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Krein-like extensions and the lower boundedness problem for elliptic operators

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For selfadjoint extensions $\widetilde{A}$ of a symmetric densely defined positive operator $A_{\min}$, the lower boundedness problem is the question of whether $\widetilde{A}$ is lower bounded *if and only if* an associated operator T in abstract boundary spaces is lower bounded. It holds when the Friedrichs extension A_γ has compact inverse (Grubb, 1974, also Gorbachuk and Mikhailets, 1976); this applies to elliptic operators A on bounded domains.
For exterior domains, A_γ^{-1} is not compact, and whereas the lower bounds satisfy $m(T) \geqslant m(\widetilde{A})$, the implication of lower boundedness from T to $\widetilde{A}$ has only been known when $m(T) > -m(A_\gamma)$. We now show it for general T.
The operator A_a corresponding to $T = aI$, generalizing the Krein–von Neumann extension A_0, appears here; its possible lower boundedness for all real a is decisive. We study this Krein-like extension, showing for bounded domains that the discrete eigenvalues satisfy $N_+(t; A_a) = c_A t^{n/2m} + O(t^{(n-1+\varepsilon)/2m})$ for $t \to \infty$.

1. Introduction

The study of extensions of a symmetric operator (or a dual pair of operators) in a Hilbert space has a long history, with prominent contributions from J. von Neumann in 1929 [50], K. Friedrichs, 1934 [19], M.G. Krein, 1947 [43], M.I. Vishik, 1952 [55], M.S. Birman, 1956 [9] and others. The present author made a number of contributions in 1968–1974 [25–28], completing the preceding theories and working out applications to elliptic boundary value problems, fully for bounded domains; further developments are found in [30,31].

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At the same time there was another, separate development of abstract extension theories, where the operator concept gradually began to be replaced by the concept of relations. This development has been aimed primarily towards applications to ODE, however including operator-valued such equations and Schrödinger operators on $\mathbb{R}^n$; keywords in this connection are: Boundary triples theory, Weyl–Titchmarsh m-functions and Krein resolvent formulas, cf. e.g. Kočubeĭ [40], Vainerman [54], Lyantze and Storozh [45], Gorbachuk and Gorbachuk [24], Derkach and Malamud [17], Arlinskii [5], Malamud and Mogilevskii [47], Brüning, Geyler and Pankrashkin [16], and their references. In recent years there have also been applications to elliptic boundary value problems, cf. e.g. Amrein and Pearson [4], Behrndt and Langer [8], Ryzhov [53], Brown, Marletta, Naboko and Wood [15], Gesztesy and Mitrea [20], and their references.

The connection between the two lines of extension theories has been clarified in a recent work of Brown, Grubb and Wood [14]. Further developments for nonsmooth domains are found in [34], Posilicano and Raimondi [52], Gesztesy and Mitrea [21], Abels, Grubb and Wood [1].

There still remain some hitherto unsolved questions, for example concerning operators over exterior (unbounded) sets, and various questions in spectral theory.

Meanwhile, there have also been developed powerful tools for PDE in microlocal analysis, beginning with pseudodifferential operators (ψdo's) and, of relevance here, going on to pseudodifferential boundary operators (ψdbo's) with or without parameters. In a modern treatment it is natural to draw on such techniques when they can be applied efficiently to solve the problems. Indeed it is the case for the problems treated in the present paper.

1.1. Lower boundedness

In the study of realizations $\widetilde{A}$ of a strongly elliptic $2m$-order differential operator A on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, it has been known since 1974 that the realization is lower bounded if and only if a certain operator T determining its boundary condition is lower bounded. (See Grubb [28]; an announcement for the symmetric case was also given by Gorbachuk and Mikhailets [23].) This proof uses the fact that the inverse of the Dirichlet realization A_γ (the Friedrichs extension [19]) is compact. It is a result in functional analysis of operators in Hilbert space, and in [28] it is primarily shown in the abstract setting of closed extensions of dual pairs of lower bounded operators $A_{\min}, A'_{\min}$ with $A_{\min} \subset A_{\max} = (A'_{\min})^*$, as developed in [25]. Then it is applied to the study of general normal boundary conditions for strongly elliptic systems on compact manifolds with boundary. A further analysis of the lower boundedness problem was given in Derkach and Malamud [17].

Assuming only positivity of A_γ, one has rather easily that lower boundedness of $\widetilde{A}$ implies lower boundedness of T, and that a conclusion in the opposite direction holds if the lower bound of T is above minus the lower bound of A_γ; the hard question is to treat large negative lower bounds of T.

In the application of the abstract theory to the case where Ω is an exterior domain (the complement of a compact smooth set in $\mathbb{R}^n$) the Dirichlet solution operator A_γ^{-1} is not compact, and it has been an open problem whether one always could conclude from lower boundedness of T to lower boundedness of $\widetilde{A}$. We shall show in this paper that it is indeed so. The proof uses that the boundary is compact, and takes advantage of principles and results for pseudodifferential boundary operators [13,31,33].

Both symmetric and nonsymmetric cases were treated in [28], but the decisive step takes place in the symmetric setting where $A_{\min} = A'_{\min}$. Once it is established there, one can follow the method of [28] (the passage from Section 2 to Section 3 there) to extend the result to dual pairs. Therefore we shall here focus the attention on the symmetric case.

The abstract theory is recalled in Section 2, its implementation for exterior domains is explained in Section 3, and the lower boundedness result is shown in Section 4.

Section 4 ends with some (easier) observations on Gårding-type inequalities, that are not tied to bounded boundaries in the same way.

1.2. Krein-like extensions

In the treatment of these lower boundedness questions, a certain family of non-elliptic realizations comes naturally into the picture. They are generalizations of the Krein–von Neumann extension [50, 43] that we shall here denote A_0; it is the restriction of $A_{\max}$ with domain $D(A_0) = D(A_{\min}) \dot{+} Z$ where $Z = \ker A_{\max}$, and has attracted much interest through the years, see e.g. the studies of its spectral properties by Alonso and Simon [2,3], Grubb [30], Ashbaugh, Gesztesy, Mitrea, Shterenberg and Teschl [6,7], with further references.

The larger family we shall consider (calling them Krein-like extensions) is the scale of selfadjoint operators A_a acting as $A_{\max}$ with domains

$$D(A_a) = \left\{u = v + aA_\gamma^{-1}z + z \;\middle|\; v \in D(A_{\min}),\ z \in Z\right\}, \tag{1.1}$$

for $a \in \mathbb{R}$. In the application to boundary value problems, they are determined by Neumann-type boundary conditions with pseudodifferential elements; however, they are non-elliptic and the domains contain L_2-functions that are not in H^s for any $s > 0$. For both interior and exterior domains, their lower boundedness is crucial for the general lower boundedness problem. Moreover, they play a role in a study [36] of perturbations of essential spectra.

In the case of a bounded domain, they will have the single point a as essential spectrum, and one can ask for the asymptotic behavior of the eigenvalue sequence converging to $+\infty$ that must exist. In the final Section 5, we deal with this question, showing that the number $N_+(t; A_a)$ of eigenvalues in $[r, t]$ (for some $r > a$) has the asymptotic behavior

$$N_+(t; A_a) - c_A t^{n/2m} = O\left(t^{(n-1+\varepsilon)/2m}\right) \quad \text{for } t \to \infty, \tag{1.2}$$

any $\varepsilon > 0$, with the same constant c_A as for the Dirichlet problem. Here we use results for singular Green operators obtained in [31]. We also show this estimate for A_0.

2. The abstract setting

We first recall how the general characterization of extensions is set up.

There is given a symmetric, closed, densely defined operator $A_{\min}$ in a complex Hilbert space H, assumed injective with closed range. Moreover, there is given an invertible selfadjoint extension A_γ, such that we have

$$A_{\min} \subset A_\gamma \subset A_{\max} \equiv (A_{\min})^*.$$

Let

$$\mathcal{M} = \{\widetilde{A} \mid A_{\min} \subset \widetilde{A} \subset A_{\max}\}.$$

To simplify notation, we write $\widetilde{A}u$ as Au, any $\widetilde{A} \in \mathcal{M}$. Since $A_{\min}$ has closed range, there is an orthogonal decomposition

$$H = R \oplus Z, \quad R = \operatorname{ran} A_{\min},\ Z = \ker A_{\max}. \tag{2.1}$$

When X is a closed subspace of H, we denote by $\operatorname{pr}_X u = u_X$ the orthogonal projection of u onto X.

The idempotent operators $\operatorname{pr}_\gamma = A_\gamma^{-1} A_{\max}$ and $\operatorname{pr}_\zeta = I - \operatorname{pr}_\gamma$ on $D(A_{\max})$ define a (non-orthogonal) decomposition of $D(A_{\max})$

$$D(A_{\max}) = D(A_\gamma) \dot{+} Z, \tag{2.2}$$

denoted $u = u_\gamma + u_\zeta = \operatorname{pr}_\gamma u + \operatorname{pr}_\zeta u$, which allows writing an "abstract Green's formula" for $u, v \in D(A_{\max})$:

$$(Au, v) - (u, Av) = \big((Au)_Z, v_\zeta\big) - \big(u_\zeta, (Av)_Z\big). \tag{2.3}$$

On the basis of (2.3) one can establish a 1–1 correspondence ([25], also described in [35, Chapter 13]) between the closed operators $\widetilde{A}$ in $\mathcal{M}$ and the closed, densely defined operators $T : V \to W$, where V and W are closed subspaces of Z, such that

$$\text{graph of } T = \big\{\big(\operatorname{pr}_\zeta u, (Au)_W\big) \bigm| u \in D(\widetilde{A})\big\}. \tag{2.4}$$

Here $V = \overline{\operatorname{pr}_\zeta D(\widetilde{A})}$ and $W = \overline{\operatorname{pr}_\zeta D(\widetilde{A}^*)}$. For a given operator $T : V \to W$, one finds the corresponding operator $\widetilde{A}$ from the formula

$$D(\widetilde{A}) = \big\{u \in D(A_{\max}) \bigm| \operatorname{pr}_\zeta u \in D(T),\ (Au)_W = T \operatorname{pr}_\zeta u\big\}. \tag{2.5}$$

In this correspondence, one has moreover:

(a) $\widetilde{A}^*$ corresponds analogously to $T^* : W \to V$. In particular, $\widetilde{A}$ is selfadjoint if and only if $V = W$ and $T = T^*$.
(b) $\widetilde{A}$ is symmetric if and only if $V \subset W$ and T is symmetric.
(c) $\ker \widetilde{A} = \ker T$; $\operatorname{ran} \widetilde{A} = \operatorname{ran} T + (H \ominus W)$.
(d) When $\widetilde{A}$ is bijective,

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V T^{-1} \operatorname{pr}_W. \tag{2.6}$$

Here i_V denotes the injection of V into H.

The analysis is related to that of Vishik [55], except that he sets the $\widetilde{A}$ in relation to operators over the nullspace going in the opposite direction of our T's and in this context focuses on those $\widetilde{A}$'s that have closed range. Our analysis covers all closed $\widetilde{A}$.

We recall furthermore that in view of (2.1), the decomposition (2.2) has the refinement

$$D(A_{\max}) = D(A_{\min}) \dotplus A_\gamma^{-1} Z \dotplus Z; \tag{2.7}$$

it allows to show that when $\widetilde{A}$ corresponds to T, then

$$D(\widetilde{A}) = \big\{u = v + A_\gamma^{-1}(Tz + f) + z \bigm| v \in D(A_{\min}),\ z \in D(T),\ f \in Z \ominus W\big\}. \tag{2.8}$$

The lower bound of an operator P is denoted by $m(P)$:

$$m(P) = \inf\big\{\operatorname{Re}(Pu, u) \bigm| u \in D(P),\ \|u\| = 1\big\} \geqslant -\infty; \tag{2.9}$$

when it is finite, P is said to be lower bounded.

Assume now moreover that $A_{\min}$ has a positive lower bound and that A_γ is the Friedrichs extension of $A_{\min}$; it has the same lower bound as $A_{\min}$. Then we have in addition the following facts, shown in [26] (also described in [35]):

(e) If $m(\widetilde{A}) > -\infty$, then $V \subset W$ and $m(T) \geqslant m(\widetilde{A})$.
(f) If $V \subset W$ and $m(T) > -m(A_\gamma)$, then $m(\widetilde{A}) \geqslant m(T)m(A_\gamma)/(m(T) + m(A_\gamma))$.

The last rule (shown by Birman [9] for selfadjoint operators $\widetilde{A}$) is based on the fact that when $V \subset W$,

$$(Au, v) = (Au_\gamma, v_\gamma) + (Tu_\zeta, v_\zeta), \quad \text{for } u, v \in D(\widetilde{A}). \tag{2.10}$$

The rule (f) does not cover low values of $m(T)$, but this was overcome in [28] when A_γ^{-1} is compact. Here the situation was set in relation to the situation where the operators are shifted by subtraction of a spectral parameter $\mu \in \varrho(A_\gamma)$ (the resolvent set), i.e., all realizations $\widetilde{A}$ are replaced by $\widetilde{A} - \mu$. Here we define

$$Z_\mu = \ker(A_{\max} - \mu), \qquad \mathrm{pr}_\gamma^\mu = (A_\gamma - \mu)^{-1}(A_{\max} - \mu), \qquad \mathrm{pr}_\zeta^\mu = I - \mathrm{pr}_\gamma^\mu, \tag{2.11}$$

which gives a decomposition

$$D(A_{\max}) = D(A_\gamma) \dotplus Z_\mu \tag{2.12}$$

(note that $D(\widetilde{A} - \mu) = D(\widetilde{A})$, $D(A_{\max} - \mu) = D(A_{\max})$, $D(A_\gamma - \mu) = D(A_\gamma)$). When μ is real we have, in the same way as in the case we started out with, a 1–1 correspondence between operators $\widetilde{A} - \mu$ and operators $T^\mu : V_\mu \to W_\mu$; here $V_\mu = \overline{\mathrm{pr}_\zeta^\mu D(\widetilde{A})}$ and $W_\mu = \overline{\mathrm{pr}_\zeta^\mu D(\widetilde{A}^*)}$, and the properties (a)–(d) have analogues for this correspondence. In particular, (d) gives a Krein-type resolvent formula when $\mu \in \varrho(\widetilde{A})$,

$$(\widetilde{A} - \mu)^{-1} = (A_\gamma - \mu)^{-1} + \mathrm{i}_{V_\mu}\big(T^\mu\big)^{-1}\mathrm{pr}_{W_\mu};$$

there is much more on this in [14].

When $\mu < m(A_\gamma)$, $A_\gamma - \mu$ has positive lower bound $m(A_\gamma) - \mu$, so also the properties (e) and (f) have analogues in the new correspondence. In particular, (f) takes the form:

(g) If $V_\mu \subset W_\mu$ and $m(T^\mu) > -(m(A_\gamma) - \mu)$, then

$$m(\widetilde{A}) - \mu \geqslant m\big(T^\mu\big)\big(m(A_\gamma) - \mu\big)/\big(m\big(T^\mu\big) + m(A_\gamma) - \mu\big). \tag{2.13}$$

(Here $V \subset W$ implies $V_\mu \subset W_\mu$, see also Proposition 2.1 below.) Note the special case:

(h) If $V_\mu \subset W_\mu$ and $m(T^\mu) \geqslant 0$, then $m(\widetilde{A}) \geqslant \mu$.

Hereby the question of whether $\widetilde{A}$ is lower bounded when T is so, is turned into the question of whether $m(T^\mu)$ becomes $\geqslant 0$ when $\mu \to -\infty$.

Define

$$E^\mu = A_{\max}(A_\gamma - \mu)^{-1} = I + \mu(A_\gamma - \mu)^{-1}; \tag{2.14}$$

it is a homeomorphism in H such that

$$F^\mu = (A_{\max} - \mu)A_\gamma^{-1} = I - \mu A_\gamma^{-1} \text{ is the inverse of } E^\mu. \tag{2.15}$$

Moreover, E^μ maps Z homeomorphically onto Z_μ (with inverse F^μ). Details are given in [28, Section 2], where the following is shown:

Proposition 2.1. *Let $\mu < m(A_\gamma)$. Define the operator G^μ in Z by*

$$G^\mu = -\mu \operatorname{pr}_Z E^\mu \mathrm{i}_Z, \tag{2.16}$$

it is a bounded selfadjoint operator in Z.

