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Existence and non-existence of solutions for a class of Monge–Ampère equations<sup>☆</sup>

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## ABSTRACT

We study the boundary value problems for Monge–Ampère equations:  $\det D^2u = e^{-u}$  in  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 1$ ,  $u|_{\partial\Omega} = 0$ . First we prove that any solution on the ball is radially symmetric by the argument of moving plane. Then we show there exists a critical radius such that if the radius of a ball is smaller than this critical value there exists a solution, and vice versa. Using the comparison between domains we can prove that this phenomenon occurs for every domain. Finally we consider an equivalent problem with a parameter  $\det D^2u = e^{-tu}$  in  $\Omega$ ,  $u|_{\partial\Omega} = 0$ ,  $t \geq 0$ . By using Lyapunov–Schmidt reduction method we get the local structure of the solutions near a degenerate point; by Leray–Schauder degree theory, a priori estimate and bifurcation theory we get the global structure.

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## 1. Introduction

The Monge–Ampère equations are a type of important fully nonlinear elliptic equations; that is, nonlinear elliptic equations that are not quasilinear [4,11,14]. In this paper, we consider the boundary value problems for a class of Monge–Ampère equations:

$$\begin{cases} \det D^2u = e^{-u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a bounded convex domain in  $\mathbb{R}^n$  ( $n \geq 1$ ) with smooth boundary, and the matrix  $D^2u = (u_{ij}) = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ ,  $i, j = 1, 2, \dots, n$ , is the Hessian of  $u$ ; in the following we will simply denote the first

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derivative by  $u_i$  ( $i = 1, 2, \dots, n$ ), second derivative by  $u_{ij}$  ( $i, j = 1, 2, \dots, n$ ) and so on; we also use  $(u_{ij})$  instead of  $D^2u$  sometimes. In this paper we only consider convex solutions of (1) in order to ensure the ellipticity of the equation. In fact, any convex solution  $u$  of Eq. (1) is smooth, negative and strictly convex on  $\overline{\Omega}$  (we can get  $C^3(\overline{\Omega})$  estimate by Theorem 17.23 in [4], then by a standard bootstrap argument and Schauder estimate we can prove higher order estimates).

This equation is an analogue of the important complex Monge–Ampère equation arising from the Kähler–Einstein metric in the case of positive first Chern class in geometry. The equation written in the local coordinates has the form:

$$\frac{\det(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j})}{\det(g_{i\bar{j}})} = e^{f-u},$$

where  $\sum_{1 \leq i, j \leq n} g_{i\bar{j}} dz^i d\bar{z}^j$  is the Kähler metric in the class of the first Chern class, and  $f$  is a known function. The equation has been studied by many mathematicians and many problems remain still open cf. [10] and references therein.

We denote  $\lambda\Omega := \{\lambda x : x \in \Omega\}$ ,  $\lambda > 0$ ; noticing that Eq. (1) is invariant under translations, in this paper we assume without loss of generality that  $0 \in \Omega$ .

The main results in Section 2 are Theorem 2.2, Corollary 2.3; the main theorems in Section 3 are Theorems 3.1 and 3.4.

We also consider a problem with a parameter  $t \geq 0$  in Section 4:

$$\begin{cases} \det u_{ij} = e^{-tu} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

This is equivalent to Eq. (1) in  $t^{\frac{1}{2}}\Omega$  for  $t > 0$  through a scaling.

Our main result is (see Theorems 4.11, 4.12 and 4.15) in Section 4:

**Theorem 1.1.** *Given a smooth bounded convex domain  $\Omega$ , there exists a critical value  $T^* > 0$  such that*

1. for  $t \in (0, T^*)$ , there exist at least two solutions of (2);
2. for  $t = T^*$ , there exists a unique solution of (2);
3. for  $t > T^*$ , there exists no solution of (2).

Moreover, we get some other results, like the local structure of the branch near the first degenerate point (see Propositions 4.7, 4.8 and Theorem 4.9).

Concerning Eq. (1), Theorem 1.1 implies:

**Theorem 1.2.** *Given a smooth bounded convex domain  $\Omega$ , there exists a critical value  $\lambda^* > 0$  such that*

1. for  $\lambda \in (0, \lambda^*)$ , there exist at least two solutions of (1) in  $\lambda\Omega$ ;
2. there exists a unique solution of (1) in  $\lambda^*\Omega$ ;
3. for  $\lambda > \lambda^*$ , there exists no solution of (1) in  $\lambda\Omega$ .

Sometimes we would like to write Eq. (1) in the form:

$$\log \det D^2 u = -u.$$

This form is more natural in this paper, because the function  $\log \det$  defined on the space of positive definite symmetric matrices has many interesting properties, like concavity (see Appendix A).

For a more general right-hand side term our method still goes through, we can have a result analogues to Theorem 1.1 for the problem

$$\begin{cases} \log \det D^2 u = -tk(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where  $t \geq 0$  is a parameter and  $k(\cdot) : (-\infty, 0] \rightarrow (-\infty, 0]$  is a  $C^2$  function satisfying some conditions (see Remark 4.16).

In this paper, in Section 2 using the argument of moving plane, we prove that any solution of (1) on the ball is radially symmetric. This method has been used by several authors in slightly different settings (see [8] and [13]). In Section 3 we can reduce the equation on the ball to an ODE, and prove there exists a critical radius such that if the radius of a ball is smaller than this critical value there exists a solution, and vice versa. Using the comparison between domains we prove that this phenomenon occurs for every domain. We calculate the one-dimensional case explicitly, which also indicates some kind of bifurcation phenomena may exist (for related results about bifurcation, see [3,5,9,16]). In Section 4, for the fixed domain, by using Lyapunov–Schmidt reduction method we get the local structure of the solutions near a degenerate point, and prove existence of at least two solutions for a certain range of parameters (see Theorem 4.9). Finally we study the global structure of the branch emerging from  $t = 0$  by the Leray–Schauder degree theory, a priori estimates and bifurcation theory. From all of these results we prove Theorem 1.1 at last. In Appendix A we collect some results concerning matrices used in our paper.

## 2. Moving plane argument

In this section we prove a symmetry result for a  $C^3(\overline{\Omega})$  solution  $u$  of Eq. (1) using the moving plane method (see [6], of course from the regularity theory we can relax the regularity assumption). With the property of being  $C^3$  continuous up to the boundary, there exist two positive constants  $\lambda$  and  $\Lambda$  such that

$$\lambda|\xi|^2 \leq g^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \xi \neq 0.$$

Here  $(g^{ij})$  is the inverse matrix of the Hessian of  $u$ .

Given three positive constants  $\lambda$ ,  $\Lambda$  and  $C_0$ , for any smooth domain  $\Gamma$  in  $\mathbb{R}^n$  of sufficiently small measure, here we give a generalization of the weak maximum principle for the elliptic operator  $Lf := h^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + f$  (here and in the sequel we will use the summation convention that repeated indices are summed from 1 to  $n$ ) if  $h^{ij}(x)$  satisfy  $\lambda|\xi|^2 \leq h^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$ ,  $\forall x \in \Gamma$ ,  $\forall \xi \in \mathbb{R}^n$ ,  $\xi \neq 0$ , and  $h^{ij}(x)$  are uniformly bounded in  $C^1(\overline{\Gamma})$  by the constant  $C_0$ .

**Lemma 2.1.** *There exists a positive constant  $\delta$ , which only depends on  $\lambda$ ,  $\Lambda$  and  $C_0$ , such that for any smooth domain  $\Gamma$  in  $\mathbb{R}^n$  of measure smaller than  $\delta$ , if  $Lf = 0$  in  $\Gamma$ , and  $f \geq 0$  on  $\partial\Gamma$ , then  $f \geq 0$  in  $\Gamma$ .*

**Proof.** Use  $f^- = \max\{-f, 0\}$  as a test function and integrate by parts to get

$$\begin{aligned} \int_{\Gamma} (f^-)^2 &= \int_{\Gamma} h^{ij} f_{ij} f^- = \int_{\Gamma} \frac{\partial}{\partial x_j} (h^{ij} f_i f^-) - f_i \frac{\partial}{\partial x_j} (h^{ij} f^-) \\ &= \int_{\Gamma} f_i^- \frac{\partial}{\partial x_j} (h^{ij} f^-) = \int_{\Gamma} h^{ij} f_i^- f_j^- + \frac{\partial}{\partial x_j} (h^{ij}) f^- f_i^-. \end{aligned}$$

Because  $h^{ij}$  are uniformly bounded in  $C^1(\Gamma)$  by a constant which is independent of the domain  $\Gamma$ , there is a constant  $C_1$  such that  $|\frac{\partial}{\partial x_j}(h^{ij})| := |\sum_j \frac{\partial h^{ij}}{\partial x_j}| \leq C_1$ , so using the uniform ellipticity and Cauchy inequality we get

$$\int_{\Gamma} (f^-)^2 \geq \int_{\Gamma} \lambda |\nabla f^-|^2 - \int_{\Gamma} \frac{\lambda}{2} |\nabla f^-|^2 - \int_{\Gamma} \frac{C^2}{2\lambda} |f^-|^2. \quad (4)$$

So we have

$$\int_{\Gamma} (f^-)^2 \geq C \int_{\Gamma} |\nabla f^-|^2, \quad (5)$$

here  $C$  only depends on  $h^{ij}$ .

If  $n > 2$  we can use Sobolev embedding theorem to obtain

$$\int_{\Gamma} |\nabla f^-|^2 \geq C \left( \int_{\Gamma} (f^-)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}. \quad (6)$$

Then from Hölder inequality we get

$$(\text{meas}(\Gamma))^{\frac{2}{n}} \left( \int_{\Gamma} (f^-)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \geq \int_{\Gamma} (f^-)^2. \quad (7)$$

Combining (5)–(7) we get

$$(\text{meas}(\Gamma))^{\frac{2}{n}} \left( \int_{\Gamma} (f^-)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \geq C \left( \int_{\Gamma} (f^-)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}. \quad (8)$$

Note that the constant in (8) is independent on  $\Gamma$ , so (8) implies  $f^- = 0$  in the case of  $\text{meas}(\Gamma)$  is small enough, that is  $f \geq 0$  in  $\Gamma$ .

