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First derivative of the period function with applications

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Abstract

Given a centre of a planar differential system, we extend the use of the Lie bracket to the determination of the monotonicity character of the period function. As far as we know, there are no general methods to study this function, and the use of commutators and Lie bracket was restricted to prove isochronicity. We give several examples and a special method which simplifies the computations when a first integral is known.
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1. Introduction

In the latest years, there have been many developments concerning the problem of centres for systems of ordinary differential equations on the plane. By one side, improvements have been done in the direction of solving the centre-focus problem (see [14] or [24] for instance, and the references therein); however, the problem is far

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to be solved. By the other side, questions about either the kind of *period annulus* or the shape of the *period function* of a centre have also been tackled (the period annulus, $\mathcal{P}$ from now on, is the greatest neighbourhood of the centre filled of periodic orbits; given a transversal section of the period annulus, the time function defined on it is called the period function).

A first question is to decide whether the centre is isochronous or not. A recent survey on this problem is given in [5]. We would like to remark that in the works of Sabatini and Villarini (see [27,29]) they settled the strong relationship between Lie brackets and isochronicity. This idea has been used fruitfully by many authors. In the recent paper [17] we have also found a full description of the link between commutators and isochronicity.

A second question is that of controlling the number of critical points of the period function. This question has been treated for special families of vector fields by several authors (Chicone–Dumortier [9,10], for some polynomial systems; Chow and Wang [11], and Gavrilov [19] for potential systems; Coppel and Gavrilov [15], Collins [13], and Gasull et al. [18], for Hamiltonian centres with homogeneous non-linearities; Rothe [26], for some Hamiltonian families; Freire et al. [16], for perturbation of isochronous centres, etc.) They mainly focus on seeking for conditions of monotonicity of the period function and seldom examples of more than one critical period are found. Maybe one of the most relevant approach to give general tools for proving the monotonicity of the period function is due to Chicone (see [7]) who gave an expression for the first derivative of the period function as a dynamical interpretation of a result of Diliberto.

In the present paper, inspired in the geometrical ideas involved in the Lie bracket, we give a method to prove that some centres have either an increasing or a decreasing period function. This method is based on a formula for computing the derivative of the period function, which is obtained from the knowledge of the set of *normalizers* of the centre. See the definitions and more detailed comments after the statement of the following theorem, which is the key point of our paper. It will be proved in Section 2.

Theorem 1. *Consider a $\mathcal{C}^1$ vector field X having a centre at a point p with period annulus $\mathcal{P}$. The following statements hold:*

1. *Let U be a vector field, $U \in \mathcal{C}^1(\mathcal{P})$, transversal to X in $\mathcal{P}\backslash\{p\}$, and such that $[X, U] = \mu X$ on $\mathcal{P}$, for some $\mathcal{C}^1$ function $\mu : \mathcal{P} \subset \mathbb{R}^2 \to \mathbb{R}$. Denote by $\psi = \psi(s)$ a trajectory of U such that $\psi(s_0) \in \mathcal{P}$. Then,*

$$T'(s_0) = \int_0^{T(s_0)} \mu(x(t), y(t)) \, dt, \qquad (1)$$

where $(x(t), y(t))$ is the orbit of X such that $(x(0), y(0)) = \psi(s_0)$ and $T(s)$ the period of the orbit of X passing through $\psi(s)$.

2. *Assume that*

(a) *the vector field $X = (P, Q)$ admits an integrating factor $V(x, y)^{-1}$ in $\mathcal{P}$; that is, there exist $V(x, y)$ and $H(x, y)$ such that $X = (P, Q) = V(-H_y, H_x)$ in $\mathcal{P}$;*

(b) *there exist scalar functions R and S such that $RH_x + SH_y = \phi(H)$, for some smooth scalar function ϕ.*

Then, by taking the vector field $U = (R, S)$, it satisfies $[X, U] = \mu X$, with

$$\mu(x, y) = \operatorname{div} U - \frac{\nabla V \cdot U}{V} - \phi'(H). \tag{2}$$

As we have already explained, the aim of Theorem 1 is to give a tool to study the shape of the period function, that is, features like its monotonicity, its number of critical periods or knowing when it is constant (isochronicity problem). To be useful we need to be able to compute μ, and control its integral. The existence of U and μ satisfying $[X, U] = \mu X$, for sufficiently regular vector fields X with a non-degenerate centre at p is already known, see for instance [1]. Note also that our expression of T' given in (1), and based on the knowledge of U, is simpler that the one obtained in [7].

Part 2 of Theorem 1 tries to give a procedure to compute μ and U when an integrating factor for X is known. It can be seen as a reciprocal of the following well known result of S. Lie: Assume that a vector field $U = (R, S)$ such that $[X, U] = \mu X$ is known. If ψ is a first integral of X or a constant—usually ψ is taken to be 1—then a solution f of the system

$$\begin{cases} f_x P + f_y Q = 0, \\ f_x R + f_y S = \psi, \end{cases} \tag{3}$$

exists and it is also a first integral of X, (see [2, p. 108]). Our result is an extension of a previous one of S. Lie, see Theorem 2.48 in [23] or Proposition 1.1 in [31], which just covers the case $\phi' = 0$.

Observe that another interpretation of part 2 of Theorem 1 is the following: if for a given Hamiltonian vector field $\nabla H^\perp = (-H_y, H_x)$ we are able to find an U such that $[\nabla H^\perp, U] = \mu_H \nabla H^\perp$ then if we consider $\mu = \mu_H - (\nabla V \cdot U)/V$ it is satisfied that $[X, U] = \mu X$, where $X = V \nabla H^\perp$.

We also want to comment that it is very easy to find a formal solution $U = (R, S)$ of $RH_x + SH_y = H$, when $\operatorname{div} X \neq 0$. It suffices to take $U = (R, S) = (-V_y, V_x)/\operatorname{div} X$. Nevertheless, in most cases U is a not well defined vector field in a neighbourhood of p and it is not useful for our purposes. The freedom to choose ϕ is a key point of the method proposed to obtain a well-defined U in $\mathcal{P}$.

The first part of this paper is devoted to prove Theorem 1. In the second part we apply it to prove the monotonicity of the period function for several families of planar systems. Hence, once an U and a μ are obtained we are interested to prove that integral (1) has constant sign. In the systems that we study it sometimes happens that the μ that we have makes difficult these computations. A second step of our way of approach is try to get a more suitable μ. We detail this idea in the sequel.

From a geometrical point of view, the vector field U is the infinitesimal generator of the Lie group of symmetries of X. As usual in Lie theory, we call the set of infinitesimal generators the *normalizer* of X, while the set of commuting vector fields is called the *centralizer*, see [30] for more details. Accordingly, our work can be seen as giving the same dynamical interpretation for normalizers than Sabatini's and Villarini's results do for centralizers. Moreover, the set of normalizers of a given vector field X has the nice structure that we show in the following proposition.

Proposition 2. *Consider the set of normalizers of X,*

$$\mathcal{N}(X) = \{U \colon [X, U] = \mu X \text{ for some } \mu\},$$

and take $U \in \mathcal{N}(X)$ that satisfies $[X, U] = \mu X$. Then, if $U^* \in \mathcal{N}(X)$, it can be written as $U^* = \psi U + gX$, where ψ is either a first integral of X or a non-zero constant and g is any C^1 function. Moreover, $[X, U^*] = \mu^* X$, with $\mu^* = (\psi\mu + \nabla g^t \cdot X)$.