Let $\widetilde{A}$ be a closed operator in $\mathcal{M}$, corresponding to $T : V \to W$. Then $\widetilde{A} - \mu$ corresponds to $T^\mu : V_\mu \to W_\mu$, determined by

$$\begin{gathered} V_\mu = E^\mu V, \qquad W_\mu = E^\mu W, \qquad D\big(T^\mu\big) = E^\mu D(T), \\ \big(T^\mu E^\mu v, E^\mu w\big) = (Tv, w) + \big(G^\mu v, w\big) \quad \textit{for } v \in D(T),\ w \in W. \end{gathered} \tag{2.17}$$

Note that in particular, if $V \subset W$,

$$\operatorname{Re}\big(T^\mu E^\mu v, E^\mu v\big) = \operatorname{Re}(Tv, v) + \big(G^\mu v, v\big) \quad \text{for } v \in D(T).$$

One then observes:

Proposition 2.2. *The following statements* (i) *and* (ii) *are equivalent*:

(i) *For any choice of $V \subset W$ and any lower bounded, closed densely defined operator $T : V \to W$ there is a $\mu < m(A_\gamma)$ such that $m(T^\mu) \geqslant 0$.*
(ii) *For any $t \geqslant 0$ there is a $\mu < m(A_\gamma)$ such that $m(G^\mu) \geqslant t$.*

Proof. Let (ii) hold, and consider a lower bounded operator $T : V \to W$; $V \subset W$. Choose μ such that $m(G^\mu) \geqslant \max\{-m(T), 0\}$. Then for $v \in D(T)$,

$$\operatorname{Re}\big(T^\mu E^\mu v, E^\mu v\big) = \operatorname{Re}(Tv, v) + \big(G^\mu v, v\big) \geqslant m(T)\|v\|^2 + m\big(G^\mu\big)\|v\|^2 \geqslant 0.$$

This shows (i).

Conversely, let (i) hold. It holds in particular for the (selfadjoint) choices $T = aI$ on Z with $a \in \mathbb{R}$; let T_a^μ denote the corresponding operator on Z_μ. By hypothesis there is a μ such that $m(T_a^\mu) \geqslant 0$. Then

$$0 \leqslant \big(T_a^\mu E^\mu v, E^\mu v\big) = (av, v) + \big(G^\mu v, v\big), \tag{2.18}$$

and hence

$$\big(G^\mu v, v\big) \geqslant -a\|v\|^2, \quad \text{for all } v \in Z.$$

To see that (ii) holds for a given $t \geqslant 0$, we just have to take $a = -t$. $\square$

Note that the proof involves the special choice $T = aI$ on Z, corresponding to the Krein-like extension A_a, cf. (1.1), (2.8). There is a formulation in terms of those operators, that can immediately be included:

Proposition 2.3. *The two statements* (i) *and* (ii) *in Proposition* 2.2 *are also equivalent with the statement*:

(iii) *For any $a \in \mathbb{R}$, the Krein-like extension A_a, corresponding to the choice $T = aI$ on Z, is lower bounded.*

Proof. The proof of Proposition 2.2 shows that when (i) holds, its application to the special cases $T = aI$ on Z gives that $m(T_a^\mu) \geqslant 0$ for $-\mu$ sufficiently large. By the rule (h), $m(A_a)$ then has lower bound $\geqslant \mu$. Since a was arbitrary, we conclude that A_a is lower bounded for any $a \in \mathbb{R}$; hence (iii) holds.

Conversely, when (iii) holds, it assures by the rule (e) applied to $A_a - \mu$, that for any a, $m(T_a^\mu) \geqslant 0$ for $-\mu$ sufficiently large. This is used in the proof of Proposition 2.2 to conclude that $m(G^\mu)$ is then $\geqslant -a$, implying (ii). $\square$

Then [28, Thm. 2.12] showed the validity of (i)–(iii) in an important case:

Theorem 2.4. *When A_γ^{-1} is a compact operator in H, then*

$$m\big(G^\mu\big) \to \infty \quad \text{for } \mu \to -\infty. \tag{2.19}$$

Consequently, (i), (ii) *and* (iii) *of Propositions* 2.2 *and* 2.3 *are valid*; *and* (f) *can be supplemented with*

(f′) *If $V \subset W$ and $m(T) > -\infty$, then $m(\widetilde{A}) > -\infty$.*

Also estimates of the type

$$\operatorname{Re}(Au, u) \geqslant c\|u\|_{\mathcal{K}}^2 - k\|u\|_H^2, \quad u \in D(\widetilde{A}), \tag{2.20}$$

were characterized in [28], when $D(A_\gamma^{1/2}) \subset \mathcal{K} \subset H$.

The proof of Theorem 2.4 involves a closer study of the Krein-like realizations A_a. We return to a further analysis of them in Section 5.

We shall now explain how the general set-up is applied to boundary value problems. Here we focus on exterior problems since problems for bounded domains were amply treated in [25–28].

3. The implementation for exterior boundary value problems

When Ω is a smooth open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$, we use the standard L_2-Sobolev spaces, with the following notation: $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) has the norm $\|v\|_s = \|\mathcal{F}^{-1}(\langle\xi\rangle^s \mathcal{F}v)\|_{L_2(\mathbb{R}^n)}$; here $\mathcal{F}$ is the Fourier transform and $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. Next, $H^s(\Omega) = r_\Omega H^s(\mathbb{R}^n)$ where r_Ω restricts to Ω, provided with the norm $\|u\|_s = \inf\{\|v\|_s \mid v \in H^s(\mathbb{R}^n),\ u = r_\Omega v\}$. Moreover, $H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}$; closed subspace of $H^s(\mathbb{R}^n)$. Spaces over the boundary, $H^s(\Sigma)$, are defined by local coordinates from $H^s(\mathbb{R}^{n-1})$, $s \in \mathbb{R}$. (There are many equally justified equivalent choices of norms there; one can choose a particular norm when convenient.) When $s > 0$, there are dense continuous embeddings

$$H^s(\Sigma) \subset L_2(\Sigma) \subset H^{-s}(\Sigma),$$

and we use the customary identification of $H^{-s}(\Sigma)$ with the antidual space of $H^s(\Sigma)$ (the space of antilinear, i.e., conjugate linear, functionals), such that the duality $(\varphi, \psi)_{-s,s}$ coincides with the $L_2(\Sigma)$-scalar product when the elements lie there.

Detailed explanations are found in many books, e.g. [44,38,35].

In the following, Ω is primarily considered to be an exterior domain, i.e., the complement of $\overline{\Omega}_0$, where Ω_0 is a nonempty smooth bounded subset of $\mathbb{R}^n$. However, the explanations in the following work equally well for interior domains and for admissible manifolds in the sense introduced in the book [33]; this includes smooth domains in $\mathbb{R}^n$ that outside of a large ball have the form of a halfspace $\mathbb{R}_+^n$ or a cone.

Let A be a symmetric elliptic operator of order $2m$ on Ω,

$$Au = \sum_{|\alpha|,|\beta|\leqslant m} D^\alpha\big(a_{\alpha,\beta}(x)D^\beta u(x)\big), \quad \overline{a_{\beta,\alpha}} = a_{\alpha,\beta}, \tag{3.1}$$

with complex coefficients $a_{\alpha,\beta}$ in $C_b^\infty(\overline{\Omega})$; here $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$, and $C_b^\infty(\overline{\Omega})$ denotes the space of C^∞-functions that are bounded with bounded derivatives of all orders. The principal symbol $a^0(x,\xi) = \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}\xi^{\alpha+\beta}$ is real. A is assumed to be uniformly strongly elliptic, i.e., a^0 satisfies, with $c_1 > 0$,

$$a^0(x,\xi) \geqslant c_1|\xi|^{2m}, \quad \text{for } x \in \overline{\Omega},\ \xi \in \mathbb{R}^n. \tag{3.2}$$

A typical case of such an operator when $m = 1$ is of the form

$$A = -\sum_{j,k=1}^n \partial_j a_{jk}(x)\partial_k + a_0(x) = \sum_{j,k=1}^n D_j a_{jk}(x) D_k + a_0(x), \tag{3.3}$$

with real coefficients satisfying $a_{jk} = a_{kj}$ and

$$\sum_{j,k} a_{jk}(x)\xi_j\xi_k \geqslant c_1|\xi|^2, \tag{3.4}$$

with $c_1 > 0$.

We let $H = L_2(\Omega)$, and as $A_{\max}$ and $A_{\min}$ we take the operators acting like A in $L_2(\Omega)$ and defined by

$$\begin{gathered} D(A_{\max}) = \big\{u \in L_2(\Omega) \bigm| Au \in L_2(\Omega) \text{ in the distribution sense}\big\}, \\ A_{\min} = \text{the closure of } A|_{C_0^\infty(\Omega)}; \end{gathered} \tag{3.5}$$

because of the symmetry, $A_{\max}$ and $A_{\min}$ are adjoints of one another. It is well known (and is accounted for e.g. in [36]) that the strong ellipticity and boundedness estimates imply that the graph-norm $(\|Au\|^2 + \|u\|^2)^{\frac12}$ and the H^{2m}-norm are equivalent on $H_0^{2m}(\Omega)$, so

$$D(A_{\min}) = H_0^{2m}(\Omega). \tag{3.6}$$

Moreover, when A_γ is taken as the Dirichlet realization of A, i.e., the restriction of $A_{\max}$ with domain $D(A_{\max}) \cap H_0^m(\Omega)$, then

$$D(A_\gamma) = H^{2m}(\Omega) \cap H_0^m(\Omega); \tag{3.7}$$

and A_γ coincides with the operator defined by variational theory (the Lax–Milgram lemma) applied to the sesquilinear form with domain $H_0^m(\Omega)$,

$$a(u,v) = \sum_{|\alpha|,|\beta|\leqslant m} \big(a_{\alpha,\beta}D^\beta u, D^\alpha v\big), \tag{3.8}$$

thus A_γ is selfadjoint.

We can assume that a large enough constant has been added to A such that

$$a(u,u) \geqslant c_0\|u\|^2, \quad \text{for } u \in H_0^m(\Omega); \tag{3.9}$$

with $c_0 > 0$; then c_0 is also a lower bound for $A_{\min}$ and A_γ, and A_γ is invertible.

The set-up of Section 2 applies readily to these choices of $A_{\min}$, $A_{\max}$ and A_γ; the operators $\widetilde{A} \in \mathcal{M}$ are called realizations of A. We shall now recall how the correspondence between a general $\widetilde{A}$ and an operator $T : V \to W$ is turned into a characterization of $\widetilde{A}$ by a boundary condition.

First we note that there is a Green's formula for A, valid for $u, v \in H^{2m}(\Omega)$:

$$(Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} = (\chi u, \gamma v)_{L_2(\Sigma)^m} - \big(\gamma u, \chi' v\big)_{L_2(\Sigma)^m}. \tag{3.10}$$

Here, with $\gamma_j u = (\vec{n} \cdot D)^j u|_\Sigma$, $\vec{n}$ denoting the interior normal to the boundary,

$$\begin{aligned} &\gamma u = \{\gamma_0 u, \ldots, \gamma_{m-1} u\}, \text{ the Dirichlet data,} \\ &\nu u = \{\gamma_m u, \ldots, \gamma_{2m-1} u\}, \text{ the Neumann data,} \\ &\chi u = \mathcal{A}_{M_0 M_1} \nu u + \mathcal{S}\gamma u, \qquad \chi' u = -\mathcal{A}^*_{M_0 M_1} \nu u + \mathcal{S}'\gamma u, \text{ Neumann-type data;} \end{aligned} \tag{3.11}$$

where $\mathcal{A}_{M_0 M_1}$ is a certain skew-triangular invertible matrix of differential operators over Σ derived from A, and $\mathcal{S}$ and $\mathcal{S}'$ are suitable matrices of differential operators; cf. [44,27]. In the second-order case (3.3), one can take χ and χ' to be the conormal derivative ν_A at the boundary,

$$\nu_A u = \sum_{j,k} a_{jk} n_j \gamma_0 \partial_k u. \tag{3.12}$$

Occasionally in the following, we shall use the notation of the calculus of pseudodifferential boundary operators (ψdbo's), as initiated by Boutet de Monvel [13] and developed further in e.g. [31,33]; there is also a detailed introduction in [35]. The calculus defines Poisson operators K (from Σ to Ω), pseudodifferential trace operators T (from Ω to Σ), singular Green operators G on Ω (including operators of the form KT) and truncated pseudodifferential operators on Ω, and their composition rules, etc. Since we shall in the present paper only use final theorems on such operators, we refrain from taking space up here with a detailed introduction.

Let us introduce the notation

$$\mathcal{H}^s = \prod_{0 \leqslant j < m} H^{s-j-\frac{1}{2}}(\Sigma), \qquad \widetilde{\mathcal{H}}^s = \prod_{0 \leqslant j < m} H^{s-2m+j+\frac{1}{2}}(\Sigma); \tag{3.13}$$

here $(\mathcal{H}^s)^* = \widetilde{\mathcal{H}}^{2m-s}$, $(\widetilde{\mathcal{H}}^s)^* = \mathcal{H}^{2m-s}$, the dualities denoted

$$\begin{aligned} &(\varphi, \psi)_{\mathcal{H}^{s*}, \mathcal{H}^s} \quad \text{or} \quad (\varphi, \psi)_{\{-s+j+\frac{1}{2}, s-j-\frac{1}{2}\}}, \\ &(\eta, \zeta)_{\widetilde{\mathcal{H}}^{s*}, \widetilde{\mathcal{H}}^s} \quad \text{or} \quad (\eta, \zeta)_{\{2m-s-j-\frac{1}{2}, s-2m+j+\frac{1}{2}\}}. \end{aligned}$$

These dualities are consistent with the scalar product in $L_2(\Sigma)^m$ when the elements lie there. Note that in particular,

$$\mathcal{H}^0 = H^{-\frac{1}{2}}(\Sigma), \qquad \widetilde{\mathcal{H}}^0 = H^{-\frac{3}{2}}(\Sigma), \qquad \big(\mathcal{H}^0\big)^* = H^{\frac{1}{2}}(\Sigma), \quad \text{when } m = 1. \tag{3.14}$$

Denote $D^s_A(\Omega)=\{u\in H^s(\Omega)\mid Au\in L_2(\Omega)\}$, with norm $(\|u\|_s^2+\|Au\|_0^2)^{\frac12}$. It is seen as in [44] that $C^\infty_{(0)}(\overline{\Omega})=r_\Omega C_0^\infty(\mathbb{R}^n)$ is dense in $D^s_A(\overline{\Omega})$, and it follows from [44] that γ, ν, χ and χ' extend to continuous maps:

$$\gamma : D^s_A(\Omega)\to \mathcal{H}^s,\qquad \nu : D^s_A(\Omega)\to \mathcal{H}^{s-m},\qquad \chi,\chi' : D^s_A(\Omega)\to \widetilde{\mathcal{H}}^s,\quad \text{for all } s\in\mathbb{R}. \tag{3.15}$$

(The mapping properties are shown in [44] for bounded domains, but this implies (3.15) when the properties are applied to $\Omega\cap B(0,R)$ for a sphere $B(0,R)$ with R so large that Σ is contained in the interior.) Moreover, Green's formula continues to hold for these extensions, when $u\in H^{2m}(\Omega)$, $v\in D(A_{\max})$:

$$(Au,v)-(u,Av)=(\chi u,\gamma v)_{\{j+\frac12,-j-\frac12\}}-\big(\gamma u,\chi' v\big)_{\{2m-j-\frac12,-2m+j+\frac12\}}. \tag{3.16}$$

Using that A_γ is invertible, one can moreover show that the nonhomogeneous Dirichlet problem is uniquely solvable: The mapping

$$\mathcal{A}_\gamma=\begin{pmatrix}A\\ \gamma\end{pmatrix} : H^s(\Omega)\to \begin{matrix}H^{s-2m}(\Omega)\\ \times\\ \mathcal{H}^s\end{matrix} \tag{3.17}$$

has for $s>m-\frac12$ the solution operator, continuous in the opposite direction,

$$\mathcal{A}_\gamma^{-1}=(R_\gamma\quad K_\gamma); \tag{3.18}$$

here R_γ is for $s=2m$ the inverse of the Dirichlet realization A_γ, and K_γ is the Poisson operator solving the Dirichlet problem $Au=0, \gamma u=\varphi$. More documentation is given in [36]. Denoting $Z^s_A(\Omega)=\{u\in H^s(\Omega)\mid Au=0\}$ (with s-norm), we have in particular the mapping property for $s>m-\frac12$:

$$\gamma : Z^s_A(\Omega)\overset{\sim}{\to}\mathcal{H}^s, \tag{3.19}$$

it extends to all $s\in\mathbb{R}$. (The extension of the inverse mapping follows from a general rule for Poisson operators; the direct mapping is treated as shown in [44], one may also consult the discussion in [35, Chapter 11].)