If  $n = 2$ , (6)–(8) do not make sense. However, we can use Hölder Inequality to obtain a similar result. Choose some  $\alpha \in (1, 2)$ , we have similarly as (6)–(8)

$$\begin{aligned} \int_{\Gamma} |\nabla f^-|^2 &\geq C \left( \int_{\Gamma} |\nabla f^-|^{\alpha} \right)^{\frac{2}{\alpha}} \geq C \left( \int_{\Gamma} (f^-)^{\frac{2\alpha}{2-\alpha}} \right)^{\frac{2-\alpha}{\alpha}}, \\ (\text{meas}(\Gamma))^{\frac{2\alpha-2}{\alpha}} \left( \int_{\Gamma} (f^-)^{\frac{2\alpha}{2-\alpha}} \right)^{\frac{2-\alpha}{\alpha}} &\geq \int_{\Gamma} (f^-)^2, \\ (\text{meas}(\Gamma))^{\frac{2\alpha-2}{\alpha}} \left( \int_{\Gamma} (f^-)^{\frac{2\alpha}{2-\alpha}} \right)^{\frac{2-\alpha}{\alpha}} &\geq C \left( \int_{\Gamma} (f^-)^{\frac{2\alpha}{2-\alpha}} \right)^{\frac{2-\alpha}{\alpha}}, \end{aligned}$$

where  $C$  is a constant depending only on  $\Omega$ . Then we get the same result too. The proof for the case  $n = 1$  is exactly the same as above  $n = 2$ .  $\square$

**Theorem 2.2.** *Let  $\Omega$  be symmetric with respect to a hyperplane, then any solution  $u \in C^3(\overline{\Omega})$  of (1) is symmetric with respect to the hyperplane too.*

**Proof.** Without loss of generality, we can assume the hyperplane is the coordinate plane  $x_1 = 0$ . We denote  $\Omega_t := \Omega \cap \{x_1 \leq t\}$  ( $t \leq 0$ ), and  $u_t(x_1, x_2, \dots, x_n) := u(2t - x_1, x_2, \dots, x_n)$  in  $\Omega_t$ . Then we have

$$D^2 u_t(x_1, x_2, \dots, x_n) = P D^2 u(2t - x_1, x_2, \dots, x_n) P^T, \quad (9)$$

where the diagonal matrix  $P = \text{diag}\{-1, 1, \dots, 1\}$  and  $P^T$  is the transpose of  $P$ . Because  $\det P = -1$ , we have

$$\begin{aligned} \det D^2 u_t(x_1, x_2, \dots, x_n) &= \det D^2 u(2t - x_1, x_2, \dots, x_n) \\ &= e^{-u(2t - x_1, x_2, \dots, x_n)} \\ &= e^{-u_t(x_1, x_2, \dots, x_n)}, \end{aligned}$$

so  $u_t$  still satisfies Eq. (1) in  $\Omega_t$ .

Now

$$\begin{aligned} -u + u_t &= \log \det(u_{ij}) - \log \det((u_t)_{ij}) \\ &= \int_0^1 \frac{d}{d\tau} \log \det(\tau u_{ij} + (1 - \tau)(u_t)_{ij}) d\tau \\ &= \left[ \int_0^1 g_\tau^{ij} d\tau \right] (u - u_t)_{ij}. \end{aligned}$$

Here  $(g_\tau^{ij})$  is the inverse matrix of  $(\tau u_{ij} + (1 - \tau)(u_t)_{ij})$ .

Let  $w_t = u - u_t$ , which satisfies the equation  $-\left[\int_0^1 g_\tau^{ij} d\tau\right](w_t)_{ij} = w_t$ . We have  $u = u_t$  on  $\partial\Omega_t \cap \{x_1 = t\}$ , and for  $t < 0$  we have  $u = 0$  and  $u_t < 0$  on  $\partial\Omega_t \cap \partial\Omega$  because the reflection of this part lies in the interior of  $\Omega$ . Here we use the fact that  $u < 0$  in the interior of  $\Omega$  because  $u$  is convex with vanishing boundary value, this fact will be used a lot in the sequel. Thus  $w_t \geq 0$  on  $\partial\Omega_t$ . Because  $\int_0^1 g_\tau^{ij} d\tau$  are uniformly elliptic with the same constant  $\lambda$  and  $\Lambda$  as for  $g^{ij}$  above and uniformly bounded in  $C^1$  with the constant which only depends on  $u$ , by Lemma 2.1 we conclude that if  $t$  is so close to  $\min\{x_1 \mid x \in \Omega\}$  that  $\text{meas}(\Omega_t)$  is small enough, then  $w_t \geq 0$  in  $\Omega_t$ . Now  $-\left[\int_0^1 g_\tau^{ij} d\tau\right](w_t)_{ij} \geq 0$ , by the strong maximum principle (see Theorem 3.5 of [4]), we get that either  $w_t > 0$  strictly in the interior point of  $\Omega_t$  or  $w_t \equiv 0$  in  $\Omega_t$ .

Now we can move the plane towards right. Define

$$T = \sup\{t < 0 \mid w_t \geq 0 \text{ in } \Omega_t\}. \quad (10)$$

If  $T < 0$ , because  $w_T > 0$  on  $\partial\Omega_T \cap \partial\Omega$ , we have  $w_T > 0$  strictly in the interior point of  $\Omega_T$ . Thus there exists a compact set  $K \subset \text{int}\Omega_T$  such that  $\text{meas}(\Omega_T \setminus K)$  is small and there exists a positive constant  $\epsilon$  such that  $w_T > \epsilon$  in  $K$ . Noticing  $w_t$  is continuous with respect to  $t$ , there exists a positive constant  $\sigma$  with  $T + \sigma < 0$  such that  $w_t > 0$  in  $K$ ,  $\forall t \in (T, T + \sigma)$ . Of course, we also have  $w_t \geq 0$  on  $\partial(\Omega_t \setminus K)$  for  $t \in (T, T + \sigma)$  (we only need to check that  $w_t \geq 0$  on  $\partial K$ , which is guaranteed by the fact  $w_t \geq 0$  in  $K$ ). Now if  $\sigma$  is so small that  $\text{meas}(\Omega_t \setminus K)$  is small enough, we can use Lemma 2.1 again to conclude that  $w_t \geq 0$  in  $\Omega_t \setminus K$ . So we get that  $w_t \geq 0$  in  $\Omega_t$ ,  $\forall t \in (T, T + \sigma)$ , which contradicts (10).

Therefore we have  $T \geq 0$ , in particular  $w_0 \geq 0$  in  $\Omega_0$ , which implies as  $x_1 < 0$

$$u(x_1, x_2, \dots, x_n) \geq u(-x_1, x_2, \dots, x_n).$$

Now we can move the plane from the right towards the left and get the reverse inequality. So we have

$$u(x_1, x_2, \dots, x_n) = u(-x_1, x_2, \dots, x_n), \quad (11)$$

which means  $u$  is symmetric with respect to the hyperplane  $x_1 = 0$ .  $\square$

From this theorem we can easily get a corollary:

**Corollary 2.3.** *If  $\Omega$  is a ball, then any solution of (1) is radially symmetric.*

**Remark 2.4.** From the proof we can see the theorem still holds in the case of (3)

$$\det D^2 u = e^{-k(u)}, \quad (12)$$

when  $k(u)$  is a Lipschitz continuous function in its domain  $[\inf u, 0]$ .

**Remark 2.5.** Solutions of the related equation in the entire space  $\mathbb{R}^n$  ( $n \geq 2$ ) may be not radially symmetric. For example, the following problem has a non-radially symmetric solution:

$$\begin{cases} \det D^2 u = e^{-u} & \text{in } \mathbb{R}^n, \\ u \geq 0 & \text{in } \mathbb{R}^n, \\ u(0) = 0. \end{cases} \quad (13)$$

Next we use the solution  $f(t)$  of this equation in one dimension to construct a non-radially symmetric solution of (13) as  $n \geq 2$ . We know

$$\begin{cases} f'' = e^{-f} & \text{in } \mathbb{R}^1, \\ f \geq 0 & \text{in } \mathbb{R}^1, \\ f(0) = 0 \end{cases} \quad (14)$$

has a unique solution  $f(t) = 2 \log(1 + e^{\sqrt{2}t}) - \sqrt{2}t - \log 4$  for  $t > 0$ , and for  $t < 0$  we have  $f(t) = f(-t)$  in  $\mathbb{R}^1$ , which is asymptotically linear. We can define

$$u(x_1, x_2, \dots, x_n) = f(x_1) + f(x_2) + \dots + f(x_n), \quad (15)$$

which is a solution of (13) and not radially symmetric. In fact because  $u$  is convex,  $\frac{u(tx)}{t}$  ( $t > 0$ ) is increasing in  $t$ , and we have  $v(x) = \lim_{t \rightarrow +\infty} \frac{u(tx)}{t} = C(|x_1| + |x_2| + \dots + |x_n|)$  for some positive constant  $C$ .

### 3. Existence and non-existence results

From Section 2 we know that any solution of Eq. (1) in the ball  $B_R(0) \subset \mathbb{R}^n$  is radially symmetric. So we may write  $u(x) = u(r)$ , here  $r = |x|$ . Moreover, 0 is the minimal point of  $u$  and  $u$  is increasing in  $[0, R]$ . Now we have

$$u_i = u'(r) \frac{x_i}{r}, \quad (16)$$

$$u_{ij} = u''(r) \frac{x_i x_j}{r^2} + u'(r) \left( \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right), \quad (17)$$

where  $\delta_{ij}$  is Kronecker  $\delta$ . By an elementary calculation we get

$$\det(u_{ij}) = \left( \frac{u'(r)}{r} \right)^{n-1} u''(r). \quad (18)$$

So Eq. (1) becomes

$$\left( \frac{u'(r)}{r} \right)^{n-1} u''(r) = e^{-u}. \quad (19)$$

Next, we try to find a solution in  $[0, R]$  which is strictly convex and satisfies  $u(R) = 0$ . Assume  $u(0) = -C$  for some positive constant  $C$ . We write the equation as

$$\frac{d}{dr} (u'(r))^n = nr^{n-1} e^{-u}. \quad (20)$$

By integration from 0 to  $r$  ( $r \in [0, R]$ ), we get (noticing  $u'(0) = 0$ )

$$(u'(r))^n = n \int_0^r s^{n-1} e^{-u(s)} ds. \quad (21)$$

Since  $u$  is increasing,  $e^{-u}$  is decreasing. Thus we have

$$(u'(r))^n \geq r^n e^{-u(r)}, \quad (22)$$

that is

$$u'(r) \geq r e^{\frac{-u(r)}{n}}, \quad (23)$$

$$\frac{d}{dr} e^{\frac{u(r)}{n}} \geq \frac{r}{n}. \quad (24)$$

By integration we get

$$e^{\frac{u(r)}{n}} \geq \frac{r^2}{2n} + e^{\frac{-C}{n}}. \quad (25)$$

In particular, since  $u(R) = 0$ , as  $r = R$  we have

$$1 \geq \frac{R^2}{2n} + e^{\frac{-C}{n}}. \quad (26)$$

Thus  $R \leq (2n)^{\frac{1}{2}}$ , which means in particular in balls with radius large enough Eq. (1) has no solution.

On the other hand, if  $R$  is small we can use  $u(x) = \frac{n}{R^2}(|x|^2 - |R|^2)$  as a sub-solution in the ball  $B_R$ , and since 0 is always a sup-solution, we can construct a solution from this sub-solution. In fact, we have  $u_{ij} = \frac{2n}{R^2} \delta_{ij}$ . So if  $R \leq (\frac{2n}{e})^{\frac{1}{2}}$ , then we have

$$\det(u_{ij}) = \left( \frac{2n}{R^2} \right)^n \geq e^n \geq e^{-u}.$$

We can use sup-solution and sub-solution method to show the existence of a solution by iteration (for the proof, see that of Lemma 3.2 below). So in balls with small radius there exists a solution.

In conclusion we have

**Theorem 3.1.** *There is no solution of Eq. (1) for  $\Omega = B_R(0)$  with  $R > 0$  large enough, and for sufficiently small  $R > 0$  there is a solution of (1).*

Now we use sub-solution and sup-solution method to construct a solution by iteration in an arbitrary domain. Notice 0 is always a sup-solution, so we just need the existence of a negative sub-solution. This is standard and well known, we include it here just for completeness.

**Lemma 3.2.** *If we have a strictly convex function  $f \in C^3(\overline{\Omega})$ , such that  $\det(f_{ij}) \geq e^{-f}$  in  $\Omega$  and  $f \leq 0$  on  $\partial\Omega$ , then (1) has a solution  $u$  in  $\Omega$ .*

**Proof.** Set  $u^0 = f$  and define the iteration as

$$\begin{cases} \det(u_{ij}^{k+1}) = e^{-u^k} & \text{in } \Omega, \\ u^{k+1} = 0 & \text{on } \partial\Omega. \end{cases} \quad (27)$$

By Theorem 17.23 of [4], we know that  $u^{k+1}$  exists with the  $C^3(\overline{\Omega})$  norm controlled by the  $C^2(\overline{\Omega})$  norm of  $u^k$ .