Proposition 2 gives a practical tool. For proving monotonicity one has to figure out in each case whether it is better to compute the value of $\int \mu$, as Theorem 1 suggests, or to find a new element of the normalizer whose corresponding μ^* is more suitable. Note that, in general, $\int \mu \neq \int \mu^*$ on the same periodic orbit of the period annulus because of the different parameterization given by the first integral ψ. However, the sign is preserved and so are the deductions on the qualitative behaviour of the period function.

We can summarize our approach to study the monotonicity of the period function in a method which, as far as we know, is a new one:

A method in three steps for proving the monotonicity of the period function:

(i) Try to compute U and μ defined in all the period annulus of p and satisfying $[X, U] = \mu X$. If X admits an integrating factor, use part 2 of Theorem 1.
(ii) Try to control the sign of the integral of μ which appears in (1). If you do not succeed then pass to the next steep.
(iii) Use Proposition 2 to get a more suitable μ. Go again to step (ii).

The last two sections of the paper contain the most interesting examples to which we have been able to apply our method. Some of the results that we get were already known but, even in these cases, we want to stress how our method enables to shorten the proofs.

In particular, in Section 3, we study Hamiltonian systems of type $H(x, y) = F(x) + G(y)$ and give some applications to physical problems. In the last section, we go through a miscellanea of examples: Lotka–Volterra centre, quadratic systems, Liénard systems and polynomial Hamiltonian systems with homogeneous non-linearities. Maybe the clearest application of our method is given in Proposition 19 of Section 4, where we prove that the period function of a family of quadratic systems is decreasing.

We end this introduction by noticing that from part 1 of Theorem 1 it can be deduced the following result on isochronicity:

Corollary 3. *Consider a C^1 vector field X having a centre at a point p and period annulus $\mathcal{P} \subset \mathbb{R}^2$. Let U be a vector field $U \in C^1(\mathcal{P})$, transversal to X in $\mathcal{P} \backslash \{p\}$, and such that $[X, U] = \mu X$ for some smooth scalar function $\mu : \mathcal{P} \to \mathbb{R}$. Let $\gamma = \{(x(t), y(t)), t \in [0, T_\gamma]\}$ be any periodic orbit of X in $\mathcal{P}$.*
Then, if there is a neighbourhood of p such that for any γ contained in it,

$$\int_0^{T_\gamma} \mu(x(t), y(t))\, dt = 0,$$

the centre is isochronous.

In [17] the converse of the above corollary is also proved and some applications of it are given.

2. Proofs and comments on Theorem 1

2.1. Proof of Theorem 1 and Proposition 2

Proof of Theorem 1. *Part* 1: Let $\gamma(t)$ be a periodic orbit of period T of X, and $p = \gamma(0) = \gamma(T)$. Take a transversal section Σ given by

$$g : (-\varepsilon, \varepsilon) \to \Sigma,$$

being $g(s)$ a solution of $x' = U(x)$ such that $g(0) = p$; that is, $g'(s) = U(g(s))$.
Consider as well the return map of X defined on Σ:

$$\pi : \Sigma_0 \subset \Sigma \to \Sigma.$$

If we call $\varphi(t, x)$ the flow defined by X, then

$$\pi(g(s)) = \varphi(T + \tau(s), g(s)).$$

Moreover, observe that in case that $\gamma(t)$ is a closed orbit of the interior of a period annulus, $T + \tau(s)$ is the period of the closed orbit passing through $g(s)$.
In this notation, it is easy to see that the monodromy matrix of the variational equation of the return map in the basis $\{X(p), U(p)\}$ is

$$\begin{pmatrix} 1 & -\tau'(0) \\ 0 & 1 \end{pmatrix}.$$

A key point of our proof is to note that the hypothesis $[X, U] = \mu X$ implies that

$$Y(t) := U(\gamma(t)) - \left\{ \int_0^t \mu(\gamma(u))\, du \right\} X(\gamma(t)),$$

is a solution of the variational equation, since

$$\frac{d}{dt} Y(t) = DU(\gamma(t))X(\gamma(t)) - \mu(\gamma(t))X(\gamma(t)) - \int_0^t \mu(\gamma(u))\, du\, DX(\gamma(t))X(\gamma(t))$$

$$= DX(\gamma(t))U(\gamma(t)) + \mu(\gamma(t))X(\gamma(t)) - \mu(\gamma(t))X(\gamma(t))$$

$$- \int_0^t \mu(\gamma(u))\, du\, DX(\gamma(t))X(\gamma(t)) = DX(\gamma(t))\, Y(t).$$

Finally, by observing that $Y(0) = U(p)$ and $Y(T) = U(p) - \int_0^T \mu(\gamma(t))\, dt X(p)$ we get that

$$\begin{pmatrix} \int_0^T -\mu(\gamma(t))\, dt \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\tau'(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so,

$$\tau'(0) = \int_0^T \mu(\gamma(t))\, dt,$$

as we wanted to prove.

Part 2: Let V^{-1} be an integrating factor of X, that is, $X = V(-H_y, H_x)$ for some Hamiltonian function H. Let us take the vector field $U = (R, S)$ satisfying $H_x R + H_y S = \phi(H)$. Then, straightforward computations give

$$[X, U] = \begin{pmatrix} R_x & R_y \\ S_x & S_y \end{pmatrix} \begin{pmatrix} -VH_y \\ +VH_x \end{pmatrix} - V \begin{pmatrix} -H_{yx} & -H_{yy} \\ H_{xx} & H_{xy} \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix}$$

$$- \begin{pmatrix} -V_xH_y & -V_yH_y \\ V_xH_x & V_yH_x \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix}$$

$$= \begin{pmatrix} -(R_x + S_y)VH_y + (RV_x + SV_y)H_y + V(R_yH_x + RH_{yx} + SH_{yy} + S_yH_y) \\ (R_x + S_y)VH_x - (RV_x + SV_y)H_x - V(R_xH_x + RH_{xx} + SH_{xy} + S_xH_y) \end{pmatrix}$$

$$= \begin{pmatrix} -\mathrm{div}(U)VH_y + (\nabla V \cdot U)H_y + V\frac{\partial}{\partial y}\phi(H) \\ \mathrm{div}(U)VH_x - (\nabla V \cdot U)H_x - V\frac{\partial}{\partial x}\phi(H) \end{pmatrix}$$

$$= \left(\mathrm{div}\, U - \frac{\nabla V \cdot U}{V} - \phi'(H) \right) X,$$

and thus the desired result.　□

Proof of Proposition 2. First of all, we observe that

$$[X, U^*] = [X, \psi U + gX] = \psi[X, U] + (\nabla \psi^t X)U + g[X, X] + (\nabla g^t X)X$$
$$= (\psi \mu + \nabla g^t X)X, \tag{4}$$

where in the last step we use that $[X, U] = \mu X$ and that ψ is either a first integral of X or a non-zero constant.

