}} \leq C t^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{q_1}) - \frac{|\alpha|}{2}}, \quad t > 0.$$

Then, since

$$(L^{2, \frac{4}{3}}(\mathbb{R}^3), L^{2, \frac{4}{3}}(\mathbb{R}^3))_{\frac{1}{2}, 1} = L^{2, 1}(\mathbb{R}^3), \quad \text{and} \quad (L^{q_0}(\mathbb{R}^3), L^{q_1}(\mathbb{R}^3))_{\frac{1}{2}, 1} = L^{q, 1}(\mathbb{R}^3),$$

real interpolation yields

$$\left\| \partial_x^\alpha e^{-tL_N} \right\|_{L^{2, 1} \rightarrow L^{q, 1}} \leq \left\| \partial_x^\alpha e^{-tL_N} \right\|_{L^{2, \frac{4}{3}} \rightarrow L^{q_0}}^{\frac{1}{2}} \left\| \partial_x^\alpha e^{-tL_N} \right\|_{L^{2, \frac{4}{3}} \rightarrow L^{q_1}}^{\frac{1}{2}} \leq C t^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{q}) - \frac{|\alpha|}{2}}, \quad t > 0.$$

This completes the proof of Lemma 3.2. \square

We next establish the $L^{p, \infty}$ -boundedness of e^{-tL_N} .

Lemma 3.3. Assume that $\mu \in \mathbb{R} \setminus \{0, \pm 1\}$ and let $1 < p < \infty$. Then there exists a positive constant $C = C(\mu, p)$ such that

$$\left\| e^{-tL_N} f \right\|_{L^{p, \infty}} \leq C(1 + Nt)^2 \|f\|_{L^{p, \infty}}$$

for all $t > 0$, $N \geq 0$ and $f \in L^{p, \infty}(\mathbb{R}^3)$ with $\tilde{\nabla} \cdot f = 0$.

Proof. Since by assumption $1 < p < \infty$, interpolation theory implies that we only need to show the L^p estimate

$$\left\| e^{-tL_N} f \right\|_{L^p} \leq C(1 + Nt)^2 \|f\|_{L^p}, \quad t > 0.$$

Recall that the semigroup e^{-tL_N} has the explicit form

$$e^{-tL_N} \varphi = e^{t\Delta} e^{iNtp_\mu(D)} P_+ \varphi + e^{t\Delta} e^{-iNtp_\mu(D)} P_- \varphi + e^{t\Delta} P_0 \varphi.$$

Since P_\pm , P_0 and $e^{t\Delta}$ are bounded in $L^p(\mathbb{R}^3)$ with $\|e^{t\Delta}\|_{L^p \rightarrow L^p} \leq 1$, it remains to show that

$$\|e^{\pm iNtp_\mu(D)} f\|_{L^p} \leq C(1 + Nt)^2 \|f\|_{L^p}. \quad (3.1)$$

Note that the Fourier multiplier $e^{\pm i N t p_\mu(\xi)}$ belongs to $C^2(\mathbb{R}^3 \setminus \{0\})$ and satisfies the condition

$$\left| \partial_\xi^\alpha e^{\pm i N t p_\mu(\xi)} \right| \leq C(1 + Nt)^2 |\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}, \quad t > 0$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^3$ with $|\alpha| \leq 2$. Hence, estimate (3.1) follows from the classical Mihlin multiplier theorem (see e.g. [14, Theorem 6.2.7]). \square

We next recall the dispersive estimates for the propagator $e^{\pm i t p_\mu(D)}$ and the behavior of the heat semigroup $e^{t\Delta}$ in homogeneous Besov spaces.

Lemma 3.4 ([18, Lemma 3.2]). *Let $\mu \in \mathbb{R} \setminus \{0, \pm 1\}$. For $2 \leq q \leq \infty$, there exists a positive constant $C = C(\mu, q)$ such that*

$$\left\| e^{\pm i t p_\mu(D)} f \right\|_{\dot{B}_{q,2}^s} \leq C(1 + |t|)^{-\frac{1}{2}\left(1 - \frac{2}{q}\right)} \|f\|_{\dot{B}_{q',2}^{s+3(1-\frac{2}{q})}}$$

for all $t \in \mathbb{R}$, $s \in \mathbb{R}$ and all $f \in \dot{B}_{q',2}^{s+3(1-\frac{2}{q})}(\mathbb{R}^3)$, where $1/q + 1/q' = 1$.

Lemma 3.5 ([24, Lemma 2.2]). *Let $-\infty < s_0 \leq s_1 < \infty$. Then, there exists a positive constant $C = C(s_0, s_1)$ such that*

$$\left\| e^{t\Delta} f \right\|_{\dot{B}_{q,r}^{s_1}} \leq C t^{-\frac{1}{2}(s_1 - s_0)} \|f\|_{\dot{B}_{q,r}^{s_0}}$$

for all $t > 0$, $1 \leq q, r \leq \infty$ and all $f \in \dot{B}_{q,r}^{s_0}(\mathbb{R}^3)$.

Combining Lemma 3.4 with Lemma 3.5 we obtain L^p – L^q estimates for $e^{t\Delta} e^{\pm i N t p_\mu(D)}$.

Lemma 3.6. *Assume that $\mu \in \mathbb{R} \setminus \{0, \pm 1\}$. Let $s \geq 0$, and let p and q satisfy $1 < p \leq q' \leq 2 \leq q < \infty$, where $1/q + 1/q' = 1$. Then, there exists a positive constant $C = C(\mu, s, p, q)$ such that*

$$\left\| (-\Delta)^{\frac{s}{2}} e^{t\Delta} e^{\pm i N t p_\mu(D)} f \right\|_{L^q} \leq C(1 + Nt)^{-\frac{1}{2}\left(1 - \frac{2}{q}\right)} t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{s}{2}} \|f\|_{L^p}$$

for all $t > 0$, $N \geq 0$ and $f \in L^p(\mathbb{R}^3)$.