Noticing

$$\begin{cases} \det(u_{ij}^1) \leq \det(u_{ij}^0) & \text{in } \Omega, \\ u^1 \geq u^0 & \text{on } \partial\Omega, \end{cases} \quad (28)$$

we have  $u^1 \geq u^0$  in  $\Omega$  by the comparison principle (Theorem 17.1 of [4]). Then  $\det(u_{ij}^1) = e^{-u^0} \geq e^{-u^1}$ . By induction we have  $u^{k+1} \geq u^k$  and  $\det(u_{ij}^{k+1}) \geq e^{-u^{k+1}}$  for any  $k$ .

From the higher order estimate of Monge–Ampère equation (Theorem 17.26 of [4]) we know that the  $C^3(\overline{\Omega})$  norm of  $u^k$  can be controlled by the  $C^3(\overline{\Omega})$  norm of  $f$ , so the sequence  $u^k$  is compact in  $C^2(\overline{\Omega})$ , and  $u^k(x)$  also converge increasingly to some  $u(x)$ ,  $\forall x \in \Omega$ , which is a convex function with vanishing boundary value. Combining these facts we know  $u^k$  converge to  $u$  in  $C^2(\overline{\Omega})$ . By taking the limit in (27), we know that  $u$  is a solution of (1).  $\square$

Next we prove a lemma concerning the comparison between domains:

**Lemma 3.3.** *Given two bounded convex domains  $\Omega_1$  and  $\Omega_2$  such that  $\Omega_1 \subset \Omega_2$ . If we have a solution  $u$  of (1) in  $\Omega_2$ , then there exists a solution  $v$  of (1) in  $\Omega_1$ , or equivalently if there is no solution of (1) in  $\Omega_1$ , then there is no solution of (1) in  $\Omega_2$ .*

**Proof.** Just take the restriction of  $u$  in  $\Omega_1$  as a sub-solution, then we can use Lemma 3.2.  $\square$

Given a bounded convex domain  $\Omega$ , a result of F. John (see [1] or [12]) says that there exists an ellipsoid  $P$  such that  $P \subset \Omega \subset nP$ .  $P$  can be transformed into a ball by a matrix  $A$  with  $\det A = 1$ , which leaves the equation invariant. Now our first main result is clear:

**Theorem 3.4.** *Given a bounded convex domain  $\Omega$ , there exists a positive constant  $\lambda^*$  such that if  $\lambda < \lambda^*$  there exists a solution of (1) in  $\lambda\Omega$ ; and if  $\lambda > \lambda^*$  there exists no solution of (1) in  $\lambda\Omega$ . Moreover, we have the estimation  $c(n)(\text{meas}(\Omega))^{-\frac{1}{n}} \leq \lambda^* \leq C(n)(\text{meas}(\Omega))^{-\frac{1}{n}}$  for some universal constants  $c(n)$  and  $C(n)$ .*



**Proof.** First from Lemma 3.3 we have if in  $\lambda\Omega$  there exists a solution then for any  $\lambda' \in (0, \lambda)$  there exists a solution in  $\lambda'\Omega$ ; and if in  $\lambda\Omega$  there exists no solution then for any  $\lambda' > \lambda$  there exists no solution in  $\lambda'\Omega$ . So we can define

$$\lambda^* = \sup\{\lambda > 0 \mid (1) \text{ has a solution in } \lambda\Omega\}. \quad (29)$$

In fact, from our previous discussion in this section we know that in small balls the equation has a solution and in large balls the equation has no solution, that is our claim is true for unit ball with the critical radius  $\lambda^*(B_1) \in [(\frac{2n}{e})^{\frac{1}{2}}, (2n)^{\frac{1}{2}}]$ . From the result of F. John (see [1] or [12]), without loss of generality we can assume that  $B_R(0) \subset \Omega \subset nB_R(0)$ , where  $B_R(0)$  is the ball with radius  $R$ . From the comparison of volume we have  $C_1(n)(\text{meas}(\Omega))^{\frac{1}{n}} \leq R \leq C_2(n)(\text{meas}(\Omega))^{\frac{1}{n}}$  for some universal constants  $C_1(n)$  and  $C_2(n)$ . Now using Lemma 3.3 again, if  $\lambda$  is so large that  $\lambda R$  is greater than  $\lambda^*(B_1)$ , then there is no solution in  $\lambda\Omega$ ; and if  $\lambda$  is so small that  $n\lambda R$  is less than  $\lambda^*(B_1)$ , then there is a solution in  $\lambda\Omega$ . Hence  $\lambda^*$  for  $\Omega$  is positive and finite. Our estimation of  $\lambda^*(\Omega)$  can be easily checked by

$$\begin{cases} \lambda^*(\Omega)R \leq \lambda^*(B_1), \\ n\lambda^*(\Omega)R \geq \lambda^*(B_1). \end{cases} \quad \square \quad (30)$$

We include here one example in  $\mathbb{R}^1$  to indicate how the solution varies with respect to the size of the domain. Note that any solution  $u$  of (1) in  $\lambda\Omega$  can be scaled to  $u^\lambda(x) = \lambda^{-2}u(\lambda x)$  defined in  $\Omega$ , which satisfies

$$\begin{aligned} \det(u_{ij}^\lambda)(x) &= \det(u_{ij})(\lambda x) \\ &= e^{-u(\lambda x)} \\ &= e^{-\lambda^2 u^\lambda(x)}. \end{aligned}$$

So we can consider an equivalent problem in a fixed domain  $\Omega$  with a parameter in the equation.

**Example 3.5.**  $u'' = e^{-tu}$  in the interval  $[-1, 1]$  with vanishing boundary value.

First we have a constant  $C \geq 2$  such that

$$t(u')^2 + 2e^{-tu} = C. \quad (31)$$

So if  $x > 0$

$$t^{\frac{1}{2}}u' = (C - 2e^{-tu})^{\frac{1}{2}}, \quad (32)$$

if  $x < 0$  we have a negative sign before the right-hand side.

Take  $f = (C - 2e^{-tu})^{\frac{1}{2}}$ , then

$$f' = \frac{1}{2}(C - 2e^{-tu})^{-\frac{1}{2}} 2te^{-tu}u' = \pm t^{\frac{1}{2}}e^{-tu} = \pm \frac{t^{\frac{1}{2}}}{2}(C - f^2).$$

So for  $x < 0$  we have

$$\frac{f'}{C - f^2} = \frac{-t^{\frac{1}{2}}}{2}. \quad (33)$$

We can integrate to obtain that for  $x < 0$

$$f(x) = C^{\frac{1}{2}} \frac{e^{-(Ct)^{\frac{1}{2}}x} - 1}{e^{-(Ct)^{\frac{1}{2}}x} + 1}. \quad (34)$$

For  $x > 0$  we have  $f(x) = f(-x)$ . From the boundary value  $f(0) = 0$  and  $f(-1) = (C - 2)^{\frac{1}{2}}$ , we have a constraint on  $C$  and  $t$ :

$$t^{\frac{1}{2}} = \frac{\log(C - 1 + (C^2 - 2C)^{\frac{1}{2}})}{C^{\frac{1}{2}}}. \quad (35)$$

If we use  $C$  as a parameter ( $C \geq 2$ ), the right-hand side has an upper bound (in fact it is increasing if  $C$  is less than some value and decreasing to 0 if  $C$  is greater than the value). So we have this result: there exists a  $t_0$  such that for  $t > t_0$  there is no  $C$  satisfying (35), therefore no solution to the original equation, and for  $t < t_0$  there are exactly two  $C$  satisfying (35) and therefore there exist two solutions to the original equation.

#### 4. The equation with a parameter

In this section we study the equation with a parameter in a fixed domain:

$$\begin{cases} \det u_{ij} = e^{-tu} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (36)$$

We know from the previous section for  $t > 0$ , (36) is equivalent to Eq. (1) in the domain  $t^{\frac{1}{2}}\Omega$ .

In this section we first extend a branch of solutions emanating from  $t = 0$ , and prove some properties of this branch, then we find that this branch degenerates at a point  $t = T$ . Moreover, we study the local structure of the branch near the degenerate point (Theorem 4.9). At last, we study the global structure of this connected component emanating from  $t = 0$ ; using a priori estimate and Leray–Schauder degree theory we can prove the main Theorem 1.1.

We first use the Implicit Function Theorem to find a branch (emanating from  $t = 0$ ) of solutions of (36). We need two function spaces:  $X$  is the space of functions in  $C^{k,\alpha}(\overline{\Omega})$  with vanishing boundary value and  $Y$  is the space of functions in  $C^{k-2,\alpha}(\overline{\Omega})$  with vanishing boundary value, here  $k$  is an integer greater than 2 and  $\alpha \in (0, 1)$ . For a function  $u \in U := \{u \in X, \text{ there exists a positive constant } \epsilon(u) \text{ such that } D^2u - \epsilon \text{Id is positive definite in } \Omega, \text{ where Id is the identity matrix}\}$  (note that  $U$  is an open set of  $X$ , in fact, the elements of  $U$  are strictly convex functions), we define a map  $F: \mathbb{R}^1 \times U \rightarrow Y$ :

$$F(t, u) = \log \det(u_{ij}) + tu. \quad (37)$$

We have the formula for the derivative:

$$D_u F(t, u)v = g^{ij}v_{ij} + tv, \quad (38)$$

here  $(g^{ij})$  is the inverse matrix of  $(u_{ij})$ . In order to check if  $D_u F(t, u)$  is surjective we need to estimate the first eigenvalue of the elliptic operator  $Lf := -g^{ij}f_{ij}$ , denoted by  $\lambda_1$ . Note that by a result of Bakelman [7] we can get

$$\int_{\Omega} g^{ij} f_{ij} f \det(u_{ij}) dx = - \int_{\Omega} g^{ij} f_i f_j \det(u_{ij}) dx, \quad (39)$$

for  $f|_{\partial\Omega} = 0$ . The calculation is simple if  $u$  is strictly convex, we present it here for readers' convenience.

**Lemma 4.1.** For a strictly convex function  $u$  and two  $C^1(\overline{\Omega})$  functions  $f$  and  $h$  with vanishing boundary values, we have

$$\int_{\Omega} g^{ij} f h_{ij} \det(u_{ij}) dx = - \int_{\Omega} g^{ij} f_i h_j \det(u_{ij}) dx, \quad (40)$$

here  $(g^{ij})$  is the inverse matrix of  $(u_{ij})$ .