Last assertion tells us that any U^* of the prescribed type is a normalizer of X, and also gives the formula for μ^*. The property that any normalizer can be written in this way follows from the fact that U and X form a basis of $\mathbb{R}^2$ just because they are transversal. Then, there exist f and g such that $U^* = fU + gX$. Equality (4) with $\psi = f$ and $U^* \in \mathcal{N}(X)$ forces $\nabla f^t X = 0$, which implies that f is either a first integral of X or a non-zero constant (if it was zero, U would not be transversal to X), as we wanted to prove. $\quad\square$

2.2. Connected issues

In this subsection we present some results and comments related with Theorem 1 and Proposition 2. The first one is about a method given in [4] to compute μ and U when the first integral of X is a polynomial.

Remark 4. Let X be a vector field having a polynomial first integral $H(x, y)$ such that $\nabla H = 0$ has finitely many solutions in $\mathbb{C}^2$. Then, the hypothesis (b) of part 2 of Theorem 1 on existence of R and S can be removed. Let us prove this assertion.

Denote by (x_i, y_i), $i = 1, \ldots, m$, the set of zeroes of ∇H in $\mathbb{C}^2$. Following [4], we can construct a new first integral just taking

$$G(H) = \prod_{i=1}^{m} (H - H(x_i, y_i)).$$

Then, the Hilbert's zeroes theorem implies that there exist polynomials $R(x, y)$, $S(x, y)$ and $r \in N$ such that

$$RH_x + SH_y = G(H)^r,$$

as we wanted to see. In [4] the authors also study the case of X having a rational first integral.

The next remark shows that in $\mathcal{P} \backslash \{p\}$ there exists always a normalizer of X orthogonal to X. This remark is used in next section, see Remark 7(2).

Remark 5. Consider a non-degenerate critical point p with period annulus $\mathcal{P}$. Remember that we denote the set of normalizers of X as

$$\mathcal{N}(X) = \{U \colon [X, U] = \mu X \text{ for some } \mu\}.$$

Once one element $U \in \mathcal{N}(X)$ is given it is possible to construct another one, defined in $\mathcal{P}\backslash\{p\}$, $U^* \in \mathcal{N}(X)$ of the form $U^* = hX^\perp$, where $X^\perp$ is the vector field orthogonal to X and $h(x)$ is a real function. It suffices to take $U^* = U + gX$ and by imposing that $\langle X, U^* \rangle = 0$ we get that $g = -\langle X, U \rangle / \langle X, X \rangle$.

3. Hamiltonian systems of type $F(x) + G(y)$

This section has three parts, the first one dealing with the general properties (finding normalizers and adapting part 2 of Theorem 1 to the specific family), the second one containing some examples and applications to physical problems and the third one with the routine computations. This family has been also studied in [12,26,28].

We start with some notation and the technicalities to look for normalizers of the vector field induced by $H(x, y) = F(x) + G(y)$.

Define the numbers $x_L = \max\{x < 0 : F'(x) = 0\}$, $x_R = \min\{x > 0 : F'(x) = 0\}$, $y_L = \max\{y < 0 : G'(y) = 0\}$, $y_R = \min\{y > 0 : G'(y) = 0\}$. If some of these sets is empty, then the corresponding number is $\pm \infty$ ($-$ for L, $+$ for R). Denote also by $\mathcal{R}$ the rectangle $\mathcal{R} = (x_L, x_R) \times (y_L, y_R) \subset \mathbb{R}^2$.

Lemma 6. *Let F and G two real analytic functions at 0, such that $F(0) = G(0) = 0$ and they have a non-degenerate minimum at 0. Then,*

1. *Let X be the vector field given by*

$$\begin{cases} \dot{x} = -G'(y), \\ \dot{y} = F'(x), \end{cases} \tag{5}$$

    and U the vector field

$$U = \begin{cases} \dot{x} = \dfrac{F(x)}{F'(x)}, \\ \dot{y} = \dfrac{G(y)}{G'(y)}, \end{cases}$$

then U is well-defined in $\mathcal{R}$ and satisfies $[X, U] = \mu X$, where

$$\mu(x,y) = \operatorname{div} U - 1 = \frac{d}{dx}\left(\frac{F(x)}{F'(x)}\right) + \frac{d}{dy}\left(\frac{G(y)}{G'(y)}\right) - 1,$$

2. *The origin of (5) is a centre, which period annulus is contained in $\mathcal{R}$, and the associated period function T satisfies:*

$$T'(s) = \int_0^{T(s)} \mu(x(t), y(t))\, dt,$$

where s refers to the parameterization of the orbits of U.

Proof of Lemma 6. The vector field U is well-defined in $\mathcal{R}$ since F and G are analytical with a non-degenerate minimum at 0. Furthermore, the non-degeneracy of functions F and G guarantees the presence of a centre. Notice that the orbits of the period annulus of the origin cannot intersect the lines that form the boundary of $\mathcal{R}$. Straightforward computations from part 2 of Theorem 1 with $V(x,y) \equiv 1$ and $\phi(x) = x$ lead to the desired result. $\square$

Remark 7. (1) If instead of system (5) we consider the vector field $\hat{X}$ given by

$$\begin{cases} \dot{x} = -V(x,y)G'(y), \\ \dot{y} = V(x,y)F'(x), \end{cases}$$

with V analytic and $V(0,0) \neq 0$ it is easy to prove that taking the same U that in the above lemma, $[\hat{X}, U] = \hat{\mu}\hat{X}$, with $\hat{\mu} = V \operatorname{div} \frac{U}{V} - 1$.

(2) As suggested by Remark 5, a different U^* orthogonal to X can be taken. In particular, we get

$$U^* = \begin{cases} \dot{x} = k(x,y)F'(x), \\ \dot{y} = k(x,y)G'(y), \end{cases} \quad \mu^*(x,y) = k(x,y)(F'(x)^2 - G'(y)^2)(G''(y) - F''(x)),$$

where $k(x,y) = \frac{F(x)+G(y)}{(F'(x)^2+G'(y)^2)^2}$. However, in this case the function μ^* is not so easy to handle because the two variables cannot be separated.

Observe that for the function μ given in Lemma 6, we can equally separate the contribution of F and G in the expression and extract some useful sufficient conditions for monotonicity avoiding integration of μ. According to this goal, given

a function F and following the previously quoted papers, we define

$$v_F(x) = F'(x)^2 - 2F(x)F''(x),$$

$$\varphi_F(x) = \left(\frac{F(x)}{F'(x)^2}\right)' = v_F(x)/F'(x)^3.$$

This notation suggests to consider the following subclasses of $\mathcal{C}^2$ real functions of one variable:

Definition 8. Let $J \in \mathcal{C}^2(\Omega, \mathbb{R})$ for some $\Omega \subseteq \mathbb{R}$. We say that J is

- *of class $\mathcal{I}$* if either $v_J \geqslant 0$ or φ_J is increasing in Ω ($v_J \not\equiv 0$),
- *of class $\mathcal{N}$* if either $v_J \equiv 0$ or φ_J is constant in Ω,
- *of class $\mathcal{D}$* if either $v_J \leqslant 0$ or φ_J is decreasing in Ω ($v_J \not\equiv 0$).

We also say that a pair of functions $\{l_1, l_2\}$ form a $\mathcal{L}_1$–$\mathcal{L}_2$ pair if l_1 is of class $\mathcal{L}_1$ and l_2 is of class $\mathcal{L}_2$, where $\mathcal{L}_j$ stands for $\mathcal{I}$, $\mathcal{N}$ or $\mathcal{D}$.

Since the initial value problem $v_F(x) = 0$ with $F(0) = F'(0) = 0$ has the only solution $F(x) = kx^2$, $k \in \mathbb{R}$, class $\mathcal{N}$ becomes quite artificial. We keep it as a class only for aesthetic purposes.

