JOURNAL OF DIFFERENTIAL EQUATIONS **142**, 434–454 (1998)
ARTICLE NO. DE973349

Semilinear Elliptic Boundary Value Problems in Chemical Reactor Theory

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Received March 6, 1997; revised June 24, 1997

Dedicated to Professor Kyuya Masuda on the Occasion of His 60th Birthday

This paper is devoted to the study of semilinear elliptic boundary value problems arising in chemical reactor theory which obey the simple Arrhenius rate law and Newtonian cooling. We prove that ignition and extinction phenomena occur in the stable steady temperature profile at some critical values of a dimensionless heat evolution rate. © 1998 Academic Press

Key Words: Semilinear elliptic boundary value problem; Arrhenius rate law; Newtonian cooling; ignition; extinction

1. INTRODUCTION AND RESULTS

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geqslant 2$, with smooth boundary ∂D; its closure $\bar{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + c(x)\, u(x)$$

be a second-order, *elliptic* differential operator with real smooth coefficients on $\bar{D}$ such that

(1) $a^{ij}(x) = a^{ji}(x)$, $1 \leqslant i, j \leqslant N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,\, j=1}^{N} a^{ij}(x)\, \xi_i \xi_j \geqslant a_0\, |\xi|^2, \qquad x \in \bar{D}, \quad \xi \in \mathbf{R}^N,$$

(2) $c(x) > 0$ in D.

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In this paper we consider the following semilinear elliptic boundary value problem stimulated by a problem of chemical reactor theory (cf. [BGW], [Co], [CL], [LW2], [Pa]):

$$\begin{cases} Au = \lambda \exp\left[\dfrac{u}{1+\varepsilon u}\right] & \text{in} \quad D, \\ Bu = a\dfrac{\partial u}{\partial \mathbf{v}} + (1-a)u = 0 & \text{on} \quad \partial D. \end{cases} \tag{$*$}_\lambda$$

Here:

(1) λ and ε are positive parameters.

(2) $a \in C^\infty(\partial D)$ and $0 \leqslant a(x') \leqslant 1$ on ∂D.

(3) $\partial/\partial \mathbf{v}$ is the conormal derivative associated with the operator A,

$$\frac{\partial}{\partial \mathbf{v}} = \sum_{i,\, j=1}^{N} a^{ij} n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, ..., n_N)$ is the unit exterior normal to the boundary ∂D.

The nonlinear term

$$f(t) = \exp\left[\frac{t}{1+\varepsilon t}\right]$$

describes the temperature dependence of reaction rate for exothermic reactions obeying the simple *Arrhenius rate law* in circumstances in which heat flow is purely conductive. In this context the parameter ε is a dimensionless ambient temperature and the parameter λ is a dimensionless heat evolution rate. The equation

$$Au = \lambda f(u) = \lambda \exp\left[\frac{u}{1+\varepsilon u}\right]$$

represents heat balance with reactant consumption ignored, where u is a dimensionless temperature excess.

On the other hand, the boundary condition

$$Bu = a\frac{\partial u}{\partial \mathbf{v}} + (1-a)u = 0$$

represents the exchange of heat at the surface of the reactant by *Newtonian cooling*. Moreover the boundary condition $Bu = 0$ is called the isothermal condition (or Dirichlet condition) if $a \equiv 0$ on ∂D, and is called the adiabatic condition (or Neumann condition) if $a \equiv 1$ on ∂D. We remark that problem $(*)_\lambda$ becomes a degenerate boundary value problem from an analytical point of view. This is due to the fact that the so-called Shapiro–Lopatinskii complementary condition is violated at the points where $a(x) = 0$. In the non-degenerate case or one-dimensional case, problem $(*)_\lambda$ has been studied by many authors (see [CL], [Co], [Pa], [LW2], [BIS]).

A function $u \in C^2(\bar{D})$ is called a *solution* of problem $(*)_\lambda$ if it satisfies the equation $Au - \lambda f(*) = 0$ and the boundary condition $Bu = 0$. A solution u is said to be *positive* if it is positive everywhere in D.

This paper is devoted to the study of the existence of positive solutions of problem $(*)_\lambda$. Our starting point is the following existence theorem for problem $(*)_\lambda$ (see [TU2, Theorem 1]):

THEOREM 0. *For each $\lambda > 0$, problem $(*)_\lambda$ has at least one positive solution. Furthermore, problem $(*)_\lambda$ has a unique positive solution if $\varepsilon \geqslant 1/4$.*

In other words, if the activation energy is so low that the parameter ε exceeds the value 1/4, then only a smooth progression of reaction rate with imposed ambient temperature can occur; such a reaction may be very rapid but it is only accelerating and lacks the discontinuous change associated with criticality and ignition. The situation may be represented schematically by Fig. 1 (cf. [BGW, Figure 6]).

The purpose of the present paper is to study the case where $0 < \varepsilon < 1/4$. First, in order to state our multiplicity theorem for problem $(*)_\lambda$, we define a function

$$v(t) = \frac{t}{f(t)} = \frac{t}{\exp[t/(1+\varepsilon t)]}, \qquad t \geqslant 0.$$

It is easy to see that if $0 < \varepsilon < 1/4$, then the function $v(t)$ has a unique local maximum at $t = t_1(\varepsilon)$,

$$t_1(\varepsilon) = \frac{1 - 2\varepsilon - \sqrt{1 - 4\varepsilon}}{2\varepsilon^2},$$

and has a unique local minimum at $t = t_2(\varepsilon)$:

$$t_2(\varepsilon) = \frac{1 - 2\varepsilon + \sqrt{1 - 4\varepsilon}}{2\varepsilon^2}.$$

