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Convergence of a Brinkman-type penalization for compressible fluid flows

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ABSTRACT

We show convergence of a Brinkman-type penalization of the compressible Navier–Stokes equation. In particular, the existence of weak solutions for the system in domains with boundaries varying in time is established.

1. Introduction

Penalization methods of various types play a significant role in theoretical studies as well as numerical analysis of flows around solid obstacles that are allowed to change in time. The Brinkman penalization method was proposed for both incompressible and compressible fluids, where the momentum equation is penalized by an extra term modeling solid obstacles as porous media, with porosity and viscous permeability approaching zero, see Angot, Bruneau and Fabrie [2], Liu and Vasilyev [13], among others. The main advantages of the method are minimal requirements on regularity of the boundary of the fluid domain, and the efficient implementation for moving solid boundaries. In this paper, we show *convergence* of the Brinkman penalization applied to the Navier–Stokes system governing the motion of a barotropic, viscous, compressible fluid. In particular, we prove existence

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of global-in-time solutions for the limit problem describing the motion of a compressible fluid in a domain varying in time.

The time evolution of the *mass density* $\varrho = \varrho(t, x)$ and the *velocity field* $\mathbf{u} = \mathbf{u}(t, x)$ of a viscous compressible fluid is governed by the NAVIER–STOKES SYSTEM of equations:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S} + \varrho \mathbf{f}, \tag{1.2}$$

where $p = p(\varrho)$ is the pressure, $\mathbf{f} = \mathbf{f}(t, x)$ is an external driving force, and the symbol $\mathbb{S}$ denotes the viscous stress tensor given through NEWTON'S RHEOLOGICAL LAW

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^t - \frac{2}{3} \operatorname{div}_x \mathbf{u}\mathbb{I}\right) + \eta \operatorname{div}_x \mathbf{u}\mathbb{I}, \tag{1.3}$$

with the *shear viscosity coefficient* $\mu > 0$, and the *bulk viscosity coefficient* $\eta \geqslant 0$.

Our aim is to study system (1.1)–(1.3) in a spatial domain, with a boundary varying in time. To this end, we introduce a vector field $\mathbf{v}_s = \mathbf{v}_s(t, x)$ and set

$$\Omega_t = \left\{x \in \mathbb{R}^3 \;\middle|\; x = \mathbf{X}(t, x_0) \text{ for a certain } x_0 \in \Omega_0\right\}, \tag{1.4}$$

where $\Omega_0 \subset \mathbb{R}^3$ is a given domain occupied by the fluid at the initial instant $t = 0$, and the vector field $\mathbf{X}$ solves the initial value problem

$$\partial_t \mathbf{X}(t, x_0) = \mathbf{v}_s\big(t, \mathbf{X}(t, x_0)\big), \qquad \mathbf{X}(0, x_0) = x_0. \tag{1.5}$$

Accordingly, we consider system (1.1)–(1.3) in a space–time domain

$$Q^f = \left\{(t, x) \;\middle|\; t \in (0, T),\ x \in \Omega_t\right\}.$$

In addition, we impose the *no-slip boundary conditions* on the solid wall, meaning,

$$\mathbf{u}(t, \cdot)|_{\partial\Omega_t} = \mathbf{v}_s(t, \cdot)|_{\partial\Omega_t} \quad \text{for } t \in (0, T). \tag{1.6}$$

The associated penalized problem is defined as follows. Similarly to Angot et al. [2], we fix a reference spatial domain $D \subset \mathbb{R}^3$ containing Ω^0 and such that

$$\mathbf{v}_s|_{\partial D} = 0.$$

Accordingly, it is enough to assume that the vector field $\mathbf{v}_s$ is defined on the set $[0, T] \times \overline{D}$.

