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Global stability of viscous contact wave for 1-D compressible Navier–Stokes equations

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ABSTRACT

The viscous contact waves for one-dimensional compressible Navier–Stokes equations has recently been shown to be asymptotically stable. The stability results are called local stability or global stability depending on whether the norms of initial perturbations are small or not. Up to now, local stability results toward viscous contact waves of compressible Navier–Stokes equations have been well established (see Huang et al., 2006, 2008, 2009 [9,10,7]), but there are few results for the global stability in the case of Cauchy problem which is the purpose of this paper. The proof is based on an elementary energy method using an inequality concerning the heat kernel (see Lemma 1 of Huang et al., 2010 [7]).

1. Introduction

Consider the one-dimensional compressible Navier–Stokes equations in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu\left(\dfrac{u_x}{v}\right)_x, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = \left(\kappa\dfrac{\theta_x}{v} + \mu\dfrac{uu_x}{v}\right)_x, \end{cases} \tag{1.1}$$

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where $x \in R^1 = (-\infty, \infty)$, $t > 0$, $v(x,t) > 0$, $u(x,t)$, $e(x,t) > 0$, $\theta(x,t) > 0$ and $p(x,t)$ are the specific volume, the velocity, the internal energy, the temperature and the pressure of the gas respectively, while $\mu > 0$ and $\kappa > 0$ denote the viscosity and the heat conductivity respectively. Here we study the ideal polytropic gas so that the pressure p, the internal energy e and the entropy s are given by

$$p = \frac{R\theta}{v}, \qquad e = \frac{R}{\gamma - 1}\theta, \qquad s = \frac{R}{\gamma - 1}\ln\theta + R\ln v,$$

where $R > 0$ is the gas constant and $\gamma > 1$ is the adiabatic exponent. We are concerned with the Cauchy problem to the system (1.1) supplemented with the following initial data and far field conditions:

$$\begin{cases} (v,u,\theta)(x,0) = (v_0,u_0,\theta_0)(x), \\ (v,u,\theta)(\pm\infty,t) = (v_\pm,u_\pm,\theta_\pm), \end{cases} \tag{1.2}$$

where $v_\pm > 0$, $u_\pm$ and $\theta_\pm > 0$ are given constants, and we impose $(v_0,u_0,\theta_0)(\pm\infty) = (v_\pm,u_\pm,\theta_\pm)$ as compatibility condition.

In this paper, we are interested in the large-time behavior of solutions to the Cauchy problem (1.1), (1.2) for one-dimensional compressible Navier–Stokes equations. It is known that the asymptotic behavior is well characterized by the solutions to the corresponding Riemann problem for the hyperbolic part of (1.1) (that is, Euler system):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = 0, \end{cases} \tag{1.3}$$

which is one of the most important examples for systems of hyperbolic conservation laws of the form

$$Z_t + f(Z)_x = 0, \quad Z = (z_1, \ldots, z_n) \in R^n. \tag{1.4}$$

It is well known that (see [26]) the system (1.4) has three basic wave patterns: two nonlinear waves (shock and rarefaction wave) and a linearly degenerate wave (contact discontinuity). These dilation invariant solutions and their superpositions in the increasing order of characteristic speed which are called Riemann solutions, govern both the local and large time asymptotic behavior of general solutions to the system of hyperbolic conservation laws (1.4) (see [16]). Since the inviscid system (1.4) is an idealization when the dissipative effects are neglected, thus it is of great importance to study the large time asymptotic behavior of solutions to the corresponding viscous systems in the form of

$$Z_t + f(Z)_x = \big(B(Z)Z_x\big)_x, \quad Z = (z_1, \ldots, z_n) \in R^n, \tag{1.5}$$

toward the viscous versions of these basic waves. In particular, such an asymptotic behavior will be important for the compressible Navier–Stokes system (1.1) which is basically the system governing viscous fluid flows when the effects of both viscosity and heat conductivity are taken into account.

Indeed, there have been intensive studies for the stability toward basic wave patterns of the system (1.5) of viscous conservation laws which is started with studies on the stability of nonlinear waves to the Cauchy problems for scalar conservation laws by Il'in and Oleinik [12] in 1960s.