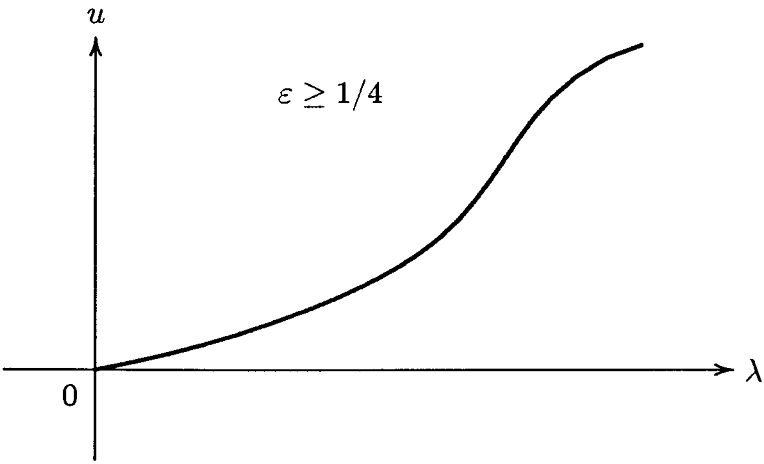

Figure 1

On the other hand, we let $\phi \in C^{\infty}(\bar{D})$ be the unique positive solution of the linear boundary value problem

$$\begin{cases} Au = 1 & \text{in} \quad D, \\ Bu = 0 & \text{on} \quad \partial D, \end{cases} \tag{1.1}$$

and let

$$\|\phi\|_{\infty} = \max_{\bar{D}} \phi(x).$$

Now we can state our multiplicity theorem for problem $(*)_\lambda$:

THEOREM 1. *We can find a constant* $\beta > 0$, *independent of* ε, *such that if* $0 < \varepsilon < 1/4$ *is so small that*

$$\frac{v(t_2(\varepsilon))}{\beta} < \frac{v(t_1(\varepsilon))}{\|\phi\|_{\infty}}, \tag{1.2}$$

then there exist at least three *distinct positive solutions of problem* $(*)_\lambda$ *for all* λ *satisfying the condition*

$$\frac{v(t_2(\varepsilon))}{\beta} < \lambda < \frac{v(t_1(\varepsilon))}{\|\phi\|_{\infty}}. \tag{1.3}$$

Theorem 1 is a generalization of [Wi, Theorem 4.3] to the degenerate case (see also [Pa], [LW2], [BIS]). We remark that, as $\varepsilon \downarrow 0$,

$$\frac{v(t_2(\varepsilon))}{\beta} \sim \frac{1}{\varepsilon^2} \exp\left[\frac{-1}{\varepsilon+\varepsilon^2}\right], \tag{1.4a}$$

$$\frac{v(t_1(\varepsilon))}{\|\phi\|_\infty} \sim \exp\left[\frac{-1}{1+\varepsilon}\right], \tag{1.4b}$$

so that condition (1.2) makes sense.

Secondly we state two existence and uniqueness theorems for problem $(*)_\lambda$. Let λ_1 be the first eigenvalue of the linear eigenvalue problem

$$\begin{cases} Au = \lambda u & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

The next two theorems assert that problem $(*)_\lambda$ is uniquely solvable for λ sufficiently small and sufficiently large if $0 < \varepsilon < 1/4$:

THEOREM 2. *Let $0 < \varepsilon < 1/4$. If the parameter λ is so small that*

$$0 < \lambda < \frac{\lambda_1 \exp\left[\dfrac{2\varepsilon - 1}{\varepsilon}\right]}{4\varepsilon^2}, \tag{1.5}$$

then problem $(*)_\lambda$ has a unique positive solution.

THEOREM 3. *Let $0 < \varepsilon < 1/4$. One can find a constant $\Lambda > 0$, independent of ε, such that if the parameter λ is so large that $\lambda > \Lambda$, then problem $(*)_\lambda$ has a unique positive solution.*

Theorems 2 and 3 are generalizations of [Wi, Theorems 2.9 and 2.6] to the degenerate case, respectively, although we only treat the nonlinear term $f(t) = \exp[t/(1+\varepsilon t)]$. Here it is worth while to point out (see condition (1.4)) that we have, as $\varepsilon \downarrow 0$,

$$\frac{v(t_2(\varepsilon))}{\beta} \sim \frac{\lambda_1 \exp\left[\dfrac{2\varepsilon - 1}{\varepsilon}\right]}{4\varepsilon^2},$$

$$\frac{v(t_1(\varepsilon))}{\|\phi\|_\infty} \sim \Lambda.$$

By virtue of Theorems 1, 2 and 3, we can define two positive numbers μ_I and μ_E by the formulas

$$\mu_I = \inf\{\mu > 0 : \text{problem}(*)_\lambda \text{ is uniquely solvable for each } \mu < \lambda\},$$
$$\mu_E = \sup\{\mu > 0 : \text{problem}(*)_\lambda \text{ is uniquely solvable for each } 0 < \lambda < \mu\}.$$

Then it is easy to see that an *ignition* phenomenon occurs at $\lambda = \mu_I$ and an *extinction* phenomenon occurs at $\lambda = \mu_E$, respectively. In other words, a small increase in λ causes a large jump in the stable steady temperature profile at $\lambda = \mu_I$ and $\lambda = \mu_E$. More precisely the minimal positive solution $\underline{u}(\lambda)$ is continuous in $\lambda > \mu_I$ but is not continuous at $\lambda = \mu_I$, while the maximal positive solution $\bar{u}(\lambda)$ is continuous in $0 < \lambda < \mu_E$ but is not continuous at $\lambda = \mu_E$. The situation may be represented schematically by Figs. 2 and 3 (cf. [BGW, Fig. 6]).