System (1.1)–(1.3) is replaced by a penalized problem

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.7}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S} + \varrho \mathbf{f} - \frac{1}{\varepsilon}\chi(\mathbf{u} - \mathbf{v}_s) \tag{1.8}$$

considered in the cylinder $(0, T) \times D$, where

$$\chi(t, x) = \left\{\begin{array}{ll} 0 & \text{if } t \in (0, T),\ x \in \Omega_t \\ 1 & \text{otherwise} \end{array}\right\}. \tag{1.9}$$

Accordingly, the function χ represents a weak solution to the transport equation

$$\partial_t \chi + \mathbf{v}_s \cdot \nabla_x \chi = 0$$

satisfying the initial condition

$$\chi(0,\cdot) = 1_D - 1_{\Omega_0}.$$

Problem (1.7)–(1.8) is supplemented with the no-slip boundary condition

$$\mathbf{u}|_{\partial D} = 0, \tag{1.10}$$

and the initial conditions

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} \geqslant 0, \qquad (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_{0,\varepsilon}. \tag{1.11}$$

For any fixed $\varepsilon > 0$, problem (1.7)–(1.11) possesses a weak solution on an arbitrary time interval $(0,T)$ and for any choice of initial data of finite energy, at least if $p(\varrho)$ depends on ϱ in a certain way (for more details see Section 2). This result was proved by Lions [12], and later extended to a larger class of physically relevant pressure-density state equations in [5,6]. Our main goal is to identify the asymptotic limit of solutions ϱ_ε, $\mathbf{u}_\varepsilon$ of problem (1.7)–(1.11) for $\varepsilon \to 0$. More specifically, we show that

$$\varrho_\varepsilon \to \varrho \quad \text{in } L^q\big((0,T)\times D\big) \text{ for a certain } q > 1, \tag{1.12}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T; W_0^{1,2}\big(D;\mathbb{R}^3\big)\big), \tag{1.13}$$

where ϱ, $\mathbf{u}$ solve (1.1)–(1.3) in Q^f. In addition,

$$\mathbf{u} = \mathbf{v}_s \quad \text{in } Q^s,$$

where we have set

$$Q^s = \big((0,T)\times D\big) \setminus \overline{Q}^f.$$

Thus, under certain technical restrictions imposed on the specific form of the pressure, we establish

- convergence of the Brinkman penalization method for the compressible Navier–Stokes system,
- existence of weak solutions for the compressible Navier–Stokes system on time-varying domains.

The paper is organized as follows. A brief review of the existence theory for system (1.7)–(1.11) is given in Section 2. The main results of the paper, along with an outline of the strategy of the proof, are presented in Section 3. In Section 4, we establish uniform estimates independent of the parameter $\varepsilon \to 0$ and show convergence toward the limit problem claimed in (1.12), (1.13). Note that similar functional-analytic methods were used by Gallouët et al. [9,10] for proving convergence of certain numerical schemes.

2. Finite energy weak solutions for the compressible Navier–Stokes system

We say that ϱ, $\mathbf{u}$ represent a finite energy weak solution to problem (1.7)–(1.11) in $(0,T)\times D$ if

$$\varrho \geqslant 0, \qquad \varrho \in L^\infty\big(0,T;L^\gamma(D)\big) \quad \text{for a certain } \gamma > 1, \tag{2.1}$$

$$\mathbf{u} \in L^2\big(0,T;W_0^{1,2}\big(D;\mathbb{R}^3\big)\big), \tag{2.2}$$

and the following integral identities hold:

$$\int_0^T\int_D \big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \big(b(\varrho)-b'(\varrho)\varrho\big)\operatorname{div}_x\mathbf{u}\varphi\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_D b(\varrho_{0,\varepsilon})\varphi(0,\cdot)\,\mathrm{d}x \tag{2.3}$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{D})$, and for $b(\varrho)=\varrho$ or $b' \in C_c^\infty[0,\infty)$, $b(0)=0$;