On the other hand, under the hypotheses of Lemma 6, the periodic orbits of the period annulus of the origin are contained in $\mathcal{R}$. Notice also that in $\mathcal{R}$, the horizontal and vertical isoclines are, respectively, the axes $x = 0$ and $y = 0$. This fact leads to the following notation.

Definition 9. Let γ be a periodic orbit of the period annulus of the origin of system (5). We denote by $(x_M, 0)$, $(0, y_M)$, $(x_m, 0)$ and $(0, y_m)$ the intersections of γ with the axes, see Fig. 1.
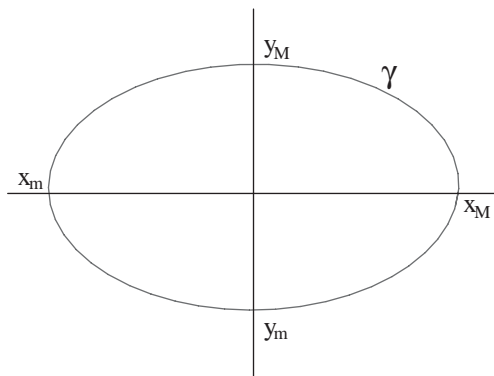

Fig. 1. Definitions of x_m, x_M, y_m and y_M.

Next proposition gives sense to these definitions and shows that the functions φ and v are suitable to find simpler ways to prove monotonicity.

Proposition 10 (see also [12] for the second part). *Consider the Hamiltonian system* (5) *generated by* $H(x, y) = F(x) + G(y)$, *with F and G two real analytic functions at* 0, *such that* $F(0) = G(0) = 0$ *and they have a non-degenerate minimum at* 0. *Then, the following hold*:

(1) *The function* μ *of Lemma* 6 *is defined in the rectangle* $\mathcal{R}$ *and can be written as*

$$\mu(x, y) = v_F(x) \frac{1}{2F'(x)^2} + v_G(y) \frac{1}{2G'(y)^2}. \tag{6}$$

(2) *By using the notation introduced in Definition* 9 *and in Fig.* 1, *the derivative of the period function in the period annulus of the origin can be written as*

$$\int_0^{T(s)} \mu(x(t), y(t)) \, dt = \frac{1}{2} \int_{y_m}^{y_M} [\varphi_F(x_+(y)) - \varphi_F(x_-(y))] \, dy$$
$$+ \frac{1}{2} \int_{x_m}^{x_M} [\varphi_G(y_+(x)) - \varphi_G(y_-(x))] \, dx. \tag{7}$$

(3) *The centre at the origin*
   (a) *is isochronous if* $\{F, G\}$ *form a* $\mathcal{N}$–$\mathcal{N}$ *pair.*
   (b) *has an increasing period function if* $\{F, G\}$ *form one of the following pairs:* $\mathcal{I}$–$\mathcal{I}$, $\mathcal{N}$–$\mathcal{I}$, $\mathcal{I}$–$\mathcal{N}$.
   (c) *has a decreasing period function if* $\{F, G\}$ *form one of the following pairs:* $\mathcal{D}$–$\mathcal{D}$, $\mathcal{N}$–$\mathcal{D}$, $\mathcal{D}$–$\mathcal{N}$.

We remark that the possibilities $\mathcal{I}$–$\mathcal{D}$ and $\mathcal{D}$–$\mathcal{I}$ are not reflected in Proposition 10. In principle, these situations could lead either to isochronous centres, or period-increasing, or period-decreasing or even more complicated behaviours, see the $\mathcal{I}$–$\mathcal{D}$ family of systems explored in Fig. 2 (the fact that these systems are of type $\mathcal{I}$–$\mathcal{D}$ is proved in Proposition 11). Of course, it is also possible that the functions F and G that define system (5), are not of any of the classes considered in Definition 8.

Proof of Proposition 10. Part 1 of the proposition follows from the equalities

$$\mu(x, y) = \left(\frac{F(x)}{F'(x)}\right)' - \frac{1}{2} + \left(\frac{G(y)}{G'(y)}\right)' - \frac{1}{2} = \frac{1}{2} - \frac{F(x)F''(x)}{F'(x)^2} + \frac{1}{2} - \frac{G(y)G''(y)}{G'(y)^2}$$
$$= v_F(x) \frac{1}{2F'(x)^2} + v_G(y) \frac{1}{2G'(y)^2}.$$

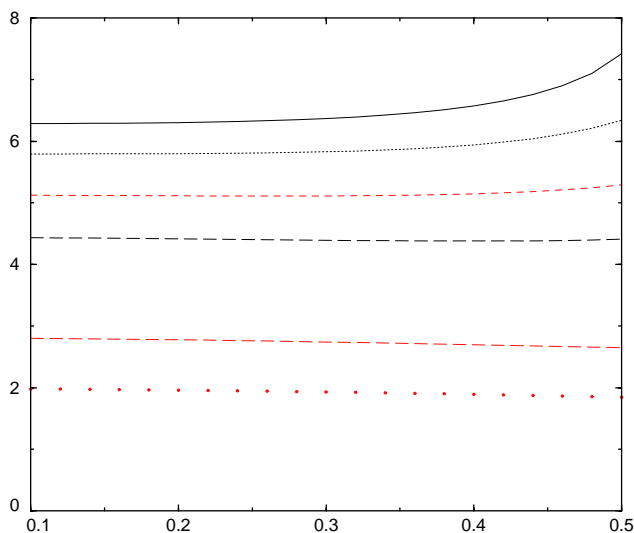Note also that $\mu(x, y) = \frac{1}{2}(\varphi_F(x)F'(x) + \varphi_G(y)G'(y))$.

Fig. 2. Numerical computations of the period function associated to $H(x,y) = k(x^2/2 + x^3/3) + y^2/2 + y^4/4$, for different values of k, $k = 1, 1.17525, 1.5, 2, 5, 10$ from above to below. While for $k = 1$ the period is increasing, from $k \approx 1.17525$ to some value it presents a minimum (so inappreciable in the scale of the figure that the centre seems to be isochronous) and it is decreasing for larger values of k like $k = 5, 10$.

To prove part 2, take a periodic orbit γ of (5), for some value h of the Hamiltonian. Call x_m, x_M, y_m and y_M the intersections of γ with the axes, as shown in Fig. 1. For each y, call $x_-(y)$ and $x_+(y)$ its two pre-images and, similarly, define $y_-(x)$ and $y_+(x)$. Then, using the hypotheses on F and G and the differential equations themselves, we obtain,

$$
\begin{aligned}
\int_0^{T(s)} \mu(x(t), y(t))\, dt &= \int_0^{T(s)} \left[\frac{1}{2}(\varphi_F(x)F'(x) + \varphi_G(y)G'(y)) \right]_{x=x(t), y=y(t)} dt \\
&= \int_{y_m}^{y_M} \left(\frac{1}{2}\varphi_F(x) \right)_{x=x_+(y)} dy - \int_{y_m}^{y_M} \left(\frac{1}{2}\varphi_F(x) \right)_{x=x_-(y)} dy \\
&\quad + \int_{x_m}^{x_M} \left(\frac{1}{2}\varphi_G(y) \right)_{y=y_+(x)} dx - \int_{x_m}^{x_M} \left(\frac{1}{2}\varphi_G(y) \right)_{y=y_-(x)} dx \\
&= \frac{1}{2} \int_{y_m}^{y_M} [\varphi_F(x_+(y)) - \varphi_F(x_-(y))]\, dy + \frac{1}{2} \int_{x_m}^{x_M} [\varphi_G(y_+(x)) - \varphi_G(y_-(x))]\, dx.
\end{aligned}
$$