By the maximum principle and the boundary point lemma, we can easily see from formula (3.2) below that the first eigenvalue $\lambda_1(a)$ satisfies the inequalities

$$\lambda_1(1) < \lambda_1(a) < \lambda_1(0),$$

and that the unique solution $\phi = \phi_{(a)}$ of problem (1.1) satisfies the inequalities

$$\phi_{(0)} < \phi_{(a)} < \phi_{(1)} \qquad \text{in} \quad D,$$

so that,

$$\frac{1}{\|\phi_{(1)}\|_\infty} < \frac{1}{\|\phi_{(a)}\|_\infty} < \frac{1}{\|\phi_{(0)}\|_\infty}.$$

Moreover it follows from formula (2.8) below that the critical value $\beta = \beta(a)$ in Theorem 1 satisfies the inequalities

$$\frac{1}{\beta(1)} \leqslant \frac{1}{\beta(a)} \leqslant \frac{1}{\beta(0)},$$

and further from formula (4.14) below that the critical value $\Lambda = \Lambda(a)$ in Theorem 3 depends essentially on the first eigenvalue $\lambda_1 = \lambda_1(a)$.

Therefore we find that the extinction phenomenon in the isothermal condition case occurs at the largest critical value $\mu_E(0)$, while the extinction phenomenon in the adiabatic condition case occurs at the smallest critical value $\mu_E(1)$. Similarly we find that the ignition phenomenon in the adiabatic condition case occurs at the smallest critical value $\mu_I(1)$, while the ignition phenomenon in the isothermal condition case occurs at the largest critical value $\mu_I(0)$.

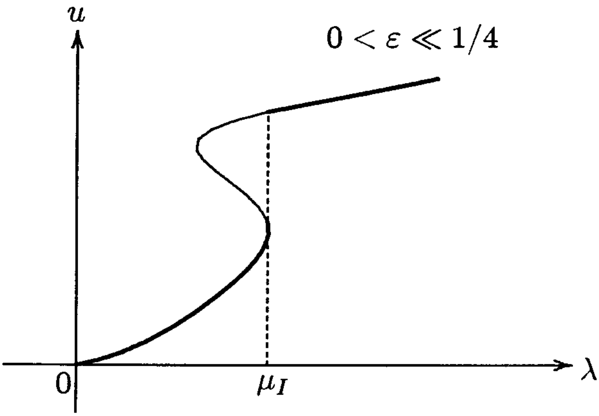

Figure 2

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. We reduce the study of problem $(*)_\lambda$ to the study of a nonlinear operator equation in an appropriate ordered Banach space as in Taira and Umezu [TU1] and [TU2]. Our proof of Theorem 1 may be carried out just as in the proof of [Wi, Theorem 4.3], by making use of the theory of positive mappings in ordered Banach spaces due to Amann [Am2]. In Section 3 we prove Theorem 2, by using a variant of variational method. In Section 4 we prove Theorem 3. Our proof of Theorem 3 is based on a method inspired by Wiebers [Wi, Theorems 2.9 and 2.6].

The authors are grateful to Kunimochi Sakamoto for fruitful conversations while working on this paper.

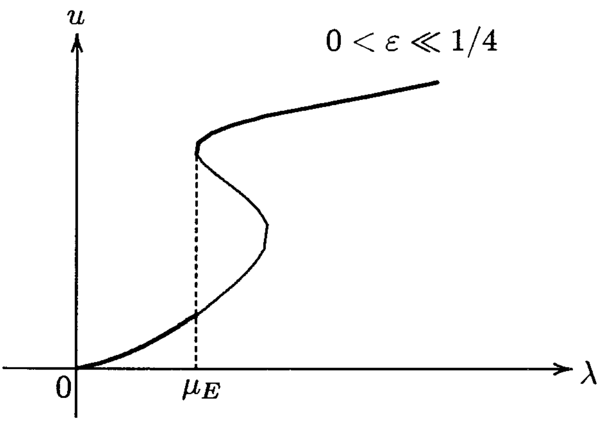

Figure 3

2. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1. First we transpose the nonlinear problem $(*)_\lambda$ into an equivalent fixed point equation for the resolvent K in an appropriate ordered Banach space, just as in Taira and Umezu ([TU1] and [TU2]).

(i) If $1 < p < \infty$, we define a closed linear subspace of the Sobolev space $W^{2,p}(D)$ by the formula

$$W_B^{2,p}(D) = \{u \in W^{2,p}(D) : Bu = 0 \text{ on } \partial D\}.$$

By [TU1, Theorem 1.1], we can introduce a continuous linear operator

$$K\colon L^p(D) \to W_B^{2,p}(D)$$

as follows: For any $g \in L^p(D)$, the function $u = Kg \in W^{2,p}(D)$ is the unique solution of the problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

Then, by the Ascoli–Arzelà theorem we find that the operator K, considered as

$$K\colon C(\bar{D}) \to C^1(\bar{D}),$$

is *compact*. Indeed it follows from an application of Sobolev's imbedding theorem that $W^{2,p}(D)$ is continuously imbedded into $C^{2-N/p}(\bar{D})$ for all $N < p < \infty$.