$$\int_0^T\int_D \big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi + p\operatorname{div}_x\varphi\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_D \left(\mathbb{S}:\nabla_x\varphi - \varrho\mathbf{f}\cdot\varphi + \frac{\chi}{\varepsilon}(\mathbf{u}-\mathbf{v}_s)\cdot\varphi\right)\mathrm{d}x\,\mathrm{d}t - \int_D (\varrho\mathbf{u})_{0,\varepsilon}\cdot\varphi(0,\cdot)\,\mathrm{d}x \tag{2.4}$$

for any $\varphi \in C_c^\infty([0,T)\times D;\mathbb{R}^3)$;

$$\int_D \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau\int_D \mathbb{S}:\nabla_x\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \int_0^\tau\int_D \left(\varrho\mathbf{f}\cdot\mathbf{u} - \frac{\chi}{\varepsilon}(\mathbf{u}-\mathbf{v}_s)\cdot\mathbf{u}\right)\mathrm{d}x\,\mathrm{d}t + \int_D \left(\frac{1}{2\varrho_{0,\varepsilon}}\big|(\varrho\mathbf{u})_{0,\varepsilon}\big|^2 + P(\varrho_{0,\varepsilon})\right)\mathrm{d}x \tag{2.5}$$

for a.a. $\tau > 0$, where we have set

$$P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z.$$

It is an implicit part of the definition that all quantities are at least integrable in $(0,T)\times D$. This is true, in particular, under the hypotheses of Theorem 3.1 below.

Note that (2.3) is nothing other than a weak formulation of the renormalized equation of continuity

$$\partial_t b(\varrho) + \operatorname{div}_x\big(b(\varrho)\mathbf{u}\big) + \big(b'(\varrho)\varrho - b(\varrho)\big)\operatorname{div}_x\mathbf{u} = 0$$

introduced by DiPerna and Lions [4], while (2.5) is the energy inequality, where we have assumed a natural compatibility condition

$$(\varrho\mathbf{u})_{0,\varepsilon} = \frac{|(\varrho\mathbf{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} = 0 \quad \text{whenever } \varrho_{0,\varepsilon} = 0. \tag{2.6}$$

As already mentioned above, the *existence* of finite energy weak solutions for system (1.7)–(1.11) for any initial data with finite energy

$$\int_D \left(\frac{1}{2\varrho_{0,\varepsilon}} \left|(\varrho\mathbf{u})_{0,\varepsilon}\right|^2 + P(\varrho_{0,\varepsilon}) \right) \mathrm{d}x < \infty$$

was established in the pioneering work of Lions [12] on condition that the pressure behaves like $p(\varrho) \approx \varrho^\gamma$, with $\gamma \geqslant 9/5$, for large values of ϱ. This result was extended to $\gamma > 3/2$ in [5,6].

2.1. A modified energy inequality for the penalized problem

Assuming that $\mathbf{v}_s$ is continuously differentiable, we can take $\varphi = \psi_n(t)\mathbf{v}_s$, $\psi_n \in C_c^\infty[0,T)$, $\psi_n \nearrow 1_{[0,\tau)}$, as a test function in (2.4). Adding the resulting expression to (2.5), we deduce a *modified energy inequality* in the form:

$$\begin{aligned}
&\int_D \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_D \mathbb{S} : \nabla_x \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^\tau \int_D \chi\,|\mathbf{u} - \mathbf{v}_s|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \int_D \left(\frac{1}{2\varrho_{0,\varepsilon}} \left|(\varrho\mathbf{u})_{0,\varepsilon}\right|^2 + P(\varrho_{0,\varepsilon}) - (\varrho\mathbf{u})_{0,\varepsilon} \cdot \mathbf{v}_s(0,\cdot) + (\varrho\mathbf{u}\cdot\mathbf{v}_s)(\tau,\cdot) \right) \mathrm{d}x \\
&\qquad + \int_0^\tau \int_D \big(\varrho\mathbf{f}\cdot(\mathbf{u} - \mathbf{v}_s) + \mathbb{S} : \nabla_x \mathbf{v}_s - \varrho\mathbf{u}\cdot\partial_t\mathbf{v}_s - \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{v}_s - p\,\mathrm{div}_x\,\mathbf{v}_s \big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{2.7}$$

for a.a. $\tau \in (0,T)$. Relation (2.7) yields uniform bounds on the sequence of solutions to the penalized problem independent of $\varepsilon \to 0$ provided the vector field $\mathbf{v}_s$ is sufficiently smooth.