Denote by γ_Z the operator acting like γ with precise domain and range

$$\gamma_Z : Z\overset{\sim}{\to}\prod_{j<m}H^{-j-\frac12}(\Sigma)=\mathcal{H}^0; \tag{3.20}$$

it has an inverse γ_Z^{-1} and an adjoint γ_Z^* that map as follows:

$$\gamma_Z^{-1} : \mathcal{H}^0\overset{\sim}{\to}Z,\qquad \gamma_Z^* : \big(\mathcal{H}^0\big)^*\overset{\sim}{\to}Z. \tag{3.21}$$

Both operators lead to Poisson operators in the ψdbo calculus when composed with i_Z; here $\mathrm{i}_Z\gamma_Z^{-1}$ equals K_γ. In the case $m=1$,

$$\gamma_Z : Z\overset{\sim}{\to}H^{-\frac12}(\Sigma),\qquad \gamma_Z^{-1} : H^{-\frac12}(\Sigma)\overset{\sim}{\to}Z,\qquad \gamma_Z^* : H^{\frac12}(\Sigma)\overset{\sim}{\to}Z.$$

For the study of general realizations $\widetilde{A}$ of A, the homeomorphism (3.20) allows us to translate the characterization in terms of operators $T : V \to W$ in Section 2 into a characterization in terms of operators L over the boundary.

For $V, W \subset Z$, let $X = \gamma V$, $Y = \gamma W$, with the notation for the restrictions of γ:

$$\gamma_V : V \overset{\sim}{\to} X, \qquad \gamma_W : W \overset{\sim}{\to} Y. \tag{3.22}$$

The map $\gamma_V : V \overset{\sim}{\to} X$ has the adjoint $\gamma_V^* : X^* \overset{\sim}{\to} V$. Here X^* denotes the antidual space of X, again with a duality coinciding with the scalar product in $L_2(\Sigma)^m$ when applied to elements that also belong to $L_2(\Sigma)^m$. The duality is written $(\psi, \varphi)_{X^*,X}$. We also write $\overline{(\psi, \varphi)_{X^*,X}} = (\varphi, \psi)_{X,X^*}$. Similar conventions are applied to Y.

When A is replaced by $A - \mu$ for $\mu < m(A_\gamma)$, there is a similar notation where Z, V and W are replaced by Z_μ, V_μ, W_μ. Since $\gamma E^\mu z = \gamma z$ (cf. (2.14)), we have that γ defines mappings

$$\gamma_{V_\mu} : V_\mu \overset{\sim}{\to} X, \qquad \gamma_{W_\mu} : W_\mu \overset{\sim}{\to} Y, \tag{3.23}$$

with *the same range spaces X and Y as when $\mu = 0$.*

We denote $K_{\gamma,X} = \mathrm{i}_V \gamma_V^{-1} : X \to V \subset H$, it is a Poisson operator when X is a product of Sobolev spaces.

Now a given $T : V \to W$ is carried over to a closed, densely defined operator $L : X \to Y^*$ by the definition

$$L = \left(\gamma_W^{-1}\right)^* T \gamma_V^{-1}, \qquad D(L) = \gamma_V D(T); \tag{3.24}$$

it is expressed in the diagram

$$\begin{tikzcd} V \arrow[r, "\sim", "\gamma_V"'] \arrow[d, "T"'] & X \arrow[d, "L"] \\ W \arrow[r, "\sim", "(\gamma_W^{-1})^*"'] & Y^*. \end{tikzcd} \tag{3.25}$$

Observe that when $v \in D(T)$ and $w \in W$ are carried over to $\varphi = \gamma_V v$ and $\psi = \gamma_W w$, then $L\varphi = (\gamma_W^*)^{-1} T v$ satisfies

$$(Tv, w) = (L\varphi, \psi)_{Y^*,Y}. \tag{3.26}$$

For the question of semiboundedness we note that when $V \subset W$, hence $X \subset Y$, then the functionals in Y^* act on the elements of X. Then when $v \in D(T) \subset V \subset W$, so that $\gamma_V v = \varphi \in D(L) \subset X \subset Y$, we may write

$$(Tv, v) = (L\varphi, \varphi)_{Y^*,Y}. \tag{3.27}$$

The L_2-norm of v is equivalent with the $\mathcal{H}^0$-norm of φ;

$$\|v\| \leqslant c_1 \|\varphi\|_{\{-j-\frac{1}{2}\}} \leqslant c_2 \|v\|, \quad \varphi = \gamma_Z v, \tag{3.28}$$

for any choice of the equivalent norms (denoted $\|\varphi\|_{\mathcal{H}^0}$ or $\|\varphi\|_{\{-j-\frac{1}{2}\}}$) on the boundary Sobolev spaces. (One could also fix the norm, e.g. by letting γ_Z be an isometry.) Then

$$\operatorname{Re}(Tv, v) \geqslant c\|v\|^2, \quad v \in D(T), \tag{3.29}$$

holds for some $c \in \mathbb{R}$ if and only if

$$\operatorname{Re}(L\varphi, \varphi)_{Y^*,Y} \geqslant c'\|\varphi\|^2_{\{-j-\frac{1}{2}\}}, \quad \varphi \in D(L), \tag{3.30}$$

holds for some $c' \in \mathbb{R}$, and here c and c' are simultaneously > 0 or $\geqslant 0$. (If we fix the norm such that γ_Z is an isometry, $c = c'$.)

The interpretation of the condition in (2.5) as a boundary condition has been explained in several places, beginning with [25], so we can do it rapidly here. Define the Dirichlet-to-Neumann operator

$$P^0_{\gamma,\chi} = \chi\gamma_Z^{-1} = \chi K_\gamma : \mathcal{H}^0 \to \widetilde{\mathcal{H}}^0; \tag{3.31}$$

it is in fact continuous from $\mathcal{H}^s$ to $\widetilde{\mathcal{H}}^s$ for all $s \in \mathbb{R}$ because of the mapping properties of χ and K_γ. It is a matrix-formed *pseudodifferential operator over Σ*; this was indicated as plausible in [25], and proved in detail in [27] on the basis of the work of Seeley on the Calderón projector. It also follows from the general ψdbo calculus. There is the analogous operator $P^0_{\gamma,\chi'}$, and when the construction is applied to $A - \mu$ instead of A we get the operator

$$P^\mu_{\gamma,\chi} = \chi\gamma_{Z_\mu}^{-1}. \tag{3.32}$$

For $m = 1$, these operators are of order 1, continuous from $H^{s-\frac{1}{2}}(\Sigma)$ to $H^{s-\frac{3}{2}}(\Sigma)$ for all s, and elliptic of order 1 when A and χ are chosen as in (3.3), (3.12). For higher m, the operators are *multi-order systems*, of the form $(P_{jk})_{0\leqslant j,k<m}$ with P_{jk} of order $2m - j - k - 1$ (continuous from $H^{s-k-\frac{1}{2}}(\Sigma)$ to $H^{s-2m+j+\frac{1}{2}}(\Sigma)$ for all s). Ellipticity is defined in relation to the multi-order. When $\mathcal{S} = 0$ in (3.11), $P^0_{\gamma,\chi}$ is elliptic, meaning that the matrix of principal symbols $\sigma_{2m-j-k-1}(P_{jk})(x', \xi')$ is regular for $\xi' \neq 0$. (This follows from the ellipticity of $P^0_{\gamma,\nu}$ shown in [27], see also [35, Chapter 11].)

We now define

$$\Gamma^0 = \chi - P^0_{\gamma,\chi}\gamma, \qquad \Gamma'^0 = \chi' - P^0_{\gamma,\chi'}\gamma, \tag{3.33}$$

also equal to $\chi A_\gamma^{-1}A_{\max}$ resp. $\chi' A_\gamma^{-1}A_{\max}$; they are trace operators in the ψdbo calculus, mapping $D(A_{\max})$ (with the graph-norm) continuously into $\widetilde{\mathcal{H}}^{2m} = (\mathcal{H}^0)^*$. They vanish on Z. With these operators there holds a modified Green's formula

$$(Au, v) - (u, Av) = \left(\Gamma^0 u, \gamma v\right)_{\{j+\frac{1}{2}, -j-\frac{1}{2}\}} - \left(\gamma u, \Gamma'^0 v\right)_{\{-j-\frac{1}{2}, j+\frac{1}{2}\}}, \tag{3.34}$$

valid for all $u, v \in D(A_{\max})$. In particular,

$$(Au, w) = \left(\Gamma^0 u, \gamma w\right)_{\{j+\frac{1}{2}, -j-\frac{1}{2}\}}, \quad \text{when } u \in D(A_{\max}),\ w \in Z. \tag{3.35}$$

When $\widetilde{A}$ corresponds to $T : V \to W$ and $L : X \to Y^*$, we can write

$$(Tu_\zeta, w) = \left(T\gamma_V^{-1}\gamma u, \gamma_W^{-1}\gamma w\right) = (L\gamma u, \gamma w)_{Y^*,Y}, \quad \text{all } u \in D(\widetilde{A}),\ w \in W. \tag{3.36}$$

The formula $(Au)_W = Tu_\zeta$ in (2.5) is then turned into

$$\big(\Gamma^0 u, \gamma w\big)_{\{j+\frac12,-j-\frac12\}} = (L\gamma u, \gamma w)_{Y^*,Y}, \quad \text{all } w \in W,$$

or, since γ maps W bijectively onto Y,

$$\big(\Gamma^0 u, \varphi\big)_{\{j+\frac12,-j-\frac12\}} = (L\gamma u, \varphi)_{Y^*,Y} \quad \text{for all } \varphi \in Y. \tag{3.37}$$

To simplify the notation, note that the injection $\mathrm{i}_Y : Y \to \mathcal{H}^0$ has as adjoint the mapping $\mathrm{i}_Y^* : (\mathcal{H}^0)^* \to Y^*$ that sends a functional ψ on $\mathcal{H}^0$ over into a functional $\mathrm{i}_Y^*\psi$ on Y by:

$$\big(\mathrm{i}_Y^*\psi, \varphi\big)_{Y^*,Y} = (\psi, \varphi)_{\{j+\frac12,-j-\frac12\}} \quad \text{for all } \varphi \in Y.$$

With this notation (also indicated in [28] after (5.23)), (3.37) may be rewritten as

$$\mathrm{i}_Y^* \Gamma^0 u = L\gamma u,$$

or, when we use that $\Gamma^0 = \chi - P^0_{\gamma,\chi}\gamma$,

$$\mathrm{i}_Y^* \chi u = \big(L + \mathrm{i}_Y^* P^0_{\gamma,\chi}\big)\gamma u. \tag{3.38}$$

We have then obtained:

Theorem 3.1. *For a closed operator $\widetilde{A} \in \mathcal{M}$, the following statements* (i) *and* (ii) *are equivalent*:

(i) *$\widetilde{A}$ corresponds to $T : V \to W$ as in Section* 2.
(ii) *$D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ that satisfy the boundary condition*

$$\gamma u \in D(L), \quad \mathrm{i}_Y^* \chi u = \big(L + \mathrm{i}_Y^* P^0_{\gamma,\chi}\big)\gamma u. \tag{3.39}$$

Here $T : V \to W$ and $L : X \to Y^*$ are defined from one another as described in (3.22)–(3.25)*.*

Note that when Y is the full space $\mathcal{H}^0$, i_{Y^*} is superfluous, and (3.39) is a *Neumann-type condition*

$$\gamma u \in D(L), \quad \chi u = \big(L + P^0_{\gamma,\chi}\big)\gamma u. \tag{3.40}$$

The whole construction can be carried out with A replaced by $A - \mu$, when $\mu < m(A_\gamma)$. We define L^μ from T^μ as in (3.24)–(3.25) with $T : V \to W$ replaced by $T^\mu : V_\mu \to W_\mu$ and use of (3.23); here

$$L^\mu = \big(\gamma^{-1}_{W_\mu}\big)^* T^\mu \gamma^{-1}_{V_\mu}, \qquad D\big(L^\mu\big) = \gamma_{V_\mu} D(T) = D(L). \tag{3.41}$$

Theorem 3.1 implies:

Corollary 3.2. *Let $\mu < m(A_\gamma)$. For a closed operator $\widetilde{A} \in \mathcal{M}$, the following statements* (i) *and* (ii) *are equivalent*:

(i) *$\widetilde{A} - \mu$ corresponds to $T^\mu : V_\mu \to W_\mu$ as in Section* 2.
(ii) *$D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ such that*

$$\gamma u \in D(L), \quad \mathrm{i}_Y^* \chi u = \big(L^\mu + \mathrm{i}_Y^* P^\mu_{\gamma,\chi}\big)\gamma u. \tag{3.42}$$

Since the boundary conditions (3.39) and (3.42) define the same realization, we obtain moreover the information that

$$\big(L^{\mu} + \mathrm{i}_Y^* P^{\mu}_{\gamma,\chi}\big)\gamma u = \big(L + \mathrm{i}_Y^* P^{0}_{\gamma,\chi}\big)\gamma u, \quad \text{for } \gamma u \in D(L),$$

i.e.,

$$L^{\mu} = L + \mathrm{i}_Y^*\big(P^{0}_{\gamma,\chi} - P^{\mu}_{\gamma,\chi}\big) \quad \text{on } D(L). \tag{3.43}$$

4. The lower boundedness question

We have shown in Section 2 that the general conclusion of lower boundedness from T to $\widetilde{A}$ (hence from L to $\widetilde{A}$ in view of (3.28)–(3.30)) hinges on whether the lower bound of G^{μ} takes arbitrary high values when $\mu \to -\infty$. Let us identify G^{μ} in terms of the operators over Σ.

Proposition 4.1. *Let $\mu < m(A_\gamma)$. We have that*

$$\big(G^{\mu} v, w\big) = \big(\big(P^{0}_{\gamma,\chi} - P^{\mu}_{\gamma,\chi}\big)\gamma_Z v, \gamma_Z w\big)_{\{j+\frac12, -j-\frac12\}}, \quad \text{for } v, w \in Z. \tag{4.1}$$

In other words,

$$G^{\mu} = \big(\gamma_Z^*\big)^{-1}\big(P^{0}_{\gamma,\chi} - P^{\mu}_{\gamma,\chi}\big)\gamma_Z^{-1}. \tag{4.2}$$

In particular, $P^{0}_{\gamma,\chi} - P^{\mu}_{\gamma,\chi}$ is continuous from $\mathcal{H}^0$ to $(\mathcal{H}^0)^* = \widetilde{\mathcal{H}}^{2m}$.

Proof. This is easily seen by use of the correspondence between realizations and operators over the boundary, applied to the Krein–von Neumann extension:

For the case $T = 0$ with $V = W = Z$ (defining the Krein–von Neumann extension), let us denote the operator corresponding to $A_0 - \mu$ in the μ-dependent setting by T_0^{μ}. Here $L = 0$, continuous from $\mathcal{H}^0$ to $(\mathcal{H}^0)^*$, and we denote the corresponding μ-dependent operator by L_0^{μ}; it is likewise continuous from $\mathcal{H}^0$ to $(\mathcal{H}^0)^*$. By (3.43),

$$L_0^{\mu} = P^{0}_{\gamma,\chi} - P^{\mu}_{\gamma,\chi} \quad \text{on } \mathcal{H}^0. \tag{4.3}$$

This shows the asserted continuity. By (2.17),

$$\big(G^{\mu} v, w\big) = \big(T_0^{\mu} E^{\mu} v, E^{\mu} w\big), \quad \text{for } v, w \in Z. \tag{4.4}$$

Then furthermore,

$$\begin{aligned}\big(T_0^{\mu} E^{\mu} v, E^{\mu} w\big) &= \big(L_0^{\mu}\gamma_{Z_\mu} E^{\mu} v, \gamma_{Z_\mu} E^{\mu} w\big)_{\{j+\frac12, -j-\frac12\}} = \big(L_0^{\mu}\gamma_Z v, \gamma_Z w\big)_{\{j+\frac12, -j-\frac12\}} \\ &= \big(\big(P^{0}_{\gamma,\chi} - P^{\mu}_{\gamma,\chi}\big)\gamma_Z v, \gamma_Z w\big)_{\{j+\frac12, -j-\frac12\}}.\end{aligned} \tag{4.5}$$

This shows (4.1), and hence (4.2). □

Remark 4.2. This was also observed in [14, Remark 3.2], formulated in the case $m=1$, for a nonsymmetric situation with general complex values of μ (then adjoints and primed operators enter).