For $u, v \in C(\bar{D})$, we write $u \leqslant v$ if $u(x) \leqslant v(x)$ in $\bar{D}$. Then the space $C(\bar{D})$ is an ordered Banach space with the linear ordering $\leqslant$, and with the positive cone

$$P = \{u \in C(\bar{D}) : u \geqslant 0\}.$$

For $u, v \in C(\bar{D})$ the notation $u \prec v$ means that $v - u \in P \backslash \{0\}$. Then it is known (see [TU1, Lemma 2.1]) that K is *strictly positive*, that is, Kg is positive everywhere in D if $g \succ 0$. Moreover it is easy to verify that a function u is a solution of problem $(*)_\lambda$ if and only if it satisfies the equation

$$u = \lambda K(f(u)) \quad \text{in} \quad C(\bar{D}). \tag{2.2}$$

(ii) The proof of Theorem 1 is based on the following result on multiple positive fixed points of nonlinear operators on ordered Banach spaces essentially due to Legget and Williams [LW1] (see [Wi, Lemma 4.4]):

LEMMA 2.1. *Let* $(X, Q, \preccurlyeq)$ *be an ordered Banach space such that the positive cone* Q *has non-empty interior. Moreover let* $\eta\colon Q \to [0, \infty)$ *be a continuous and concave functional and let* G *be a compact mapping of* $Q_\tau := \{w \in Q\colon \|w\| \leqslant \tau\}$ *into* Q *for some constant* $\tau > 0$ *such that*

$$\|G(w)\| < \tau \qquad \text{for all } w \in Q_\tau \text{ satisfying } \|w\| = \tau. \tag{2.3}$$

Assume that there exist constants $0 < \delta < \tau$ *and* $\sigma > 0$ *such that the set*

$$W = \{w \in \mathring{Q}_\tau\colon \eta(w) > \sigma\} \tag{2.4}$$

is non-empty, where $\mathring{A}$ *denotes the interior of a subset* A *of* Q, *and that*

$$\|G(w)\| < \delta \qquad \text{for all } \quad w \in Q_\delta \text{ satisfying } \|w\| = \delta, \tag{2.5}$$

$$\eta(w) < \sigma \qquad \text{for all } \quad w \in Q_\delta, \tag{2.6}$$

and

$$\eta(G(w)) > \sigma \qquad \text{for all } \quad w \in Q_\tau \text{ satisfying } \eta(w) = \sigma. \tag{2.7}$$

Then the mapping G *has at least* three *distinct fixed points.*

(iii) *End of Proof of Theorem* 1. The proof of Theorem 1 may be carried out just as in the proof of [Wi, Theorem 4.3.]

Let $\mathscr{B}$ be the set of all subdomains Ω of D with smooth boundary such that $\operatorname{dist}(\Omega, \partial D) > 0$, and let

$$\beta = \sup_{\Omega \in \mathscr{B}} C_\Omega, \qquad C_\Omega = \inf_{x \in \Omega} (K\chi_\Omega)(x), \tag{2.8}$$

where χ_A denotes the characteristic function of a set A. It is easy to see that the constant β is positive, since the resolvent K of problem (2.1) is strictly positive.

Since $\lim_{t \to \infty} v(t) = \lim_{t \to \infty} t/f(t) = \infty$, one can find a constant $\bar{t}_1(\varepsilon)$ such that

$$\bar{t}_1(\varepsilon) = \min\{t > t_2(\varepsilon) : v(t) = v(t_1(\varepsilon))\}.$$

Then we remark that

$$t_1(\varepsilon) < t_2(\varepsilon) < \bar{t}_1(\varepsilon),$$

and

$$v(t_1(\varepsilon)) = v(\bar{t}_1(\varepsilon)) = \frac{\bar{t}_1(\varepsilon)}{f(\bar{t}_1(\varepsilon))}. \tag{2.9}$$

Now we shall apply Lemma 2.1 with

$$
\begin{aligned}
X &:= C(\bar{D}),\\
Q &:= P = \{u \in C(\bar{D}) : u \geqslant 0\},\\
G(\cdot) &:= \lambda K(f(\cdot)),\\
\delta &:= t_1(\varepsilon), \qquad \sigma := t_2(\varepsilon), \qquad \tau := \bar{t}_1(\varepsilon).
\end{aligned}
$$

To do so, it suffices to verify that the conditions of Lemma 2.1 are fulfilled for all λ satisfying condition (1.3).

(iii-a) If $t > 0$, we let

$$
P(t) = \{u \in P : \|u\|_\infty \leqslant t\}.
$$

If $u \in P(\bar{t}_1(\varepsilon))$ and $\|u\|_\infty = \bar{t}_1(\varepsilon)$ and if $\phi = K1$ is the unique solution of problem (1.1), then it follows from condition (1.3) and formula (2.9) that

$$
\begin{aligned}
\|\lambda K(f(u))\|_\infty &< \frac{v(t_1(\varepsilon))}{\|\phi\|_\infty}\|K(f(u))\|_\infty\\
&\leqslant \frac{v(t_1(\varepsilon))}{\|\phi\|_\infty}\, f(\bar{t}_1(\varepsilon))\, \|K1\|_\infty\\
&= v(t_1(\varepsilon))\, f(\bar{t}_1(\varepsilon))\\
&= \bar{t}_1(\varepsilon),
\end{aligned}
$$

since $f(t)$ is increasing. This proves that the mapping $\lambda K(f(\cdot))$ satisfies condition (2.3) with $Q_\tau := P(\bar{t}_1(\varepsilon))$.

Similarly one can verify that if $u \in P(t_1(\varepsilon))$ and $\|u\|_\infty = t_1(\varepsilon)$, then we have

$$
\|\lambda K(f(u))\|_\infty < t_1(\varepsilon).
$$

This proves that the mapping $\lambda K(f(\cdot))$ satisfies condition (2.5) with $Q_\delta := P(t_1(\varepsilon))$.

(iii-b) If $\Omega \in \mathscr{B}$, we let

$$
\eta(u) = \inf_{x \in \Omega} u(x).
$$

Then it is easy to see that η is a continuous and concave functional of P. If $u \in P(t_1(\varepsilon))$, then we have

$$
\eta(u) \leqslant \|u\|_\infty \leqslant t_1(\varepsilon) < t_2(\varepsilon).
$$

This verifies condition (2.6) for the functional η.