3. Main result

For the sake of simplicity, we assume that the pressure is given by an *isentropic equation of state*

$$p(\varrho) = a\varrho^\gamma, \quad a > 0,\ \gamma > 1; \tag{3.1}$$

whence the function P appearing in the energy inequality (2.7) can be taken in the form

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma. \tag{3.2}$$

Note that, in accordance with the boundary condition (1.10), the total mass

$$M_\varepsilon = \int_D \varrho(t,\cdot)\,\mathrm{d}x = \int_D \varrho_{0,\varepsilon}\,\mathrm{d}x \tag{3.3}$$

is a constant of motion even in the class of weak solutions satisfying (2.3).

Our main result reads as follows.

Theorem 3.1. *Let $\Omega_0 \subset \overline{\Omega}_0 \subset D \subset \mathbb{R}^3$ be bounded domains, with boundaries of class $C^{2+\nu}$, $\nu > 0$. Assume that the pressure p is given by* (3.1)*, with $\gamma > 3/2$, and that $\mathbf{f}$ belongs to $L^\infty((0,T) \times D; \mathbb{R}^3)$. Let $\mathbf{v}_s$ be a given vector field belonging to $C^{2+\nu}([0,T] \times \overline{D}; \mathbb{R}^3)$,*

$$\mathbf{v}_s|_{\partial D} = 0.$$

Finally, we suppose that the initial data satisfy (2.6) *and*

$$\varrho_{0,\varepsilon} \to \varrho_0 \quad \text{in } L^\gamma(D), \qquad \varrho_0|_{\Omega_0} \geqslant 0, \qquad \varrho_0|_{D \setminus \Omega_0} = 0, \tag{3.4}$$

$$(\varrho \mathbf{u})_{0,\varepsilon} \to (\varrho \mathbf{u})_0 \quad \text{in } L^1\big(D; \mathbb{R}^3\big), \qquad (\varrho \mathbf{u})_0|_{D \setminus \Omega_0} = 0, \tag{3.5}$$

$$\int_D \frac{|(\varrho \mathbf{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} \, \mathrm{d}x < c, \tag{3.6}$$

where c is independent of $\varepsilon \to 0$.

Then any sequence $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon > 0}$ of finite energy weak solutions of problem (2.1)–(2.5) *contains a subsequence such that*

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\text{weak}}\big([0,T]; L^\gamma(D)\big) \cap L^\gamma\big(Q^f\big), \tag{3.7}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{in } L^2\big(0,T; W_0^{1,2}\big(D; \mathbb{R}^3\big)\big), \qquad \mathbf{u} = \mathbf{v}_s \quad \text{in } Q^s, \tag{3.8}$$

where the limit functions ϱ, $\mathbf{u}$ are distributional solutions of the equation of continuity (1.1) *in $(0,T) \times D$ and of the momentum equation* (1.2) *in Q^f.*

Remark 3.1. Here the symbol $C_{\text{weak}}(0,T;X)$ denotes the space of functions ranging in a Banach space X continuous for $t \in [0,T]$ with respect to the weak topology on X.

The rest of the paper is devoted to the proof of Theorem 3.1. It is worth noting that we assume the initial density ϱ_0 to vanish outside Ω_0. As the limit velocity $\mathbf{u}$ coincides with $\mathbf{v}_s$ in Q^s, we have $\varrho = 0$ in Q^s. Finally, since the densities ϱ_ε are non-negative, this in turn implies pointwise (a.a.) convergence $\varrho_\varepsilon \to \varrho$ in Q^s, see Section 4 below. This property is indispensable in the proof of pointwise convergence of the densities in the whole set $(0,T) \times D$.