The statements of part 3 mainly follow from the fact that the two variables play separate roles. Let us suppose, for instance, that $F(x)$ is of class $\mathcal{I}$; both if $v_F \geqslant 0$ and if φ_F is increasing, the term $\frac{1}{2} \int_{y_m}^{y_M} (\varphi_F(x_+(y)) - \varphi_F(x_-(y)))\, dy$ will be strictly positive. Similar reasonings apply for $G(y)$ and for the other two different classes of functions, $\mathcal{N}$ and $\mathcal{D}$.   □

Bearing in mind the definitions of classes $\mathcal{I}$, $\mathcal{N}$ and $\mathcal{D}$, in the next result we group all the functions that we will need from now on along the section so that the remaining results will not need detailed proofs. The list does not pretend to be exhaustive and tries to show the strength and clearness of the method.

Proposition 11. (1) *The following functions are of class $\mathcal{I}$ in Ω:*

(a) $I_1(z) = e^z - z - 1$, $\Omega = \mathbb{R}$.

(b) $I_2(z) = z^3/3 + z^2/2$, $\Omega = (-5/2, +\infty)$.

(c) $I_3(z) = -z^2\left(\frac{z^2}{4} + \frac{a-1}{3}z - \frac{a}{2}\right)$, *with* $0 < a < 1$ *and* $\Omega = (-a, 1)$.

(d) $I_4(z) = z^2\left(\frac{z^2}{4} + \frac{a+1}{3}z + \frac{a}{2}\right)$, *with* $0 < a \leqslant 1$ *and* $\Omega = (-a, +\infty)$.

(e) $I_5(z) = \frac{z^6}{6} - \frac{z^4}{2} + \frac{z^2}{2}$, $\Omega = \mathbb{R}\backslash\{-1, 1\}$.

(f) $I_6(z) = 1 - \cos z$, $\Omega = \mathbb{R}$.

(g) $I_7(z) = (p + qz)^\alpha - p^\alpha$ *with* p, q *positive real numbers and* $\alpha \notin [0, 1)$, $\Omega = \mathbb{R}$.

(h) $I_8(z) = \frac{z^2}{1+z^2}$, $\Omega = \mathbb{R}$.

(i) $I_9(z) = z \arctan z - \frac{1}{2}\ln(1 + z^2)$, $\Omega = \mathbb{R}$.

(2) *The following functions are of class $\mathcal{D}$ in Ω:*

(a) $D_1(z) = \alpha_m \frac{z^{2m}}{2m} + \alpha_n \frac{z^{2n}}{2n}$, *with* $\alpha_n > 0$, $\alpha_m \geqslant 0$ *and* $m > n \geqslant 1$, $\Omega = \mathbb{R}$.

(b) $D_2(z) = (p + qz)^\alpha - p^\alpha$ *with* p, q *positive real numbers and* $\alpha \in (0, 1)$, $\Omega = \mathbb{R}$.

The proof of the last proposition is given in Section 3.2. As a consequence of it, we can state:

Theorem 12. *The* 54 *parametric families of Hamiltonian systems associated either to* $H(x, y) = cI_i(x) + kI_j(y)$, *for* $i = 1, \ldots, 9$, $i \leqslant j \leqslant 9$; *or to* $H(x, y) = cI_i(x) + ky^2$, *for* $i = 1, \ldots, 9$ *and* $c > 0$, $k > 0$, *have increasing period function in the period annulus of the origin.*

The 5 *parametric families of Hamiltonian systems associated either to* $H(x, y) = cD_i(x) + kD_j(y)$, *for* $i = 1, \ldots, 2$, $i \leqslant j \leqslant 2$; *or to* $H(x, y) = cD_i(x) + ky^2$, *for* $i = 1, \ldots, 2$ *and* $c > 0$, $k > 0$, *have decreasing period function in the period annulus of the origin.*

Proof of Theorem 12. The theorem follows directly from Propositions 10(3) and 11. Only a nuance in the case of function D_1 must be underlined: note that when $n > 1$ the centre is degenerate, which breaks the first condition of Lemma 6. However, both the transversal vector field U, and the function μ are well-defined and the proofs and conclusions are still valid. Observe that in this case—as in any degenerate centre— the period function tends to infinity when the periodic orbits tend to the critical point. $\quad\square$

3.1. Distinguished examples from Theorem 12

The general family of Hamiltonian systems (5) treated in this section has connections with many physical problems and other well-known examples. Among the 59 cases presented in Theorem 12, we would like to stress how our method works for the non-forced pendulum, some applications to celestial mechanics and to relativistic mechanics, the Lotka–Volterra model and a number of potential systems. First of all, using function I_6, we get:

Example 13. The non-forced pendulum, the Hamiltonian system with

$$H(x,y) = \frac{y^2}{2} - \cos x + 1,$$

has increasing period.

A less trivial potential Hamiltonian arises when using function D_1:

Example 14. The potential Hamiltonian systems with $H(x,y) = \frac{y^2}{2} + \alpha_m \frac{x^{2m}}{2m} + \alpha_n \frac{x^{2n}}{2n}$, with $\alpha_m \geqslant 0$, $\alpha_n > 0$ and $m > n \geqslant 1$ have decreasing periods.

The features of $I_7(z) = D_2(z) = (p + qz)^\alpha - p^\alpha$ provide two interesting applications.

When $\alpha = \frac{1}{2}$, the resulting Hamiltonian is used in relativistic mechanics, where the problem of finding *constant period oscillators* (isochronous centres) has some interest, see [21] and the references therein. In that paper, the authors find numerical approximations of a function V such that the Hamiltonian $H(x,y) = V(x) + K(y)$, where $K(y) = \sqrt{m^2 + y^2/c^2} - m$, is isochronous. We think also that a nice way to find isochronous centres would be looking for V such that v_V compensates v_K. Here we give an example of decreasance of the period function.

Example 15. The period function associated to the centre of the Hamiltonian system given by $H(x,y) = \frac{1}{2}x^2 + \sqrt{m^2 + y^2/c^2} - m$ is decreasing.

The function $I_7(z)$ when $\alpha = -\frac{1}{2}$ leads to a Hamiltonian used in celestial mechanics to study the Sitnikov motion problem, see [3].

Example 16. The period function associated to the centre of the Hamiltonian system given by $H(x,y) = \frac{1}{2}y^2 - \frac{1}{\sqrt{x^2 + \frac{1}{4}}} + 2$ is increasing.