In the case where the Riemann solution of the system (1.4) consists of only shock waves, the viscous versions of shock waves corresponding to the system (1.5) are the so-called viscous shock waves satisfying a system of ordinary differential equations with two given end-states. In different settings, the local stability of viscous shock waves has been established. We refer to [22,18,14,8] for the compressible Navier–Stokes equations (1.1) and in [17,5,27,19] for the system of viscous conservation laws with artificial viscosity, respectively.

In the case where the Riemann solution of the compressible Euler equations (1.3) consists of only rarefaction waves, the local and global stability results of rarefaction waves were obtained in [23,15,20,24,28,25,3] for different settings. Moreover, Nishihara, Yang and Zhao [25] and Duan, Liu and Zhao [3] showed the H^1-global stability results in the case of general gases and also L_∞-global stability was established for the perfect fluids provided that the adiabatic exponent γ is close to 1 or for the isentropic perfect fluids. Here, the H^1- (or L_∞-) global stability means that H^1- (or L_∞-) norms of initial perturbations are large. Since it does not require the strength of the rarefaction waves to be small, these results show the nonlinear stability of strong rarefaction waves for the one-dimensional compressible Navier–Stokes equations. For the local stability toward rarefaction waves of the system of viscous conservation laws, we refer to [29,28,2].

However, compared to the works on the stability of nonlinear waves (shock and rarefaction waves), the stability of contact discontinuities is more subtle and began to be studied since the middle of 1990s. The local stability of a weak contact discontinuity for the compressible Euler equations with uniform viscosity was first studied by Xin [30]. This was later generalized by Liu and Xin [21] to show the local stability of the contact discontinuities for the system (1.5) of viscous conservation laws with artificial viscosity. Recently, the local stability of the superposition of contact discontinuities and shock waves for the system (1.5) of viscous conservation laws with artificial viscosity was proved by Zeng [31]. But these methods do not apply to the compressible Navier–Stokes system because the viscosity matrix in (1.1) is only semi-positive definite. The more satisfactory answers were obtained in [9,10,7]. It is shown by Huang, Matsumura and Xin [9] that a smooth viscous contact wave for the compressible Navier–Stokes system which approximates the given contact discontinuity for the compressible Euler equations on any finite time interval is locally stable provided that the integral of initial perturbations is zero. Here, the stability is in sup-norm and a convergence rate is also obtained. Later, this result was improved by Huang, Xin and Yang [10] in which the assumption that the integral of initial perturbation is zero is removed. The elementary energy method different from the anti-derivative method in [9,10] was recently proposed by Huang, Li and Matsumura [7]. In [7], the local stability of the superposition of contact discontinuity and rarefaction waves was proved without introducing the anti-derivative variables by a new estimates on the heat kernel.

Although considerable progress has been obtained for the stability of viscous contact waves, however most of these results are obtained for the case where the initial perturbations are small. Thus, a natural question arises: can we show similar stability of viscous contact wave for large initial perturbation? In this paper, based on the new estimates on the heat kernel in [7], we give some positive answers to this question for the Cauchy problem (1.1) and (1.2), see Theorems 2.1 and 2.2 below for details.

The rest of the paper will be arranged as follows. In the next section, we state two main results in this paper. In Section 3, we proved the first main theorem (Theorem 2.1) and the last section is devoted to prove the second main result (Theorem 2.2).

Notation. Throughout the rest of this paper, $O(1), c$ or C will be used to denote a generic positive constant independent of t and x and $c_i(\cdot,\cdot)$ or $C_i(\cdot,\cdot)$ $(i \in Z_+)$ stands for some generic constants depending only on the quantities listed in the parentheses. As long as no confusion arises, denote the usual Sobolev space with norm $\|\cdot\|_{H^k}$ by $H^k := H^k(R^1)$ and $\|\cdot\|_{H^0} = \|\cdot\|$ will be used to denote the usual L_2-norm. Finally, $\|\cdot\|_{L_p}$ and $\int \cdot\, dx$ are used to denote $\|\cdot\|_{L_p(R^1)}$ and $\int_{R^1} \cdot\, dx$, respectively.