**Proof.** It is just an integration by parts, but some terms can be canceled:

$$\begin{aligned} \int_{\Omega} g^{ij} f_i h_j \det(u_{ij}) dx &= \int_{\Omega} \frac{\partial}{\partial x_i} (g^{ij} f h_j \det(u_{ij})) - \int_{\Omega} g^{ij} f h_{ij} \det(u_{ij}) \\ &\quad - \int_{\Omega} \frac{\partial}{\partial x_i} (g^{ij}) f h_j \det(u_{ij}) - \int_{\Omega} g^{ij} f h_j \frac{\partial}{\partial x_i} \det(u_{ij}). \end{aligned}$$

On the right-hand side, the first term is the divergence of a vector field which vanishes on the boundary. For the last two terms we have

$$\frac{\partial}{\partial x_i} g^{ij} = -g^{iq} g^{pj} u_{pqi}, \quad (41)$$

and

$$\frac{\partial}{\partial x_i} \det(u_{ij}) = g^{kl} u_{kli} \det(u_{ij}). \quad (42)$$

Now we have

$$g^{iq} g^{pj} u_{pqi} = g^{ij} g^{kl} u_{kli}, \quad (43)$$

which can be seen by changing  $i, p, q$  into  $k, i, l$  respectively in the left-hand side of (43), because these are used to be summed. So the last two terms cancel each other and only the second term which we want is left.  $\square$

Concerning the spectrum of the elliptic operator  $Lf := -g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$  with Dirichlet boundary condition (here  $(g^{ij})$  is the inverse matrix of  $(u_{ij})$  and  $u \in U$ ), we have a result analogues to the Laplace operator:

**Lemma 4.2.** The spectrum of the elliptic operator  $L$  is real, and the first eigenvalue  $\lambda_1 > 0$  has a positive eigenfunction. Moreover,  $\lambda_1$  is simple.

**Proof.** We need two spaces:  $X$  is the completion of the  $C_0^\infty(\Omega)$  function with the norm  $\|f\|^2 = \int_{\Omega} g^{ij} f_i f_j \det(u_{ij}) dx$  and  $Y$  is the completion of the  $C_0^\infty(\Omega)$  function with the norm  $\|f\|^2 = \int_{\Omega} f^2 \det(u_{ij}) dx$ . (In fact because  $u$  is in  $U$  and strictly convex these two space are  $H_0^1(\Omega)$  and  $L^2(\Omega)$  respectively with an equivalent norm.)  $X$  can be embedded into  $Y$  compactly. So the inverse of  $L$  is a self-adjoint, compact, positive definite operator on  $Y$ .

The first eigenvalue can be characterized by

$$\lambda_1 = \inf_{f \in X, f \neq 0} \frac{\int_{\Omega} g^{ij} f_i f_j \det(u_{ij}) dx}{\int_{\Omega} f^2 \det(u_{ij}) dx}. \quad (44)$$

There exists a non-negative minimizer  $f$ , which satisfies the equation  $Lf = \lambda_1 f$ . Then from the strong maximum principle (see Theorem 3.5 of [4]) we know  $f$  is positive in the interior of  $\Omega$ . The simplicity of the first eigenvalue is the same as the Laplace case.  $\square$

It is well known that for  $t = 0$  there exists a unique smooth convex solution  $u_0$  of (36). Of course, the first eigenvalue  $\lambda_{1,0}$  is positive. Then we know from the Implicit Function Theorem that there exist a constant  $T_0 > 0$  and a  $C^1$  map  $u : [0, T_0) \rightarrow U$  such that  $F(t, u_t) = 0$ . In the sequel we denote the first eigenvalue associated with  $u_t$  by  $\lambda_{1,t}$ .

Here we need a lemma:

**Lemma 4.3.** *The first eigenvalue  $\lambda_1$  of  $L$  is continuous in  $u$  with respect to the  $C^2(\overline{\Omega})$  norm.*

**Proof.** We need to prove that if  $u_k$  converges to  $u$  in  $C^2(\overline{\Omega})$  norm, then the first eigenvalue  $\lambda_{1,k}$  of the elliptic operator  $L_k f := -g_k^{ij} f_{ij}$  converges to the first eigenvalue  $\lambda_1$  of the elliptic operator  $L f := -g^{ij} f_{ij}$ , where  $g_k^{ij}$  is the inverse matrix of  $D^2 u_k$  and  $g^{ij}$  is the inverse matrix of  $D^2 u$ . Denote the first (positive) eigenfunction of  $L$  by  $f$  which satisfies  $\int_{\Omega} f^2 \det(u_{ij}) dx = 1$ , and the first (positive) eigenfunction of  $L_k$  by  $f_k$  which satisfies  $\int_{\Omega} f_k^2 \det(u_k)_{ij} dx = 1$ .

First we have

$$\begin{aligned} \lambda_1 &= \frac{\int_{\Omega} g^{ij} f_i f_j \det(u_{ij}) dx}{\int_{\Omega} f^2 \det(u_{ij}) dx} \\ &= \lim_{k \rightarrow +\infty} \frac{\int_{\Omega} g_k^{ij} f_i f_j \det((u_k)_{ij}) dx}{\int_{\Omega} f^2 \det((u_k)_{ij}) dx} \\ &\geq \overline{\lim}_{k \rightarrow +\infty} \lambda_{1,k}. \end{aligned}$$

If there exists a subsequence of  $\lambda_{1,k}$ , still denoted by  $\lambda_{1,k}$ , which converges to  $\lambda_1 - \delta$  for some positive constant  $\delta$ , then there exists a subsequence of  $f_k$ , still denoted by  $f_k$ , converging weakly in the Sobolev space  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$ . Denote the limit function as  $h$ , then by taking the limit we have

$$\int_{\Omega} h^2 \det(u_{ij}) dx = 1, \quad (45)$$

and

$$\int_{\Omega} g^{ij} h_i h_j \det(u_{ij}) dx \leq \lambda_1 - \delta, \quad (46)$$

which contradicts the definition of  $\lambda_1$ .  $\square$

Next we estimate the first eigenvalue  $\lambda_{1,t}$  associated with  $u_t$  above, to get the maximum interval for  $t$  such that the branch exists. We show that the branch extends until a point at which the curve is not differentiable in  $t$ .

**Proposition 4.4.** *Given the  $C^1$  map  $u : [0, T_0) \rightarrow U$  starting from  $u_0$  such that  $F(t, u_t) = 0$  above, we have  $\lambda_{1,t} > t$ .*

**Proof.** First from the discussion above we have  $\lambda_{1,0} > 0$ . Assume there exists  $t_0 \in (0, T_0)$  such that  $\lambda_{1,t_0} = t_0$  with a first positive eigenfunction as  $f$ , that is

$$-g_{t_0}^{ij} f_{ij} = t_0 f. \quad (47)$$

Without loss of generality we can assume for all  $t < t_0$  we have  $\lambda_{1,t} > t$ . Because  $u_t$  is differentiable in  $t$ , we differentiate the equation  $F(t, u_t) = 0$  in  $t \in (0, T_0)$ :

$$-g_t^{ij} \left( \frac{\partial u_t}{\partial t} \right)_{ij} = t \frac{\partial u_t}{\partial t} + u_t, \quad (48)$$

where  $g_t^{ij}$  is the inverse matrix of  $D^2 u_t$ .

Now we prove that the non-negative part  $(\frac{\partial u_t}{\partial t})^+ \equiv 0$ , where  $(\frac{\partial u_t}{\partial t})^+ := \max\{0, \frac{\partial u_t}{\partial t}\}$ .

In fact, if  $(\frac{\partial u_t}{\partial t})^+ \not\equiv 0$ , using  $(\frac{\partial u_t}{\partial t})^+$  as a test function, then we obtain by integration by parts (note that  $u_t$  is convex and equal to 0 on the boundary, so  $u_t < 0$  in  $\Omega$  and  $\frac{\partial u_t}{\partial t} \equiv 0$  on  $\partial\Omega$ , and here  $t < t_0$ , so by our assumption  $t < \lambda_{1,t}$ )

$$\begin{aligned} & \int_{\Omega} g_t^{ij} \left( \frac{\partial u_t}{\partial t} \right)_i \left( \frac{\partial u_t}{\partial t} \right)_j^+ \det((u_t)_{ij}) dx \\ &= - \int_{\Omega} g_t^{ij} \left( \frac{\partial u_t}{\partial t} \right)_{ij} \left( \frac{\partial u_t}{\partial t} \right)^+ \det((u_t)_{ij}) dx \\ &= t \int_{\Omega} \left[ \left( \frac{\partial u_t}{\partial t} \right)^+ \right]^2 \det((u_t)_{ij}) dx + \int_{\Omega} \left( \frac{\partial u_t}{\partial t} \right)^+ u_t \det((u_t)_{ij}) dx \\ &\leq t \int_{\Omega} \left[ \left( \frac{\partial u_t}{\partial t} \right)^+ \right]^2 \det((u_t)_{ij}) dx \\ &< \lambda_{1,t} \int_{\Omega} \left[ \left( \frac{\partial u_t}{\partial t} \right)^+ \right]^2 \det((u_t)_{ij}) dx, \end{aligned}$$

but by the definition of  $\lambda_{1,t}$ , it is impossible. So  $(\frac{\partial u_t}{\partial t})^+ \equiv 0$ . Thus we have  $\frac{\partial u_t}{\partial t} \leq 0$ . Then from (48) we have

$$-g_t^{ij} \left( \frac{\partial u_t}{\partial t} \right)_{ij} < 0 \quad \text{in } \Omega, \quad (49)$$

so we have  $\frac{\partial u_t}{\partial t} < 0$  in  $\Omega$  by the strong maximum principle (see Theorem 3.5 of [4]).

Now because  $\frac{\partial u_t}{\partial t}$  is continuous in  $t$ , we have  $\frac{\partial u_t}{\partial t}(t_0) \leq 0$ . Then by the Hopf Lemma we have  $\frac{\partial u_t}{\partial t}(t_0) < 0$  in  $\Omega$  and  $\frac{\partial u_t}{\partial t}(t_0)$  has no vanishing gradient on the boundary. Denote  $v = -\frac{\partial u_t}{\partial t}(t_0)$ , then by (48) we have

$$-g_{t_0}^{ij} v_{ij} > t_0 v. \quad (50)$$

Because  $f = 0$  on  $\partial\Omega$  and  $v$  has no vanishing gradient on the boundary, for any point  $x \in \partial\Omega$ , we can define  $\frac{f(x)}{v(x)} := \frac{\partial f}{\partial v} / \frac{\partial v}{\partial v}$ , where  $v$  denotes the exterior unit normal of  $\partial\Omega$  at  $x$ . We get that

$C = \sup_{x \in \Omega} \frac{f(x)}{v(x)}$  is a positive finite number, which is attained in  $\overline{\Omega}$ . Now combining (47) and (50) we get

$$-g_{t_0}^{ij}(f - Cv)_{ij} < t_0(f - Cv) \leq 0.$$

By the definition of  $C$ ,  $f - Cv$  either has 0 as a maximum which is attained in the interior of  $\Omega$  or has vanishing gradient at some point of  $\partial\Omega$ . So by Hopf Lemma (see Lemma 3.4 of [4]) we have  $f - Cv$  is a constant, that is 0. Now substitute  $\frac{\partial u_t}{\partial t}(t_0) = -C^{-1}f$  into (48), we get  $u_{t_0} \equiv 0$ , which contradicts  $u_t < 0$  for  $t \in [0, T_0)$  in  $\Omega$ . So our assertion is proved.  $\square$

**Remark 4.5.** We have another simple proof of this proposition, but the method in the proof above is more valuable because it can give us more information. In fact, first multiplying (48) by the positive first eigenfunction  $f$  and by integration by parts we get

$$\begin{aligned} \int_{\Omega} \left( t \frac{\partial u_t}{\partial t} + u_t \right) f \det((u_t)_{ij}) dx &= - \int_{\Omega} g_t^{ij} \left( \frac{\partial u_t}{\partial t} \right)_{ij} f \det((u_t)_{ij}) dx \\ &= - \int_{\Omega} g_t^{ij} f_{ij} \frac{\partial u_t}{\partial t} \det((u_t)_{ij}) dx \\ &= \lambda_{1,t} \int_{\Omega} f \frac{\partial u_t}{\partial t} \det((u_t)_{ij}) dx. \end{aligned}$$

So if  $\lambda_{1,t} = t$ , we must have  $\int_{\Omega} u_t f \det((u_t)_{ij}) dx = 0$ , which is impossible because in  $\Omega$  we have  $u_t < 0$  and  $f > 0$ .