(iii-c) If we let

$$W = \{u \in \mathring{P}(\bar{t}_1(\varepsilon)) : \eta(u) > t_2(\varepsilon)\},$$

then we find that

$$W \supset \left\{ u \in P : \frac{\bar{t}_1(\varepsilon)}{2} \leqslant u < \bar{t}_1(\varepsilon) \quad \text{on} \quad \bar{D}, \quad \eta(u) > t_2(\varepsilon) \right\} \neq \emptyset,$$

since $t_2(\varepsilon) < \bar{t}_1(\varepsilon)$. This verifies condition (2.4) for the functional η.

(iii-d) Now, since $\lambda > v(t_2(\varepsilon))/\beta$, by formula (2.8) one can find a subdomain $\Omega \in \mathscr{B}$ such that

$$\lambda > \frac{v(t_2(\varepsilon))}{C_\Omega}.$$

If $u \in P(\bar{t}_1(\varepsilon))$ and $\eta(u) = t_2(\varepsilon)$, then we have

$$\begin{aligned} \eta(\lambda K(f(u))) &= \inf_{x \in \Omega} \lambda K(f(u))(x) \\ &\geqslant \inf_{x \in \Omega} \lambda K(f(u)\chi_\Omega)(x) \\ &> \frac{v(t_2(\varepsilon))}{C_\Omega} \inf_{x \in \Omega} K(f(u)\chi_\Omega)(x). \end{aligned} \tag{2.10}$$

However, since $\inf_\Omega u = \eta(u) = t_2(\varepsilon)$ and $f(t)$ is increasing, it follows that

$$\begin{aligned} \frac{v(t_2(\varepsilon))}{C_\Omega} \inf_{x \in \Omega} K(f(u)\chi_\Omega)(x) &\geqslant \frac{v(t_2(\varepsilon))}{C_\Omega} \inf_{x \in \Omega} K(f(t_2(\varepsilon))\chi_\Omega)(x) \\ &= \frac{v(t_2(\varepsilon))}{C_\Omega} f(t_2(\varepsilon)) \inf_{x \in \Omega} (K\chi_\Omega)(x) \\ &= v(t_2(\varepsilon))\, f(t_2(\varepsilon)) \\ &= t_2(\varepsilon). \end{aligned} \tag{2.11}$$

Therefore, combining inequalities (2.10) and (2.11) we obtain that

$$\eta(\lambda K(f(u))) > t_2(\varepsilon).$$

This verifies condition (2.7) for the mapping $\lambda K(f(\cdot))$.

The proof of Theorem 1 is now complete. ∎

3. PROOF OF THEOREM 2

We let

$$f(t) = \exp\left[\frac{t}{1+\varepsilon t}\right], \qquad t \geqslant 0.$$

If u_1 and u_2 are two positive solutions of problem $(*)_\lambda$, then we have, by the mean value theorem,

$$\begin{aligned}\int_D A(u_1-u_2)\cdot(u_1-u_2)\,dx &= \int_D \lambda(f(u_1)-f(u_2))(u_1-u_2)\,dx \\ &= \lambda \int_D G(x)(u_1-u_2)^2\,dx, \qquad (3.1)\end{aligned}$$

where

$$G(x) = \int_0^1 f'(u_2(x)+\theta(u_1(x)-u_2(x)))\,d\theta.$$

We shall prove Theorem 2 by using a variant of variational method. To do so, we introduce an unbounded linear operator $\mathfrak{A}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$D(\mathfrak{A}) = \{u \in W^{2,2}(D) : Bu = 0\}.$$

(b) $\mathfrak{A}u = Au$, $u \in D(\mathfrak{A})$.

Then it is known (see [Ta1, Theorems 7.3 and 7.4], [Um, Theorem 2]) that the operator $\mathfrak{A}$ is a positive and self-adjoint operator in $L^2(D)$, and has a compact resolvent. Hence we obtain that the first eigenvalue λ_1 of $\mathfrak{A}$ is characterized by the following formula:

$$\lambda_1 = \min\left\{\int_D Au(x)\cdot\overline{u(x)}\,dx : u \in W^{2,2}(D),\ \int_D |u(x)|^2\,dx = 1,\ Bu = 0\right\}. \qquad (3.2)$$

Thus it follows from formulas (3.2) and (3.1) that

$$\begin{aligned}\lambda_1 \int_D (u_1-u_2)^2\,dx &\leqslant \int_D A(u_1-u_2)\cdot(u_1-u_2)\,dx \\ &= \lambda \int_D G(x)(u_1-u_2)^2\,dx \\ &\leqslant \lambda \sup f'(t) \int_D (u_1-u_2)^2\,dx. \qquad (3.3)\end{aligned}$$

However it is easy to see that

$$\sup f'(t) = f'\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) = 4\varepsilon^2 \exp\left[\frac{1-2\varepsilon}{\varepsilon}\right].$$

Hence, combining this fact with inequality (3.3) we obtain that

$$\lambda_1 \int_D (u_1 - u_2)^2 \, dx \leqslant 4\lambda\varepsilon^2 \exp\left[\frac{1-2\varepsilon}{\varepsilon}\right] \int_D (u_1 - u_2)^2 \, dx.$$

Therefore we find that $u_1 \equiv u_2$ in D, if the parameter λ is so small that condition (1.5) is satisfied, that is, if we have

$$\lambda_1 - 4\lambda\varepsilon^2 \exp\left[\frac{1-2\varepsilon}{\varepsilon}\right] > 0.$$

The proof of Theorem 2 is complete. ∎

4. PROOF OF THEOREM 3

This section is devoted to the proof of Theorem 3. Our proof of Theorem 3 is based on a method inspired by Wiebers [Wi, Theorems 2.9 and 2.6].