2. Main results

We first recall the viscous contact wave of the system (1.1). The Riemann problem of system (1.3) with initial data

$$(v, u, \theta)(x, 0) = (v_\pm, u_\pm, \theta_\pm), \qquad \pm x > 0$$

admits a contact discontinuity

$$(\overline{V}, \overline{U}, \overline{\Theta})(x,t) = \begin{cases} (v_-, u_-, \theta_-), & x < 0,\ t > 0, \\ (v_+, u_+, \theta_+), & x > 0,\ t > 0, \end{cases} \tag{2.1}$$

provided that

$$v_- \neq v_+, \qquad u_- = u_+, \qquad p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}. \tag{2.2}$$

Without loss of generality, we can assume that $u_- = u_+ = 0$ from now on. In the setting of compressible Navier–Stokes system (1.1), the wave (V, U, Θ) corresponding to the contact discontinuity $(\overline{V}, \overline{U}, \overline{\Theta})$ becomes smooth and behaves as a diffusion wave due to the dissipation effect. We call this wave a "viscous contact wave". The viscous contact wave (V, U, Θ) is constructed as follows. Motivated by (2.2), we expect

$$\frac{R\Theta}{V} \approx p_+, \qquad |U|^2 \ll 1. \tag{2.3}$$

Then the leading order of energy equation $(1.1)_3$ is

$$\frac{R}{\gamma - 1}\theta_t + p_+ u_x = \kappa \left(\frac{\theta_x}{v}\right)_x. \tag{2.4}$$

Using (2.4) and the mass equation $(1.1)_1$, we get a nonlinear diffusion equation

$$\Theta_t = a\left(\frac{\Theta_x}{\Theta}\right)_x, \quad \Theta(\pm\infty, t) = \theta_\pm,\ a = \frac{\kappa p_+(\gamma - 1)}{\gamma R^2} > 0, \tag{2.5}$$

which has a unique self-similarity solution $\Theta(x,t) = \Theta(\xi)$, $\xi = \frac{x}{\sqrt{1+t}}$ due to [1,4]. Furthermore, on one hand, $\Theta(\xi)$ is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_- > \theta_+$; on the other hand, Θ satisfies

$$\begin{cases} c_1|\theta_+ - \theta_-| \leqslant (\gamma - 1)^{\frac{1}{2}} \left|\Theta_\xi(0)\right| \leqslant c_2|\theta_+ - \theta_-|, \\ (\gamma - 1)^{\frac{k-1}{2}} \left|\partial_\xi^k \Theta\right| \leqslant c_3 \left|\Theta_\xi(0)\right| \exp\left(-\dfrac{c_4 \xi^2}{\gamma - 1}\right), \quad \text{as } |\xi| \to \infty,\ k \geqslant 1, \end{cases} \tag{2.6}$$

where c_i, $i = 1, \ldots, 4$, are positive constants depending only on $\theta_\pm$. Once Θ is determined, we define V and U by

$$V = \frac{R}{p_+}\Theta, \qquad U = \frac{\kappa(\gamma - 1)}{\gamma R}\frac{\Theta_x}{\Theta}. \tag{2.7}$$

We are now in a position to state our first main results. Let

$$(\phi, \psi, \zeta) = (v - V, u - U, \theta - \Theta)$$

and for interval $I \subset [0, \infty)$, we define a function space $X(I)$ as

$$X(I) = \left\{(\phi, \psi, \zeta) \in C\left(I; H^1\right) \,\middle|\, \phi_x \in L_2(I; L_2),\ (\psi_x, \zeta_x) \in L_2\left(I; H^1\right)\right\}.$$

Then we have:

Theorem 2.1. *Let (V, U, Θ) be the viscous contact wave of compressible Navier–Stokes equations defined in* (2.7)*. Then, for any $M_0 > 0$, there exist positive constants δ_0 and ε_0 such that for any $\delta = |\theta_+ - \theta_-| \leqslant \delta_0$, if*

$$\begin{cases} (\phi, \psi, \zeta)(\cdot, 0) \in H^1, \\ \left\|(\phi, \psi, \zeta)(\cdot, 0)\right\| \leqslant \varepsilon_0, \quad \left\|(\phi_x, \psi_x, \zeta_x)(\cdot, 0)\right\| \leqslant M_0, \end{cases} \tag{2.8}$$

then the Cauchy problem (1.1), (1.2) *has a unique global solution $(v, u, \theta)(x, t)$ satisfying $(\phi, \varphi, \zeta) \in X([0, \infty))$ and*

$$\lim_{t\to\infty} \sup_{x\in R^1} \left|(v - V, u - U, \theta - \Theta)(x, t)\right| = 0. \tag{2.9}$$