An alternative proof that does not refer to the correspondence between realizations and operators over the boundary goes as follows: Recalling that $E^\mu = A_{\max}(A_\gamma-\mu)^{-1}$ maps Z homeomorphically onto Z_μ, we have for $v,w\in Z$, $\varphi=\gamma_Z v$, $\psi=\gamma_Z w$,

$$\begin{aligned}\big(G^\mu v,w\big)&=-\mu\big(E^\mu v,w\big)=-\mu\big(A(A_\gamma-\mu)^{-1}v,w\big)\\&=-\mu\big(\chi(A_\gamma-\mu)^{-1}v,\gamma_Z w\big)_{\{j+\frac12,-j-\frac12\}}\\&=-\mu\big(\chi(A_\gamma-\mu)^{-1}v,\psi\big)_{\{j+\frac12,-j-\frac12\}},\end{aligned}\tag{4.6}$$

where we have used Green's formula (3.16) and the fact that $\gamma(A_\gamma-\mu)^{-1}=0$. Now if $v\in H^{2m}(\Omega)$, we can use that $-\mu(A_\gamma-\mu)^{-1}=I-A(A_\gamma-\mu)^{-1}$ to write

$$\begin{aligned}\big(G^\mu v,w\big)&=\big(\chi\big(I-A(A_\gamma-\mu)^{-1}\big)v,\psi\big)_{\{j+\frac12,-j-\frac12\}}\\&=(\chi v,\psi)_{\{j+\frac12,-j-\frac12\}}-\big(\chi A(A_\gamma-\mu)^{-1}v,\psi\big)_{\{j+\frac12,-j-\frac12\}}\\&=\big(\chi\gamma_Z^{-1}\varphi,\psi\big)_{\{j+\frac12,-j-\frac12\}}-\big(\chi E^\mu\gamma_Z^{-1}\varphi,\psi\big)_{\{j+\frac12,-j-\frac12\}}\\&=\big(P^0_{\gamma,\chi}\varphi,\psi\big)_{\{j+\frac12,-j-\frac12\}}-\big(\chi\gamma_{Z_\mu}^{-1}\varphi,\psi\big)_{\{j+\frac12,-j-\frac12\}}\\&=\big(\big(P^0_{\gamma,\chi}-P^\mu_{\gamma,\chi}\big)\varphi,\psi\big)_{\{j+\frac12,-j-\frac12\}}.\end{aligned}$$

This shows the identity for smooth functions v in the nullspace. Since the smooth null-solutions are dense in Z, the general statement follows by approximation.

We note in passing that since $\chi(A_\gamma-\mu)^{-1}$ is the adjoint of the μ-dependent Poisson operator K^μ_γ (by Green's formula), (4.6) also leads to the alternative formula

$$P^0_{\gamma,\chi}-P^\mu_{\gamma,\chi}=-\mu\big(K^\mu_\gamma\big)^*K_\gamma.\tag{4.7}$$

The question of the behavior of the lower bound of G^μ is hereby turned into the question of the lower bound of $P^0_{\gamma,\chi}-P^\mu_{\gamma,\chi}$, in relation to the norm on $\mathcal{H}^0$. Note that this difference is a multi-order system of ψdo's where the entries are of order $2m$ lower than the entries in $P^0_{\gamma,\chi}$.

Now this will be set in relation to a similar family of operators in a situation where the domain Ω is replaced by a bounded set. Choose a large open ball $B(0,R)$ containing $\mathbb{R}^n\setminus\Omega$ in its interior. Let $\Omega_<=\Omega\cap B(0,R)$; its boundary $\Sigma_<$ consists of the two disjoint pieces Σ and $\Sigma'=\partial B(0,R)$. When the whole construction is applied to A on $\Omega_<$, we get a family of matrix-formed Dirichlet-to-Neumann operators $P^\mu_{\gamma,\chi_<}$ on $\Sigma_<=\Sigma\cup\Sigma'$.

Proposition 4.3. *For the pseudodifferential operators $P^\mu_{\gamma,\chi_<}$ on $\Sigma_<$, we have*

$$\big(\big(P^0_{\gamma,\chi_<}-P^\mu_{\gamma,\chi_<}\big)\varphi,\varphi\big)_{\{j+\frac12,-j-\frac12\}}\geqslant C(\mu)\|\varphi\|^2_{\{-j-\frac12\}}$$

for $\varphi\in\mathcal{H}^0_<=\prod_{j<m}H^{-j-\frac12}(\Sigma_<)$, with

$$C(\mu)\to\infty\quad\text{for }\mu\to-\infty.$$

Proof. This follows from Theorem 2.4, applied to the operator $G^\mu_<$ defined for this case. The information on the lower bound of $G^\mu_<$ carries over to the assertion for $P^0_{\gamma,\chi_<} - P^\mu_{\gamma,\chi_<}$, since they are related as in Proposition 4.1; recall also (3.28)–(3.30). $\square$

(This is of course a qualitative statement, which is independent of how the norm in $\mathcal{H}^0$ has been chosen.)

Define

$$Q^\mu = P^0_{\gamma,\chi} - P^\mu_{\gamma,\chi}, \qquad Q^\mu_1 = r_\Sigma\big(P^0_{\gamma,\chi_<} - P^\mu_{\gamma,\chi_<}\big)e_\Sigma, \tag{4.8}$$

where e_Σ extends distributions on Σ by 0 on Σ'. Since Σ and Σ' are disjoint closed manifolds, both Q^μ and Q^μ_1 are (matrix-formed) ψdo's on Σ, continuous from $\mathcal{H}^0$ to $\widetilde{\mathcal{H}}^{2m}$.

Theorem 4.4. *The operator norm from $\mathcal{H}^0$ to $\widetilde{\mathcal{H}}^{2m}$ of the difference*

$$Q^\mu - Q^\mu_1 = P^0_{\gamma,\chi} - P^\mu_{\gamma,\chi} - r_\Sigma\big(P^0_{\gamma,\chi_<} - P^\mu_{\gamma,\chi_<}\big)e_\Sigma \tag{4.9}$$

is bounded for $\mu \to -\infty$.

Proof. In this proof we use microlocal details from the pseudodifferential calculus. Introductions to ψdo's can be found in many textbooks, e.g. in [35, Chapters 7–8].

The use of ψdo's on the manifold Σ is somewhat technical, because they are defined first by Fourier transformation formulas in $\mathbb{R}^{n-1}$ and then carried over to Σ by local coordinates; in this process there appear a lot of remainder terms that have to be handled too. The heart of our proof lies in the fact that the remainder terms have much better asymptotic properties than the given operators (are "negligible"); this is an aspect of the fact that ψdo's are *pseudo-local*.

When $P^0_{\gamma,\chi}$ is constructed from A and the trace operators, the construction of its symbol takes place in the neighborhood of each point (x',ξ'), $x' \in \Sigma$ (localized) and $\xi' \in \mathbb{R}^{n-1}$. The same holds for $P^0_{\gamma,\chi_<}$ on Σ. But in the localizations at points of Σ, A, γ and χ are *the same* for the two operators, and therefore the resulting complete symbols of $P^0_{\gamma,\chi}$ and $P^0_{\gamma,\chi_<}$ at a point of Σ must be the same, modulo symbols of order $-\infty$. (This uses that also the constructions in the ψdbo calculus are the same for Ω and $\Omega_<$ at points of Σ.) It follows that

$$P^0_{\gamma,\chi} - r_\Sigma P^0_{\gamma,\chi_<} e_\Sigma \text{ is of order } -\infty, \tag{4.10}$$

i.e., the localized symbol of $P^0_{\gamma,\chi} - r_\Sigma P^0_{\gamma,\chi_<} e_\Sigma$ and all its derivatives are $O((1+|\xi'|)^{-N})$ for all $N \in \mathbb{N}$. Then the operator is bounded as an operator from any m-tuple of Sobolev spaces over Σ to any other; in particular, it is bounded as an operator from $\mathcal{H}^0$ to $\widetilde{\mathcal{H}}^{2m}$.

Now consider the μ-dependent symbols. There is the difficulty here that the individual operators $P^\mu_{\gamma,\chi}$ and $P^\mu_{\gamma,\chi_<}$ have norms that grow with $|\mu|$ (even as operators from $\mathcal{H}^0$ to $\widetilde{\mathcal{H}}^0$); this is demonstrated by the simple example of $1-\Delta$ on a half-space (considered in [35, Chapter 9]), where $P^\mu_{\gamma,\chi}$ has symbol $-(1+|\xi'|^2+\mu)^{\frac{1}{2}}$. We shall then use a sharper version of the device used for $P^0_{\gamma,\chi} - r_\Sigma P^0_{\gamma,\chi_<} e_\Sigma$. Namely that the operators, being constructed out of the elliptic differential operator $A-\mu$ and the differential trace operators, have μ-dependent symbols that are ψdo symbols in the n cotangent variables (ξ',η_n) where $\eta_n = |\mu|^{\frac{1}{2}}$. (This is the "easy" parameter-dependent case, said to be *of regularity* $+\infty$ in [33], *strongly polyhomogeneous* in [37].)

Again, the local constructions of symbols of $P^\mu_{\gamma,\chi}$ and $P^\mu_{\gamma,\chi_<}$ have identical ingredients at the points of Σ, and we now deduce that the symbols differ by a symbol in the parameter-dependent class of order $-\infty$, so that it is $O((1+|\xi'|+|\mu|^{\frac{1}{2}})^{-N})$ for all $N \in \mathbb{N}$, with all its derivatives. Then the symbol and its derivatives are also $O((1+|\xi'|)^{-N'}(1+|\mu|)^{-N''})$ for all $N', N'' \in \mathbb{N}$. It follows that

$$P^{\mu}_{\gamma,\chi} - r_\Sigma P^{\mu}_{\gamma,\chi_<} e_\Sigma \text{ is of order } -\infty, \text{ with norm } O\big((1+|\mu|)^{-N}\big), \text{ for any } N, \tag{4.11}$$

as an operator from an arbitrary m-tuple of Sobolev spaces to another. In particular, it is bounded as an operator from $\mathcal{H}^0$ to $\widetilde{\mathcal{H}}^{2m}$ with a bound independent of μ.

The assertion on

$$Q^{\mu} - Q^{\mu}_1 = \big(P^0_{\gamma,\chi} - r_\Sigma P^0_{\gamma,\chi_<} e_\Sigma\big) - \big(P^{\mu}_{\gamma,\chi} - r_\Sigma P^{\mu}_{\gamma,\chi_<} e_\Sigma\big)$$

now follows by adding the two parts. □

We can then conclude:

Theorem 4.5. *In the situation of exterior domains, the pseudodifferential operators $P^{\mu}_{\gamma,\chi}$ on Σ satisfy*

$$\big((P^0_{\gamma,\chi} - P^{\mu}_{\gamma,\chi})\varphi, \varphi\big)_{\{j+\frac12, -j-\frac12\}} \geqslant C(\mu)\|\varphi\|^2_{\{-j-\frac12\}} \quad \text{for } \varphi \in \mathcal{H}^0, \tag{4.12}$$

for some function $C(\mu)$ satisfying

$$C(\mu) \to \infty \quad \text{for } \mu \to -\infty. \tag{4.13}$$

It follows that $m(G^{\mu}) \to \infty$ for $\mu \to -\infty$, and hence:

In the correspondence described in Theorem 3.1, *$X \subset Y$ and L is lower bounded, if and only if $\widetilde{A}$ is lower bounded.*

Proof. Using Proposition 4.3 and Theorem 4.4 we have for $\varphi \in \mathcal{H}^0$ that

$$\begin{aligned}\big((P^0_{\gamma,\chi} - P^{\mu}_{\gamma,\chi})\varphi, \varphi\big)_{\{j+\frac12, -j-\frac12\}} &= \big(Q^{\mu}_1\varphi, \varphi\big)_{\{j+\frac12, -j-\frac12\}} - \big((Q^{\mu}_1 - Q^{\mu})\varphi, \varphi\big)_{\{j+\frac12, -j-\frac12\}} \\ &\geqslant C(\mu)\|\varphi\|^2_{\{-j-\frac12\}} - C_1\|\varphi\|^2_{\{-j-\frac12\}} \geqslant C'(\mu)\|\varphi\|^2_{\{-j-\frac12\}};\end{aligned}$$

where $C'(\mu)$ behaves as in (4.13). In view of (4.1) and (3.28), we conclude that $m(G^{\mu}) \to \infty$ for $\mu \to -\infty$. Then the statements (i) and (ii) of Proposition 2.2 are valid.

Let $\widetilde{A}$ correspond to $T: V \to W$ as in the beginning of Section 2, and to $L: X \to Y^*$ as in Theorem 3.1. As noted earlier, $V \subset W$ and T is lower bounded, if and only if $X \subset Y$ and L is lower bounded. We have from rule (e) that lower boundedness of $\widetilde{A}$ implies $V \subset W$ and lower boundedness of T. We can now complete the argument in the converse direction: When $X \subset Y$ and L is lower bounded, hence $V \subset W$ and T is lower bounded, then by Proposition 2.2(i), there is a $\mu \in \mathbb{R}$ such that $m(T^{\mu}) \geqslant 0$, and hence by rule (h), $m(\widetilde{A}) \geqslant \mu$. □

By Proposition 2.3, we have in particular for the Krein-like extensions:

Corollary 4.6. *In the exterior domain case one has for any $a \in \mathbb{R}$ that the Krein-like extension A_a defined by* (1.1) *is lower bounded.*

We recall that this was already known to hold for bounded domains.

Remark 4.7. The above theorem says nothing about the size of $C(\mu)$. In [28] for the interior domain case, we conjectured that $C(\mu)$ may possibly be shown to be of the order of magnitude $|\mu|^{1/2m}$. Calculations on second-order cases where A has a structure like $D_n^2 + B^2$ in product coordinates near Σ, confirm that $C(\mu)$ is of the order of magnitude $|\mu|^{1/2}$ then. Such calculations might solve the problem also for domains with unbounded boundary, provided suitable uniform ellipticity conditions are satisfied. We may possibly return to this in detail elsewhere.

Let us end this section by some remarks on other lower boundedness estimates. It is used in the above proofs that the boundary Σ is compact. There is a more restricted type of lower boundedness, that can be shown to hold for $\widetilde{A}$ and L simultaneously, in uniformly elliptic situations regardless of compactness of the boundary, namely m-coerciveness, also known as the Gårding inequality.