From the proof (see the statement below (49)) we also know  $u_t$  is decreasing in  $t$ . Now with this estimate we can extend the  $C^1$  map  $u_t$  to be defined on a maximal interval  $[0, T)$ ,  $T > 0$ . By Theorem 3.4 and rescaling we know that (36) has no solution for  $t > (\lambda^*)^2$  where  $\lambda^*$  is the critical value as in Theorem 3.4, so  $0 < T < +\infty$ . We conclude that either

- (i)  $u_t$  converges to  $-\infty$  as  $t$  approaches  $T$ , or
- (ii)  $u_t$  converges decreasingly to some convex function  $u_T$  as  $t$  approaches  $T$  (we can prove  $u_t$  converges to  $u_T$  in  $C^2(\overline{\Omega})$ , using the higher order estimate, see the proof of Lemma 3.2), which is a solution of (36) at  $t = T$ , but  $u_t$  is not left differentiable in  $t$  at  $T$ , that is the solution  $u_T$  is degenerated.

In fact the case (i) cannot happen on any smooth convex domain  $\Omega$ , because we have a priori estimate:

**Lemma 4.6.** *Given a positive constant  $t_0 > 0$ , any solution of (36) with  $t > t_0$ , must satisfy  $\sup_{\Omega} |u| \leq C(t_0)$  for some constant  $C(t_0)$  which depends on  $t_0$  and  $\Omega$  only.*

**Proof.** Denote  $M := \sup_{\Omega} |u|$ . From the arithmetic–geometric mean value inequality (see [15, Theorem 6.6.9, p. 154]) we have

$$\frac{1}{n} \Delta u \geq (\det(u_{ij}))^{\frac{1}{n}} = e^{-\frac{tu}{n}}.$$

Moreover, there exists a Dirichlet Green function  $G(x, y) > 0$  in  $\Omega$  such that by the Green formula we have

$$\begin{aligned}
 u(x) &= - \int_{\Omega} G(x, y) \Delta u(y) dy \\
 &\leq -n \int_{\Omega} G(x, y) e^{-\frac{tu(y)}{n}} dy.
 \end{aligned}$$

Noticing that there exists a unique positive function  $\phi(x)$  satisfying  $-\Delta\phi = 1$  in  $\Omega$  with vanishing boundary value, we get (integrating the inequality above in  $\Omega$ )

$$\begin{aligned}
 \text{meas}(\Omega)M &\geq - \int_{\Omega} u(x) dx \\
 &\geq n \int_{\Omega} \int_{\Omega} G(x, y) e^{-\frac{tu(y)}{n}} dy dx \\
 &= n \int_{\Omega} e^{-\frac{tu(y)}{n}} \int_{\Omega} G(x, y) dx dy \\
 &= n \int_{\Omega} e^{-\frac{tu(y)}{n}} \phi(y) dy.
 \end{aligned}$$

Now assume  $x_0$  is the minimal point of  $u$ , that is,  $u(x_0) = -M$ . Without loss of generality, we can assume  $x_0 = 0$  by translation. We define a function  $\psi$  to be a cone over  $\Omega$ , that is,  $\psi(0) = -M$ ,  $\psi = 0$  on  $\partial\Omega$  and  $\psi(x) = -(1-t)M$  where  $t$  is characterized uniquely by  $\frac{x}{t} \in \partial\Omega$ . Because  $u$  is convex with vanishing boundary value, we have  $\psi(x) \geq u(x)$ .

Now  $A := \{x: \psi(x) \leq -\frac{M}{2}\} = \frac{1}{2}\Omega$ . Take a small positive constant  $\epsilon$  such that with  $\Omega_{\epsilon} := \{x: \phi(x) \leq \epsilon\}$ , we have  $\text{meas}(\Omega_{\epsilon}) \leq 4^{-n} \text{meas}(\Omega)$  and

$$\begin{aligned}
 \int_{\Omega} e^{-\frac{tu(y)}{n}} \phi(y) dy &\geq \int_{(\Omega \cap A) \setminus \Omega_{\epsilon}} e^{-\frac{tu(y)}{n}} \phi(y) dy \\
 &\geq \epsilon e^{\frac{tM}{2n}} \text{meas}((\Omega \cap A) \setminus \Omega_{\epsilon}) \\
 &\geq \epsilon e^{\frac{tM}{2n}} [\text{meas}(\Omega) - (1 - 2^{-n}) \text{meas}(\Omega) - \text{meas}(\Omega_{\epsilon})] \\
 &\geq C e^{\frac{tM}{2n}},
 \end{aligned}$$

where  $C$  is a constant only depending on  $\Omega$ . Now combining the two inequalities above we get

$$M \geq C e^{\frac{tM}{2n}} \geq C e^{\frac{t_0 M}{2n}}.$$

This implies a uniform bound  $C(t_0)$  such that  $M \leq C(t_0)$ , here  $C(t_0)$  depends on  $t_0$  and  $\Omega$  only.  $\square$

This lemma also guarantees that the branch extending from  $t = 0$  always stays in  $U$  and the branch has some compactness property.

We can use the estimation of the first eigenvalue  $\lambda_{1,t} > t$  to get some more results:

**Proposition 4.7.** *If there exists another solution  $v$  of (36) for  $t \in [0, T)$ , then  $v < u_t$  (here  $u_t$  is the solution in the branch extended from  $t = 0$ ). Moreover, the first eigenvalue of the operator  $Lf := -g^{ij} f_{ij}$  (here  $g^{ij}$  is the inverse matrix of  $D^2 v$ ) is smaller than  $t$ .*

**Proof.** From the concavity of the log det function (see Proposition A.4 of Appendix A), we have

$$\begin{aligned}\log \det([\tau v + (1 - \tau)u_t]_{ij}) &\geq \tau \log \det(v_{ij}) + (1 - \tau) \log \det((u_t)_{ij}) \\ &= -\tau t v - (1 - \tau) t u_t,\end{aligned}$$

for any  $\tau \in [0, 1]$  at any fixed point  $x \in \Omega$ , which becomes an equality as  $\tau = 0$  or  $\tau = 1$ . The left-hand side is a concave function  $h(\tau)$ ,  $\tau \in [0, 1]$ . So we can compare the derivative in  $\tau$  of both sides at  $\tau = 0$  and obtain

$$-g_t^{ij}(v - u_t)_{ij} \leq t v - t u_t, \quad (51)$$

here  $(g_t^{ij})$  is the inverse matrix of  $((u_t)_{ij})$ . Comparing the derivative in  $\tau$  of both sides at  $\tau = 1$  we obtain

$$-g_t^{ij}(v - u_t)_{ij} \geq t v - t u_t, \quad (52)$$

here  $(g_t^{ij})$  is the inverse matrix of  $(v_{ij})$ . Here inequalities (51) or (52) become equalities if and only if the concave function  $h''(\tau) \equiv 0$ , that is, the matrix  $((v - u_t)_{ij})(x) = 0$  (see Proposition A.4 of Appendix A).

Noticing  $\lambda_{1,t} > t$ , we can proceed as in the proof of Proposition 4.4 (using an integration by parts) to prove that  $v \leq u_t$ .

We assume by contradiction that  $(v - u_t)^+ \not\equiv 0$ , then we have

$$\begin{aligned}\int_{\Omega} g_t^{ij}(v - u_t)_i(v - u_t)_j^+ \det((u_t)_{ij}) dx &= - \int_{\Omega} g_t^{ij}(v - u_t)_{ij}(v - u_t)^+ \det((u_t)_{ij}) dx \\ &\leq t \int_{\Omega} [(v - u_t)^+]^2 \det((u_t)_{ij}) dx \\ &< \lambda_{1,t} \int_{\Omega} [(v - u_t)^+]^2 \det((u_t)_{ij}) dx,\end{aligned}$$

which contradicts the definition of  $\lambda_{1,t}$ , so  $(v - u_t)^+ \equiv 0$ , that is  $v \leq u_t$ . Now we have

$$\det((u_t)_{ij}) = e^{-t u_t} \leq e^{-t v} = \det(v_{ij}).$$

Write this in another form so that we can use the strong maximum principle (see Theorem 3.5 of [4]):

$$\begin{aligned}0 &\leq \log \det(v_{ij}) - \log \det((u_t)_{ij}) \\ &= \int_0^1 \frac{d}{d\tau} \log \det(\tau v_{ij} + (1 - \tau)(u_t)_{ij}) d\tau \\ &= \left[ \int_0^1 g_{\tau}^{ij} d\tau \right] (v - u_t)_{ij},\end{aligned}$$

here  $(g_{\tau}^{ij})$  is the inverse matrix of  $(\tau v_{ij} + (1 - \tau)(u_t)_{ij})$ . Now we can conclude that either  $v \equiv u_t$  or  $v < u_t$  in  $\Omega$  by the strong maximum principle (see Theorem 3.5 of [4]).



In fact by the Hopf Lemma (see Lemma 3.4 of [4]) we have  $v - u_t < 0$  in  $\Omega$  and  $v - u_t$  has no vanishing gradient on the boundary. Denote  $\varphi := -v + u_t$ , then by (52) we have

$$-g^{ij}\varphi_{ij} \leq t\varphi. \quad (53)$$

Now assume the first eigenvalue of the operator  $L$ :  $\lambda_1 \geq t$  with a positive eigenfunction  $f$ . In fact we cannot have  $\lambda_1 > t$ , because otherwise we can proceed as the proof above to get that  $v \geq u_t$ . (Note above we just use the first eigenvalue to prove that  $u_t \geq v$ .) So with our assumption we must have  $\lambda_1 = t$ . Because  $\varphi = 0$  on  $\partial\Omega$  and  $f$  has no vanishing gradient on the boundary, for any point  $x \in \partial\Omega$ , we can define  $\frac{\varphi(x)}{f(x)} := \frac{\partial\varphi}{\partial\nu} / \frac{\partial f}{\partial\nu}$ , where  $\nu$  denotes the exterior unit normal of  $\partial\Omega$  at  $x$ . We get that  $C = \sup_{x \in \Omega} \frac{\varphi(x)}{f(x)}$  is a positive finite number, which is attained in  $\overline{\Omega}$ . Now from (53) and  $-g^{ij}f_{ij} = tf$  we have

$$-g^{ij}(\varphi - Cf)_{ij} \leq t\varphi - tCf \leq 0.$$

By the definition of  $C$ ,  $\varphi - Cf$  either has 0 as a maximum which is attained in the interior of  $\Omega$  or has vanishing gradient at some point of  $\partial\Omega$ . So by Hopf Lemma (see Lemma 3.4 of [4]) we have  $\varphi - Cf$  is a constant, that is 0. Now (53), hence (52), becomes an equality for any  $x \in \Omega$ . So we must have  $D^2v \equiv D^2u_t$ , which implies  $v \equiv u_t$  by the boundary condition. This is a contradiction and our assertion is proved.  $\square$

The method in the proof of the first part of Proposition 4.7 can also be used to prove the following result:

**Proposition 4.8.** *If there is a solution  $v$  of (36) for  $t > T$ , then we have  $v < u_s$  for any  $s \in [0, T)$ .*

**Proof.** For any  $s < T$ , from the concavity of the log det function (see Appendix A), we have

$$\begin{aligned} \log \det([\tau v + (1 - \tau)u_s]_{ij}) &\geq \tau \log \det(v_{ij}) + (1 - \tau) \log \det((u_s)_{ij}) \\ &= -\tau tv - (1 - \tau)su_s, \end{aligned}$$

for any  $\tau \in [0, 1]$ , which becomes an equality as  $\tau = 0$  or  $\tau = 1$ . So we can compare the derivative in  $\tau$  of both sides at  $\tau = 0$  and obtain

$$-g_s^{ij}(v - u_s)_{ij} \leq tv - su_s \leq s(v - u_s).$$

The last inequality is because  $t > s$  and  $v \leq 0$ . Here  $g_s^{ij}$  is the inverse matrix of  $((u_s)_{ij})$ .