4.1. *An a Priori Estimate*

In this subsection we shall establish an *a priori* estimate for positive solutions of problem $(*)_\lambda$ which will play an important role in the proof of Theorem 3.

First we introduce another ordered Banach subspace of $C(\bar{D})$ for the fixed point equation (2.2) which combines the good properties of the resolvent K of problem (2.1) with the good properties of the natural ordering of $C(\bar{D})$.

Let $\phi = K1$ be the unique solution of problem (1.1). Then it follows from an application of [TU1, Lemma 2.1] that the function ϕ belongs to $C^\infty(\bar{D})$ and satisfies the conditions

$$\phi(x) \begin{cases} > 0 & \text{if} \quad \text{either } x \in D \text{ or } a(x) > 0, \\ = 0 & \text{if} \quad a(x) = 0, \end{cases}$$

and

$$\frac{\partial \phi}{\partial \mathbf{v}}(x) < 0 \qquad \text{if} \quad a(x) = 0.$$

By using the function ϕ, we can introduce a subspace of $C(\bar{D})$ as follows:

$C_\phi(\bar{D}) = \{u \in C(\bar{D}) :$ there exists a constant $c > 0$ such that $-c\phi \leqslant u \leqslant c\phi\}$.

The space $C_\phi(\bar{D})$ is given a norm by the formula

$$\|u\|_\phi = \inf\{c > 0 : -c\phi \leqslant u \leqslant c\phi\}.$$

If we let

$$P_\phi = C_\phi(\bar{D}) \cap P = \{u \in C_\phi(\bar{D}) : u \geqslant 0\},$$

then it is easy to see that the space $C_\phi(\bar{D})$ is an ordered Banach space having the positive cone P_ϕ with nonempty interior. For u, $v \in C_\phi(\bar{D})$, the notation $u \ll v$ means that $v - u$ is an interior point of P_ϕ. We know (see [TU1, Proposition 2.2]) that K maps $C_\phi(\bar{D})$ compactly into itself, and that K is *strongly positive*, that is, $Kg \gg 0$ for all $g \in P_\phi \backslash \{0\}$.

It is easy to see that a function u is a solution of problem $(*)_\lambda$ if and only if it satisfies the equation

$$u = \lambda K(f(u)) \qquad \text{in} \quad C_\phi(\bar{D}). \tag{4.1}$$

Recall (see [Ta3, Theorem 1]) that the first eigenvalue λ_1 of $\mathfrak{A}$ is positive and simple and that the corresponding eigenfunction φ_1 is positive everywhere in D. Without loss of generality, one may assume that

$$\max_{\bar{D}} \varphi_1(x) = 1.$$

We let

$$\gamma = \min \left\{ \frac{f(t_1(\varepsilon))}{t_1(\varepsilon)} : 0 < \varepsilon \leqslant \frac{1}{4} \right\}. \tag{4.2}$$

Here we remark that $t_1(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$, so that the constant γ is positive.

Then we have the following *a priori* estimate for all positive solutions u of problem $(*)_\lambda$:

PROPOSITION 4.1. *One can find a constant $0 < \varepsilon_0 \leqslant 1/4$ such that if $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leqslant \varepsilon_0$, then we have, for all positive solutions u of problem $(*)_\lambda$,*

$$u \geqslant \lambda \varepsilon^{-2} \varphi_1.$$

Proof. (i) Let c be a parameter satisfying $0 < c < 1$. Then we have

$$A(\lambda c\varepsilon^{-2}\varphi_1) - \lambda f(\lambda c\varepsilon^{-2}\varphi_1) = \lambda c\varepsilon^{-2}\varphi_1\left(\lambda_1 - \lambda\,\frac{f(\lambda c\varepsilon^{-2}\varphi_1)}{\lambda c\varepsilon^{-2}\varphi_1}\right) \qquad \text{in } D.$$

However, since we have

$$\frac{f(t)}{t} \to 0 \qquad \text{as} \quad t \to \infty,$$

$$\frac{f(t)}{t} \to \infty \qquad \text{as} \quad t \to 0,$$

it follows that

$$\frac{f(\lambda c\varepsilon^{-2}\varphi_1)}{\lambda c\varepsilon^{-2}\varphi_1} \geqslant \min\left\{\frac{f(t_1(\varepsilon))}{t_1(\varepsilon)},\ \frac{f(\lambda\varepsilon^{-2})}{\lambda\varepsilon^{-2}}\right\}. \tag{4.3}$$

First we obtain from formula (4.2) that, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon < 1/4$,

$$\lambda_1 - \lambda\,\frac{f(t_1(\varepsilon))}{t_1(\varepsilon)} \leqslant \lambda_1 - \lambda\gamma < 0. \tag{4.4}$$

Secondly we have, for all $\lambda > \lambda_1/\gamma$,

$$\begin{aligned}\lambda_1 - \lambda\,\frac{f(\lambda\varepsilon^{-2})}{\lambda\varepsilon^{-2}} &= \lambda_1 - \varepsilon^2 \exp\left[\frac{1}{\varepsilon + \varepsilon^2/\lambda}\right]\\ &\leqslant \lambda_1 - \varepsilon^2 \exp\left[\frac{1}{\varepsilon + \varepsilon^2\gamma/\lambda_1}\right].\end{aligned}$$

However one can find a constant $\varepsilon_0 \in (0, 1/4]$ such that, for all $0 < \varepsilon \leqslant \varepsilon_0$,

$$\lambda_1 - \varepsilon^2 \exp\left[\frac{1}{\varepsilon + \varepsilon^2\gamma/\lambda_1}\right] < 0.$$