Consider a case where Ω is admissible in the sense of [33], as mentioned in the beginning of Section 3. This assures that $\overline{\Omega}$ is covered by a finite system of local coordinates, some of them for bounded pieces, some of them for unbounded pieces, carried over to subsets of $\mathbb{R}^n$ where the part in $\overline{\Omega}$ resp. $\partial\Omega$ carries over to bounded resp. unbounded subsets of $\overline{\mathbb{R}}^n_+$ resp. $\mathbb{R}^{n-1}$, in a controlled way. Detailed explanations are given in [33], including the still more general situation of admissible manifolds. All that was described in Section 2 works in this case; let us also in addition mention the trace mapping property

$$\gamma : H^r(\Omega) \to \mathcal{H}^r \quad \text{continuously for } r > m - \frac{1}{2}, \tag{4.14}$$

and the interpolation property: When $0 < r < m$, there is for any $\varepsilon > 0$ a positive constant $c(\varepsilon)$ such that

$$\|u\|_r^2 \leqslant \varepsilon\|u\|_m^2 + c(\varepsilon)\|u\|_0^2, \quad \text{for } u \in H^m(\Omega). \tag{4.15}$$

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be an admissible domain, and let $\widetilde{A}$ correspond to $T : V \to W$ and $L : X \to Y^*$ as in Sections* 2–3. *Then the following statements* (*with positive constants c, c', c''*) *are equivalent*:

(i) *$D(\widetilde{A}) \subset H^m(\Omega)$ and $\widetilde{A}$ satisfies the Gårding inequality*

$$\operatorname{Re}(\widetilde{A}u, u) \geqslant c\|u\|_m^2 - k\|u\|_0^2, \quad \textit{for } u \in D(\widetilde{A}). \tag{4.16}$$

(ii) *$D(T) \subset Z \cap H^m(\Omega) = Z_A^m(\Omega)$, $V \subset W$, and T satisfies the Gårding inequality*

$$\operatorname{Re}(Tz, z) \geqslant c'\|z\|_m^2 - k'\|z\|_0^2, \quad \textit{for } z \in D(T). \tag{4.17}$$

(iii) *$D(L) \subset \mathcal{H}^m$, $X \subset Y$, and L satisfies the Gårding inequality*

$$\operatorname{Re}(L\varphi, \varphi)_{Y^*,Y} \geqslant c''\|\varphi\|^2_{\{m-j-\frac{1}{2}\}} - k''\|\varphi\|^2_{\{-j-\frac{1}{2}\}}. \tag{4.18}$$

Proof. This is a straightforward generalization of the proof for the case of bounded domains in [26, Prop. 2.7], to admissible domains.

Note first that the statements in (ii) and (iii) are equivalent in view of (3.27) and the homeomorphisms (3.19).

Next, we note that (i) implies in particular that $\widetilde{A}$ is lower bounded. Then (i) implies that $V \subset W$ and hence $X = \gamma V \subset \gamma W = Y$, in view of property (e) in Section 2. Thus (2.10) holds. When (4.16) is valid and $z \in D(T)$, we can approximate $A_\gamma^{-1}Tz$ in m-norm by a sequence of functions $v^j \in D(A_{\min})$, since A_γ is the Friedrichs extension of $A_{\min}$. Let $u^j = -v^j + A_\gamma^{-1}Tz + z$, then $u^j \in D(\widetilde{A})$ in view of (2.8), with $u^j_\gamma = -v^j + A_\gamma^{-1}Tz$, $u^j_\zeta = z$. Clearly, $u^j \to z$ in $H^m(\Omega)$ and $u^j_\gamma = -v^j + A_\gamma Tz \to 0$ in $H^m(\Omega)$. We combine (2.10) with the inequality (4.16) to see that

$$\operatorname{Re}\bigl(Au^j, u^j\bigr) = \bigl(Au^j_\gamma, u^j_\gamma\bigr) + \operatorname{Re}(Tz, z) \geqslant c\bigl\|u^j\bigr\|_m^2 - k\bigl\|u^j\bigr\|_0^2.$$

Here the term $(Au_\gamma^j, u_\gamma^j)$ is equivalent with $\|u_\gamma^j\|_m^2$, so it goes to 0 for $j \to \infty$, so we conclude that

$$\operatorname{Re}(Tz, z) \geqslant c\|z\|_m^2 - k\|z\|_0^2.$$

Thus (i) implies (ii) and hence also (iii).

Now assume that (ii) and (iii) hold. Using (2.10), we find for $u \in D(\widetilde{A})$ that

$$\begin{aligned}\operatorname{Re}(Au, u) &= (Au_\gamma, u_\gamma) + \operatorname{Re}(Tu_\zeta, u_\zeta)\\ &\geqslant c\|u_\gamma\|_m^2 + c'\|u_\zeta\|_m^2 - k'\|u_\zeta\|_0^2 \geqslant c''\|u\|_m^2 - k'\|u_\zeta\|_0^2,\end{aligned} \tag{4.19}$$

where we have again used that (Au_γ, u_γ) is equivalent with $\|u_\gamma\|_m^2$. To handle the last term, note that choosing r with $m - \frac{1}{2} < r < m$, we have that

$$\begin{aligned}k'\|u_\zeta\|_0^2 &\leqslant c_1\|\gamma u_\zeta\|^2_{\{-j-\frac{1}{2}\}} = c_1\|\gamma u\|^2_{\{-j-\frac{1}{2}\}} \leqslant c_2\|\gamma u\|^2_{\{r-j-\frac{1}{2}\}}\\ &\leqslant c_3\|u\|_r^2 \leqslant \varepsilon c_3\|u\|_m^2 + c(\varepsilon)c_3\|u\|_0^2,\end{aligned} \tag{4.20}$$

where we used (3.19), (4.14) and (4.15). Then (4.19) implies

$$\operatorname{Re}(Au, u) \geqslant \big(c'' - \varepsilon c_3\big)\|u\|_m^2 - c(\varepsilon)c_3\|u\|_0^2,$$

which shows (i) when ε is taken sufficiently small. □

The papers [27] and [28] give a full analysis of the analytical details required to have (iii) in cases of normal boundary conditions, for bounded domains and compact manifolds. This involves a condition for m-coerciveness that is a special case of ellipticity of the boundary condition (the Shapiro–Lopatinskii condition). The analysis can be extended to admissible sets with suitable precautions on uniformity of estimates.

We underline that the discussion of lower bounds as in Theorem 4.5 is valid for much more general realizations, and is *not* linked with ellipticity of the boundary condition. An interesting consequence for questions of spectral asymptotics is that also for non-elliptic boundary conditions, lower boundedness of L (or T) assures that there is no eigenvalue sequence going to $-\infty$. (For spectral asymptotics of resolvent differences, see e.g. Birman [10], Birman and Solomyak [12], Grubb [31,36], Malamud [46], and their references.)

Estimates with other spaces $\mathcal{K}$ in lieu of $H^m(\Omega)$ are also treated in our early papers.

5. Krein-like extensions and their spectral asymptotics on bounded domains

We here make a closer study of the Krein-like extensions A_a defined in (1.1), corresponding to the choice $T = aI$ in Z.

Proposition 5.1. *The realization A_a represents the boundary condition*

$$\chi u = C\gamma u, \quad \textit{with } C = a\big(\gamma_Z^{-1}\big)^*\gamma_Z^{-1} + P^0_{\gamma,\chi}, \tag{5.1}$$

in the sense that

$$D(A_a) = \big\{u \in D(A_{\max}) \bigm| \chi u = C\gamma u\big\}. \tag{5.2}$$

Here $(\gamma_Z^{-1})^*\gamma_Z^{-1}$ is a pseudodifferential operator continuous from $\mathcal{H}^s$ to $\widetilde{\mathcal{H}}^{s+2m}$, for all $s \in \mathbb{R}$ (and elliptic as such); it is of order $2m$ steps lower than $P^0_{\gamma,\chi}$.

Proof. We see from (3.24) that A_a corresponds to

$$L_a = a\big(\gamma_Z^{-1}\big)^*\gamma_Z^{-1}, \qquad D(L) = \mathcal{H}^0, \tag{5.3}$$

so that A_a is defined by the boundary condition in (5.1).

To account for the properties of $(\gamma_Z^{-1})^*\gamma_Z^{-1}$ (for the interested reader), we use the ψdbo calculus. Note that $(\gamma_Z^{-1})^*\gamma_Z^{-1}$ has the asserted continuity property for $s = 0$, is bijective, and acts like $(\gamma_Z^{-1})^*\operatorname{pr}_Z \operatorname{i}_Z\gamma_Z^{-1}$. Here $\operatorname{i}_Z\gamma_Z^{-1}$ is the Poisson operator K_γ, as noted earlier, and its adjoint $K_\gamma^* = (\gamma_Z^{-1})^*\operatorname{pr}_Z$ is a trace operator of class 0 in the ψdbo calculus. Then, by the composition rules,

$$\big(\gamma_Z^{-1}\big)^*\gamma_Z^{-1} = K_\gamma^* K_\gamma$$

is a pseudodifferential operator on Σ; and it has the asserted continuity property for all s since it has it for $s = 0$. It is elliptic as an operator from $\mathcal{H}^s$ to $\widetilde{\mathcal{H}}^{s+2m}$, because it is bijective. $\square$

Remark 5.2. It should be noted that *the boundary condition* (5.1) *is not elliptic* (does not satisfy the appropriate Shapiro–Lopatinskii condition). In fact, for pseudodifferential Neumann-type boundary conditions $\chi u = C\gamma u$ it is known that ellipticity holds if and only if the ψdo $L = C - P^0_{\gamma,\chi}$ is elliptic as an operator from $\mathcal{H}^s$ to $\widetilde{\mathcal{H}}^s$. The actual L equals $aK_\gamma^* K_\gamma$, which has principal symbol 0 as an operator from $\mathcal{H}^s$ to $\widetilde{\mathcal{H}}^s$, since it is of lower order.

For $m = 1$, C is of order 1, continuous from $H^{s-\frac12}(\Sigma)$ to $H^{s-\frac32}(\Sigma)$, and $L = aK_\gamma^* K_\gamma$ is of order -1, continuous from $H^{s-\frac12}(\Sigma)$ to $H^{s+\frac12}(\Sigma)$, for all s.

We henceforth take $a \in \mathbb{R}\setminus\{0\}$. From (2.6) we then have

$$A_a^{-1} = A_\gamma^{-1} + a^{-1}\operatorname{pr}_Z. \tag{5.4}$$

(We here read pr_X as a mapping in H instead of as a mapping from H to X; this will often be the case in the following, and the meaning should be clear from the context.)

Let us assume from now on, instead of the primary hypothesis for Sections 3–4, that Ω is a bounded smooth subset of $\mathbb{R}^n$ with boundary Σ; aside from this we keep the notation. As remarked in the beginning of Section 3, the explanations there hold also for this case (are in fact easier to verify).

Since the embedding of $D(A_\gamma) = H^{2m}(\Omega)\cap H_0^m(\Omega)$ into $L_2(\Omega)$ is compact, the inverse A_γ^{-1} is a compact operator in $L_2(\Omega)$, so A_γ has a discrete spectrum consisting of eigenvalues going to ∞. It is well known (cf. e.g. Hörmander [39, Chapter 29.3]), that the counting function $N(t; A_\gamma)$, counting the number of eigenvalues of A_γ in $[0,t]$ with multiplicities, has the asymptotic behavior

$$N(t; A_\gamma) - c_A t^{n/2m} = O\big(t^{(n-1)/2m}\big) \quad \text{for } t\to\infty; \tag{5.5}$$

here

$$c_A = (2\pi)^{-n}\int_{x\in\Omega,\, a^0(x,\xi)<1} dx\,d\xi. \tag{5.6}$$

Equivalently, the j'th eigenvalue $\mu_j(A_\gamma^{-1})$ of A_γ^{-1} satisfies

$$\mu_j\big(A_\gamma^{-1}\big) - c_A' j^{-2m/n} = O\big(j^{-(2m+1)/n}\big) \quad \text{for } j\to\infty;\ \text{with } c_A' = c_A^{2m/n}. \tag{5.7}$$

(The passage between counting function estimates and eigenvalue estimates is recalled below in Lemma 5.4 and its corollary.)

Since Z is infinite dimensional, $a^{-1}\,\mathrm{pr}_Z$ has the point a^{-1} as essential spectrum, so A_a^{-1} has essential spectrum consisting of the points a^{-1} and 0, and A_a has the essential spectrum $\{a\}$. Since A_a is selfadjoint and not upper bounded (since it extends $A_{\min}$), there must be a sequence of discrete eigenvalues (with finite dimensional eigenspaces) above a going to ∞. We shall investigate this sequence.

The Krein–von Neumann extension A_0 has essential spectrum $\{0\}$ and an eigenvalue sequence going to ∞, and the question of the asymptotic behavior of that sequence was raised in Alonso and Simon [2] and answered in Grubb [30]. The result was a rather precise estimate of the function $N_+(t; A_0)$ counting the number of eigenvalues in $]0, t]$:

$$N_+(t; A_0) - c_A t^{n/2m} = O\left(t^{(n-\theta)/2m}\right) \quad \text{for } t \to \infty; \tag{5.8}$$

here c_A is the same constant as for the Dirichlet problem and

$$\theta = \max\left\{\frac{1}{2} - \varepsilon, 2m/(2m-n+1)\right\}. \tag{5.9}$$

We note in passing that the value $\frac{1}{2} - \varepsilon$ came from the application of an estimate announced by Kozlov in [41], whereas his later paper [42], not available to the author when [30] was written, has the value $\frac{1}{2}$, so (5.9) can immediately be replaced by

$$\theta = \max\left\{\frac{1}{2}, 2m/(2m-n+1)\right\}. \tag{5.10}$$

We show at the end of this section that the estimate can be improved even further, to $\theta = 1 - \varepsilon$ (following up on a remark at the end of [30]). This comes after our deduction of a similar estimate for the operators A_a, $a \neq 0$.

The proof of (5.8) was based on a transformation of the eigenvalue equation

$$A_0 u = \lambda u, \quad \text{with } \lambda \neq 0,\ u \neq 0, \tag{5.11}$$

into the problem for the $4m$-order operator A^2:

$$A^2 v = \lambda A v \quad \text{for } v \in H_0^{2m}(\Omega), \tag{5.12}$$

where u and v are recovered from one another by

$$v = A_\gamma^{-1} A u, \qquad u = \frac{1}{\lambda} A v. \tag{5.13}$$

There were earlier eigenvalue estimates for implicit eigenvalue problems as in (5.12) (as initiated by Pleijel [51], surveyed in Birman and Solomyak [11]) giving the principal asymptotics, and the sharper estimates in (5.8) were obtained by turning the problem into the study of eigenvalues of the compact operator

$$S_0 = R_\varrho^{1/2} A R_\varrho^{1/2}, \tag{5.14}$$

where R_ϱ is the solution operator for the Dirichlet problem for A^2. (Further developments of the implicit eigenvalue problem are described in [33, Chapter 4.6].)

The study of A_0 has been taken up again recently by Ashbaugh, Gesztesy, Mitrea, Shterenberg and Teschl [6,7], also for nonsmooth domains, with much additional information. In particular they observe that when $A=-\Delta$, (5.12) is of interest as the "buckling problem" in elasticity.

Unfortunately, in the case of A_a, we do not have an equally simple reduction of the eigenvalue problem. Let $u=v+aA_\gamma^{-1}z+z$ as in (1.1); then applications of powers of A give

$$\begin{aligned}
&Au-\lambda u=Av+az-\lambda\big(v+aA_\gamma^{-1}z+z\big)=(A-\lambda)v+\big(a-\lambda-a\lambda A_\gamma^{-1}\big)z,\\
&A^2u-\lambda Au=A^2v-\lambda(Av+az)=\big(A^2-\lambda A\big)v-a\lambda z,\\
&A^3u-\lambda A^2u=A^3v-\lambda A^2v.
\end{aligned}\tag{5.15}$$

We see from the third line that in order for u to be an eigenvector, v must be an eigenvector of a certain implicit problem for A^3. Here A^3 is of order $6m$, and the information $v\in H_0^{2m}(\Omega)$ does not give enough boundary conditions to define an elliptic realization of A^3. But there is a supplementing boundary condition depending on λ:

Theorem 5.3. *Let $u\in D(A_a)$, with $u=v+aA_\gamma^{-1}z+z$, $v\in H_0^{2m}(\Omega)$, $z\in Z$. Then u is a nonzero eigenfunction for A_a with eigenvalue $\lambda\neq a$ if and only if v is a nonzero solution of the elliptic problem*

$$A^3v=\lambda A^2v,\qquad \gamma v=\nu v=0,\qquad \gamma A^2v=\lambda^2(\lambda-a)^{-1}\gamma Av,\tag{5.16}$$

and

$$z=K_\gamma(\lambda-a)^{-1}\gamma Av.\tag{5.17}$$

In particular, u, v and z are in $C^\infty(\overline{\Omega})$ then.