The hypotheses concerning regularity of $\partial\Omega_0$ and $\mathbf{v}_s$ are quite strong and could be relaxed. However, the uniform bounds on the pressure in the set Q^f require certain minimal regularity of the boundary, both in space and time.

The proof of Theorem 3.1 is based on uniform bounds on the sequence $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ that may be deduced directly from the energy inequality (2.7). Uniform bounds on the pressure $p(\varrho_\varepsilon) = a\varrho_\varepsilon^\gamma$ in Q^f represent a more delicate issue. Indeed in order to pass to the limit in the momentum equation (2.4) we have to establish:

- equi-integrability of the sequence $\{p(\varrho_\varepsilon)\}_{\varepsilon>0}$ in $L^1(Q^f)$;
- pointwise (a.a.) convergence $\varrho_\varepsilon \to \varrho$ implying $p(\varrho_\varepsilon) \to p(\varrho)$.

Since the underlying spatial domain changes in time, a straightforward application of the so-called *Bogovskii operator* that would yield the desired pressure estimates (cf. [7]) is not obvious. Instead we use a combination of local estimates obtained in the same way as in [12], with a suitably chosen test functions that control integrability of the pressure at the spatial boundary of Q^f, see also Kukučka [11].

Finally, the pointwise convergence $\varrho_\varepsilon \to \varrho$ in the domain Q^f is shown by means of the concept of *oscillations defect measure* introduced in [5].

4. Uniform bounds and convergence

In the remaining part of the paper, we set $\mathbf{f} = 0$ as the principal steps of the proof require only minor and entirely obvious modifications to accommodate the presence of a bounded driving force. As already observed in (3.3), the total mass of the fluid is a constant of motion. Consequently, it follows from hypothesis (3.4) that

$$\{\varrho_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^\infty\big(0,T;L^1(D)\big). \tag{4.1}$$

Since $\partial_t \mathbf{v}_s$, $\nabla_x \mathbf{v}_s$ are bounded, a short examination of the energy inequality (2.7) gives rise to the uniform bounds

$$\{\varrho_\varepsilon\}_{\varepsilon>0} \quad \text{bounded in } L^\infty\big(0,T;L^\gamma(D)\big), \tag{4.2}$$

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \quad \text{bounded in } L^\infty\big(0,T;L^2\big(D;\mathbb{R}^3\big)\big), \tag{4.3}$$

$$\left\{\nabla_x \mathbf{u}_\varepsilon + (\nabla_x \mathbf{u}_\varepsilon)^t - \frac{2}{3}\operatorname{div}_x \mathbf{u}_\varepsilon \mathbb{I}\right\}_{\varepsilon>0} \quad \text{bounded in } L^2\big(0,T;L^2\big(D;\mathbb{R}^{3\times 3}\big)\big), \tag{4.4}$$

where (4.4), combined with (1.10) and Korn's inequality, yields

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \quad \text{bounded in } L^2\big(0,T;W_0^{1,2}\big(D;\mathbb{R}^3\big)\big). \tag{4.5}$$

Finally,

$$\int_0^T\int_D \chi|\mathbf{u}_\varepsilon - \mathbf{v}_s|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{Q^s} |\mathbf{u}_\varepsilon - \mathbf{v}_s|^2 \leqslant \varepsilon c, \tag{4.6}$$

with c independent of ε.