Proof. Since $2 \leq q < \infty$, it follows from the continuous embedding $\dot{B}_{q,2}^0(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ and Lemma 3.4 that

$$\begin{aligned} \left\| (-\Delta)^{\frac{s}{2}} e^{t\Delta} e^{\pm i N t p_\mu(D)} f \right\|_{L^q} &\leq C \left\| e^{t\Delta} e^{\pm i N t p_\mu(D)} f \right\|_{\dot{B}_{q,2}^s} \\ &\leq C(1 + Nt)^{-\frac{1}{2}\left(1 - \frac{2}{q}\right)} \left\| e^{t\Delta} f \right\|_{\dot{B}_{q',2}^{s+3(1-\frac{2}{q})}}, \quad t > 0. \end{aligned} \quad (3.2)$$

By Lemma 3.5 and the embedding $L^{q'}(\mathbb{R}^3) \hookrightarrow \dot{B}_{q',2}^0(\mathbb{R}^3)$, we obtain

$$\begin{aligned} \|e^{t\Delta} f\|_{\dot{B}_{q',2}^{s+3(1-\frac{2}{q})}} &\leq C t^{-\frac{3}{2}(1-\frac{2}{q})-\frac{s}{2}} \|e^{\frac{t}{2}\Delta} f\|_{\dot{B}_{q',2}^0} \\ &\leq C t^{-\frac{3}{2}(1-\frac{2}{q})-\frac{s}{2}} \|e^{\frac{t}{2}\Delta} f\|_{L^{q'}} \\ &\leq C t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s}{2}} \|f\|_{L^p}, \quad t > 0. \end{aligned} \quad (3.3)$$

Substituting (3.3) into (3.2), we obtain the desired estimate. \square

We now transfer the estimate given in Lemma 3.6 to weak Lebesgue spaces.

Lemma 3.7. Assume that $\mu \in \mathbb{R} \setminus \{0, \pm 1\}$. Let $s \geq 0$, and let p and q satisfy $1 < p < q' < 2 < q < \infty$, where $1/q + 1/q' = 1$. Then, there exists a positive constant $C = C(\mu, s, p, q)$ such that

$$\left\| (-\Delta)^{\frac{s}{2}} e^{t\Delta} e^{\pm iNtp_\mu(D)} f \right\|_{L^{q,\infty}} \leq C(1+Nt)^{-\frac{1}{2}(1-\frac{2}{q})} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s}{2}} \|f\|_{L^{p,\infty}}$$

for all $t > 0$, $N \geq 0$ and $f \in L^{p,\infty}(\mathbb{R}^3)$.

4. Linear and nonlinear inhomogeneous estimates

In this section, we estimate the Duhamel terms stated in (2.7) and (2.8).

We start by proving linear inhomogeneous estimates for the external forces.

Lemma 4.1. (i) There exists a positive constant C such that

$$\sup_{t>0} \left\| \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \leq C \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}}$$

for all $N \geq 0$ and all $F \in BC(\mathbb{R}_+; L^{\frac{3}{2},\infty}(\mathbb{R}^3))$.

(ii) There exists a positive constant C such that

$$\sup_{t \in \mathbb{R}} \left\| \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \leq C \sup_{t \in \mathbb{R}} \|F(t)\|_{L^{\frac{3}{2},\infty}}$$

for all $N \geq 0$ and all $F \in BC(\mathbb{R}; L^{\frac{3}{2},\infty}(\mathbb{R}^3))$.

Proof. (i) Since $P_j \mathbb{P} = P_j$ for $j = \pm, 0$, we see that for $\varphi \in L^{\frac{3}{2},1}(\mathbb{R}^3)$

$$\left\langle \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau, \varphi \right\rangle = \int_0^t \left\langle e^{(t-\tau)\Delta} e^{iN(t-\tau)p_\mu(D)} P_+ \widetilde{\nabla} \cdot F(\tau), \varphi \right\rangle d\tau$$

$$\begin{aligned}
& + \int_0^t \left\langle e^{(t-\tau)\Delta} e^{-iN(t-\tau)p_\mu(D)} P_- \tilde{\nabla} \cdot F(\tau), \varphi \right\rangle d\tau \\
& + \int_0^t \left\langle e^{(t-\tau)\Delta} P_0 \tilde{\nabla} \cdot F(\tau), \varphi \right\rangle d\tau \\
& = \int_0^t \left\langle F(\tau), \tilde{\nabla} e^{-(t-\tau)L'_N} \varphi \right\rangle d\tau,
\end{aligned} \tag{4.1}$$

where $e^{-tL'_N}$ is given by

$$e^{-tL'_N} \varphi = e^{t\Delta} e^{-iNtp_\mu(D)} P_+ \varphi + e^{t\Delta} e^{iNtp_\mu(D)} P_- \varphi + e^{t\Delta} P_0 \varphi.$$

Hence, we have

$$\left| \left\langle \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau, \varphi \right\rangle \right| \leq \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \int_0^\infty \|\nabla e^{-tL'_N} \varphi\|_{L^{3,1}} dt. \tag{4.2}$$