Proof. Assume that $Au=\lambda u$, $\lambda\neq a$. It follows from (5.15) that then $A^3v=\lambda A^2v$. Since $v\in H_0^{2m}(\Omega)$, $\gamma v=\nu v=0$ (recall (3.11)). From the first line in (5.15) it is seen that

$$Av=\lambda\big(v+aA_\gamma^{-1}z+z\big)-az,$$

which implies

$$\gamma Av=(\lambda-a)\gamma z,\quad \text{hence } \gamma z=(\lambda-a)^{-1}\gamma Av.\tag{5.18}$$

Moreover,

$$A^2v=A\big(\lambda v+\lambda aA_\gamma^{-1}z+(\lambda-a)z\big)=\lambda Av+\lambda az,$$

and hence

$$\gamma A^2v=\lambda\gamma Av+\lambda a\gamma z=\big(\lambda+\lambda a(\lambda-a)^{-1}\big)\gamma Av=\lambda^2(\lambda-a)^{-1}\gamma Av.$$

This shows the last boundary condition in (5.16) for v. We also see from (5.18) that z is determined from v by $z=K_\gamma(\lambda-a)^{-1}\gamma Av$, showing (5.17). Clearly $u\neq 0$ implies $v\neq 0$.

Conversely let v be a nontrivial solution of (5.16), define z by (5.17) and let $u=v+aA_\gamma^{-1}z+z$. By the third line of (5.15), the function $f=A^2u-\lambda Au$ satisfies $Af=0$; moreover, by the second line,

$$\gamma f = \gamma\big(A^2 v - \lambda A v - a\lambda z\big) = \gamma A^2 v - \lambda\gamma A v - a\lambda(\lambda - a)^{-1}\gamma A v = 0;$$

where we used (5.17) and the last boundary condition in (5.16). Then by the unique solvability of the Dirichlet problem, $f = 0$.

Now let $g = Au - \lambda u$, then $Ag = f = 0$, and, by the first line of (5.15),

$$\gamma g = \gamma(A - \lambda)v + \gamma(a - \lambda)z = \gamma A v + (a - \lambda)(\lambda - a)^{-1}\gamma A v = 0,$$

so $g = 0$. This shows that $Au = \lambda u$.

The problem is elliptic, since it is a perturbation by lower-order terms of the problem

$$A^3 v = 0, \qquad \gamma v = \nu v = 0, \qquad \gamma A^2 v = 0,$$

which only has the zero solution (indeed, $A^3 v = 0$ and $\gamma A^2 v = 0$ imply $A^2 v = 0$, and then $\gamma v = \nu v = 0$ implies $v = 0$). Then since there are $3m$ boundary conditions of different orders, the problem is elliptic. In particular, the solution of (5.16) is in $C^\infty(\overline{\Omega})$. $\square$

There may possibly be a strategy to find spectral asymptotics formulas for the very implicit eigenvalues λ of (5.16). But rather than pursuing this, we shall apply functional analytical methods to A_a combined with ψdbo results, using perturbation theory for the identity (5.4).

Let us first show how the asymptotic behavior of the counting functions for positive eigenvalues is related to the asymptotic behavior of positive eigenvalues of the inverse operator.

Lemma 5.4. *Let P be a selfadjoint invertible operator whose spectrum on $\mathbb{R}_+$ is discrete, consisting of a nondecreasing sequence of positive eigenvalues $\lambda_{j,+}(P)$ going to ∞ for $j \to \infty$ (repeated according to multiplicities). Let $N_+(t; P)$ denote the number of eigenvalues in $[0, t]$, and let $\mu_{j,+}(P^{-1}) = \lambda_{j,+}(P)^{-1}$. Let $C > 0$ and let $\beta > \alpha > 0$.*

There exists $c_1 > 0$ such that

$$\big|\mu_{j,+}\big(P^{-1}\big) - Cj^{-\alpha}\big| \leqslant c_1 j^{-\beta} \quad \text{for all } j \in \mathbb{N}, \tag{5.19}$$

if and only if there exists $c_2 > 0$ such that

$$\big|N_+(t; P) - C^{1/\alpha} t^{1/\alpha}\big| \leqslant c_2 t^{(1+\alpha-\beta)/\alpha} \quad \text{for all } t > 0. \tag{5.20}$$

Proof. This goes as in the proof for the compact case in [29, Lemma 6.2] (a very detailed version is given in [33, Lemma A.5]): Rewrite (5.19) as

$$\big|C^{-1} j^\alpha \mu_{j,+}\big(P^{-1}\big) - 1\big| \leqslant c_3 j^{\alpha-\beta},$$

$c_3 = c_1 C^{-1}$. Since $1 - \varepsilon \leqslant (1 + \varepsilon)^{-1} \leqslant (1 - \varepsilon)^{-1} \leqslant 1 + 2\varepsilon$ for $\varepsilon \in [0, \frac{1}{2}]$, this is equivalent with the existence of a constant c_4 such that

$$\big|Cj^{-\alpha}\lambda_{j,+}(P) - 1\big| \leqslant c_4 j^{\alpha-\beta},$$

which is rewritten, with $c_5 = C^{-1}c_4$, as

$$\big|\lambda_{j,+}(P) - C^{-1} j^\alpha\big| \leqslant c_5 j^{2\alpha-\beta}. \tag{5.21}$$

Next we note that the functions $j \to \lambda_{j,+}(P)$ and $t \to N_+(t; P)$ are essentially inverses of one another (in the sense that $N_+(t; P)$ is a step-function and $j \mapsto \lambda_{j,+}(P)$ should be filled out at non-integer arguments to have the reflected graph; both are monotone nondecreasing). To see how one passes from inequalities for one of them to the other, consider e.g. the inequality

$$\lambda_{j,+}(P) \leqslant C^{-1} j^{\alpha} + c_5 j^{2\alpha-\beta}.$$

Define $\varphi(j) = C^{-1} j^{\alpha} + c_5 j^{2\alpha-\beta}$. Let $t = \varphi(j)$ for some $j \in \mathbb{N}$, then

$$N_+(t; P) \geqslant N_+\big(\lambda_{j,+}(P); P\big) \geqslant j.$$

Now $t = C^{-1} j^{\alpha} + c_5 j^{2\alpha-\beta}$ implies $t \leqslant c_6 j^{\alpha}$ (since $2\alpha - \beta < \alpha$) and

$$(Ct)^{1/\alpha} = \big(j^{\alpha} + Cc_5 j^{2\alpha-\beta}\big)^{1/\alpha} = j\big(1 + Cc_5 j^{\alpha-\beta}\big)^{1/\alpha}.$$

Hence

$$\begin{aligned} j &= (Ct)^{1/\alpha}\big(1 + Cc_5 j^{\alpha-\beta}\big)^{-1/\alpha} \geqslant (Ct)^{1/\alpha}\big(1 - c_7 j^{\alpha-\beta}\big) \\ &\geqslant (Ct)^{1/\alpha}\big(1 - c_7\big(c_6^{-1} t\big)^{(\alpha-\beta)/\alpha}\big) = C^{1/\alpha} t^{1/\alpha} - c_8 t^{(1+\alpha-\beta)/\alpha}; \end{aligned}$$

for j so large that $Cc_5 j^{\alpha-\beta} \leqslant \frac{1}{2}$; here we have used the general inequality, valid for $s \in \mathbb{R}$,

$$1 - c_s|x| \leqslant (1+x)^s \leqslant 1 + c_s|x|, \quad \text{for } |x| \leqslant \frac{1}{2}. \tag{5.22}$$

This shows that for $t = \varphi(j)$, j sufficiently large,

$$N_+(t; P) \geqslant C^{1/\alpha} t^{1/\alpha} - c_8 t^{(1+\alpha-\beta)/\alpha},$$

giving part of the implication from (5.21) to (5.20). The other needed implications are shown in a similar way. □

We shall mainly use the special case where $\alpha = M/n$, $\beta = (M+\theta)/n$ for some $\theta > 0$ and some positive integer M, corresponding to $(1+\alpha-\beta)/\alpha = (n-\theta)/M$:

Corollary 5.5. *Let $\theta > 0$, $C_P > 0$. In the setting of Lemma 5.4, there exists $c_1 > 0$ such that*

$$\big|\mu_{j,+}\big(P^{-1}\big) - C_P^{M/n} j^{-M/n}\big| \leqslant c_1 j^{-(M+\theta)/n} \quad \text{for all } j \in \mathbb{N}, \tag{5.23}$$

if and only if there exists $c_2 > 0$ such that

$$\big|N_+(t; P) - C_P t^{n/M}\big| \leqslant c_2 t^{(n-\theta)/M} \quad \text{for all } t > 0. \tag{5.24}$$

For the study of the eigenvalues of A_a, we note that using the orthogonal decomposition (2.1) we can write the identity (5.4) in the form

$$\begin{aligned}
A_a^{-1} &= \mathrm{pr}_R\ A_\gamma^{-1}\,\mathrm{pr}_R + \mathrm{pr}_R\ A_\gamma^{-1}\,\mathrm{pr}_Z + \mathrm{pr}_Z\ A_\gamma^{-1}\,\mathrm{pr}_R + \mathrm{pr}_Z\ A_\gamma^{-1}\,\mathrm{pr}_Z + a^{-1}\,\mathrm{pr}_Z \\
&= B_1 + B_2 + S, \quad \text{with} \\
B_1 &= \mathrm{pr}_R\ A_\gamma^{-1}\,\mathrm{pr}_R, \\
B_2 &= a^{-1}\,\mathrm{pr}_Z, \\
S &= \mathrm{pr}_R\ A_\gamma^{-1}\,\mathrm{pr}_Z + \mathrm{pr}_Z\ A_\gamma^{-1}\,\mathrm{pr}_R + \mathrm{pr}_Z\ A_\gamma^{-1}\,\mathrm{pr}_Z\,.
\end{aligned} \tag{5.25}$$

For the part $B_1 + B_2$, where the two terms act separately in the two orthogonal subspaces R and Z, we see that $B = B_1 + B_2$ has the spectrum

$$\sigma(B_1 + B_2) = \sigma(B_1) \cup \sigma(B_2), \tag{5.26}$$

consisting of a sequence of positive eigenvalues $\mu_{j,+}(B_1)$ (since B_1 is compact nonnegative), the point 0 (in the essential spectrum) and an eigenvalue a^{-1} of infinite multiplicity. The essential spectrum consists of the two points 0 and a^{-1}. Since A_a^{-1} is a perturbation of $B_1 + B_2$ by a compact operator S, its essential spectrum again consists of 0 and a^{-1}. As noted earlier, A_a is unbounded above, so it has a sequence of eigenvalues going to infinity, corresponding to a positive eigenvalue sequence for A_a^{-1} going to 0.

In the detailed analysis, we shall again take advantage of the calculus of pseudodifferential boundary operators, using some composition rules and an important result shown in [31]. The main point is to identify certain terms as *singular Green operators*, which have a better spectral behavior than the pseudodifferential terms on Ω. We refer to [31] for details (introductions to the ψdbo calculus are also given in [33] and [35]).

The following result was shown in [14, Prop. 3.5] in the second-order case:

Proposition 5.6. *The orthogonal projection* pr_R *in* $H = L_2(\Omega)$ *acts as*

$$\mathrm{pr}_R = AR_\varrho A = I - \mathrm{pr}_Z,$$

where R_ϱ *is the solution operator for the Dirichlet problem for* A^2*. Here* pr_Z *is a singular Green operator on* Ω *of order and class* 0.

Proof. The proof, formulated in [14] for the nonselfadjoint second-order case with a spectral parameter, goes over verbatim to the $2m$-order case, when γ_0, γ_1 are replaced by γ, ν. □

In particular, pr_R and pr_Z are continuous in $H^s(\Omega)$ for all $s > -\frac{1}{2}$.

It follows that all the ingredients in (5.25) are in the ψdbo calculus:

Proposition 5.7. 1° *The operators*

$$\mathrm{pr}_R\ A_\gamma^{-1}\,\mathrm{pr}_Z, \qquad \mathrm{pr}_Z\ A_\gamma^{-1}\,\mathrm{pr}_R \quad \textit{and} \quad \mathrm{pr}_Z\ A_\gamma^{-1}\,\mathrm{pr}_Z,$$

hence also their sum S*, cf.* (5.25)*, are singular Green operators on* Ω *of order* $-2m$ *and class* 0.

2° *For any positive integer* N*,*

$$\begin{aligned}
A_\gamma^{-N} &= \mathrm{pr}_R\ A_\gamma^{-N}\,\mathrm{pr}_R + S_{1,N}, \\
A_a^{-N} &= B_{1,N} + B_{2,N} + S_N, \quad \textit{with } B_{1,N} = \mathrm{pr}_R\ A_\gamma^{-N}\,\mathrm{pr}_R,\ B_{2,N} = a^{-N}\,\mathrm{pr}_Z,
\end{aligned} \tag{5.27}$$

where $S_{1,N}$ *and* S_N *are singular Green operators on* Ω *of order* $-2mN$ *and class* 0.

Proof. 1° It is well known from the ψdbo calculus that $A_\gamma^{-1} = A_+^{(-1)} + G_\gamma$, where $A_+^{(-1)}$ is the truncated operator $r^+ A^{(-1)} e^+$ and G_γ is a singular Green operator on Ω of order $-2m$ and class 0. Here $A^{(-1)}$ is a pseudodifferential parametrix of A extended to $\mathbb{R}^n$, r^+ restricts from $\mathbb{R}^n$ to Ω and e^+ extends by zero on $\mathbb{R}^n \setminus \Omega$. Since pr_Z is a singular Green operator of order and class 0 by Proposition 5.6, the compositions with pr_Z lead to singular Green operators of order $-2m$ and class 0. Since $\mathrm{pr}_R = I - \mathrm{pr}_Z$, composition with it preserves the order and the property of being a singular Green operator of class 0.

2° The statement for the first line of (5.27) has already been shown for $N = 1$; for general N, it follows by similar arguments applied to A_γ^{-N}. For the second line of (5.27), we calculate:

$$\begin{aligned} A_a^{-N} &= \left(\mathrm{pr}_R\, A_\gamma^{-1}\, \mathrm{pr}_R + a^{-1}\, \mathrm{pr}_Z + S\right)^N = \left(\mathrm{pr}_R\, A_\gamma^{-1}\, \mathrm{pr}_R\right)^N + a^{-N}\, \mathrm{pr}_Z + \text{s.g.o.s} \\ &= \mathrm{pr}_R\, A_\gamma^{-N}\, \mathrm{pr}_R + a^{-N}\, \mathrm{pr}_Z + \text{s.g.o.s}, \end{aligned}$$

by the ψdbo rules of calculus, where the s.g.o.s stand for singular Green operators of class 0 and order $-2mN$. $\square$

A main result of [31] was the following asymptotic estimate of s-numbers of singular Green operators. When Q is a compact operator, its s-numbers are the positive eigenvalues of $|Q| = (Q^*Q)^{1/2}$, $s_j(Q) = \mu_j(|Q|)$, arranged nonincreasingly and repeated according to multiplicity.

Theorem 5.8. *When G is a singular Green operator on Ω of negative order $-M$ and class* 0, *then it is compact in $L_2(\Omega)$ with s-numbers satisfying*

$$s_j(G)\, j^{M/(n-1)} \to c\left(g^0\right) \quad \text{for } j \to \infty, \tag{5.28}$$

where $c(g^0)$ is a nonnegative constant defined from the principal symbol g^0 of G.

The remarkable feature here is that the spectral asymptotics formula involves the boundary dimension $n-1$ rather than the interior dimension n.

An application to the operators in Proposition 5.7 gives:

Corollary 5.9. *The asymptotic property*

$$s_j(G)\, j^{2mN/(n-1)} \to c\left(g^0\right) \quad \text{for } j \to \infty, \tag{5.29}$$

holds for the singular Green operators S_N and $S_{1,N}$ considered in Proposition 5.7.

It is seen that A_a^{-N} has several ingredients with different spectral asymptotics properties. Therefore we need a theorem on how eigenvalue asymptotics formulas with remainder asymptotics are perturbed when operators are added together.

This builds on a variant of a result of Ky Fan [18].

Lemma 5.10. *If Q, B, and S are bounded selfadjoint operators whose spectra on $\mathbb{R}_+$ are discrete, and $Q = B + S$, then one has for the positive eigenvalues $\mu_{j,+}$, arranged nonincreasingly and repeated according to multiplicity*:

$$\mu_{j+k-1,+}(B+S) \leqslant \mu_{j,+}(B) + \mu_{k,+}(S), \tag{5.30}$$

for all j, k such that the eigenvalues exist.