Remark 17. Theorem 12 covers many of the examples of a paper of Chow and Wang, see [11], where they study, for potential Hamiltonian systems, not only the first derivative of the period function but also give an expression for the second
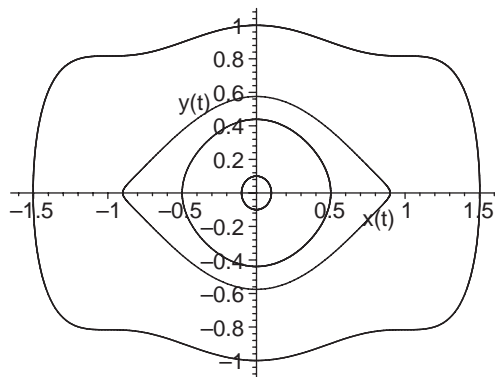
Fig. 3. The phase portrait of the Hamiltonian system derived from $H(x,y) = \frac{z^6}{6} - \frac{z^4}{2} + \frac{z^2}{2} + \frac{y^2}{2}$.

derivative. In the current context, potential Hamiltonian systems are equivalent to $G(y) = y^2/2$. In particular, taking $F(x) = I_1(x)$, $I_2(x)$, $I_3(x)$, $I_4(x)$ and $I_5(x)$, we obtain the increasing periods showed in [11] in Examples 1, 2, 3.a, 3.b and 5, respectively; and taking $F(x) = D_1(x)$ with $\{\alpha_m = 0, n = 2\}$ and $\{m = 4, n = 2, \alpha_m = \alpha_n = 1\}$ we obtain the decreasing periods given in [11] in Examples 3.d and 3.c $(b = 0)$. These are all the examples in that paper where they succeed to prove monotonicity.

The case when $F(x) = I_5(x) = \frac{z^6}{6} - \frac{z^4}{2} + \frac{z^2}{2}$ and $G(y) = \frac{y^2}{2}$ deserves some attention. The vector field has exactly three critical points: the centre at the origin and two cusps at $(\pm 1, 0)$. All the orbits of the vector field are closed, except for the two heteroclinics that link the two cusps, see Fig. 3. We have proved that the period function of the origin's period annulus is increasing; moreover, it must go to infinity as it approaches to those heteroclinics. Outside the heteroclinics, the normalizer U (see Lemma 6) is no longer transversal to the vector field and so, we cannot deduce that the period function is increasing. Indeed, there are strong numerical evidences that it is decreasing as the orbits go to infinity.

The increasance of periods for the Lotka–Volterra predator–prey system is one of the most known results related to periods in planar ODEs. It was first stated by Hsu [20], but some gap was found in the proof. Afterwards, it has been proved by several authors, see [25,28,32], sometimes being the main purpose of the paper.

Example 18. The centre of the classical Lotka–Volterra predator–prey system,

$$\begin{cases} \dot{x} = x(\alpha - \beta y), \\ \dot{y} = y(\gamma x - m), \end{cases} \tag{8}$$

has an increasing period function.

Here, we give a short proof of such a fact. By means of a change of variables $u = \log((\gamma x)/m)$, $v = \log((\beta y)/\alpha)$, the Lotka–Volterra system can be transformed into a Hamiltonian system of type $H(u,v) = F(u) + G(v)$, with $F(u) = \alpha(e^u - u - 1)$ and $G(v) = m(e^v - v - 1)$. Then, Theorem 12 with function I_1 gives the result. However, an advantage of our method is that we do not need to do any transformation and we can apply it directly to the original system, see Section 4.2.

We devote the rest of this section to prove all the cases listed in Proposition 11. The proof is quite technical and straightforward. So, the reader not interested in such details can jump directly to Section 4.

3.2. Proof of Proposition 11

To avoid cumbersome notations, in the whole proof we drop the subscripts for φ and v.

1. Functions of class $\mathcal{I}$.
   (a) For $I_1(z) = e^z - z - 1$, $\frac{d}{dz}\varphi(z) = -\frac{e^z}{(e^z-1)^4}(-e^{2z} - 4e^z + 4ze^z + 2z + 5)$.

   The function $-e^{2z} - 4e^z + 4ze^z + 2z + 5$ is always a negative function and so φ increasing.
   (b) For $I_2(z) = z^3/3 + z^2/2$, $\frac{d}{dz}\varphi(z) = \frac{1}{3}\frac{2z+5}{(z+1)^4}$.
   (c) Consider $I_3(z) = -z^2\left(\frac{z^2}{4} + \frac{a-1}{3}z - \frac{a}{2}\right)$, with $0 < a < 1$ and $-a < z < 1$. Elementary computations give:

   $$\frac{d}{dz}\varphi(z) = -\frac{1}{6}\frac{P(z,a)}{(z+a)^4(z-1)^4},$$

   where $P(z,a) = (-10 + 4z)a^3 + (11 - 42z + 16z^2)a^2 + (24z^3 - 10 - 68z^2 + 42z)a - 4z - 24z^3 + 16z^2 + 9z^4$. The proof is finished in (d) together with that of I_4.
   (d) For $I_4(z) = z^2\left(\frac{z^2}{4} + \frac{a+1}{3}z + \frac{a}{2}\right)$, with $0 < a \leqslant 1$ and $-a < z < +\infty$, we obtain in a similar way:

   $$\frac{d}{dz}\varphi(z) = \frac{1}{6}\frac{P(-z,-a)}{(z+a)^4(z+1)^4},$$

   so that to prove that both I_3 and I_4 are of class $\mathcal{I}$, we need
   - $P(z,a) \leqslant 0$ for all $(z,a) \in R_1 := (-a, 1) \times (0, 1)$,
   - $P(z,a) \geqslant 0$ for all $(z,a) \in R_2 := (-\infty, -a) \times [-1, 0)$.
       From standard computations it is easy to see that:
   (i) The restriction of $P(z,a)$ to ∂R_1 is negative except at $(z,a) = (0,0)$ and $(z,a) = (-1,1)$, where it is zero.
   (ii) The restriction of $P(z,a)$ to ∂R_2 is positive except at $(z,a) = (0,0)$, where it is zero.

(iii) $\frac{\partial}{\partial a} P(z,a)$ never vanishes in $R_1 \cup R_2$.

    Then, the result follows.

(e) For $I_5(z) = \frac{z^6}{6} - \frac{z^4}{2} + \frac{z^2}{2}$, $\frac{d}{dz}\varphi(z) = \frac{1}{3}\frac{10z^6 - 39z^4 + 60z^2 + 9}{(z^2-1)^6}$.

    The polynomial $10w^3 - 39w^2 + 60w + 9$ has two non-real roots and one real negative, so $10z^6 - 39z^4 + 60z^2 + 9$ has no real roots and φ' turns out to be positive everywhere it is defined.

(f) For $I_6(z) = 1 - \cos z$, $v(z) = (1 - \cos z)^2 \geq 0$.

(g) Since $I_7(z) = (p + qz)^\alpha - p^\alpha$ and $D_2(z)$ are the same function we are going to give the proof together.

    For the sake of simplicity, we write M instead of I_7 or D_2. We first compute $v = v_M$:

$$v(z) = M'(z)^2 - 2M(z)M''(z) = 4\alpha\frac{p}{q}p^{2\alpha}w^{\alpha-2}h(w),$$

where $w = (1 + pz^2/q) \geq 1$ and

$$h(w) = (\alpha - 1)w^{\alpha+1} + (2 - \alpha)w^\alpha + (1 - 2\alpha)w + (2\alpha - 2).$$

This expression tells us that all the cases behave as $p = q = 1$, that is, $M(z) = (1 + z^2)^\alpha - 1$, because it reduces the study of the sign of v_M to that of $h(w)$.