Now, following the strategy of [36, Corollary 2.3] and [13, Lemma 2.1], we consider the sub-linear operator from $L^{\frac{6}{5},1}(\mathbb{R}^3) + L^{2,1}(\mathbb{R}^3)$ to the real-valued function V defined by $V(t) = \|\nabla e^{-tL'_N} \varphi\|_{L^{3,1}}$ for $\varphi \in L^{\frac{6}{5},1}(\mathbb{R}^3) + L^{2,1}(\mathbb{R}^3)$. By Lemma 3.2,

$$V(t) \leq Ct^{-\frac{5}{4}} \|\varphi\|_{L^{\frac{6}{5},1}}, \quad V(t) \leq Ct^{-\frac{3}{4}} \|\varphi\|_{L^{2,1}}, \quad t > 0,$$

which imply that $V \in L^{\frac{4}{5},\infty}(0,\infty) \cap L^{\frac{4}{3},\infty}(0,\infty)$ with $\|V\|_{L^{\frac{4}{5},\infty}(0,\infty)} \leq C\|\varphi\|_{L^{\frac{6}{5},1}}$ and $\|V\|_{L^{\frac{4}{3},\infty}(0,\infty)} \leq C\|\varphi\|_{L^{2,1}}$. Then, since

$$(L^{\frac{6}{5},1}(\mathbb{R}^3), L^{2,1}(\mathbb{R}^3))_{\frac{1}{2},1} = L^{\frac{3}{2},1}(\mathbb{R}^3), \quad (L^{\frac{4}{5},\infty}(0,\infty), L^{\frac{4}{3},\infty}(0,\infty))_{\frac{1}{2},1} = L^1(0,\infty),$$

the real interpolation theorem [5, Theorem 5.3.2] yields $V \in L^1(0,\infty)$ and we obtain

$$\int_0^\infty \|\nabla e^{-tL'_N} \varphi\|_{L^{3,1}} dt \leq C\|\varphi\|_{L^{\frac{3}{2},1}}. \tag{4.3}$$

The desired estimate follows from (4.2), (4.3) and by the duality $(L^{\frac{3}{2},1}(\mathbb{R}^3))^* = L^{3,\infty}(\mathbb{R}^3)$.

(ii) Similarly to (4.1), we have for $\varphi \in L^{\frac{3}{2},1}(\mathbb{R}^3)$

$$\left\langle \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau, \varphi \right\rangle = \int_{-\infty}^t \left\langle F(\tau), \tilde{\nabla} e^{-(t-\tau)L'_N} \varphi \right\rangle d\tau,$$

which implies that

$$\left| \left\langle \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau, \varphi \right\rangle \right| \leq \sup_{t \in \mathbb{R}} \|F(t)\|_{L^{\frac{3}{2}, \infty}} \int_0^\infty \|\nabla e^{-tL'_N} \varphi\|_{L^{3,1}} dt$$

for all $t \in \mathbb{R}$. Estimate (4.3) and the duality $(L^{\frac{3}{2},1}(\mathbb{R}^3))^* = L^{3,\infty}(\mathbb{R}^3)$ complete the proof of Lemma 4.1. \square

Lemma 4.2. Let $\mu \in \mathbb{R} \setminus \{0, \pm 1\}$. Assume that $s, p > 0$ satisfy

$$0 < s < \frac{1}{2}, \quad \max \left\{ \frac{2-s}{3}, \frac{5+2s}{9} \right\} < \frac{1}{p} < \frac{2}{3}.$$

(i) There exists a positive constant $C = C(\mu, s, p)$ such that

$$\sup_{t>0} \left\| \int_0^t e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \leq CN^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)} \sup_{t>0} \|F(t)\|_{L^{p,\infty}}$$

for all $N > 0$ and all $F \in BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3))$.

(ii) There exists a positive constant $C = C(\mu, s, p)$ such that

$$\sup_{t \in \mathbb{R}} \left\| \int_{-\infty}^t e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \leq CN^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)} \sup_{t \in \mathbb{R}} \|F(t)\|_{L^{p,\infty}}$$

for all $N > 0$ and all $F \in BC(\mathbb{R}; L^{p,\infty}(\mathbb{R}^3))$.

Proof. (i) We set

$$\frac{1}{q} = \frac{1}{3} + \frac{s}{3}.$$

Then, p and q satisfy the relations

$$2 < q < 3, \quad \frac{2-s}{3} = \frac{1}{q'} < \frac{1}{p} < 1.$$

Moreover, we remark that $(-\Delta)^{-\frac{s}{2}}$ is bounded from $L^{q,\infty}(\mathbb{R}^3)$ to $L^{3,\infty}(\mathbb{R}^3)$ by the Hardy–Littlewood–Sobolev inequality and the Hunt interpolation theorem [31, Theorem IX.19]. Hence, Lemma 3.7 implies

$$\begin{aligned}
& \left\| \int_0^t e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \\
& \leq C \int_0^t \|(-\Delta)^{\frac{s}{2}} \widetilde{\nabla} \cdot e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} F(\tau)\|_{L^{q,\infty}} d\tau \\
& \leq C \int_0^t (t-\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{s+1}{2}} \{1+N(t-\tau)\}^{-\frac{1}{2}\left(1-\frac{2}{q}\right)} \|F(\tau)\|_{L^{p,\infty}} d\tau \\
& \leq C \sup_{t>0} \|F(t)\|_{L^{p,\infty}} \int_0^\infty t^{-\frac{3}{2p}} (1+Nt)^{-\frac{1}{6}+\frac{s}{3}} dt. \tag{4.4}
\end{aligned}$$

Since by assumption $\frac{5+2s}{9} < \frac{1}{p} < \frac{2}{3}$, we see that

$$\int_0^\infty t^{-\frac{3}{2p}} (1+Nt)^{-\frac{1}{6}+\frac{s}{3}} dt = N^{-1+\frac{3}{2p}} \int_0^\infty t^{-\frac{3}{2p}} (1+t)^{-\frac{1}{6}+\frac{s}{3}} dt < \infty. \tag{4.5}$$

Substituting (4.5) into (4.4) yields the desired estimate.