Combining (4.2), (4.5) with Eq. (2.3) we may infer that

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\text{weak}}\big([0,T];L^\gamma(D)\big), \tag{4.7}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T;W_0^{1,2}\big(D;\mathbb{R}^3\big)\big), \tag{4.8}$$

passing to subsequences if necessary. Moreover, as a consequence of (4.6),

$$\mathbf{u} = \mathbf{v}_s \quad \text{in } Q^s. \tag{4.9}$$

4.1. Pressure estimates

The pressure plays a crucial role in the analysis of the limit for $\varepsilon \to 0$. Obviously, we cannot control the pressure in the set Q^s, where the momentum equation contains a singular term. On the other hand, local pressure estimates in Q^f can be obtained in the same way as in the monograph of Lions [12], namely by taking the quantities

$$\varphi(t,x) = \psi(t,x)\nabla_x\Delta_x^{-1}\big[1_D\varrho_\varepsilon^\nu\big],$$

where $\nu > 0$ is a small positive number, ψ is a smooth function, $\operatorname{supp}[\psi] \subset Q^f$, and the symbol Δ_x denotes the Laplace operator considered on the whole Euclidean space $\mathbb{R}^3$. Since $\gamma > 3/2$ and the estimates (4.1)–(4.5) hold, we get, after a bit tedious but straightforward computation,

$$a \int_K \varrho_\varepsilon^{\gamma+\nu} \, \mathrm{d}x \, \mathrm{d}t = \int_K p(\varrho_\varepsilon) \varrho_\varepsilon^\nu \, \mathrm{d}x \, \mathrm{d}t \leqslant c(K) \quad \text{for any compact } K \subset Q^f. \tag{4.10}$$

The pressure estimates can be extended "up to the boundary" provided we are able to construct suitable test functions in the momentum equation (2.4). More specifically, we need $\varphi = \varphi(t, x)$ such that

- $\partial_t \varphi$, $\nabla_x \varphi$ belong to $L^q(Q^f)$ for a given (large) $q \ll 1$;
- $\varphi(t, \cdot) \in W_0^{1,q}(\Omega_t; \mathbb{R}^3)$ for any $t \in (0, T)$;
- $\varphi(T; \cdot) = 0$;
- $$\operatorname{div}_x \varphi(t, x) \to \infty \quad \text{for } x \to \partial\Omega_t \text{ uniformly for } t \text{ in compact subsets of } (0, T). \tag{4.11}$$

Indeed using such a φ as a test function in (2.4) would yield a uniform bound

$$\int_{Q^f \cap ([0,\tau] \times D)} p(\varrho_\varepsilon) \operatorname{div}_x \varphi \, \mathrm{d}x \, \mathrm{d}t \leqslant c(\tau) \quad \text{for any } \tau < T,$$

which, together with (4.2), (4.10), and (4.11), implies the desired conclusion

$$\{p(\varrho_\varepsilon)\}_{\varepsilon > 0} \quad \text{equi-integrable in } L^1(Q^f). \tag{4.12}$$

In order to construct the test function φ, we introduce the distance function

$$d(t, x) = \operatorname{dist}[x, \partial\Omega_t] \quad \text{for } t \in [0, T],\ x \in \Omega_t,$$

where dist is understood in the 3D Euclidean space $\mathbb{R}^3$. In accordance with the hypotheses of Theorem 3.1, the lateral boundary of the set Q^f is of class C^2, therefore there exists an open neighborhood $\mathcal{U}$ of $\partial Q^f \cap ((\tau_1, \tau_2) \times D)$ such that

$$d \in C^2(\mathcal{U}) \quad \text{for any } 0 < \tau_1 < \tau_2 < T,$$

see Foote [8] and Delfour and Zolésio [3].

Now, choose a function h such that

$$h(z) = \begin{cases} z^\alpha & \text{for } z \in [0, \delta/2), \\ \text{non-negative and smooth} & \text{in } [\delta/2, \delta], \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the functions φ can be taken in the form

$$\varphi(t, x) = \psi(t) h(d) \nabla_x d, \qquad \psi \in C_c^\infty(0, T), \qquad \psi \geqslant 0, \qquad \psi = 1 \quad \text{in } [\tau_1, \tau_2].$$

For $\delta > 0$, $\alpha = \alpha(q) \in (0, 1)$ small enough, the functions φ are bounded and their first derivatives belong to L^q for a given $q \ll 1$. Moreover, as $\nabla_x d(t, x)$ is a unit vector pointing to the nearest point to x on $\partial\Omega_t$, (4.11) follows.