    Some elementary calculus gives the following properties: $h(1) = h'(1) = 0$ and $h''(w) = w^{\alpha-2}(\alpha(\alpha^2 - 1)w + \alpha(\alpha - 1)(2 - \alpha))$. Then, if h'' does not change sign, the function h also keeps the same sign. We can easily see that:

- when $\alpha > 1$, $h''(w) \geq 0 \Leftrightarrow w > (\alpha - 2)/(\alpha + 1)$;
- when $0 < \alpha < 1$, $h''(w) \leq 0 \Leftrightarrow w > (\alpha - 2)/(\alpha + 1)$;
- when $\alpha < 0$, $h''(w) \geq 0 \Leftrightarrow w < (\alpha - 2)/(\alpha + 1)$.

    For the function $(\alpha - 2)/(\alpha + 1)$, it is straightforward to see that the last three inequalities on w are true and so, $h(w) \geq 0$ for all w if $\alpha \notin (0, 1)$ and $h(w) \leq 0$ for all w if $\alpha \in (0, 1)$. The first and the third give the statement referred to function I_7 while the second one leads to that of D_2.

(h) For $I_8(z) = \frac{z^2}{1+z^2}$, $v(z) = 12\frac{z^4}{(1+z^2)^4} \geq 0$.

(i) For $I_9(z) = z\arctan z - \frac{1}{2}\ln(1 + z^2)$, $v(z) = \frac{\arctan^2(z)(1-z)^2 + \ln(1+z^2)}{1+z^2} \geq 0$.

2. Functions of class $\mathcal{D}$.

(a) Consider $D_1(z) = \alpha_m\frac{z^{2m}}{2m} + \alpha_n\frac{z^{2n}}{2n}$, with $\alpha_n \neq 0$, $\alpha_m/\alpha_n \geq 0$ and $m > n \geq 1$. Denoting $w = w(z) = z^{2(m-n)}$ it turns out that

$$v(z) = z^{4n-2}(Aw^2 + Bw + C),$$

with $A := \alpha_m^2(-1 + \frac{1}{m})$, $B := \frac{\alpha_m\alpha_n}{mn}(m - 2m^2 + n + 2mn - 2n^2)$ and $C := \alpha_n^2(-1 + \frac{1}{n})$. For $w = 0$ the value of the second degree polynomial is

$\alpha_n^2(-1+\frac{1}{n}) \leqslant 0$. Now, if we prove that it does not have positive solutions, we are done. So, we impose $\frac{B}{2A} \geqslant 0$ and $AC \geqslant 0$. The last inequality always holds since $AC = \alpha_m^2 \alpha_n^2(-1+\frac{1}{m})(-1+\frac{1}{n}) \geqslant 0$. On the other hand,

$$\frac{B}{2A} = \frac{\alpha_m}{2n(1-m)\alpha_n}(m - 2m^2 + n + 2mn - 2n^2).$$

Since $\quad m - 2m^2 + n + 2mn - 2n^2 = m(1-n) + n(1-m) - 2(m-n)^2 < 0$, $B/(2A) \geqslant 0$ reduces then to $\alpha_m/\alpha_n \geqslant 0$, which is true by hypothesis. Finally, although it is not necessary for D_1 being of class $\mathcal{D}$, we need to assume that α_n is positive so that the origin is a centre.

(b) For $D_2(z)$ see the proof of $I_7(z)$. $\square$

4. Other examples

4.1. A quadratic system with decreasing periods

This subsection is devoted to a new result about a family of quadratic systems with a decreasing period function. A bigger family of quadratic systems including the next one was treated in [4] as Example 2. Despite they obtain a general expression for $\mu(x,y)$, it is too difficult to handle for our purposes. We have considered the following case, which is also a Loud's system, see [22] and also [8, Ex. 5.21].

Proposition 19. *The quadratic system*

$$\begin{cases} \dot{x} = -y + 2Dx^2 - Dy^2, \\ \dot{y} = x + Dxy, \end{cases} \tag{9}$$

has a decreasing period function.

Proof of Proposition 19. First of all, notice that the change of variables $\tilde{x} = Dx$, $\tilde{y} = Dy$ eliminates the parameter D in (9) and so we can consider only the case $D = 1$:

$$\begin{cases} \dot{x} = -y + 2x^2 - y^2, \\ \dot{y} = x + xy. \end{cases} \tag{10}$$

A first integral for (10) is

$$H(x,y) = \frac{1}{2}\frac{x^2}{(1+y)^4} - \frac{1}{6}\frac{(1+3y)}{(1+y)^3} + \frac{1}{6},$$

which associated integrating factor is $1/V$, where $V = (1+y)^5$.

It is not difficult to see that the periodic orbits γ_h corresponding to the period annulus of the origin are included in the sets $\{H = h, 0 < h < 1/6\}$ (they are one of the two connected components of the level sets). When $h \to 1/6$, the periodic orbits approach to the curve $x^2 = y^2 + 4/3y + 1/3$.

Note that for H of the special form $H(x, y) = A(y) + B(y)x^2$ it is easy to prove that

$$U = \left(\left(1 - \frac{A(y)}{A'(y)} \frac{B'(y)}{B(y)} \right) \frac{x}{2}, \frac{A(y)}{A'(y)} \right)$$

is a normalizer of $\nabla H^\perp$, see also [17]. Arguing as in Remark 7 (1), we have that the same U is also a normalizer for any system of the form $V(x, y) \nabla H^\perp$. By performing these computations in our case we can take $U = (R, S)$ where

$$R(x, y) = x(\tfrac{1}{2} + \tfrac{1}{3}y(3 + y)) \quad \text{and} \quad S(x, y) = \tfrac{1}{6}y(3 + y)(1 + y).$$

Furthermore, $RH_x + SH_y = H$. By using part 2 of Theorem 1 we have that a μ associated to system (10) is

$$\mu = (1 + \tfrac{7}{3}y + \tfrac{5}{6}y^2) - (\tfrac{5}{6}y(3 + y)) - 1 = -\tfrac{1}{6}y$$

and hence

$$T'(s) = \int_0^{T(s)} \mu(x(t), y(t)) \, dt = -\frac{1}{6} \int_0^{T(s)} y(t) \, dt.$$

Fixed y and s, the first integral H tells us that there exists only a pair of values of x, $-x_-(y) = x_+(y) > 0$, such $(x, y) \in \gamma_s$. For a fixed s, define y_m and y_M the two intersection of γ_s with the y-axis, see also Fig. 1.

At this point the integration could be cumbersome and perhaps not possible. We make use, now, of Proposition 2. It turns out that taking $g(x, y) = -\frac{x}{6(1+y)}$, and defining

$$\mu^*(x, y) = \mu(y) + \nabla g^t \cdot X = -\frac{1}{6} \frac{x^2}{1 + y},$$

we can compute T' as

$$T'(w) = -\frac{1}{6} \left(\int_{y_m}^{y_M} \frac{x_+(y)}{(1 + y)^2} \, dy - \int_{y_m}^{y_M} \frac{x_-(y)}{(1 + y)^2} \, dy \right) = -\frac{1}{3} \left(\int_{y_m}^{y_M} \frac{x_+(y)}{(1 + y)^2} \, dy \right),$$

because of the symmetry on x. Clearly, the argument of the last integral is always positive and so, we can assert that the period is decreasing. $\square$

Remark 20. It is easy to see that the periods of the orbits of system (10) move in a narrow range. As we have already seen, the period annulus is unbounded and not

global. The "finite" part of its boundary is given by the algebraic curve $3(x^2 - y^2) - 4y - 1 = 0$. An easy computation shows that the time to travel through this curve is $T^* = 6$:

$$T^* = \int_{-\infty}^{\infty} \frac{dx}{-y + 2x^2 - y^2} = \int_{-\infty}^{\infty} \frac{9\,dx}{(1+9x^2)^{1/2} + 1 + 9x^2} = \frac{(1+9x^2)^{1/2} - 1}{x}\bigg|_{-\infty}^{\infty} = 6.$$

Since we do not consider the time spent through infinity, we just can state that for all the closed orbits of this centre, the period T satisfies $2\pi > T > 6$. In fact, the period function starts with the value 2π and decreases, tending to some value which is greater or equal than 6.