If S has a finite number $K \geqslant 0$ of positive eigenvalues, then

$$\mu_{j+K,+}(B+S) \leqslant \mu_{j,+}(B), \tag{5.31}$$

for all j such that the eigenvalues exist.

Proof. The l'th positive eigenvalue of Q is characterized by

$$\mu_{l,+}(Q) = \min_{u_1,\ldots,u_{l-1}\in H} \max\bigl\{(Qu,u) \bigm| \|u\|=1,\ u \perp u_1,\ldots,u_{l-1}\bigr\}, \tag{5.32}$$

as long as this expression is positive; it is reached when the $u_1,\ldots,u_{l-1}$ are an orthogonal system of eigenvectors for the first $l-1$ positive eigenvalues. Let $x_1,\ldots,x_{j-1}$ be an orthogonal system of eigenvectors for the first $j-1$ positive eigenvalues of B, and let $y_1,\ldots,y_{k-1}$ be an orthogonal system of eigenvectors for the first $k-1$ positive eigenvalues of S. Then since $Q=B+S$, we have in view of (5.32):

$$\begin{aligned}\mu_{j+k-1,+}(Q) &\leqslant \max\bigl\{(Qu,u) \bigm| \|u\|=1,\ u\perp x_1,\ldots,x_{j-1},y_1,\ldots,y_{k-1}\bigr\}\\ &\leqslant \max\bigl\{(Bu,u) \bigm| \|u\|=1,\ u\perp x_1,\ldots,x_{j-1}\bigr\}\\ &\quad+\max\bigl\{(Su,u) \bigm| \|u\|=1,\ u\perp y_1,\ldots,y_{k-1}\bigr\}\\ &=\mu_{j,+}(B)+\mu_{k,+}(S),\end{aligned} \tag{5.33}$$

showing (5.30). The last statement in case $K=0$ follows from (5.32), since $(Su,u)\leqslant 0$ then. For $K>0$ it follows from the calculation in (5.33) with $k-1=K$. $\square$

We use this to show, as a variant of [29, Prop. 6.1]:

Proposition 5.11. *Let Q, B, and S be bounded selfadjoint operators such that $Q=B+S$, where the spectrum of B in $\mathbb{R}_+$ is discrete, with eigenvalues $\mu_{j,+}(B)\searrow 0$, and S is compact. Assume that, with $\beta>\alpha>0$, $\gamma>\alpha$, and a positive constant C,*

$$\mu_{j,+}(B)-Cj^{-\alpha} \text{ is } O\bigl(j^{-\beta}\bigr) \quad \text{for } j\to\infty, \tag{5.34}$$

$$s_j(S) \text{ is } O\bigl(j^{-\gamma}\bigr) \quad \text{for } j\to\infty. \tag{5.35}$$

Then

$$\mu_{j,+}(Q)-Cj^{-\alpha} \text{ is } O\bigl(j^{-\beta'}\bigr) \quad \text{for } j\to\infty, \tag{5.36}$$

with

$$\beta'=\min\bigl\{\beta,\gamma(1+\alpha)/(1+\gamma)\bigr\}; \tag{5.37}$$

here $\beta'\in\,]\alpha,\beta]$.

Proof. By hypothesis, B has infinitely many positive eigenvalues. If S has so too, we proceed as in [29, Prop. 6.1]: Let $d\in\,]0,1[$, to be chosen later. For each $l\in\mathbb{N}$, let $k=[l^d]+1$ and let $j=l-[l^d]$ in (5.30). Then (5.34)–(5.35) imply by use of (5.22):

$$\begin{aligned}\mu_{l,+}(Q) &\leqslant C\big(l-\big[l^d\big]\big)^{-\alpha}+c_2\big(l-\big[l^d\big]\big)^{-\beta}+c_3\big(\big[l^d\big]+1\big)^{-\gamma}\\ &\leqslant Cl^{-\alpha}\big(1-\big[l^d\big]/l\big)^{-\alpha}+c_2l^{-\beta}\big(1-\big[l^d\big]/l\big)^{-\beta}+c_3l^{-d\gamma}\\ &\leqslant Cl^{-\alpha}+c_2l^{-\beta}+c_4l^{d-\alpha-1}+c_5l^{d-\beta-1}+c_3l^{-d\gamma}\\ &\leqslant Cl^{-\alpha}+c_6l^{-\beta'},\end{aligned}$$

where $\beta'=\min\{\beta,\alpha-d+1,\beta-d+1,d\gamma\}$. Taking $d=(1+\alpha)/(1+\gamma)$, we have (5.37).

If S has a finite number K of positive eigenvalues, we have if $K=0$ that

$$\mu_{j,+}(Q)\leqslant\mu_{j,+}(B)\leqslant Cj^{-\alpha}+c_1j^{-\beta},\tag{5.38}$$

and if $K>0$, for $j\geqslant K$, by (5.31),

$$\begin{aligned}\mu_{j,+}(Q)-Cj^{-\alpha}&\leqslant\mu_{j-K,+}(B)-Cj^{-\alpha}\leqslant C(j-K)^{-\alpha}-Cj^{-\alpha}+c_1(j-K)^{-\beta}\\ &=Cj^{-\alpha}\big[(1-K/j)^{-\alpha}-1\big]+c_1j^{-\beta}(1-K/j)^{-\beta}\\ &\leqslant c_2j^{-\alpha-1}+c_1j^{-\beta}+c_3j^{-\beta-1}\leqslant c_4j^{-\beta''},\end{aligned}\tag{5.39}$$

with $\beta''=\min\{\alpha+1,\beta\}>\beta'$, since $\alpha+1>\gamma(\alpha+1)/(\gamma+1)$.

This shows the desired upper estimate. A similar lower estimate is obtained by noting that Lemma 5.10 applied to $B=Q+(-S)$ gives

$$\mu_{j,+}(Q)\geqslant\mu_{j+k-1,+}(B)-\mu_{k,+}(-S).\qquad\square$$

If needed, one can of course use the finer estimates (5.38) or (5.39) in appropriate situations.

The results will first be used to give an eigenvalue estimate for $\mathrm{pr}_R\,A_\gamma^{-N}\,\mathrm{pr}_R$:

Proposition 5.12. $B_{1,N}=\mathrm{pr}_R\,A_\gamma^{-N}\,\mathrm{pr}_R$ *is a nonnegative compact selfadjoint operator whose positive eigenvalues satisfy, with* $c'_A=c_A^{2mN/n}$, c_A *defined by* (5.6):

$$\mu_{j,+}(B_{1,N})-c'_Aj^{-2mN/n}\ \textit{is}\ O\big(j^{-(2mN+\theta_N)/n}\big)\quad\textit{for}\ j\to\infty,\tag{5.40}$$

where $\theta_N=2mN/(2mN+n-1)$.

Proof. Since pr_R is bounded and A_γ^{-N} is compact, $B_{1,N}$ is compact. The nonnegativity follows since $A_\gamma^{-N}\geqslant 0$ so that

$$(B_{1,N}u,u)=\big(\mathrm{pr}_R\,A_\gamma^{-N}\,\mathrm{pr}_R\,u,u\big)=\big(A_\gamma^{-N}\,\mathrm{pr}_R\,u,\mathrm{pr}_R\,u\big)\geqslant 0,$$

for all $u\in H$. For the eigenvalue asymptotics, we use the decomposition in the first line of (5.27), where A_γ^{-N} has the spectral behavior inferred from (5.7):

$$\mu_j\big(A_\gamma^{-N}\big)-c_A^{2mN/n}j^{-2mN/n}=O\big(j^{-(2mN+1)/n}\big)\quad\text{for } j\to\infty,$$

and $S_{1,N}$ has the spectral behavior (5.29), by Corollary 5.9. We can then apply Proposition 5.11 with

$$\alpha=2mN/n,\qquad\beta=(2mN+1)/n,\qquad\gamma=2mN/(n-1).\tag{5.41}$$

Since

$$\frac{\gamma(1+\alpha)}{1+\gamma} = \frac{2mN}{n-1}\frac{1+2mN/n}{1+2mN/(n-1)} = \frac{2mN+2mN/(2mN+n-1)}{n} < \beta = \frac{2mN+1}{n},$$

we have that

$$\beta' = (2mN+\theta_N)/n \quad \text{with } \theta_N = 2mN/(2mN+n-1). \qquad \square$$

Next, we treat the full operator A_a^{-N}. The study is the easiest to complete when $a<0$.

Theorem 5.13. *Consider A_a^{-N}; it equals $B+S_N$ with $B=B_{1,N}+B_{2,N}$ and S_N as in Proposition* 5.7. *Assume that $a<0$. Then when N is odd,*

$$\mu_{j,+}\big(A_a^{-N}\big) - c'_A j^{-2mN/n} \text{ is } O\big(j^{-(2mN+\theta_N)/n}\big) \quad \text{for } j\to\infty, \tag{5.42}$$

with $\theta_N = 2mN/(2mN+n-1)$, $c'_A = c_A^{2mN/n}$, c_A defined in (5.6).

Proof. For $B_{1,N}$ we have the asymptotic eigenvalue estimate in Proposition 5.12. We add $B_{2,N}$ to $B_{1,N}$, which just adjoins the negative eigenvalue a^{-N} with infinite multiplicity. With $B=B_{1,N}+B_{2,N}$, we now apply Proposition 5.11 to the sum $A_a^{-N}=Q=B+S_N$, with $\beta=(2mN+\theta_N)/n$. This gives (5.36), with

$$\beta' = \min\left\{\beta, \frac{2mN}{n-1}\frac{1+2mN/n}{1+2mN/(n-1)}\right\} = \beta. \qquad \square$$

The cases where $a>0$, or N is even so that $a^N>0$, are handled by transforming the problem into one where the eigenvalue sequence we want to describe runs outside the interval containing the essential spectrum.

Theorem 5.14. *The conclusion of Theorem* 5.13 *holds also when N is even and when $a>0$.*

Proof. It remains to treat the cases where $a^N>0$. Let b be a point in the interval $]0,a^{-N}[$ which is in the resolvent set of both B and $Q=B+S_N$. Replace B and Q by

$$B' = b^2(b-B)^{-1} - b = bB(b-B)^{-1}, \qquad Q' = b^2(b-Q)^{-1} - b = bQ(b-Q)^{-1}. \tag{5.43}$$

Then the point a^{-N} in the essential spectrum is moved to $ba^{-N}(b-a^{-N})^{-1}<0$, whereas the point 0 is preserved, and the sequence of positive eigenvalues $\mu_{j,+}(B)$ decreasing to 0 in the interval $]0,b[$ is turned into the sequence of positive eigenvalues

$$\mu_{j,+}\big(B'\big) = b\mu_{j,+}(B)\big(b-\mu_{j,+}(B)\big)^{-1} \searrow 0. \tag{5.44}$$

The operators B' and Q' are of the type treated in Lemma 5.10, their difference being the compact operator

$$S'_N = Q' - B' = b^2(b-B-S_N)^{-1} - b^2(b-B)^{-1} = b^2(b-B-S_N)^{-1}S_N(b-B)^{-1}. \tag{5.45}$$

Concerning their asymptotic eigenvalue properties, we have that (5.34) implies

$$\begin{aligned}\mu_{j,+}(B') - Cj^{-\alpha} &= b\mu_{j,+}(B)\big(b - \mu_{j,+}(B)\big)^{-1} - Cj^{-\alpha} \\ &= \mu_{j,+}(B) - Cj^{-\alpha} + \mu_{j,+}(B)\big[b\big(b - \mu_{j,+}(B)\big)^{-1} - 1\big] \\ &= \mu_{j,+}(B) - Cj^{-\alpha} + \mu_{j,+}(B)^2\big(b - \mu_{j,+}(B)\big)^{-1} \\ &= O\big(j^{-\beta}\big) + O\big(j^{-2\alpha}\big). \end{aligned} \tag{5.46}$$

This will be used with $C = c'_A$ and exponents as in (5.40), $\alpha = 2mN/n$ and $\beta = (2mN + \theta_N)/n$. Clearly $2\alpha > \beta$, so then

$$\mu_{j,+}\big(B'\big) - c'_A j^{-2mN/n} = O\big(j^{-(2mN+\theta_N)/n}\big).$$

Since S'_N equals S_N composed with bounded operators, the estimate (5.35) implies a similar estimate for S'_N. Now Proposition 5.11 can be applied, with α and β as already indicated, and $\gamma = 2mN/(n-1)$, showing that the positive eigenvalues of Q' have the behavior

$$\mu_{j,+}\big(Q'\big) - c'_A j^{-2mN/n} = O\big(j^{-(2mN+\theta_N)/n}\big) \quad \text{for } j \to \infty. \tag{5.47}$$

Finally this is carried over to the desired behavior of the eigenvalue sequence $\mu_{j,+}(Q)$ by a calculation similar to (5.46), using that

$$Q = bQ'\big(Q' + b^{-1}\big)^{-1}. \qquad \square$$

This has the following implications for the counting functions for eigenvalues of A_a^N going to ∞:

Theorem 5.15. *Let N be a positive integer, and let $r^N > a^N$. The number $N_{+,r^N}(t; A_a^N)$ of eigenvalues of A_a^N in $[r^N, t]$ behaves asymptotically as follows*:

$$N_{+,r^N}\big(t; A_a^N\big) - c_A t^{n/2mN} = O\big(t^{(n-\theta_N)/2mN}\big) \quad \textit{for } t \to \infty, \tag{5.48}$$

with $\theta_N = 2mN/(2mN + n - 1)$, c_A defined by (5.6).

Proof. When $a^N < 0$, the spectrum of A_a^N is discrete on $\mathbb{R}_+$, and we can apply Corollary 5.5 directly to (5.42), concluding (5.48) for $r = 0$. A replacement of 0 by some other $r^N > a^N$ only shifts N_+ by a fixed finite number, and does not change the asymptotic property.

Now let $a^N > 0$ and take an $r > |a|$, such that r^{-N} is not in the spectra of A_a^{-N} and $\mathrm{pr}_R A_\gamma^{-N} \mathrm{pr}_R$. For this r, the number $N_{+,r^N}(t; A_a^N)$ is the number of eigenvalues of $A_a^N - r^N$ in $[0, t - r^N]$.

Observe that when we take $b = r^{-N}$ in the proof of Theorem 5.14, then

$$Q = A_a^{-N}, \qquad Q' = r^{-N} A_a^{-N}\big(r^{-N} - A_a^{-N}\big)^{-1} = \big(A_a^N - r^N\big)^{-1}.$$

For Q' we have the asymptotic estimate (5.47). Then we can apply Corollary 5.5 to $A_a^N - r^N$ and its inverse Q', concluding that

$$N_{+,r^N}\big(t; A_a^N\big) - c_A\big(t - r^N\big)^{n/2mN} = O\big(\big(t - r^N\big)^{(n-\theta_N)/2mN}\big) \quad \text{for } t \to \infty.$$

This implies (5.48), since $(t - r^N)^s = t^s(1 - r^N/t)^s = t^s + O(t^{s-1})$ by (5.22). $\square$

We can finally conclude an improved estimate for A_a itself:

Theorem 5.16. *Let $r > a$. The number $N_{+,r}(t; A_a)$ of eigenvalues of A_a in $[r, t]$ behaves asymptotically as follows, for any $\varepsilon > 0$:*

$$N_{+,r}(t; A_a) - c_A t^{n/2m} = O\big(t^{(n-1+\varepsilon)/2m}\big) \quad \textit{for } t \to \infty, \tag{5.49}$$

with c_A defined in (5.6).

Proof. It suffices to consider $r > |a|$. Since the number of eigenvalues of A_a in $[r, t]$ is the same as the number of eigenvalues of A_a^N in $[r^N, t^N]$, we conclude from Theorem 5.15 that

$$\begin{aligned} N_{+,r}(A_a; t) - c_A t^{n/2m} &= N_{+,r^N}\big(A_a^N; t^N\big) - c_A\big(t^N\big)^{n/2mN} = O\big(\big(t^N\big)^{(n-\theta_N)/2mN}\big) \\ &= O\big(t^{(n-\theta_N)/2m}\big). \end{aligned}$$

Here N can be taken arbitrarily large. Since $\theta_N = 1 - (n-1)/(2mN + n - 1) \to 1$ for $N \to \infty$, it can for any $\varepsilon > 0$ be obtained to be $> 1 - \varepsilon$, which shows the statement in the theorem. □

This ends our study of eigenvalue asymptotics for A_a, $a \neq 0$.