4.2. Pointwise convergence of the density

Pointwise (a.a.) convergence $\varrho_\varepsilon \to \varrho$ plays a central role in the existence theory for the compressible Navier–Stokes system. Our approach is based on the property of *weak sequential stability* of the effective viscous flux established by Lions [12], combined with the concept of *oscillation defect measure* introduced in [5]. We proceed by several steps:

4.2.1. Convergence in the set Q^s

In accordance with hypothesis (3.4), we have $\varrho_0|_{D\setminus\Omega_0} = 0$. Moreover, by virtue of (4.7), (4.8), it is easy to check that ϱ, $\mathbf{u}$ solve the equation of continuity

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \tag{4.13}$$

in the sense of distributions in the set $(0,T) \times D$. Moreover, as the limit velocity coincides with $\mathbf{v}_s$ in Q^s, equation (4.13) is satisfied in the whole space $(0,T) \times R^3$ provided $\mathbf{u}$, ϱ were extended to be zero outside D. In other words, we may assume that

$$\int_0^T \int_{\mathbb{R}^3} (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi)\, \mathrm{d}x\, \mathrm{d}t = -\int_{\mathbb{R}^3} \varrho_0 \varphi(0,\cdot)\, \mathrm{d}x \tag{4.14}$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^3)$.

Since the motion of $\partial\Omega_t$ is governed by the velocity field $\mathbf{v}_s$ and $\varrho_0 = 0$ outside Ω_0, we conclude, at least formally, that

$$\varrho(t,x) = 0 \quad \text{for a.a. } (t,x) \in Q^s. \tag{4.15}$$

This argument can be made rigorous, even in the class of weak solutions, by means of the regularization procedure introduced by DiPerna and Lions [4].

Finally, as ϱ_ε are non-negative and (4.7) holds, relation (4.15) implies

$$\varrho_\varepsilon \to \varrho \quad \text{in } L^q(Q^s) \text{ for any } 1 \leqslant q < \gamma. \tag{4.16}$$

4.2.2. Effective viscous pressure

The next step is to revoke the result of Lions [12] on the effective viscous pressure. To this end, we first introduce a family of cut-off functions

$$T_k(\varrho) = \min\{\varrho, k\}, \quad k \geqslant 0.$$

Next observe that the bounds (4.1)–(4.5), together with (4.10), allow us to pass to the limit in the momentum equation (2.4) in the domain Q^f to obtain

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \overline{p(\varrho)} = \operatorname{div}_x \mathbb{S} \quad \text{in } Q^f \tag{4.17}$$

in the sense of distributions, where $\overline{p(\varrho)}$ denotes a weak limit of $\{p(\varrho_\varepsilon)\}_{\varepsilon>0}$.

Now, Lions' result asserts a remarkable identity

$$(4/3\mu + \eta)\big(\overline{T_k(\varrho)\operatorname{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\operatorname{div}_x \mathbf{u}\big) = \overline{p(\varrho)T_k(\varrho)} - \overline{p(\varrho)}\,\overline{T_k(\varrho)} \quad \text{in } Q^f. \tag{4.18}$$

In particular, as the pressure p is an increasing function of the density, the expression on the left-hand side of (4.18) is non-negative. Identity (4.18) holds locally and can be deduced, in a highly non-trivial way, from the expressions resulting from (2.4) tested on $\varphi\nabla_x\Delta_x^{-1}[1_D T_k(\varrho)]$, and from (4.17) tested on $\varphi\nabla_x\Delta_x^{-1}[1_D \overline{T_k(\varrho)}]$, where φ is a smooth function compactly supported in Q^f. An alternative proof of (4.18) based on div–curl lemma can be found in [6].