(ii) Similarly to (4.4), we have

$$\begin{aligned}
& \left\| \int_{-\infty}^t e^{(t-\tau)\Delta} e^{\pm iN(t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \\
& \leq C \int_{-\infty}^t (t-\tau)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{s+1}{2}} \{1+N(t-\tau)\}^{-\frac{1}{2}\left(1-\frac{2}{q}\right)} \|F(\tau)\|_{L^{p,\infty}} d\tau \\
& \leq C \sup_{t \in \mathbb{R}} \|F(t)\|_{L^{p,\infty}} \int_0^\infty t^{-\frac{3}{2p}} (1+Nt)^{-\frac{1}{6}+\frac{s}{3}} dt.
\end{aligned}$$

Hence, the desired estimate follows from (4.5). \square

Next, we consider the bilinear estimates for the nonlinear terms. We put

$$\begin{aligned}
B_1(u, v)(t) &:= \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot (u \otimes v)(\tau) d\tau \quad t > 0, \\
B_2(u, v)(t) &:= \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot (u \otimes v)(\tau) d\tau \quad t \in \mathbb{R}.
\end{aligned}$$

The subsequent bilinear estimates follow from Lemma 4.1 and the Hölder inequality in weak Lebesgue spaces.

Lemma 4.3. (i) *There exists a constant $C > 0$ such that*

$$\sup_{t>0} \|B_1(u, v)(t)\|_{L^{3,\infty}} \leq C \sup_{t>0} \|u(t)\|_{L^{3,\infty}} \sup_{t>0} \|v(t)\|_{L^{3,\infty}}$$

for all $N \geq 0$ and all $u, v \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3))$.

(ii) *There exists a constant $C > 0$ such that*

$$\sup_{t \in \mathbb{R}} \|B_2(u, v)(t)\|_{L^{3,\infty}} \leq C \sup_{t \in \mathbb{R}} \|u(t)\|_{L^{3,\infty}} \sup_{t \in \mathbb{R}} \|v(t)\|_{L^{3,\infty}}$$

for all $N \geq 0$ and $u, v \in BC(\mathbb{R}; L^{3,\infty}(\mathbb{R}^3))$.

Proof. By Lemma 4.1, we obtain for $j = 1, 2$,

$$\sup_t \|B_j(u, v)(t)\|_{L^{3,\infty}} \leq C \sup_t \|u \otimes v(t)\|_{L^{\frac{3}{2},\infty}} \leq C \sup_t \|u(t)\|_{L^{3,\infty}} \sup_t \|v(t)\|_{L^{3,\infty}}.$$

This gives the desired estimate. \square

5. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1 by using a fixed point argument. To this end, we first consider the linear equations

$$\begin{cases} \partial_t u - \Delta u + N \mathbb{P} J_\mu \mathbb{P} u = \widetilde{\nabla} \cdot F & t > 0, x \in \mathbb{R}^3, \\ \widetilde{\nabla} \cdot u = 0 & t > 0, x \in \mathbb{R}^3, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3. \end{cases} \quad (5.1)$$

Making use of the Duhamel principle, we construct mild solutions to (5.1) which satisfy the following integral equations

$$u(t) = e^{-tL_N} u_0 + \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau. \quad (5.2)$$

Our result on the existence and uniqueness of time periodic solutions to (5.2) reads as follows.

Theorem 5.1 (Periodic solutions to the linear equation). *Assume that $\mu \in \mathbb{R} \setminus \{0, \pm 1\}$. Let $s, p > 0$ satisfy*

$$0 < s < \frac{1}{2}, \quad \max \left\{ \frac{2-s}{3}, \frac{5+2s}{9} \right\} < \frac{1}{p} < \frac{2}{3},$$

and let $F \in BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$ be a T -periodic function for some $T > 0$. Then, there exists a unique $u_0 = u_0(F) \in L^{3,\infty}(\mathbb{R}^3)$ and a unique T -periodic solution $u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3))$ to (5.2) satisfying

$$\sup_{t>0} \|u(t)\|_{L^{3,\infty}} \leq C_1 \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \quad (5.3)$$

and

$$\begin{aligned} \sup_{t>0} \|u(t)\|_{L^{3,\infty}} &\leq C_2 N^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})} \left\{ \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C_1 \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}}, \end{aligned} \quad (5.4)$$

with some positive constants C_1 and $C_2 = C_2(\mu, s, p)$.

Proof. Following the strategy in [13, Theorem 2.4], we first show the unique existence of $u_0 = u_0(F) \in L^{3,\infty}(\mathbb{R}^3)$ satisfying

$$(Id - e^{-TL_N})u_0 = \int_0^T e^{-(T-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau. \quad (5.5)$$

Then, the solution $u = u(t)$ defined by

$$u(t) := e^{-tL_N} u_0 + \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau \quad (5.6)$$

gives the T -periodic solution to (5.2).

For the construction of such a u_0 , we consider the sequence $\{a_k\}_{k \in \mathbb{N}}$ defined by

$$a_k := \int_0^{(k-1)T} e^{-(kT-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau.$$

We show that there exists $a \in L^{3,\infty}(\mathbb{R}^3) \cap L^{6,\infty}(\mathbb{R}^3)$, which satisfies

$$\lim_{k \rightarrow \infty} a_k = a \quad \text{in } L^{6,\infty}(\mathbb{R}^3), \quad w^* - \lim_{k \rightarrow \infty} a_k = a \quad \text{in } L^{3,\infty}(\mathbb{R}^3).$$

Then we prove that $u_0 := a + G$ satisfies (5.5), where $G = G(F)$ is defined by

$$G := \int_0^T e^{-(T-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau.$$