Actually, some of the above techniques can also be used to improve the result of [30] for A_0, so we include this here.

Theorem 5.17. *For the discrete eigenvalue sequence of the Krein–von Neumann extension A_0, the number $N_+(t; A_0)$ of eigenvalues in $]0, t]$ satisfies, for any $\varepsilon > 0$,*

$$N_+(t; A_0) - c_A t^{n/2m} = O\big(t^{(n-1+\varepsilon)/2m}\big) \quad \textit{for } t \to \infty. \tag{5.50}$$

Proof. We here use some further rules for eigenvalues and s-numbers, found e.g. in Gohberg and Krein [22]. Denote the positive eigenvalues $\lambda_j(A_0)$, $j = 1, 2, \ldots$. It is shown in [30] that their inverses are the eigenvalues $\mu_j(S_0)$, where $S_0 = R_\varrho^{1/2} A R_\varrho^{1/2}$ as recalled in (5.14); this was used in [30] to show the estimate (5.8). Here $R_\varrho^{\frac12}$ maps $L_2(\Omega)$ bijectively onto $H_0^{2m}(\Omega)$, and the factor A is really $A_{\min}$ mapping $H_0^{2m}(\Omega)$ bijectively onto $R = \operatorname{ran} A_{\min}$, where one can apply $R_\varrho^{\frac12}$. They also define mappings between the spaces intersected with higher-order Sobolev spaces.

In addition to S_0 we shall study iterates of S_0. For $2N$'th powers we can write

$$S_0^{2N} = \big(R_\varrho^{\frac12} A R_\varrho^{\frac12}\big)^{2N} = R_\varrho^{\frac12}(AR_\varrho)^{2N-1} A R_\varrho^{\frac12} = B_N A R_\varrho A \check{B}_N,$$

where

$$B_N = R_\varrho^{\frac12}(AR_\varrho)^{N-1}, \qquad \check{B}_N = (R_\varrho A)^{N-1} R_\varrho^{\frac12}.$$

Here we recognize $AR_\varrho A$ as the projection $\mathrm{pr}_R = I - \mathrm{pr}_Z$, cf. Proposition 5.6. Then

$$S_0^{2N} = B_N(I - \mathrm{pr}_Z)\check{B}_N = B_N\check{B}_N - B_N \,\mathrm{pr}_Z\, \check{B}_N. \tag{5.51}$$

The first term is a compact nonnegative operator whose positive eigenvalues satisfy:

$$\mu_j(B_N\check{B}_N) = \mu_j(\check{B}_N B_N) = \mu_j\big((R_\varrho A)^{N-1} R_\varrho (AR_\varrho)^{N-1}\big).$$

The operator $(R_\varrho A)^{N-1}R_\varrho(AR_\varrho)^{N-1}$ is of the form $A_+^{(-2N)}+G_{2N}$, where $A_+^{(-2N)}$ is the truncation to Ω of a parametrix $A^{(-2N)}$ of A^{2N} (as used earlier in the proof of Proposition 5.7), and G_{2N} is a singular Green operator of order $-4mN$ and class 0. Then by Corollary 4.5.6 of [33] we have the asymptotic eigenvalue estimate (in view of Corollary 5.5):

$$\mu_j(B_N\check{B}_N)=\mu_j\big(A_+^{(-2N)}+G_{2N}\big)=c'_A j^{-4mN/n}+O\big(j^{-(4mN+1-\varepsilon)/n}\big)\quad\text{for } j\to\infty,$$

for any $\varepsilon>0$, with $c'_A=c_A^{4mN/n}$. (It is used here that A is a scalar differential operator, see the discussion in [33, Rem. 4.5.5] concerning systems.)

For the second term $B_N\operatorname{pr}_Z\check{B}_N$ we use that there exists a homeomorphism

$$\Lambda^{2m}_{-,+}:H^{2m+s}(\Omega)\xrightarrow{\sim}H^s(\Omega),\quad\text{with inverse }\Lambda^{-2m}_{-,+},\ \text{for any } s\in\mathbb{R},$$

belonging to the ψdbo calculus, as introduced in [32] (also explained in Section 2.5 of [33]). Then

$$\begin{aligned}B_N\operatorname{pr}_Z\check{B}_N&=R_\varrho^{\frac12}\Lambda^{2m}_{-,+}\Lambda^{-2m}_{-,+}(AR_\varrho)^{N-1}\operatorname{pr}_Z(R_\varrho A)^{N-1}\Lambda^{-2m}_{-,+}\Lambda^{2m}_{-,+}R_\varrho^{\frac12}\\&=R_\varrho^{\frac12}\Lambda^{2m}_{-,+}\widetilde{G}_{2N}\Lambda^{2m}_{-,+}R_\varrho^{\frac12},\end{aligned}$$

where $\widetilde{G}_{2N}$ is a singular Green operator of order $-4mN$ and class 0. The operators $R_\varrho^{\frac12}\Lambda^{2m}_{-,+}$ and $\Lambda^{2m}_{-,+}R_\varrho^{\frac12}$ are bounded in $L_2(\Omega)$. Using Theorem 5.8 for $\widetilde{G}_{2N}$ together with the general rule $s_j(EGF)\leqslant\|E\|s_j(G)\|F\|$, we find:

$$s_j(B_N\operatorname{pr}_Z\check{B}_N)\leqslant Cs_j(\widetilde{G}_{2N})\leqslant C'j^{-4mN/(n-1)}.$$

Now the perturbation result Proposition 5.11 applied to the decomposition in (5.51) gives (as in the proof of Theorem 5.13):

$$\mu_j\big(S_0^{2N}\big)-c_A^{4mN/n}j^{-4mN/n}=O\big(j^{-(4mN+\theta_{2N})/n}\big)\quad\text{for } j\to\infty;\ \text{with } c'_A=c_A^{4mN/n},$$

with the usual $\theta_{2N}=4mN/(4mN+n-1)$, and hence (as in the proof of Theorem 5.16)

$$N_+\big(t;S_0^{-1}\big)=N_+\big(t^{2N};S_0^{-2N}\big)=c_At^{n/2m}+O\big(t^{(n-\theta_{2N})/2m}\big),\quad\text{for } t\to\infty.$$

Since $\theta_{2N}\to 1$ for $N\to\infty$, and N can be taken arbitrarily large, the assertion of the theorem follows. $\square$

The validity of the improved estimate (5.50) has been announced by Mikhailets in [48]; we have recently been informed that proof details are in [49].

The spectral results in this section are formulated for a bounded domain Ω in $\mathbb{R}^n$, but the methods work for general compact manifolds with boundary, as in [33], so the results are valid for such cases too.

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References

[1] H. Abels, G. Grubb, I. Wood, Extension theory and Kreĭn-type resolvent formulas for nonsmooth boundary value problems, in press, arXiv:1008.3281.
[2] A. Alonso, B. Simon, The Birman–Krein–Vishik theory of selfadjoint extensions of semibounded operators, J. Operator Theory 4 (1980) 251–270.
[3] A. Alonso, B. Simon, Addenda to "The Birman–Krein–Vishik theory of selfadjoint extensions of semibounded operators", J. Operator Theory 6 (1981) 461.
[4] W.O. Amrein, D.B. Pearson, M operators: a generalisation of Weyl–Titchmarsh theory, J. Comput. Appl. Math. 171 (2004) 1–26.
[5] Yu.M. Arlinskii, On functions connected with sectorial operators and their extensions, Integral Equations Operator Theory 33 (1999) 125–152.
[6] M.S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, G. Teschl, The Krein–von Neumann extension and its connection to an abstract buckling problem, Math. Nachr. 283 (2010) 165–179.
[7] M.S. Ashbaugh, F. Gesztesy, M. Mitrea, G. Teschl, Spectral theory for perturbed Krein Laplacians in nonsmooth domains, Adv. Math. 223 (2010) 1372–1467.
[8] J. Behrndt, M. Langer, Boundary value problems for elliptic partial differential operators on bounded domains, J. Funct. Anal. 243 (2007) 536–565.
[9] M.S. Birman, On the theory of self-adjoint extensions of positive definite operators, Mat. Sb. (N.S.) 38 (80) (1956) 431–450 (in Russian).
[10] M.S. Birman, Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions, Vestn. Leningr. Univ. 17 (1962) 22–55; English translation in: Spectral Theory of Differential Operators, Amer. Math. Soc. Transl. Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 19–53.
[11] M.S. Birman, M.Z. Solomyak, Asymptotic properties of the spectrum of differential equations, Akad. Nauk SSSR Vsesojuz. Inst. Naucn. i Tehn. Informacii, Moscow 14 (1977) 5–58; English translation in: J. Sov. Math. 12 (1979) 247–283.
[12] M.S. Birman, M.Z. Solomyak, Asymptotics of the spectrum of variational problems on solutions of elliptic equations in unbounded domains, Funktsional. Anal. i Prilozhen. 14 (1980) 27–35; English translation in: Funct. Anal. Appl. 14 (1981) 267–274.
[13] L. Boutet de Monvel, Boundary problems for pseudodifferential operators, Acta Math. 126 (1971) 11–51.
[14] B.M. Brown, G. Grubb, I.G. Wood, M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems, Math. Nachr. 282 (2009) 314–347.
[15] B.M. Brown, M. Marletta, S. Naboko, I.G. Wood, Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices, J. Lond. Math. Soc. 77 (2008) 700–718.
[16] J. Brüning, V. Geyler, K. Pankrashkin, Spectra of self-adjoint extensions and applications to solvable Schrödinger operators, Rev. Math. Phys. 20 (2008) 1–70.
[17] V.A. Derkach, M.M. Malamud, Generalized resolvents and the boundary value problems for Hermitian operators with gaps, J. Funct. Anal. 95 (1991) 1–95.
[18] Ky Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, Proc. Natl. Acad. Sci. USA 37 (1951) 760–766.
[19] K. Friedrichs, Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren, Math. Ann. 109 (1934) 465–487.
[20] F. Gesztesy, M. Mitrea, Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains, in: D. Mitrea, M. Mitrea (Eds.), Perspectives in Partial Differential Equations, Harmonic Analysis and Applications: A Volume in Honor of Vladimir G. Maz'ya's 70th Birthday, in: Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 105–173.
[21] F. Gesztesy, M. Mitrea, A description of all selfadjoint extensions of the Laplacian and Kreĭn-type resolvent formulas in nonsmooth domains, J. Anal. Math. 113 (2011) 53–172.
[22] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, translated from the Russian by A. Feinstein, Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI, 1969, 378 pp.
[23] M.L. Gorbachuk, V.A. Mikhailets, Semibounded selfadjoint extensions of symmetric operators, Dokl. Akad. Nauk SSSR 226 (1976); English translation in: Sov. Math. Dokl. 17 (1976) 185–186.
[24] V.I. Gorbachuk, M.L. Gorbachuk, Boundary Value Problems for Operator Differential Equations, Kluwer, Dordrecht, 1991.
[25] G. Grubb, A characterization of the non-local boundary value problems associated with an elliptic operator, Ann. Sc. Norm. Super. Pisa 22 (1968) 425–513.
[26] G. Grubb, Les problèmes aux limites généraux d'un opérateur elliptique, provenant de la théorie variationnelle, Bull. Sci. Math. 94 (1970) 113–157.
[27] G. Grubb, On coerciveness and semiboundedness of general boundary value problems, Israel J. Math. 10 (1971) 32–95.
[28] G. Grubb, Properties of normal boundary problems for elliptic even-order systems, Ann. Sc. Norm. Super. Pisa (Ser. IV) 1 (1974) 1–61.
[29] G. Grubb, Remainder estimates for eigenvalues and kernels of pseudo-differential elliptic systems, Math. Scand. 43 (1978) 275–307.
[30] G. Grubb, Spectral asymptotics for the "soft" selfadjoint extension of a symmetric elliptic differential operator, J. Operator Theory 10 (1983) 9–20.
[31] G. Grubb, Singular Green operators and their spectral asymptotics, Duke Math. J. 51 (1984) 477–528.
[32] G. Grubb, Pseudo-differential boundary problems in L_p spaces, Comm. Partial Differential Equations 15 (1990) 289–340.

[33] G. Grubb, Functional Calculus of Pseudodifferential Boundary Problems, second edition, Progr. Math., vol. 65, Birkhäuser, Boston, 1996.
[34] G. Grubb, Krein resolvent formulas for elliptic boundary problems in nonsmooth domains, Rend. Semin. Mat. Univ. Politec. Torino 66 (2008) 13–39.
[35] G. Grubb, Distributions and Operators, Grad. Texts in Math., vol. 252, Springer, New York, 2009.
[36] G. Grubb, Perturbation of essential spectra of exterior elliptic problems, J. Appl. Anal. 90 (2011) 103–123.
[37] G. Grubb, R.T. Seeley, Weakly parametric pseudodifferential operators and Atiyah–Patodi–Singer boundary problems, Invent. Math. 121 (1995) 481–529.
[38] L. Hörmander, Linear Partial Differential Operators, Grundlehren Math. Wiss., vol. 116, Springer-Verlag, Berlin, 1963.
[39] L. Hörmander, The Analysis of Linear Partial Differential Operators IV, Fourier Integral Operators, Grundlehren Math. Wiss., vol. 275, Springer-Verlag, Berlin, 1985.
[40] A.N. Kočubeĭ, Extensions of symmetric operators and symmetric binary relations, Math. Notes (1) 17 (1975) 25–28.
[41] V.A. Kozlov, Estimation of the remainder in a formula for the asymptotic behavior of the spectrum of nonsemibounded elliptic systems, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. (1979) 112–113.
[42] V.A. Kozlov, Remainder estimates in formulas for the asymptotic behavior of the spectrum for linear operator pencils, Funktsional. Anal. i Prilozhen. 17 (1983) 80–81; English translation in: Funct. Anal. Appl. 17 (1983) 147–149.
[43] M.G. Krein, The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I, Mat. Sb. 20 (1947) 431–495 (in Russian).
[44] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol. 1, Éditions Dunod, Paris, 1968.
[45] V.E. Lyantze, O.G. Storozh, Methods of the Theory of Unbounded Operators, Naukova Dumka, Kiev, 1983 (in Russian).
[46] M.M. Malamud, Spectral theory of elliptic operators in exterior domains, Russ. J. Math. Phys. 17 (2010) 96–125.
[47] M.M. Malamud, V.I. Mogilevskii, Kreĭn type formula for canonical resolvents of dual pairs of linear relations, Methods Funct. Anal. Topology (4) 8 (2002) 72–100.
[48] V.A. Mikhailets, Distribution of the eigenvalues of finite multiplicity of Neumann extensions of an elliptic operator, Differ. Uravn. 30 (1994) 178–179; English translation in: Differential Integral Equations 30 (1994) 167–168.
[49] V.A. Mikhailets, Discrete spectrum of the extreme nonnegative extension of the positive elliptic differential operator, in: Proceedings of the Ukrainian Mathematical Congress 2001, Section 7, Nonlinear Analysis, Kyiv, 2006, pp. 80–94 (in Russian).
[50] J. von Neumann, Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann. 102 (1929) 49–131.
[51] Å. Pleijel, Certain Indefinite Differential Eigenvalue Problems – The Asymptotic Distribution of Their Eigenfunctions, Part. Diff. Equ. and Continuum Mech., Wisconsin Press, Madison, 1961, pp. 19–37.
[52] A. Posilicano, L. Raimondi, Krein's resolvent formula for self-adjoint extensions of symmetric second-order elliptic differential operators, J. Phys. A 42 (2009) 015204, 11 pp.
[53] V. Ryzhov, A general boundary value problem and its Weyl function, Opuscula Math. 27 (2007) 305–331.
[54] L.I. Vainerman, On extensions of closed operators in Hilbert space, Math. Notes 28 (1980) 871–875.
[55] M.I. Vishik, On general boundary value problems for elliptic differential operators, Trudy Mosc. Mat. Obsv. 1 (1952) 187–246; English translation in: Amer. Math. Soc. Transl. (2) 24 (1963) 107–172.