Remark 2.1. In Theorem 1 of [7], the authors assume that the H^1-norm of initial perturbations is small, but we require only that the L_2-norm of initial perturbations is small. Therefore, Theorem 2.1 is a generalization of Theorem 1 of [7]. Also, we proved in [6] the stability of viscous contact wave for Cauchy problem (1.1), (1.2) in the case where L_2-norm of initial perturbations is small and that of their derivative and integral can be large. Notice that in this paper, we remove the condition on the integral of initial perturbations. It is essentially based on the elementary inequality concerning the heat kernel (see Proposition 3.1).

In Theorem 2.1, even though the H^1-norm of the initial perturbation can be large, one can conclude by employing Sobolev's inequality that the L_∞-norm of the initial perturbation is small. This implies that the nonlinear stability result obtained in Theorem 2.1 is essentially a local stability one. Thus a natural question is how to get the global stability for large perturbations in both the H^1-norm and the L_∞-norm. The second theorem shows that such a stability result holds in the case when the adiabatic exponent γ is close to 1.

Theorem 2.2. *Let $|\theta_+ - \theta_-| \leqslant m_0(\gamma - 1)$ for some constant m_0, (V, U, Θ) be the viscous contact wave defined in* (2.7) *and assume that there exist positive constants m_v, m_θ satisfying*

$$\begin{cases} (\phi, \psi, \zeta)(\cdot, 0) \in H^1, \\ 0 < m_v^{-1} \leqslant v_0(x), \quad V(x,t) \leqslant m_v, \\ 0 < m_\theta^{-1} \leqslant \theta_0(x), \quad \Theta(x,t) \leqslant m_\theta \end{cases} \tag{2.10}$$

for all $(t, x) \in [0, \infty) \times (-\infty, \infty)$. Then, there exists a positive constant δ_0 such that if $\gamma - 1 < \delta_0$, then the Cauchy problem (1.1), (1.2) *has a unique global solution $(v, u, \theta)(x, t)$ satisfying $(\phi, \varphi, \zeta) \in X([0, \infty))$ and*

$$\lim_{t\to\infty} \sup_{x\in(-\infty,\infty)} \left|(v - V, u - U, \theta - \Theta)(x, t)\right| = 0. \tag{2.11}$$

Remark 2.2. The difference $|\theta_+ - \theta_-|$ is naturally bounded by $\gamma - 1$ multiplying some constant m_0 from the physical point of view.

Remark 2.3. The same result as one in Theorem 2.2 was obtained by Huang and Zhao [11] for the initial–boundary value problem of the compressible Navier–Stokes system with a free boundary condition (see Theorem 1.2 of [11]). However, the approach cannot be applied here since the analysis in [11] depends crucially on the availability of a Poincaré-type inequality, which cannot be true for Cauchy problems.

3. Preliminaries

We first state an elementary inequality derived in Lemma 1 of Huang, Li and Matsumura [7] which will play an essential role later. Here, we state Lemma 1 of [7] by choosing special coefficient depending on $\gamma-1$ which is important in our analysis.

Proposition 3.1. *For $0<T\leqslant+\infty$, suppose that $h(x,t)$ satisfies*

$$h\in L_\infty(0,T;L_2),\qquad h_x\in L_2(0,T;L_2),\qquad h_t\in L_2\big(0,T;H^{-1}\big).$$

Then

$$\int_0^T\!\!\int h^2\omega^2\,dx\,dt\leqslant 4\pi\big\|h(0)\big\|^2+4\pi\frac{\gamma-1}{\alpha}\int_0^T\big\|h_x(t)\big\|^2\,dt+8\frac{\alpha}{\gamma-1}\int_0^T\big\langle h_t(t),hg^2(t)\big\rangle\,dt,\tag{3.1}$$

where for $\alpha>0$

$$\omega(x,t)=(1+t)^{-\frac12}\exp\left\{-\frac{\alpha x^2}{(\gamma-1)(1+t)}\right\},\qquad g(x,t)=\int_{-\infty}^x\omega(y,t)\,dy,\tag{3.2}$$

and $\langle\cdot,\cdot\rangle$ denotes the dual product between H^{-1} and H^1.