Because  $\lambda_{1,s} > s$ , we can proceed as in the proof of Proposition 4.7 (using an integration by parts) to prove that  $v \leq u_s$ .

We assume by contradiction that  $(v - u_s)^+ \not\equiv 0$ , then we have

$$\begin{aligned} &\int_{\Omega} g_s^{ij}(v - u_s)_i(v - u_s)_j^+ \det((u_s)_{ij}) dx \\ &= - \int_{\Omega} g_s^{ij}(v - u_s)_{ij}(v - u_s)^+ \det((u_s)_{ij}) dx \\ &\leq s \int_{\Omega} [(v - u_s)^+]^2 \det((u_s)_{ij}) dx \end{aligned}$$

$$< \lambda_{1,s} \int_{\Omega} [(v - u_s)^+]^2 \det((u_s)_{ij}) dx,$$

which contradicts the definition of  $\lambda_{1,s}$ , so  $(v - u_s)^+ \equiv 0$ , that is  $v \leq u_s$ . Thus

$$\begin{aligned} \det((u_s)_{ij}) &= e^{-su_s} \\ &\leq e^{-tv} \\ &= \det(v_{ij}). \end{aligned}$$

We can proceed as the proof of Proposition 4.7 to conclude that  $v < u_s$  in  $\Omega$ .  $\square$

Next we study the structure of the branch near the degenerate point. We have a smooth and strictly convex function  $u_T$  satisfying the following equation (for convenience, in the following we use  $u$  instead of  $u_T$ ):

$$\begin{cases} \det(u_{ij}) = e^{-Tu} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lambda_1 = T, \end{cases} \quad (54)$$

where  $\lambda_1$  is the first eigenvalue of the operator  $Lf := -g^{ij}f_{ij}$ ,  $g^{ij}$  is the inverse matrix of  $D^2u$ . We also have a positive eigenfunction  $f$  corresponding to  $\lambda_1$ .

We obtain the following result about the local structure of the degenerate point of (36).

**Theorem 4.9.** *For  $u$  satisfying (54), we have a unique family  $u_s = u + sf + o(s)$  near  $u$ , satisfying*

$$\det((u_s)_{ij}) = e^{-(T+r(s))u(s)}, \quad (55)$$

where  $r(s)$  is a continuously differentiable function defined in a small open neighborhood of 0 in  $\mathbb{R}^1$ . Moreover,  $r(0) = 0$ ,  $r(s) \leq 0$ ,  $u_s$  is increasing in  $s$ . That implies near  $T$  and  $u$  for  $t < T$  there are two solutions of (36) and no solutions for  $t > T$ .

**Proof.** In this case the Implicit Function Theorem is invalid. We need to use the Lyapunov–Schmidt reduction method (for the general introduction please see [7]).  $X$  can be split as a direct sum  $\text{span}\{f\} \oplus W$ , where  $W = \{v: v \in X, \int_{\Omega} f v \det(u_{ij}) dx = 0\}$ .  $Y$  has a similar decomposition  $Y = \text{span}\{f\} \oplus Z$ , where  $Z = \{v: v \in Y, \int_{\Omega} f v \det(u_{ij}) dx = 0\}$ .

We write the map  $F$  of (37) near  $u$  for  $(r, s, w) \in \mathbb{R}^1 \times \mathbb{R}^1 \times W$  as

$$F(r, s, w) = \log \det(u + sf + w)_{ij} + (T + r)(u + sf + w), \quad (56)$$

this map is well defined in a small neighborhood of  $(0, 0, 0)$ . Now the equation  $F(r, s, w) = 0$  can be written as

$$\begin{cases} P_1[\log \det(u + sf + w)_{ij} + (T + r)(u + sf + w)] = 0, \\ P_2[\log \det(u + sf + w)_{ij} + (T + r)(u + sf + w)] = 0, \end{cases} \quad (57)$$

here  $P_1$  is the projection from  $Y$  to  $\text{span}\{f\}$  and  $P_2$  is the projection from  $Y$  to  $Z$ . The derivative

$$D_w(P_2 F)(0, 0, 0)v = g^{ij}v_{ij} + Tv, \quad \forall v \in W, \quad (58)$$

is a linear operator from  $W$  to  $Z$  with a bounded inverse operator.

Then by the Implicit Function Theorem the second equation of (57) can be solved, that is, there exists a continuously differentiable map  $w$  from a neighborhood of  $(0, 0)$  in  $\mathbb{R}^2$  to a neighborhood of  $0$  in  $W$  such that  $w(0, 0) = 0$  and

$$F(r, s, w(r, s)) = \lambda(r, s)f, \quad (59)$$

for some continuously differentiable function  $\lambda(r, s)$  defined in a neighborhood of  $(0, 0)$  in  $\mathbb{R}^2$ . For  $w$  in this neighborhood of  $0$  in  $W$ ,  $F(r, s, w) = 0$  is equivalent to  $w = w(r, s)$  for some  $(r, s)$  and  $\lambda(r, s) = 0$ .

Now let us look at the structure of  $\lambda(r, s) = 0$ . First, differentiating (59) with  $r$  at  $(0, 0)$  we obtain

$$\begin{aligned} \frac{\partial \lambda(r, s)}{\partial r} f &= \frac{\partial F}{\partial r} + D_w F \left( \frac{\partial w}{\partial r} \right) \\ &= u + g^{ij} \left( \frac{\partial w}{\partial r} \right)_{ij} + T \frac{\partial w}{\partial r}. \end{aligned}$$

Multiply both sides by  $f$  and integrate by parts to get at  $(0, 0)$  (note  $g^{ij} f_{ij} + T f = 0$ )

$$\begin{aligned} \frac{\partial \lambda}{\partial r}(0, 0) \int_{\Omega} f^2 \det(u_{ij}) dx &= \int_{\Omega} f u \det(u_{ij}) dx + \int_{\Omega} \left[ g^{ij} \left( \frac{\partial w}{\partial r} \right)_{ij} + T \frac{\partial w}{\partial r} \right] f \det(u_{ij}) dx \\ &= \int_{\Omega} f u \det(u_{ij}) dx + \int_{\Omega} [g^{ij} f_{ij} + T f] \frac{\partial w}{\partial r} \det(u_{ij}) dx \\ &= \int_{\Omega} f u \det(u_{ij}) dx \\ &< 0. \end{aligned}$$

So at  $(0, 0)$  we get

$$\frac{\partial \lambda}{\partial r} < 0. \quad (60)$$

Similarly we obtain the formula for  $\frac{\partial \lambda(r, s)}{\partial s}$ :

$$\begin{aligned} \frac{\partial \lambda}{\partial s}(0, 0) f &= \frac{\partial F}{\partial s} + D_w F \left( \frac{\partial w}{\partial s} \right) \\ &= g^{ij} f_{ij} + T f + g^{ij} \left( \frac{\partial w}{\partial s} \right)_{ij} + T \frac{\partial w}{\partial s} \\ &= g^{ij} \left( \frac{\partial w}{\partial s} \right)_{ij} + T \frac{\partial w}{\partial s}. \end{aligned}$$

Multiply both sides by  $f$  and integrate by parts to get at  $(0, 0)$  (note  $g^{ij} f_{ij} + T f = 0$ )

$$\frac{\partial \lambda(r, s)}{\partial s} \int_{\Omega} f^2 \det(u_{ij}) dx = \int_{\Omega} \left[ g^{ij} \left( \frac{\partial w}{\partial s} \right)_{ij} + T \frac{\partial w}{\partial s} \right] f \det(u_{ij}) dx$$

$$\begin{aligned}
&= \int_{\Omega} [g^{ij} f_{ij} + T f] \frac{\partial w}{\partial s} \det(u_{ij}) dx \\
&= 0.
\end{aligned}$$

So at  $(0, 0)$  we have

$$\frac{\partial \lambda}{\partial s} = 0. \quad (61)$$

Furthermore, this implies at the origin

$$g^{ij} \left( \frac{\partial w}{\partial s} \right)_{ij} + T \frac{\partial w}{\partial s} = 0. \quad (62)$$

Since  $T$  is the first eigenvalue and the first eigenfunction is unique modulo a constant, we have a constant  $c$  such that  $\frac{\partial w}{\partial s}(0, 0) = cf$ . However, by the definition of  $W$  we also have  $\int_{\Omega} w(r, s) f \det(u_{ij}) dx = 0$ , which implies  $\int_{\Omega} \frac{\partial w}{\partial s}(0, 0) f \det(u_{ij}) dx = 0$ . So  $c = 0$  and  $\frac{\partial w}{\partial s}(0, 0) \equiv 0$ .

Now we know that there exist a small open neighborhood of 0 in  $\mathbb{R}^1$  and a unique continuously differentiable function  $r(s)$  defined in it such that  $\lambda(r(s), s) = 0$  and  $r(0) = 0$ .

Differentiating  $\lambda(r(s), s) = 0$  with  $s$ , we get

$$\frac{\partial \lambda}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial \lambda}{\partial s} = 0, \quad (63)$$

which implies

$$\frac{\partial r}{\partial s}(0) = 0.$$

Differentiate (63) with  $s$  again

$$\frac{\partial^2 \lambda}{\partial^2 r} \left( \frac{\partial r}{\partial s} \right)^2 + 2 \frac{\partial^2 \lambda}{\partial r \partial s} \frac{\partial r}{\partial s} + \frac{\partial \lambda}{\partial r} \frac{\partial^2 r}{\partial s^2} + \frac{\partial^2 \lambda}{\partial s^2} = 0. \quad (64)$$

Taking values at  $(0, 0)$ , we have

$$\frac{\partial \lambda}{\partial r}(0, 0) \frac{\partial^2 r}{\partial s^2}(0) + \frac{\partial^2 \lambda}{\partial s^2}(0, 0) = 0. \quad (65)$$

Next we want to calculate  $\frac{\partial^2 \lambda}{\partial^2 s}(0, 0)$ . We have by differentiating (59) in  $s$  twice

$$\frac{\partial^2 \lambda}{\partial^2 s} f = \frac{\partial^2 F}{\partial^2 s} + D_{ww} F \left( \frac{\partial w}{\partial s}, \frac{\partial w}{\partial s} \right) + D_w F \left( \frac{\partial^2 w}{\partial^2 s} \right) + \left( \frac{\partial}{\partial s} D_w F \right) \left( \frac{\partial w}{\partial s} \right) + D_w \left( \frac{\partial}{\partial s} F \right) \left( \frac{\partial w}{\partial s} \right). \quad (66)$$

The first term can be calculated by taking the second derivative along the line  $(0, s, 0)$ :

$$\begin{aligned}
\frac{\partial^2 F}{\partial^2 s}(0, 0, 0) &= \frac{d^2}{ds^2} [\log \det(u + sf)_{ij} + T(u + sf)] \Big|_{s=0} \\
&= -g^{iq} g^{pj} f_{pq} f_{ij} \\
&< 0.
\end{aligned}$$

The second term, the fourth term and the last term are 0 at the origin because  $\frac{\partial w}{\partial s}(0, 0) \equiv 0$ . The third term is

$$D_w F\left(\frac{\partial^2 w}{\partial^2 s}\right)(0, 0) = g^{ij}\left(\frac{\partial^2 w}{\partial^2 s}\right)_{ij} + T\left(\frac{\partial^2 w}{\partial^2 s}\right). \quad (67)$$

Now we can multiply (66) with  $f$  and integrate (note that by (67) and integration by parts the third term has no contribution) to obtain  $\frac{\partial^2 \lambda}{\partial^2 s}(0, 0) < 0$ . So from (65) we have  $\frac{\partial^2 r}{\partial^2 s}(0) < 0$ , which in particular implies  $r(s) < 0$  in a neighborhood of 0.