4.2.3. Oscillations defect measure

Following [5], we introduce the *oscillations defect measure*

$$\mathbf{osc}_q[\varrho_\varepsilon \to \varrho](O) = \sup_{k\geqslant 1} \limsup_{\varepsilon\to 0} \int_O \left|T_k(\varrho_\varepsilon) - T_k(\varrho)\right|^q \,\mathrm{d}x\,\mathrm{d}t \tag{4.19}$$

for any subset $O \subset Q^f$, and any weakly converging sequence $\varrho_\varepsilon \to \varrho$.

Rewriting conveniently the expression on the right-hand side of (4.18), we obtain

$$\mathbf{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho](O) \leqslant c\big(|O|\big) \tag{4.20}$$

for any compact $O \subset Q^f$, see [5]. However, as the constant c depends only on the Lebesgue measure of O, we obtain that (4.20) holds on in the whole set Q^f. Finally, by virtue of (4.16), we conclude that

$$\mathbf{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho]\big((0,T)\times D\big) \leqslant c. \tag{4.21}$$

As shown in [5, Proposition 7.1], relation (4.21), together with the uniform bounds (4.1)–(4.5), imply that the limit ϱ, $\mathbf{u}$ satisfy the renormalized equation (2.3) in $(0,T)\times D$.

4.2.4. Strong convergence of the density in Q^f

Since the limit functions ϱ, $\mathbf{u}$ satisfy the renormalized equation (2.3) in $(0,T)\times D$, we may infer that

$$\begin{aligned}
&\int_D \big(\overline{L_k(\varrho)} - L_k(\varrho)\big)(\tau)\,\mathrm{d}x + \int_0^\tau\!\!\int_D \big(\overline{T_k(\varrho)\operatorname{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_D \big(\overline{L_k(\varrho_0)} - L_k(\varrho_0)\big)\,\mathrm{d}x + \int_0^\tau\!\!\int_D \big(T_k(\varrho)\operatorname{div}_x\mathbf{u} - \overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u}\big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.22}$$

for any $\tau \in (0,T)$, where we have set

$$L_k(\varrho) = \varrho\int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z.$$

Now observe, by virtue of (4.16), (4.18), that

$$\overline{T_k(\varrho)\operatorname{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u} \geqslant 0,$$

while hypothesis (3.4) implies that

$$\int_D \big(\overline{L_k(\varrho_0)} - L_k(\varrho_0)\big)\,\mathrm{d}x = 0.$$

Finally, it follows from (4.21) that

$$\int_0^\tau\!\!\int_D \big(T_k(\varrho)\operatorname{div}_x\mathbf{u} - \overline{T_k(\varrho)}\operatorname{div}_x\mathbf{u}\big)\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } k\to\infty$$

for any τ. Letting $k \to \infty$ in (4.22), we conclude that

$$\int_D \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho)\right)(\tau)\,\mathrm{d}x = 0 \quad \text{for any } \tau \geqslant 0, \tag{4.23}$$

which implies strong convergence $\varrho_\varepsilon - \varrho$ in $L^q((0,T) \times D)$ for any $1 \leqslant q < \gamma$.

Theorem 3.1 has been proved.

5. Concluding remarks

As already mentioned in the introductory part, optimal regularity assumptions on the velocity field $\mathbf{v}_s$ were not an issue in the present paper. It is easy to check that the proof of convergence remains basically the same provided $\mathbf{v}_s$ is regular enough to keep valid the energy estimates resulting from (2.7), and to ensure the existence of characteristics, in particular, $\mathrm{div}_x \mathbf{v}_s \in L^1(0,T; L^\infty(D))$. In such a case, however, the pressure estimates deduced in Section 4.1 may not hold, in general, and the convergence of the pressure can be therefore established only locally in Q^f. For more recent results on transport equations with irregular velocity fields see Ambrosio [1]. For related results on the *incompressible* Navier–Stokes system see [14].

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