It is easy to check that

$$4\alpha g_t=(\gamma-1)\omega_x,\qquad \big\|g(\cdot,t)\big\|_{L_\infty}=\sqrt{\pi}(\gamma-1)^{\frac12}\alpha^{-\frac12}.\tag{3.3}$$

Next, we state the local existence of the solution to the Cauchy problem (1.1), (1.2) (see Lemma 3.1 in [3]).

Proposition 3.2. *Assume that the initial data (v_0,u_0,θ_0) satisfy*

$$2\underline{m}\leqslant v_0(x),\theta_0(x)\leqslant\frac12\overline{m},\qquad(\phi,\psi,\zeta)(x,0)\in H^1\big(R^1\big).\tag{3.4}$$

Then, the Cauchy problem (1.1), (1.2) admits a unique solution $(\phi,\psi,\zeta)\in X([0,t_1])$ for some sufficiently small $t_1>0$ and $(\phi,\psi,\zeta)(x,t)$ satisfies

$$\begin{cases}\underline{m}\leqslant v(x,t),\theta(x,t)\leqslant\overline{m},\\ \big\|(\phi,\psi,\varphi)(t)\big\|^2\leqslant 2\big\|(\phi,\psi,\varphi)(0)\big\|^2,\\ \big\|(\phi_x,\psi_x,\varphi_x)(t)\big\|^2\leqslant 2\big\|(\phi_x,\psi_x,\varphi_x)(0)\big\|^2,\end{cases}\tag{3.5}$$

for all $0\leqslant t\leqslant t_1$, where $\varphi=s(v,\theta)-s(V,\Theta)$ and $\underline{m}$, $\overline{m}$ are positive constants independent of x. Here t_1 depends only on $\|(\phi,\psi,\zeta)(0)\|_{H^1}$.

It is easy to check that the viscous contact wave defined in (2.7) satisfies the system

$$\begin{cases} V_t - U_x = 0, \\ U_t + \left(\dfrac{R\Theta}{V}\right)_x = \mu\left(\dfrac{U_x}{V}\right)_x + R_1, \\ \dfrac{R}{\gamma-1}\Theta_t + p(V,\Theta)U_x = \left(\kappa\dfrac{\Theta_x}{V}\right)_x + \mu\dfrac{U_x^2}{V} + R_2, \end{cases} \tag{3.6}$$

where

$$R_1 = U_t - \mu\left(\frac{U_x}{V}\right)_x, \qquad R_2 = -\mu\frac{U_x^2}{V}. \tag{3.7}$$

Let us consider two lemmas for the properties of the viscous contact wave (V, U, Θ) which will be used in Sections 4 and 5, respectively.

Lemma 3.1. *Assume that $\delta = |\theta_+ - \theta_-| \leqslant \delta_0$ for a small positive constant δ_0. Then the viscous contact wave (V, U, Θ) has the following properties*:

$$\begin{gathered} |V - v_\pm| + |\Theta - \theta_\pm| \leqslant O(1)\delta e^{-\frac{cx^2}{1+t}}, \\ \left|\partial_x^k V\right| + \left|\partial_x^{k-1} U\right| + \left|\partial_x^k \Theta\right| \leqslant O(1)\delta(1+t)^{-\frac{k}{2}} e^{-\frac{cx^2}{1+t}}, \quad k \geqslant 1. \end{gathered} \tag{3.8}$$

Therefore, we have

$$R_1 = O(1)\delta(1+t)^{-\frac{3}{2}} e^{-\frac{cx^2}{1+t}}, \qquad R_2 = O(1)\delta(1+t)^{-2} e^{-\frac{cx^2}{1+t}}. \tag{3.9}$$