Note that a_k may be written as

$$\begin{aligned} a_k &= \sum_{j=1}^{k-1} \int_{(j-1)T}^{jT} e^{-(kT-\tau)L_N} \mathbb{P}\tilde{\nabla} \cdot F(\tau) d\tau = \sum_{j=1}^{k-1} e^{-(k-j)TL_N} \int_0^T e^{-(T-\tau)L_N} \mathbb{P}\tilde{\nabla} \cdot F(\tau) d\tau \\ &= \sum_{j=1}^{k-1} e^{-jTL_N} G \end{aligned} \quad (5.7)$$

for $k \in \mathbb{N}$, and Lemma 3.2 (ii) gives that

$$\begin{aligned} \|e^{-jTL_N} G\|_{L^{6,\infty}} &\leq \int_0^T \|e^{-(j+1)T-\tau)L_N} \mathbb{P}\tilde{\nabla} \cdot F(\tau)\|_{L^{6,\infty}} d\tau \\ &\leq C \int_0^T ((j+1)T-\tau)^{-\frac{5}{4}} \|\mathbb{P}F(\tau)\|_{L^{\frac{3}{2},\infty}} d\tau \\ &\leq CT^{-\frac{1}{4}} j^{-\frac{5}{4}} \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}}, \quad j \in \mathbb{N}. \end{aligned} \quad (5.8)$$

Hence, it follows from (5.7) and (5.8) that $\{a_k\}_{k \in \mathbb{N}}$ strongly converges with $a = \lim_{k \rightarrow \infty} a_k$ in $L^{6,\infty}(\mathbb{R}^3)$. Furthermore, we have by Lemmas 3.3 and 4.1 (i) that

$$\begin{aligned} \|a_k\|_{L^{3,\infty}} &= \left\| e^{-TL_N} \int_0^{(k-1)T} e^{((k-1)T-\tau)L_N} \mathbb{P}\tilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \\ &\leq C(1+NT)^2 \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}}. \end{aligned}$$

Hence, $\{a_k\}_{k \in \mathbb{N}}$ is bounded in $L^{3,\infty}(\mathbb{R}^3)$, and then $w^* - \lim_{k \rightarrow \infty} a_k = a$ in $L^{3,\infty}(\mathbb{R}^3)$ (see [13, Lemma 2.3]). Similarly, we have $w^* - \lim_{k \rightarrow \infty} e^{-TL_N} a_k = e^{-TL_N} a$ in $L^{3,\infty}(\mathbb{R}^3)$ and

$$w^* - \lim_{k \rightarrow \infty} (Id - e^{-TL_N})(a_k + G) = (Id - e^{-TL_N})(a + G) \quad \text{in } L^{3,\infty}(\mathbb{R}^3). \quad (5.9)$$

Next, since

$$(Id - e^{-TL_N})a_k = \sum_{j=1}^{k-1} (e^{-jTL_N} - e^{-(j+1)TL_N})G = e^{-TL_N}G - e^{-kTL_N}G$$

by (5.7), it follows from (5.8) that

$$\begin{aligned} \|(Id - e^{-TL_N})a_k - e^{-TL_N}G\|_{L^{6,\infty}} &= \|e^{-kTL_N}G\|_{L^{6,\infty}} \\ &\leq CT^{-\frac{1}{4}} k^{-\frac{5}{4}} \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \longrightarrow 0 \quad (k \longrightarrow \infty). \end{aligned}$$

Hence, $\lim_{k \rightarrow \infty} (Id - e^{-TL_N})a_k = e^{-TL_N}G$ in $L^{6,\infty}(\mathbb{R}^3)$. Also, since $\{(Id - e^{-TL_N})a_k\}_{k \in \mathbb{N}}$ is bounded in $L^{3,\infty}(\mathbb{R}^3)$, we obtain

$$w^* - \lim_{k \rightarrow \infty} (Id - e^{-TL_N})(a_k + G) = e^{-TL_N}G + (Id - e^{-TL_N})G = G \quad \text{in } L^{3,\infty}(\mathbb{R}^3). \quad (5.10)$$

By (5.9) and (5.10), we see that $u_0 := a + G$ satisfies the desired equality (5.5).

Next, we show the uniqueness of the T -periodic mild solution $u = u(t)$. To this end, we follow the strategy in [13, Remark 2.5 (d)]. Let u and v be T -periodic mild solutions to (5.1) of the form (5.6) with the initial data u_0 and v_0 , respectively. Then, since u and v are also nT -periodic for $n \in \mathbb{N}$, u_0 and v_0 must satisfy

$$(Id - e^{-nTL_N})u_0 = \int_0^{nT} e^{-(nT-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau$$

and

$$(Id - e^{-nTL_N})v_0 = \int_0^{nT} e^{-(nT-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau,$$

respectively. Hence, we have

$$u_0 - v_0 = e^{-nTL_N}(u_0 - v_0). \quad (5.11)$$

Let us take σ and r so that

$$\sigma > 0, \quad r > 3, \quad \sigma - \frac{3}{r} > 3.$$

Then, (5.11), the $L^{3,\infty}$ - $L^{r,\infty}$ smoothing estimate for the heat semigroup $e^{t\Delta}$ and Lemma 3.3 give that

$$\begin{aligned} & \|u_0 - v_0\|_{\dot{W}_{r,\infty}^\sigma} \\ &= \|(-\Delta)^{\frac{\sigma}{2}} e^{nT\Delta} (e^{inNTp_\mu(D)} P_+ + e^{-inNTp_\mu(D)} P_- + P_0)(u_0 - v_0)\|_{L^{r,\infty}} \\ &\leq C(nT)^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{r})-\frac{\sigma}{2}} \|(e^{inNTp_\mu(D)} P_+ + e^{-inNTp_\mu(D)} P_- + P_0)(u_0 - v_0)\|_{L^{3,\infty}} \\ &\leq C(nT)^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{r})-\frac{\sigma}{2}} (1 + nNT)^2 \|u_0 - v_0\|_{L^{3,\infty}} \\ &\leq CT^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{r})-\frac{\sigma}{2}} (n^{-1} + NT)^2 n^{-\frac{1}{2}(\sigma-\frac{3}{r}-3)} \|u_0 - v_0\|_{L^{3,\infty}} \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which yields the uniqueness $u_0 = v_0$.