Hence it follows that, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leqslant \varepsilon_0$,

$$\lambda_1 - \lambda\,\frac{f(\lambda\varepsilon^{-2})}{\lambda\varepsilon^{-2}} < 0. \tag{4.5}$$

Therefore, combining inequalities (4.3), (4.4) and (4.5) we obtain that, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leqslant \varepsilon_0$,

$$
\begin{aligned}
A(\lambda c\varepsilon^{-2}\varphi_1) - \lambda f(\lambda c\varepsilon^{-2}\varphi_1) &= \lambda c\varepsilon^{-2}\varphi_1\left(\lambda_1 - \lambda\frac{f(\lambda c\varepsilon^{-2}\varphi_1)}{\lambda c\varepsilon^{-2}\varphi_1}\right)\\
&\leqslant \lambda c\varepsilon^{-2}\varphi_1\left(\lambda_1 - \lambda \min\left\{\frac{f(t_1(\varepsilon))}{t_1(\varepsilon)}, \frac{f(\lambda\varepsilon^{-2})}{\lambda\varepsilon^{-2}}\right\}\right)\\
&< 0 \qquad \text{in } D.
\end{aligned}
$$

By applying the resolvent K to the both sides, we have, for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leqslant \varepsilon_0$,

$$
\lambda K(f(c\lambda\varepsilon^{-2}\varphi_1)) \gg c\lambda\varepsilon^{-2}\varphi_1. \tag{4.6}
$$

(ii) Now we need the following lemma (see [Wi, Lemma 1.3]):

Lemma 4.2. *If there exist a function $\tilde{u} \gg 0$ and a constant $s_0 > 0$ such that $\lambda K(f(s\tilde{u})) \gg s\tilde{u}$ for all $0 \leqslant s < s_0$, then we have, for each fixed point u of the mapping $\lambda K(f(u))$,*

$$
u \geqslant s_0 \tilde{u}.
$$

(iii) Since $0 \ll \lambda K(f(0))$ and estimate (4.6) holds for all $0 < c < 1$, it follows from an application of Lemma 4.2 with $\tilde{u} := \lambda\varepsilon^{-2}\varphi_1$, $s_0 := 1$ and $s := c$ (and also equation (4.1)) that every positive solution u of problem $(*)_\lambda$ satisfies the estimate

$$
u \geqslant \lambda\varepsilon^{-2}\varphi_1
$$

for all $\lambda > \lambda_1/\gamma$ and $0 < \varepsilon \leqslant \varepsilon_0$.

The proof of Proposition 4.1 is complete. ∎

4.2. *End of Proof of Theorem* 3

(I) First we define a function

$$
F(t) = f(t) - f'(t)t = \frac{\varepsilon^2 t^2 + (2\varepsilon - 1)t + 1}{(1+\varepsilon t)^2} \exp\left[\frac{t}{1+\varepsilon t}\right] \qquad \text{for} \quad t \geqslant 0.
$$

Then we have the following:

Lemma 4.3. *Let $0 < \varepsilon < 1/4$. Then the function $F(t)$ has the following properties*:

$$
F(t) \begin{cases} > 0 & \text{if either} \quad 0 \leqslant t < t_1(\varepsilon) \quad \text{or} \quad t > t_2(\varepsilon),\\ = 0 & \text{if} \quad t = t_1(\varepsilon) \quad \text{and} \quad t = t_2(\varepsilon),\\ < 0 & \text{if} \quad t_1(\varepsilon) < t < t_2(\varepsilon). \end{cases}
$$

Moreover the function $F(t)$ *is decreasing in the interval* $(0, (1-2\varepsilon)/2\varepsilon^2)$ *and is increasing in the interval* $((1-2\varepsilon)/2\varepsilon^2, \infty)$, *and has a minimum at* $t=(1-2\varepsilon)/2\varepsilon^2$.

(II) The next proposition is an essential step in the proof of Theorem 3:

PROPOSITION 4.4. *Let* $0<\varepsilon<1/4$. *Then there exists a constant* $\alpha>0$, *independent of* ε, *such that we have, for all* $u \succcurlyeq \alpha\varepsilon^{-2}\varphi_1$,

$$K(F(u)) \gg 0. \tag{4.7}$$

Proof. Our proof mimics that of [Am1, Lemma 7.8].
Since $t_2(\varepsilon)<2\varepsilon^{-2}$, we find from Lemma 4.3 that

$$F(t) \geqslant F(2\varepsilon^{-2})>0, \qquad t \geqslant 2\varepsilon^{-2}.$$

We define two functions

$$z_-(u)(x)=\begin{cases} -F(u(x)) & \text{if} \quad u(x) \geqslant 2\varepsilon^{-2}, \\ 0 & \text{if} \quad u(x)<2\varepsilon^{-2}, \end{cases}$$

and

$$z_+(u)(x)=F(u(x))+z_-(u)(x).$$

Moreover, we define two sets

$$M=\{x \in \bar{D} : \varphi_1(x)>\tfrac{1}{2}\},$$

and

$$L=\{x \in \bar{D} : u(x) \geqslant 2\varepsilon^{-2}\}.$$

Then we have $M \subset L$ for all $u \succcurlyeq 4\varepsilon^{-2}\varphi_1$, and so

$$z_-(u) \leqslant -F(2\varepsilon^{-2})\chi_L \leqslant -F(2\varepsilon^{-2})\chi_M.$$

By using Friedrichs' mollifiers, we can construct a function $v \in C^\infty(\bar{D})$ such that $v \succ 0$ and

$$z_-(u) \leqslant -F(2\varepsilon^{-2})v. \tag{4.8}$$

On the other hand, by Lemma 4.3 we remark that

$$\min\{F(t) : 0 \leqslant t \leqslant 2\varepsilon^{-2}\}=F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right)<0.$$