Lemma 3.2. *Let $|\theta_+ - \theta_-| \leqslant m_0(\gamma - 1)$ for a fixed positive constant m_0 and assume that $\gamma - 1$ is small. Then the viscous contact wave (V, U, Θ) has the following properties*:

$$\begin{gathered} |V - v_\pm| + |\Theta - \theta_\pm| \leqslant O(1)(\gamma - 1) e^{-\frac{cx^2}{(\gamma-1)(1+t)}}, \\ \left|\partial_x^k V\right| + \left|\partial_x^k \Theta\right| \leqslant O(1)(\gamma-1)^{\frac{2-k}{2}}(1+t)^{-\frac{k}{2}} e^{-\frac{cx^2}{(\gamma-1)(1+t)}}, \quad k \geqslant 1, \\ \left|\partial_x^{k-1} U\right| \leqslant O(1)(\gamma-1)\left|\partial_x^k \Theta\right| \leqslant O(1)(\gamma-1)^{\frac{4-k}{2}}(1+t)^{-\frac{k}{2}} e^{-\frac{cx^2}{(\gamma-1)(1+t)}}, \quad k \geqslant 1. \end{gathered} \tag{3.10}$$

Therefore, we have

$$R_1 = O(1)(\gamma-1)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}} e^{-\frac{cx^2}{(\gamma-1)(1+t)}}, \qquad R_2 = O(1)(\gamma-1)^2(1+t)^{-2} e^{-\frac{cx^2}{(\gamma-1)(1+t)}}. \tag{3.11}$$

4. Proof of Theorem 2.1

Due to (3.6), we can rewrite the Cauchy problem (1.1), (1.2) as

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + (p - p_+)_x = \mu\left(\dfrac{u_x}{v} - \dfrac{U_x}{V}\right)_x - R_1, \\ \dfrac{R}{\gamma-1}\zeta_t + pu_x - p_+U_x = \kappa\left(\dfrac{\theta_x}{v} - \dfrac{\Theta_x}{V}\right)_x + \mu\left(\dfrac{u_x^2}{v} - \dfrac{U_x^2}{V}\right) - R_2, \\ (\phi, \psi, \zeta)(x, 0) = (\phi_0, \psi_0, \zeta_0)(x), \quad x \in (-\infty, \infty). \end{cases} \tag{4.1}$$

To prove Theorem 2.1, a crucial step is to show the following a priori estimate by the combination with the local existence of the solution in Proposition 3.2.

Proposition 4.1 *(A priori estimate). For $0 < T < \infty$, let $(\phi, \psi, \zeta) \in X([0, T])$ be the solution to the Cauchy problem* (1.1), (1.2) *and assume that*

$$\begin{gathered}
\|(\phi, \psi, \zeta)(\cdot, t)\| \leqslant \varepsilon_0, \qquad \|(\phi_x, \psi_x, \zeta_x)(\cdot, t)\| \leqslant M_0, \\
\underline{m} \leqslant v(x, t), \theta(x, t) \leqslant \overline{m} \quad \big(x \in R^1,\ t \in [0, T]\big), \\
\sup_{0 \leqslant t \leqslant T} \|(\phi, \psi, \zeta)(\cdot, t)\| \leqslant M_1,
\end{gathered} \tag{4.2}$$

for some positive constants $\varepsilon_0 (\leqslant 1)$, M_0, $\underline{m}$, $\overline{m}$ and M_1. Then there exist positive constants δ_0, ε_1 and c depending on $\underline{m}$, $\overline{m}$ and M_1 but independent of T, ε_0 and M_0 such that if $\delta = |\theta_+ - \theta_-| \leqslant \delta_0$ and $\sup_{0 \leqslant t \leqslant T} \|(\phi, \psi, \zeta)(\cdot, t)\|_{L^\infty} \leqslant \varepsilon_1$, the following estimates hold

$$\begin{gathered}
\|(\phi, \psi, \zeta)(\cdot, t)\|^2 + \int_0^t \|(\psi_x, \zeta_x)(\cdot, s)\|^2 \, ds \leqslant c\big(\delta^{\frac{1}{2}} + \varepsilon_0^2\big), \\
\|(\phi_x, \psi_x, \zeta_x)(\cdot, t)\|^2 + \int_0^t \|(\phi_x, \psi_{xx}, \zeta_{xx})(\cdot, s)\|^2 \, ds \leqslant c\big(1 + M_0^2\big),
\end{gathered} \tag{4.3}$$

for all $t \in [0, T]$.

Proposition 4.1 is an easy consequence of Lemmas 4.1–4.4 below.