Moreover, if we define  $u_s = u + sf + w(s, r(s))$ , we have

$$\det(u_s)_{ij} = e^{-(T+r(s))u_s}. \quad (68)$$

But

$$\begin{aligned} \frac{\partial}{\partial s} w(s, r(s)) \Big|_{s=0} &= \frac{\partial w}{\partial s}(0, 0) + \frac{\partial w}{\partial r}(0, 0) \frac{\partial r}{\partial s}(0) \\ &= 0, \end{aligned}$$

and we have  $f > 0$ , so at least for  $s$  small  $u + sf + w(s, r(s)) = u + s(f + \frac{w(s, r(s))}{s})$  is increasing with respect to  $s$ . So near  $u$  there exist two family solutions of Eq. (36) for those  $t < T$  and no solution for  $t > T$ , that is, the branch  $u_t$  turns to the left at  $t = T$ .  $\square$

Now we study the global structure of the branch, using the Leray–Schauder degree theory (see [2]). Define the space  $E := C_0^2(\overline{\Omega})$  (that is,  $C^2$  function on  $\overline{\Omega}$  with vanishing boundary value) and a map  $K : E \rightarrow E$  which is uniquely defined by  $\log \det D^2 K(f) = f$ . We know that for each  $f \in E$  there exists a unique  $K(f) \in C^3(\overline{\Omega})$ , which depends on  $f$  continuously in  $E$ . Moreover, we have  $\sup_{x \in \Omega} (|K(f)| + |DK(f)| + |D^2 K(f)| + |D^3 K(f)|)(x) \leq C$ , where  $C$  is a constant depending on  $\|f\|_{C^2}$  and  $\Omega$  (see Theorems 9.2, 17.21, 17.20, 17.26 in [4]). So  $K : E \rightarrow E$  is a continuous compact map. With these definitions, we can write Eq. (36) in the following form (note if we define  $f := \log \det D^2 u$ , then  $u = K(f)$ ):

$$T_t(f) = 0, \quad (69)$$

where  $T_t(f) = f + tK(f)$ .

Define  $\Sigma := \{(f, t) \in E \times [0, +\infty) : T_t f = 0\}$ , we know that  $(0, 0) \in \Sigma$  (note that such  $t$  is unique for  $f$  if it exists). Since  $K$  is a continuous compact map, we know that  $\Sigma$  is a closed locally compact set. In fact, for any bounded set  $B \subset E \times [0, +\infty)$ ,  $B \cap \Sigma$  is a compact set.

Define  $\Sigma'$  to be the connected component of  $\Sigma$  containing  $(0, 0)$  and by the following Theorem 3.5.3 (Leray–Schauder) of [2] we have:

**Lemma 4.10.**  $\Sigma'$  is an unbounded set in  $E \times \mathbb{R}^1$ .

For convenience, we give Theorem 3.5.3 here.

**Theorem 3.5.3 of [2].** Let  $X$  be a real Banach space,  $T : X \times \mathbb{R}^1 \rightarrow X$  be a compact map satisfying  $T(x, 0) = \theta$ , and  $f(x, \lambda) = x - T(x, \lambda)$ . Let  $S = \{(x, \lambda) \in X \times \mathbb{R}^1 \mid f(x, \lambda) = \theta\}$ , and let  $\zeta$  be the component of  $S$  passing through  $(\theta, 0)$ . If  $\zeta^\pm = \zeta \cap (X \times \mathbb{R}_\pm^1)$ , then both  $\zeta^+$  and  $\zeta^-$  are unbounded.

By Theorem 3.4 we can define  $T^* := \sup\{t > 0 : \exists (f, t) \in \Sigma', T_t f = 0\}$ , which is a finite positive number. (In fact  $T^* = (\lambda^*)^2$  by the following Theorem 4.11, where  $\lambda^*$  is the critical value in Theorem 3.4.) The following Theorem 4.11 also implies that the branch  $\Sigma'$  extends to the maximum  $t$

such that (69) (or (36)) has a solution. We know that from the local compactness and closed property of  $\Sigma$  there exists a solution of (69) at  $T^*$ .

**Theorem 4.11.** *For  $t > T^*$ , (69) (or (36)) has no solution.*

**Proof.** Assume there exists a solution  $v$  of (36) for some  $t_0 > T^*$ .

Define  $\Sigma'' := \{(f, t) : (f, t) \in \Sigma' \text{ and } K(f) \geq v\}$ , here  $K(f) = u$  is the solution of (36) by the definition of  $T_t(f)$ .

We want to prove that  $\Sigma''$  is nonempty, open and closed relatively to  $\Sigma'$ , so we can get  $\Sigma'' = \Sigma'$ . If this is proved, then there exists a sequence  $(f_k, t_k)$  in  $\Sigma'$  such that  $K(f_k)$  diverges to  $-\infty$  in  $E = C_0^2(\overline{\Omega})$ , we have  $\inf_{\Omega} K(f_k)$  diverges to  $-\infty$  too, because otherwise we will have a uniform bound for  $K(f_k)$  in  $C_0^2(\overline{\Omega})$  norm by a priori estimate. Thus, we must have  $\inf_{x \in \Omega} v(x) = -\infty$ , which is a contradiction.

First, by Proposition 4.8 or direct comparison we have  $(0, 0) \in \Sigma''$ , so  $\Sigma''$  is nonempty.

The closeness of  $\Sigma''$  is obvious by the continuity of the operator  $K$ .

Next if  $(f_0, s_0) \in \Sigma''$  for some  $s_0 \leq T^*$ , let  $K(f_0) = w$ , we get  $\det D^2 w = e^{-s_0 w}$  and  $w \geq v$ , then we have

$$\begin{aligned} \det D^2 w &= e^{-s_0 w} \\ &\leq e^{-t_0 v} \\ &= \det D^2 v, \end{aligned}$$

by  $0 \geq w \geq v$  and  $t_0 \geq s_0$ . Then by the strong maximum principle, unless  $w \equiv v$  (which is impossible here because  $t_0 > s_0$ ), we must have  $w > v$  strictly in  $\Omega$ , and  $\frac{\partial w}{\partial n} < \frac{\partial v}{\partial n}$  on  $\partial\Omega$ , where  $n$  is the outer normal vector of  $\partial\Omega$  (see the proof of Proposition 4.7).

We can choose a small open neighborhood  $B \subset E \times \mathbb{R}^1$  of  $(f_0, s_0)$  such that for any  $(f, s) \in B \cap \Sigma'$  we have  $|s - s_0| < \epsilon$ ,  $\sup_{x \in \overline{\Omega}} (|w - u| + |D(w - u)| + |D^2(w - u)|)(x) < \epsilon$  for  $u = K(f)$  and for some  $\epsilon > 0$  by the continuity of  $K$ . Moreover, we have  $u \geq v$  in  $\Omega_\epsilon := \{x : w(x) \geq v(x) + \epsilon\}$ . While for  $\epsilon$  small enough, in  $\Omega \setminus \Omega_\epsilon$  we have

$$\begin{aligned} u(x) &\geq u(\pi x) - \frac{\partial u}{\partial n}(\pi x)|x - \pi x| \\ &\geq -\left(\frac{\partial w}{\partial n}(\pi x) + \epsilon\right)|x - \pi x| \\ &\geq -(1 - \epsilon)\frac{\partial v}{\partial n}(\pi x)|x - \pi x| \\ &\geq v(x), \end{aligned}$$

here  $\pi x$  is the projection of  $x$  onto  $\partial\Omega$ , which is a smooth map near  $\partial\Omega$ . We also use the fact that  $u, v$  and  $w$  are convex functions with vanishing boundary value. So  $\Sigma''$  is open relatively to  $\Sigma'$ .  $\square$

Now we know by Lemma 4.10 and Theorem 4.11 that  $\Sigma'$  starts from  $t = 0$  and reaches  $T^*$ , and at last diverges to infinity as and only as  $t$  approaches 0 (by Lemma 4.6, we know that if a sequence  $(f_k, t_k) \in \Sigma'$  satisfying  $T_{t_k} f_k = 0$  tends to infinity in  $E \times \mathbb{R}^1$  (by an a priori estimate, it also tends to infinity in  $C(\overline{\Omega}) \times \mathbb{R}^1$ ), then  $t_k$  tends to 0). So for  $0 < t < T^*$ , now we prove the existence of at least two solutions for (69) (or (36)):

**Theorem 4.12.** *For  $t \in (0, T^*)$ , (69) (or (36)) has at least two solutions.*

**Proof.** Let

$$T_1 := \sup\{t_1 > 0: \forall 0 < t \leq t_1, \exists f_1 \neq f_2 \text{ such that } (f_1, t) \in \Sigma', (f_2, t) \in \Sigma'\}.$$

We know that  $T_1 \geq T > 0$ , where  $T$  is the first degenerate point of the branch emanating from  $t = 0$ , introduced in p. 2862, before Lemma 4.6. We prove that  $T_1 = T^*$ .

Assuming that  $T_1 < T^*$ , take a  $T_2 \in [T_1, T^*)$  such that there exists a unique  $f$  such that  $t(f) = T_2$ ,  $(f, t(f)) \in \Sigma'$ . We define

$$\Sigma_1 := \{(f, t(f)) \in \Sigma': t(f) \leq T_2\}.$$

Since  $t(f)$  depends continuously on  $f$ ,  $\Sigma_1$  is a closed subset of  $\Sigma'$ . Now by the following Lemma 4.13 we get that  $\Sigma_1$  is connected. We can repeat the proof of Theorem 4.11 to prove that there exists no solution of (36) for  $t > T_2$ , which is a contradiction with  $\Sigma'$  reaches  $T^* > T_2$ , so our claim follows.  $\square$

**Lemma 4.13.**  $\Sigma_1$  is connected.

**Proof.** If  $\Sigma_1$  is not connected, there exist two disjoint closed subset  $K_1, K_2$  of  $\Sigma'$  such that  $\Sigma_1 = K_1 \cup K_2$ .

Because for  $T_2$  there exists a unique  $f$  such that  $t(f) = T_2$ ,  $(f, T_2) \in \Sigma'$ , and without loss of generality, we can assume  $(f, T_2) \in K_1$ , then  $\forall (f', t(f')) \in K_2$  we have  $t(f') < T_2$  strictly. By Lemma 4.6 we know that  $\{(f', t(f')) \in K_2: t(f') \geq \frac{T_1}{2}\}$  is a compact set. So by the compactness, we have that there exists  $\delta > 0$  such that  $t(f') \leq T_2 - \delta$ ,  $\forall (f', t(f')) \in K_2$ .