It remains to prove the a priori estimates (5.3) and (5.4). Since

$$e^{-tL_N}a_k = \int_0^{(k-1)T} e^{-(kT+t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau,$$

similarly to (4.1), (4.2) and (4.3), we obtain for $\varphi \in L^{\frac{3}{2},1}(\mathbb{R}^3)$

$$\begin{aligned} \left| \left\langle e^{-tL_N} a_k, \varphi \right\rangle \right| &= \left| \int_0^{(k-1)T} \left\langle F(\tau), \tilde{\nabla} e^{-(kT+t-\tau)L'_N} \varphi \right\rangle d\tau \right| \\ &\leq \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \int_0^{(k-1)T} \|\nabla e^{-(kT+t-\tau)L'_N} \varphi\|_{L^{3,1}} d\tau \\ &\leq \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \int_0^\infty \|\nabla e^{-tL'_N} \varphi\|_{L^{3,1}} dt \\ &\leq C \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \|\varphi\|_{L^{\frac{3}{2},1}}, \end{aligned}$$

and also

$$\begin{aligned} \left| \left\langle e^{-tL_N} G, \varphi \right\rangle \right| &= \left| \int_0^T \left\langle F(\tau), \tilde{\nabla} e^{-(T+t-\tau)L'_N} \varphi \right\rangle d\tau \right| \\ &\leq \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \int_0^T \|\nabla e^{-(T+t-\tau)L'_N} \varphi\|_{L^{3,1}} d\tau \\ &\leq \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \int_0^\infty \|\nabla e^{-tL'_N} \varphi\|_{L^{3,1}} dt \\ &\leq C \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \|\varphi\|_{L^{\frac{3}{2},1}}. \end{aligned}$$

These imply that

$$\|e^{-tL_N} a_k\|_{L^{3,\infty}} + \|e^{-tL_N} G\|_{L^{3,\infty}} \leq C \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \quad (5.12)$$

for all $t > 0$ and all $k \in \mathbb{N}$. Therefore, by (5.12) and Lemma 4.1 (i)

$$\begin{aligned} \|u(t)\|_{L^{3,\infty}} &\leq \liminf_{k \rightarrow \infty} \|e^{-tL_N} a_k\|_{L^{3,\infty}} + \|e^{-tL_N} G\|_{L^{3,\infty}} + \left\| \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \\ &\leq C \sup_{t>0} \|F(t)\|_{L^{\frac{3}{2},\infty}} \end{aligned}$$

for all $t > 0$, which gives (5.3). Finally, we show (5.4). By (2.5), we decompose $e^{-tL_N} a_k$ as

$$\begin{aligned}
e^{-tL_N} a_k &= \int_0^{(k-1)T} e^{-(kT+t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau \\
&= \int_0^{(k-1)T} e^{(kT+t-\tau)\Delta} e^{iN(kT+t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot P_+ F(\tau) d\tau \\
&\quad + \int_0^{(k-1)T} e^{(kT+t-\tau)\Delta} e^{-iN(kT+t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot P_- F(\tau) d\tau \\
&\quad + \int_0^{(k-1)T} e^{(kT+t-\tau)\Delta} \widetilde{\nabla} \cdot P_0 F(\tau) d\tau.
\end{aligned} \tag{5.13}$$

Concerning the third term in (5.13), it follows from Lemma 4.1 (i) with $N = 0$ that

$$\begin{aligned}
\left\| \int_0^{(k-1)T} e^{(kT+t-\tau)\Delta} \widetilde{\nabla} \cdot P_0 F(\tau) d\tau \right\|_{L^{3,\infty}} &= \left\| e^{(T+t)\Delta} \int_0^{(k-1)T} e^{((k-1)T-\tau)\Delta} \widetilde{\nabla} \cdot P_0 F(\tau) d\tau \right\|_{L^{3,\infty}} \\
&\leq C \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}}.
\end{aligned} \tag{5.14}$$

For the first and the second term in (5.13), set $\frac{1}{q} = \frac{1}{3} + \frac{s}{3}$. Then, similarly to (4.4) and (4.5), we obtain by Lemma 3.7

$$\begin{aligned}
&\left\| \int_0^{(k-1)T} e^{(kT+t-\tau)\Delta} e^{\pm iN(kT+t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot P_\pm F(\tau) d\tau \right\|_{L^{3,\infty}} \\
&\leq C \int_0^{(k-1)T} \|(-\Delta)^{\frac{s}{2}} e^{(kT+t-\tau)\Delta} e^{\pm iN(kT+t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot P_\pm F(\tau)\|_{L^{q,\infty}} d\tau \\
&\leq C \sup_{t>0} \|P_\pm F(t)\|_{L^{p,\infty}} \int_0^{(k-1)T} (kT+t-\tau)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s+1}{2}} \{1+N(kT+t-\tau)\}^{-\frac{1}{2}(1-\frac{2}{q})} d\tau \\
&\leq C N^{-1+\frac{3}{2p}} \sup_{t>0} \|P_\pm F(t)\|_{L^{p,\infty}} \int_0^\infty t^{-\frac{3}{2p}} (1+t)^{-\frac{1}{6}+\frac{s}{3}} dt.
\end{aligned} \tag{5.15}$$