Using the local existence (see Proposition 3.2) and the above a priori estimate (see Proposition 4.1), one can prove Theorem 2.1 by the continuum process. Note that the local existence in Proposition 3.2 holds for arbitrarily H^1 initial perturbation provided the initial volume and temperature functions are lower and upper bounded. While in a priori estimate in Proposition 4.1, the L^2-norm of the initial perturbation is sufficiently small. Thus we can prove Theorem 2.1 by the combination of Propositions 3.2 and 4.1. We omit the details for brevity.

We first estimate the L_2-norm of the perturbation (ϕ, ψ, ζ).

Lemma 4.1. *Under the assumptions of Proposition* 4.1*, the following estimate holds*

$$\|(\phi, \psi, \zeta)(t)\|^2 + \int_0^t \|(\psi_x, \zeta_x)(s)\|^2 \, ds \leqslant c\big(\delta + \|(\phi, \psi, \zeta)(0)\|^2\big) + c\delta \int_0^t \|\phi_x(s)\|^2 \, ds,$$

for all $0 \leqslant t \leqslant T$, where $c = c(\underline{m}, \overline{m}, M_1) > 0$.

Proof. Similar to [11], multiplying $(4.1)_1$ by $-R\Theta(v^{-1} - V^{-1})$, $(4.1)_2$ by ψ and $(4.1)_3$ by $\zeta\theta^{-1}$, then adding the resulting equations together, we have

$$\left(\frac{1}{2}\psi^2 + R\Theta\Phi\left(\frac{v}{V}\right) + \frac{R}{\delta}\Theta\Phi\left(\frac{\theta}{\Theta}\right)\right)_t + \frac{\mu\Theta}{v\theta}\psi_x^2 + \frac{\kappa\Theta}{v\theta^2}\zeta_x^2 + H_x + Q = -R_1\psi - R_2\frac{\zeta}{\theta}, \tag{4.4}$$

where $\Phi(\sigma)=\sigma-1-\ln\sigma$, $\sigma>0$ and

$$H=(p-p_+)\psi-\mu\left(\frac{u_x}{v}-\frac{U_x}{V}\right)\psi-\frac{\zeta}{\theta}\left(\frac{\theta_x}{v}-\frac{\Theta_x}{V}\right),$$

$$\begin{aligned}Q={}&p_+\Phi\left(\frac{V}{v}\right)U_x+\frac{p_+}{\gamma-1}\Phi\left(\frac{\Theta}{\theta}\right)U_x+\mu\psi_xU_x\left(\frac{1}{v}-\frac{1}{V}\right)\\&-\frac{\zeta}{\theta}(p-p_+)U_x-\frac{\kappa\Theta_x}{v\theta^2}\zeta\zeta_x-\frac{\kappa\Theta\Theta_x}{vV\theta^2}\phi\zeta_x\\&+\frac{\kappa\Theta_x^2}{vV\theta^2}\phi\zeta-\frac{\mu\zeta U_x^2}{\theta}\left(\frac{1}{v}-\frac{1}{V}\right)-\frac{2\mu\zeta}{v\theta}\psi_xU_x.\end{aligned}\tag{4.5}$$

It is easy to check that $\Phi(1)=\Phi'(1)=0$, $\Phi''(s)=s^{-2}>0$. This yields that

$$c_1\phi^2\leqslant\Phi\left(\frac{v}{V}\right)+\left(\frac{V}{v}\right)\leqslant c_2\phi^2,\qquad c_1\zeta^2\leqslant\Phi\left(\frac{\theta}{\Theta}\right)+\Phi\left(\frac{\Theta}{\theta}\right)\leqslant c_2\zeta^2,\tag{4.6}$$

for some positive constants $c_i=c_i(\underline{m},\overline{m})$, $i=1,2$. Using Lemma 3.1, (4.3) and (4.6), we get from (4.5)

$$|Q|\leqslant\frac{\mu\Theta}{4v\theta}\psi_x^2+\frac{\kappa\Theta}{4v\theta^2}\zeta_x^2+c(\overline{m},\underline{m})\big(\phi^2+\zeta^2\big)\big(|U_x|+\Theta_x^2\big),\tag{4.7}$$