Since $z_+(u)(x)=0$ if $x \in L$ and $z_+(u)(x)=F(u(x))$ if $x \notin L$, it follows that

$$z_+(u) \geqslant F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) \chi_{\bar{D}\setminus L}.$$

If α is a constant such that $\alpha > 4$, we define a set

$$M_\alpha = \left\{ x \in \bar{D} : \varphi_1(x) < \frac{2}{\alpha} \right\}.$$

Then we have, for all $u \geqslant \alpha\varepsilon^{-2}\varphi_1$,

$$\bar{D}\setminus L = \{ x \in \bar{D} : u(x) < 2\varepsilon^{-2} \} \subset M_\alpha,$$

and so

$$z_+(u) \geqslant F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) \chi_{M_\alpha}. \tag{4.9}$$

Hence, combining inequalities (4.8) and (4.9) we obtain that, for all $u \geqslant \alpha\varepsilon^{-2}\varphi_1$,

$$K(F(u)) = K(z_+(u) - z_-(u)) \geqslant F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) K(\chi_{M_\alpha}) + F(2\varepsilon^{-2})\, Kv. \tag{4.10}$$

However, by [TU1, estimate (2.4)] it follows that there exists a constant $c > 0$ such that

$$Kv \geqslant c\varphi_1. \tag{4.11}$$

Furthermore, since $\chi_{M_\alpha} \to 0$ in $L^p(D)$ as $\alpha \to \infty$, it follows that $K(\chi_{M_\alpha}) \to 0$ in $C^1(\bar{D})$ and so $K(\chi_{M_\alpha}) \to 0$ in $C_\phi(\bar{D})$. Hence, for any positive integer k one can choose the constant α so large that

$$K(\chi_{M_\alpha}) \leqslant \frac{c}{k}\, \varphi_1. \tag{4.12}$$

Thus, carrying inequalities (4.11) and (4.12) into the right-hand side of inequality (4.10) we obtain that, for all $u \geqslant \alpha\varepsilon^{-2}\varphi_1$,

$$\begin{aligned} K(F(u)) &= K(z_+(u) - z_-(u)) \\ &\geqslant F\left(\frac{1-2\varepsilon}{2\varepsilon^2}\right) \frac{c}{k}\, \varphi_1 + F(2\varepsilon^{-2})\, c\varphi_1 \\ &= F(2\varepsilon^{-2})\, c\varphi_1 \left(1 + \frac{F((1-2\varepsilon)/2\varepsilon^2)}{F(2\varepsilon^{-2})}\, \frac{1}{k}\right). \end{aligned} \tag{4.13}$$

However we have, as $\varepsilon \downarrow 0$,

$$\frac{F((1-2\varepsilon)/2\varepsilon^2)}{F(2\varepsilon^{-2})} = \frac{(4\varepsilon-1)(\varepsilon+2)^2}{\varepsilon^2+4\varepsilon+2} \exp\left[\frac{-2\varepsilon-3}{\varepsilon+2}\right] \to -2e^{-3/2}.$$

Therefore inequality (4.7) follows from inequality (4.13) if we take the positive integer k so large that

$$k > -\min_{0<\varepsilon<1/4} \frac{F((1-2\varepsilon)/2\varepsilon^2)}{F(2\varepsilon^{-2})}.$$

The proof of Proposition 4.4 is complete. ∎

Proposition 4.4 implies the following important property of the mapping $K(f(\cdot))$ (see [Wi, Lemma 2.2]):

PROPOSITION 4.5. *Let $0<\varepsilon<1/4$ and let α be the same constant as in Proposition 4.4. Then we have, for all $u \geqslant \alpha\varepsilon^{-2}\varphi_1$ and all $s>1$,*

$$sK(f(u)) \gg K(f(su)).$$

(III) Now we let

$$\Lambda = \max\left\{\frac{\lambda_1}{\gamma}, \alpha\right\}. \tag{4.14}$$

If u_1 and u_2 are two positive solutions of $(*)_\lambda$ with $\lambda > \Lambda$ and $0<\varepsilon\leqslant\varepsilon_0$, then combining Propositions 4.1 and 4.5 we find that, for all $s>1$,

$$sK(f(u_i)) \gg K(f(su_i)), \qquad i=1, 2,$$

so that

$$su_i = s\lambda K(f(u_i)) \gg \lambda K(f(su_i)), \qquad i=1, 2.$$

Hence we obtain that $u_1 = u_2$, by applying the following lemma (see [Wi, Lemma 1.3]):

LEMMA 4.6. *If there exists a function $\tilde{u} \gg 0$ such that $s\tilde{u} \gg \lambda K(f(s\tilde{u}))$ for all $s>1$, then $\tilde{u} \geqslant u$ for each fixed point u of the mapping $\lambda K(f(u))$.*

Finally it remains to consider the case where $\varepsilon_0 < \varepsilon < 1/4$. If u is a positive solution of problem $(*)_\lambda$, then we have

$$A\left(u - \frac{\lambda}{\lambda_1}\varphi_1\right) = \lambda f(u) - \lambda\varphi_1 \geqslant \lambda(1-\varphi_1) \geqslant 0 \qquad \text{in } D.$$

By the strong maximum principle and the boundary point lemma (see [PW]), it follows that

$$u \geqslant \frac{\lambda}{\lambda_1} \varphi_1.$$

By combining this assertion with Proposition 4.5, we can prove that the uniqueness result holds for all

$$\lambda \geqslant \frac{\alpha \lambda_1}{\varepsilon^2},$$

just as in the case $0 < \varepsilon \leqslant \varepsilon_0$.

The proof of Theorem 3 is now complete. ▮

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