Now the set  $K_3 := \{(f, t(f)) \in \Sigma': t(f) \geq T_2\}$  is also a compact set. From the above discussion we know that  $K_3 \cap K_2 = \emptyset$ . So we have  $\Sigma' = (K_1 \cup K_3) \cup K_2$ , and  $(K_1 \cup K_3)$  and  $K_2$  are disjoint closed sets. This contradicts with the connectedness of  $\Sigma'$ .  $\square$

Next we want to investigate the exact number of the solutions at  $t = T^*$ . First, we give a lemma, which is kind of the inverse of Proposition 4.7.

**Lemma 4.14.** Given  $t > 0$ , if there exist two solutions  $u$  and  $v$  of (36) with  $u \geq v$ , then the first eigenvalue of the operator  $Lf := -g^{ij}f_{ij}$  satisfies that  $\lambda_1 > t$ , where  $(g^{ij})$  is the inverse matrix of  $(u_{ij})$ .

**Proof.** Using the comparison principle as before, we know that  $u > v$  strictly in the interior of  $\Omega$  and  $\frac{\partial u}{\partial \nu} < \frac{\partial v}{\partial \nu}$  on  $\partial\Omega$ .

From the concavity of the logdet function (see Appendix A, Proposition A.4), we have

$$\begin{aligned} \log \det([\tau v + (1 - \tau)u]_{ij}) &\geq \tau \log \det(v_{ij}) + (1 - \tau) \log \det(u_{ij}) \\ &= -\tau t v - (1 - \tau) t u, \end{aligned}$$

for any  $\tau \in [0, 1]$ , which becomes an equality as  $\tau = 0$  or  $\tau = 1$ . So we can compare the derivative in  $\tau$  of both sides at  $\tau = 0$  and obtain

$$g^{ij}(v - u)_{ij} \geq -tv + tu, \quad (70)$$

where  $(g^{ij})$  is the inverse matrix of  $(u_{ij})$ .

Denote  $h = u - v$ , which is a positive function and satisfies

$$-g^{ij}h_{ij} \geq th. \quad (71)$$

We also know that the first eigenvalue  $\lambda_1$  of the operator  $Lf := -g^{ij}f_{ij}$  has a positive eigenfunction  $f$ , that is

$$-g^{ij}f_{ij} = \lambda_1 f. \quad (72)$$

Now we use Bakelman's formula (40) to integrate by parts

$$\begin{aligned} t \int_{\Omega} h f \det(u_{ij}) dx &\leq - \int_{\Omega} g^{ij} h_{ij} f \det(u_{ij}) dx \\ &= - \int_{\Omega} g^{ij} f_{ij} h \det(u_{ij}) dx \\ &= \lambda_1 \int_{\Omega} h f \det(u_{ij}) dx. \end{aligned}$$

Because the integrand is positive, we get  $\lambda_1 \geq t$ .

Moreover, if  $\lambda_1 = t$ , the inequality above becomes an equality, so from (71) we must have

$$-g^{ij}h_{ij} = th. \quad (73)$$

Noticing that

$$K(\tau) := \log \det([\tau v + (1 - \tau)u]_{ij}) - \tau \log \det(v_{ij}) - (1 - \tau) \log \det(u_{ij})$$

is concave for  $\tau \in [0, 1]$  too by Proposition A.4 in Appendix A and its proof, and moreover that  $K(0) = K(1) = 0$  and that (73) means that  $K'(0) = 0$ , we get that

$$\log \det([\tau v + (1 - \tau)u]_{ij}) = \tau \log \det(v_{ij}) + (1 - \tau) \log \det(u_{ij}). \quad (74)$$

By the strict concavity of the function  $\log \det$  (see Proposition A.4 in Appendix A), this means we must have  $D^2u \equiv D^2v$  in  $\Omega$ . Thus  $u \equiv v$ , which contradicts our assumption, so we must have  $\lambda_1 > t$ .  $\square$

**Theorem 4.15.** *For  $t = T^*$ , (36) has exactly one solution. Moreover, we have  $T^* = T$ , where  $T$  is the first degenerate point of the branch emanating from  $t = 0$ , introduced in p. 2862, before Lemma 4.6.*

**Proof.** Concerning the case  $t = T$ , the existence of a solution  $u_T$  is discussed in p. 2862. Moreover, from Proposition 4.7, we know that this  $u_T$  is the maximal solution, that is, if there exists another solution  $v$  such that

$$\det(v_{ij}) = e^{-T^*v}, \quad v|_{\partial\Omega} = 0 \quad (75)$$

then  $u_T \geq v$ . Noticing that the first eigenvalue  $\lambda_T$  associated with  $u_T$  equals  $T$ , we must have  $u_T \equiv v$ , in view of Lemma 4.14. That is, the solution (36) at  $t = T$  is unique.

Of course  $T \leq T^*$ , then combining Theorem 4.12 and the above discussion, we get  $T^* = T$ .  $\square$

Now Theorem 1.1 is proved by the above Theorems 4.11, 4.12, 4.15, and Theorem 1.2 is equivalent to Eq. (36) through a scaling (see the discussion before Example 3.5).



**Remark 4.16.** We assume

(k1)  $k(0) = 0$ ;

(k2)  $k'(u) \geq c$  for some positive constant  $c$ , in particular,  $k(u) \leq cu$  for  $u < 0$  and  $k$  is increasing.

Here, the condition (k2) is essential in our proof, but the condition (k1) can always be satisfied by adding a constant and multiplying  $u$  by an appropriate constant. However, Propositions 4.7, 4.8 and Theorem 4.9 (which is not needed in proving the main Theorem 1.1) need some more conditions:

(k3)  $k$  is concave, or  $k''(u) \leq 0$ .

Moreover, Lemma 4.6 is true for Eq. (3), and the proof of Theorems 4.11 and 4.12 can be easily modified in the case of Eq. (3). For example, in Theorem 3.1, (24) can be modified into

$$\begin{aligned} \frac{d}{dr} e^{\frac{k(u(r))}{n}} &\geq \frac{1}{n} e^{\frac{k(u(r))}{n}} k'(u(r)) u'(r) \\ &\geq \frac{c}{n} r, \end{aligned}$$

where the constant  $c$  is as in the condition (k2). The calculation below (24) is almost the same.

**Remark 4.17.** For those  $t \in (0, T)$  (here  $T$  is the first degenerate point of the branch emanating from  $t = 0$ , introduced in p. 2862, before Lemma 4.6. From the discussion above, we have  $T = T^*$ ), noticing Lemma 4.6, there exists another method to prove the existence of the second solution by mountain pass lemma (see [11]).

First, we know (see Bakelman [7]) critical points of the functional defined on the space of smooth convex functions:

$$I(u) := - \int_{\Omega} \left( \frac{1}{n+1} u \det D^2 u + \frac{1}{t} e^{-tu} \right) dx \quad (76)$$

are weak solutions of Eq. (36).

For each  $t \in (0, T)$  we have constructed a solution  $u_t$ . Moreover, we have  $\lambda_{1,t} > t$ . This implies  $u_t$  is a local minimizer of  $I$ . We also have for fixed  $u(x) < 0$

$$I(\tau u) = - \int_{\Omega} \left( \tau^{n+1} \frac{1}{n+1} u \det D^2 u + \frac{1}{t} e^{-\tau t u} \right) dx,$$

which diverges to  $-\infty$  as  $\tau \rightarrow +\infty$ . So there exists a mountain pass structure and we can use the logarithmic gradient heat flow to prove the existence of a mountain pass type critical point  $v_t$  (cf. [11]).

Note that by Proposition 4.7,  $v_t$  must have  $\lambda_1 < t$ , but its Morse index is 1, so we must have  $\lambda_2 \geq t$ .

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## Appendix A

In this appendix we collect some formulas for matrix analysis we used in this paper.

**Proposition A.1.** Let  $F(A) := A^{-1}$  be defined on the invertible matrix space, then its (Fréchet) differential is  $DF(A)B = -A^{-1}BA^{-1}$  for any matrix  $B$ .

**Proof.** Because  $F(A)A = \text{Id}$ , where  $\text{Id}$  is the identity matrix, take differential we obtain  $(DF(A)B)A + F(A)B = 0$ . Rewrite this and we get the formula.  $\square$

**Proposition A.2.** Let  $G(A) = \log \det A$  be defined on the positive definite symmetric matrix space, then its (Fréchet) differential is  $DG(A)B = A^{ij}B_{ij}$  (repeated indices are summed) for any symmetric matrix  $B$ , where  $(A^{ij})$  is the inverse matrix of  $A$ .

**Proof.** We need to calculate  $\frac{d}{dt}G(A(t))|_{t=0}$  where  $A(t)$  is a curve near  $A$  with  $A(0) = A$  and  $\frac{d}{dt}A(t)|_{t=0} = B$ .

Because  $A$  is a positive definite symmetric matrix, there exists a nonsingular matrix  $P$  with  $\det P > 0$  such that  $A = PP^T$ , where  $P^T$  is the transpose of  $P$ . We have

$$\log \det A(t) = \log \det(P^{-1}A(t)(P^T)^{-1}) + \log \det P + \log \det P^T. \quad (\text{A.1})$$

Now we have

$$P^{-1}A(t)(P^T)^{-1} = \text{Id} + tP^{-1}B(P^T)^{-1} + o(t), \quad (\text{A.2})$$

so

$$\det(P^{-1}A(t)(P^T)^{-1}) = 1 + t\text{Tr}(P^{-1}B(P^T)^{-1}) + o(t), \quad (\text{A.3})$$

here  $\text{Tr}$  is the trace. This implies

$$\log \det A(t) = \log \det P + \log \det P^T + t\text{Tr}(P^{-1}B(P^T)^{-1}) + o(t). \quad (\text{A.4})$$

So

$$\frac{d}{dt} \log \det A(t) \Big|_{t=0} = \text{Tr}(P^{-1}B(P^T)^{-1}). \quad (\text{A.5})$$

Since  $\text{Tr}(Q_1Q_2) = \text{Tr}(Q_2Q_1)$  for any matrices  $Q_1, Q_2$ , we have

$$\begin{aligned} \frac{d}{dt} \log \det A(t) \Big|_{t=0} &= \text{Tr}(B(P^T)^{-1}P^{-1}) \\ &= \text{Tr}(BA^{-1}). \end{aligned}$$

If we write this using coefficients of the matrices it is the form in the proposition.  $\square$

**Corollary A.3.** Let  $H(A) = \det A$  be defined on the positive definite symmetric matrix space, then its (Fréchet) differential is  $DH(A)(B) = A^{ij}B_{ij} \det A$  for any symmetric matrix  $B$ , where  $(A^{ij})$  is the inverse matrix of  $A$  and repeated indices are summed.

**Proof.** Because we have  $H(A) = e^{G(A)}$ ,  $DH(A) = e^{G(A)}DG(A)$ .  $\square$

**Proposition A.4.** Let  $F(A) = \log \det A$  be defined on the positive definite symmetric matrix space, then it is strictly concave.

**Proof.** We need to prove for two positive definite symmetric matrix  $A$  and  $B$ , we have  $f(t) = \log \det(tA + (1-t)B)$  is concave for  $t \in [0, 1]$ , or  $f''(t) \leq 0$ . First we have by Proposition A.2

$$f'(t) = A_t^{ij}(A - B)_{ij}, \quad (\text{A.6})$$

where  $A_t^{ij}$  is the inverse matrix of  $tA + (1-t)B$ . Then we have by Proposition A.1

$$f''(t) = -A_t^{iq}(A - B)_{pq}A_t^{pj}(A - B)_{ij}, \quad (\text{A.7})$$

which is non-positive for each  $t$  which can be seen by diagonalizing  $A_t$  (in fact negative unless  $A = B$ ).  $\square$

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