Hence, it follows from (5.13), (5.14) and (5.15) that

$$\begin{aligned} \sup_{t>0} \|e^{-tL_N} a_k\|_{L^{3,\infty}} &\leq CN^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)} \left\{ \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} \end{aligned} \quad (5.16)$$

for all $k \in \mathbb{N}$. Since

$$\begin{aligned} e^{-tL_N} G &= \int_0^T e^{(T+t-\tau)\Delta} e^{iN(T+t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot P_+ F(\tau) d\tau \\ &\quad + \int_0^T e^{(T+t-\tau)\Delta} e^{-iN(T+t-\tau)p_\mu(D)} \widetilde{\nabla} \cdot P_- F(\tau) d\tau \\ &\quad + \int_0^T e^{(T+t-\tau)\Delta} \widetilde{\nabla} \cdot P_0 F(\tau) d\tau, \end{aligned}$$

we obtain similarly to (5.14) and (5.15)

$$\begin{aligned} \sup_{t>0} \|e^{-tL_N} G\|_{L^{3,\infty}} &\leq CN^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)} \left\{ \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}}. \end{aligned} \quad (5.17)$$

Hence, we obtain by (5.16), (5.17), Lemmas 4.1 and 4.2 that

$$\begin{aligned} \|u(t)\|_{L^{3,\infty}} &\leq \liminf_{k \rightarrow \infty} \|e^{-tL_N} a_k\|_{L^{3,\infty}} + \|e^{-tL_N} G\|_{L^{3,\infty}} + \left\| \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau \right\|_{L^{3,\infty}} \\ &\leq CN^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}\right)} \left\{ \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} \right\} + C \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}}. \end{aligned}$$

This completes the proof of Theorem 5.1. \square

Remark 5.2. The proof of Theorem 5.1 shows that there exists a unique solution map S_T from a given T -periodic external force $F \in BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$ to a T -periodic mild solution $u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3))$ of (5.1) satisfying (5.3) and (5.4). This solution operator S_T is defined as

$$S_T F(t) := e^{-tL_N} u_0 + \int_0^t e^{-(t-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau, \quad t > 0, \quad (5.18)$$

where $u_0 = u_0(F) = a + G$ with

$$a := w^* - \lim_{k \rightarrow \infty} \int_0^{(k-1)T} e^{-(kT-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau \quad \text{in } L^{3,\infty}(\mathbb{R}^3), \quad (5.19)$$

$$G := \int_0^T e^{-(T-\tau)L_N} \mathbb{P} \widetilde{\nabla} \cdot F(\tau) d\tau. \quad (5.20)$$

Note that S_T is linear and that $S_T(F_1 + F_2) = S_T F_1 + S_T F_2$ for T -periodic external forces $F_1, F_2 \in BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$.

We are now in the position to give a proof of Theorem 2.1.

Proof of Theorem 2.1. Let s and p satisfy (2.9) and let F be a T -periodic external force in the class $BC(\mathbb{R}_+; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$. We define the map Φ_T and the solution space $(X, \|\cdot\|_X)$ by

$$\Phi_T(u)(t) := S_T(F - u \otimes u)(t) = S_T(F)(t) - S_T(u \otimes u)(t),$$

$$X := \left\{ u \in BC(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3)) \mid u \text{ is } T\text{-periodic, } \|u\|_X := \sup_{t>0} \|u(t)\|_{L^{3,\infty}} \leq K \right\},$$

where S_T is the solution operator defined as (5.18)–(5.20) in Remark 5.2 and K is a real constant to be specified later.

We prove that Φ_T maps X into itself and is a contraction on X for sufficiently small $K > 0$.

For $u \in X$, it follows from (5.3) and (5.4) that

$$\begin{aligned} \|\Phi_T(u)\|_X &\leq \|S_T(F)\|_X + \|S_T(u \otimes u)\|_X \\ &\leq C N^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})} \left\{ \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} + C \sup_{t>0} \|(u \otimes u)(t)\|_{L^{\frac{3}{2},\infty}} \\ &\leq C_1 N^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})} \left\{ \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C_2 \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} + C_3 K^2 \end{aligned} \quad (5.21)$$

with some positive constants $C_1 = C_1(\mu, s, p)$, C_2 and C_3 . Furthermore, since $F - u \otimes u$ is T -periodic, $\Phi_T(u)$ is also T -periodic by Theorem 5.1. Next, for $u, v \in X$, we have by (5.3)

$$\begin{aligned} \|\Phi_T(u) - \Phi_T(v)\|_X &= \|S_T(u \otimes u) - S_T(v \otimes v)\|_X \\ &\leq C \sup_{t>0} \|u \otimes (u - v)(t) + (u - v) \otimes v(t)\|_{L^{\frac{3}{2},\infty}} \end{aligned}$$

$$\begin{aligned} &\leq C \left\{ \sup_{t>0} \|u(t)\|_{L^{3,\infty}} + \sup_{t>0} \|v(t)\|_{L^{3,\infty}} \right\} \sup_{t>0} \|u(t) - v(t)\|_{L^{3,\infty}} \\ &\leq 2C_4 K \|u - v\|_X \end{aligned} \quad (5.22)$$

with some positive constant C_4 . Choosing $K > 0$ small enough so that

$$0 < K \leq \min \left\{ \frac{1}{3C_3}, \frac{1}{4C_4} \right\}, \quad (5.23)$$

and assuming that the external force F satisfies the size condition

$$\begin{aligned} \sup_{t>0} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t>0} \|P_- F(t)\|_{L^{p,\infty}} &\leq \frac{K}{3C_1} N^{\frac{3}{2}(\frac{2}{3}-\frac{1}{p})}, \\ \sup_{t>0} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} &\leq \frac{K}{3C_2}, \end{aligned}$$

we obtain by (5.21), (5.22) and (5.23) that

$$\|\Phi_T(u)\|_X \leq K, \quad \|\Phi_T(u) - \Phi_T(v)\|_X \leq \frac{1}{2} \|u - v\|_X$$

for all $u, v \in X$. Hence, the contraction mapping principle yields the existence of unique fixed point $u \in X$ of Φ_T , and this fixed point u is the desired unique T -periodic mild solution of (1.1) in X . \square

6. Proofs of Theorems 2.2 and 2.3

In this section, we give a detailed proof of Theorem 2.3. Theorem 2.2 can be shown then similarly.