4.2. The Lotka–Volterra system (a second proof)

We give another proof of the monotonicity of the period function for the Lotka–Volterra system

$$\begin{cases} \dot{x} = x(\alpha - \beta y) = -xy H_y(x,y), \\ \dot{y} = y(\gamma x - m) = xy H_x(x,y), \end{cases} \tag{11}$$

which works directly on (11), without changing variables. Here $H(x,y) = F(x) + G(y)$ where

$$F(x) = \gamma x - m\left(\ln\left(\frac{\gamma x}{m}\right) + 1\right) \quad \text{and} \quad G(y) = \beta y - \alpha\left(\ln\left(\frac{\beta y}{\alpha}\right) + 1\right).$$

By Remark 7(1), $U = \left(\frac{F(x)}{F'(x)}, \frac{G(y)}{G'(y)}\right)$ is a normalizer of (11) and μ is

$$\mu(x,y) = 1 - \frac{\gamma x F(x)}{(\gamma x - m)^2} - \frac{\beta y G(y)}{(\beta y - \alpha)^2}.$$

Now, we want to see that $T'(s) > 0$, where s is the parameter of some orbit of U, because this parameter increases forward from the critical point.

We perform the integration in the following way:

$$\int_0^{T(s)} \mu(x(t), y(t))\,dt = \int_{y_m}^{y_M} \left[\frac{1}{y(\gamma x - m)}\left(\frac{1}{2} - \frac{\gamma x F(x)}{(\gamma x - m)^2}\right)\right]_{x_-(y)}^{x_+(y)} dy$$

$$+ \int_{x_m}^{x_M} \left[\frac{1}{x(\alpha - \beta y)}\left(\frac{1}{2} - \frac{\beta y G(y)}{(\alpha - \beta y)^2}\right)\right]_{y_+(x)}^{y_-(x)} dx.$$

Then, making the change of variables $u = \ln(\gamma x/m)$, $v = \ln(\beta y/\alpha)$ in both integrals we obtain

$$\int_0^{T(s)} \mu(x(t), y(t))\, dt = \frac{1}{m} \int_{v_m}^{v_M} \left[\frac{1 + 2ue^u - e^{2u}}{2(e^u - 1)^3} \right]_{u_-(v)}^{u_+(v)} dv + \frac{1}{\alpha} \int_{u_m}^{u_M} \left[\frac{1 + 2ve^v - e^{2v}}{2(1 - e^v)^3} \right]_{v_+(u)}^{v_-(u)} du.$$

If we denote

$$H(\xi) = \frac{1 + 2\xi e^\xi - e^{2\xi}}{2(e^\xi - 1)^3},$$

it turns out that

$$H'(\xi) = -\frac{1}{2} \frac{e^\xi(-4e^\xi + 5 + 4\xi e^\xi + 2\xi - e^{2\xi})}{(e^\xi - 1)^4}.$$

The function in parenthesis in the numerator is negative (it is the same function that the one that appears in the proof of Proposition 11, function I_1). Then, $H(\xi)$ is an increasing function. This fact and the preceding computations clearly imply the result. $\square$

4.3. A family of Liénard systems

In the next result we prove that a subfamily of Liénard systems—which includes the quadratic one with $A(x) = x^2/2$ studied in [6]—has an increasing period function in the period annulus of the origin.

Proposition 21. *The family of Liénard equations*

$$\begin{cases} \dot{x} = -y + A(x), \\ \dot{y} = A'(x), \end{cases} \tag{12}$$

with A an smooth function satisfying $A(0) = A'(0) = 0$, has a centre at the origin. Furthermore, if $A(x) = kI_i(x)$, for some $i = 1, \ldots, 9$ and $k > 0$ where I_i are the functions which appear in Proposition 11 or $A(x) = kx^2$, then the period function of (12) is increasing in the period annulus of the origin.

Proof of Proposition 21. By using the change of variables $(u, v) = (x, y - A(x))$ we get the new system

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = A'(u)(1 + v). \end{cases}$$

Applying the new change $(z, w) = (u, \log(1 + v))$ we arrive to

$$\begin{cases} \dot{z} = 1 - e^w, \\ \dot{w} = A'(z), \end{cases}$$

which is of the form of the systems for which Theorem 12 applies. By applying it with the Hamiltonians $I_1(w) + kI_j(z)$, with $j = 1, \ldots, 9$, or with the Hamiltonian $I_1(w) + kz^2$ the result follows. $\square$

4.4. Polynomial Hamiltonian with homogeneous non-linearities

To finish, we give an overview to one of the families where the period function is better understood: that of Hamiltonian systems obtained from

$$H(x, y) = \tfrac{1}{2}(x^2 + y^2) + H_{n+1}(x, y),$$

where H_{n+1} is a homogeneous polynomial of degree $n + 1$. It has been shown, see [18] and the references therein, that the period function of the centre at the origin is always increasing when n is even and has at most one critical period when n is odd. Here, we will see how formula (9) for the derivative of the period function used in [18] can be obtained using our method.

To achieve this goal it is convenient to express the system in polar coordinates. We remark that the Lie bracket does not depend on the chosen variables. Therefore, the vector field is

$$\begin{cases} \dot{r} = -r^n g'(\theta), \\ \dot{\theta} = 1 + (n + 1)r^{n-1}g(\theta), \end{cases}$$

while the Hamiltonian writes now as $H(r, \theta) = \tfrac{1}{2}r^2 + r^{n+1}g(\theta)$.

Aiming to use part 2 of Theorem 1, we search for $R = R(r, \theta)$ and $S = S(r, \theta)$ such that $RH_x + SH_y = H$. We observe first that $R_1 = r/2$ and $S_1 = (1 - \tfrac{n}{2})g(\theta)/g'(\theta)$ satisfy $R_1 H_r + S_1 H_\theta = H$. Then, using that $H_r = H_x \cos\theta + H_y \sin\theta$ and $H_\theta = -rH_x \sin\theta + rH_y \cos\theta$, it turns out that

$$R = R_1 \cos\theta - S_1 r \sin\theta,$$

$$S = R_1 \sin\theta + S_1 r \cos\theta,$$

satisfy the required relation. Hence, from our main theorem, we know that

$$\mu(\theta) = \frac{1}{2}\frac{g'(\theta)^2(1 - n) + g''(\theta)g(\theta)(n - 1)}{g'(\theta)^2} = \frac{1 - n}{2}\frac{d}{d\theta}\left(\frac{g(\theta)}{g'(\theta)}\right).$$

Integrating by parts it becomes (except for a positive constant) the same formula used in [8].

Acknowledgments

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