Proof of Theorem 2.3. Let s and p satisfy (2.9) and let F be an almost periodic external force in $BC(\mathbb{R}; L^{p,\infty}(\mathbb{R}^3) \cap L^{\frac{3}{2},\infty}(\mathbb{R}^3))$. We define the map Φ and the solution space $(Y, \|\cdot\|_Y)$ by

$$\begin{aligned} \Phi(u)(t) &:= \int_{-\infty}^t e^{-(t-\tau)L_N} \widetilde{\mathbb{P}} \widetilde{\nabla} \cdot (F - u \otimes u)(\tau) d\tau, \\ Y &:= \left\{ u \in AP(\mathbb{R}; L^{3,\infty}(\mathbb{R}^3)) \mid \|u\|_Y := \sup_{t \in \mathbb{R}} \|u(t)\|_{L^{3,\infty}} \leq K \right\}, \end{aligned}$$

where $K > 0$ is a real constant to be chosen later. We will show that Φ is a contraction in Y provided K is sufficiently small.

Let $u \in Y$. Then, since $w := F - u \otimes u$ is almost periodic in $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, for every $\varepsilon > 0$, there exists a $L_\varepsilon > 0$ such that for each $a \in \mathbb{R}$ we may choose $T \in [a, a + L_\varepsilon)$ so that

$$\sup_{t \in \mathbb{R}} \|w(t+T) - w(t)\|_{L^{\frac{3}{2},\infty}} < \varepsilon.$$

Hence, it follows from Lemmas 4.1 (ii) that

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|\Phi(u)(t+T) - \Phi(u)(t)\|_{L^{3,\infty}} &= \sup_{t \in \mathbb{R}} \left\| \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot \{w(\tau+T) - w(\tau)\} d\tau \right\|_{L^{3,\infty}} \\ &\leq C \sup_{t \in \mathbb{R}} \|w(t+T) - w(t)\|_{L^{\frac{3}{2},\infty}} \\ &< C\varepsilon, \end{aligned}$$

which implies that $\Phi(u)$ is also almost periodic in $L^{3,\infty}(\mathbb{R}^3)$.

Next, since $P_j \mathbb{P} = P_j$ for $j \in \{\pm, 0\}$, we have by Lemmas 4.1 and 4.2

$$\begin{aligned} \|\Phi(u)\|_Y &= \left\| \int_{-\infty}^t e^{-(t-\tau)L_N} \tilde{\nabla} \cdot \{P_+ F + P_- F + P_0 F - \mathbb{P}(u \otimes u)\}(\tau) d\tau \right\|_Y \\ &\leq C N^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})} \left\{ \sup_{t \in \mathbb{R}} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t \in \mathbb{R}} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C \sup_{t \in \mathbb{R}} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} + C \sup_{t \in \mathbb{R}} \|(u \otimes u)(t)\|_{L^{\frac{3}{2},\infty}} \\ &\leq C_1 N^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})} \left\{ \sup_{t \in \mathbb{R}} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t \in \mathbb{R}} \|P_- F(t)\|_{L^{p,\infty}} \right\} \\ &\quad + C_2 \sup_{t \in \mathbb{R}} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} + C_3 K^2 \end{aligned} \quad (6.1)$$

for $u \in Y$ with some positive constants $C_1 = C_1(\mu, s, p)$, C_2 and C_3 . Furthermore, Lemma 4.1 yields for all $u, v \in Y$

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_Y &= \left\| \int_{-\infty}^t e^{-(t-\tau)L_N} \mathbb{P} \tilde{\nabla} \cdot \{u \otimes (u-v) + (u-v) \otimes v\}(\tau) d\tau \right\|_Y \\ &\leq C \left\{ \sup_{t \in \mathbb{R}} \|u(t)\|_{L^{3,\infty}} + \sup_{t \in \mathbb{R}} \|v(t)\|_{L^{3,\infty}} \right\} \sup_{t \in \mathbb{R}} \|u(t) - v(t)\|_{L^{3,\infty}} \\ &\leq 2C_4 K \|u - v\|_Y, \end{aligned} \quad (6.2)$$

with some constant $C_4 > 0$. Finally choosing $K > 0$ sufficiently small so that

$$0 < K \leq \min \left\{ \frac{1}{3C_3}, \frac{1}{4C_4} \right\}, \quad (6.3)$$

and if the external force F satisfies the size condition

$$\sup_{t \in \mathbb{R}} \|P_+ F(t)\|_{L^{p,\infty}} + \sup_{t \in \mathbb{R}} \|P_- F(t)\|_{L^{p,\infty}} \leq \frac{K}{3C_1} N^{\frac{3}{2}(\frac{2}{3}-\frac{1}{p})},$$

$$\sup_{t \in \mathbb{R}} \|P_0 F(t)\|_{L^{\frac{3}{2},\infty}} \leq \frac{K}{3C_2},$$

we obtain by (6.1), (6.2) and (6.3)

$$\|\Phi(u)\|_Y \leq K, \quad \|\Phi(u) - \Phi(v)\|_Y \leq \frac{1}{2} \|u - v\|_Y$$

for all $u, v \in Y$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in Y$ satisfying the integral equation (2.8) for all $t \in \mathbb{R}$. \square

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