$$\begin{aligned}\int\left(|R_1\psi|+\left|\frac{R_2\zeta}{\theta}\right|\right)dx&\leqslant c(\overline{m},\underline{m})\big(\|R_1\|_{L_1}+\|R_2\|_{L_1}\big)\big(\|\psi\|^{\frac12}\|\psi_x\|^{\frac12}+\|\zeta\|^{\frac12}\|\zeta_x\|^{\frac12}\big)\\&\leqslant c(\overline{m},\underline{m},M_1)\delta\big(\big\|(\psi_x,\zeta_x)\big\|^2+(1+t)^{-\frac43}\big).\end{aligned}\tag{4.8}$$

Integrating (4.4) with respect to x, t and using (4.7), (4.8) and (3.8), we have

$$\begin{aligned}&\big\|(\phi,\psi,\zeta)(t)\big\|^2+\int_0^t\big\|(\psi_x,\zeta_x)(s)\big\|^2ds\\&\quad\leqslant c\big(\delta+\big\|(\phi,\psi,\zeta)(0)\big\|^2\big)+c\delta\int_0^t\!\!\int(1+s)^{-1}\big(\phi^2+\zeta^2\big)e^{-\frac{\hat{c}x^2}{1+s}}\,dx\,ds,\end{aligned}\tag{4.9}$$

where $c=c(\underline{m},\overline{m},M_1)>0$ and $\hat{c}$ is the positive constant in Lemma 3.1.

Lemma 4.1 follows directly from (4.9) and the following Lemma 4.2. $\square$

Lemma 4.2. *Under the assumptions of Proposition* 4.1, *the following estimate holds*

$$\int_0^t\!\!\int\big|(\phi,\psi,\zeta)\big|^2\omega^2\,dx\,ds\leqslant c+c\int_0^t\big\|(\phi_x,\psi_x,\zeta_x)\big\|^2ds,\tag{4.10}$$

where ω is the function in (3.2) *by choosing* $\alpha=\frac{\hat{c}}{2(\gamma-1)}$ *and* $c=c(\underline{m},\overline{m},M_1)>0$.

Proof. After getting the following two estimates

$$\int_0^t\!\!\int \big(R\zeta+(\gamma-1)p_+\phi\big)^2\omega^2\,dx\,ds \leqslant c\big(1+\eta^{-1}\big)\int_0^t \big\|(\phi_x,\psi_x,\zeta_x)\big\|^2\,ds + c + c(\delta+\eta)\int_0^t\!\!\int \big(\phi^2+\zeta^2\big)\omega^2\,dx\,ds, \quad \text{for any } \eta>0, \tag{4.11}$$

$$\int_0^t\!\!\int \big((R\zeta-p_+\phi)^2+\psi^2\big)\omega^2\,dx\,ds \leqslant c + c\int_0^t \big\|(\phi_x,\psi_x,\zeta_x)\big\|^2\,ds + c\delta\int_0^t\!\!\int \big(\phi^2+\zeta^2\big)\omega^2\,dx\,ds, \tag{4.12}$$

adding (4.11) and (4.12) yields (4.10).

We first prove (4.11). From $(4.1)_3$ and $(4.1)_1$, we get

$$\left(\frac{R}{\gamma-1}\zeta+p_+\phi\right)_t = -\frac{R\zeta-p_+\phi}{v}(\psi_x+U_x)+\kappa\left(\frac{V\zeta_x-\phi\Theta_x}{vV}\right)_x+G, \tag{4.13}$$

where $G=\mu v^{-1}(U_x+\psi_x)^2$. Taking $h=R\zeta+(\gamma-1)p_+\phi$, we have from (4.13)

$$\frac{1}{\gamma-1}\langle h_t, hg^2\rangle = -\int \frac{R\zeta-p_+\phi}{v}h\psi_x g^2\,dx - \int \frac{R\zeta-p_+\phi}{v}hU_x g^2\,dx - \int \frac{V\zeta_x-\phi\Theta_x}{vV}\big(hg^2\big)_x\,dx + \int Ghg^2\,dx = \sum_{i=1}^4 J_i, \tag{4.14}$$

where g is the function defined in (3.2).

Now we estimate J_i $(i=1,\dots,4)$ term by term. Noticing that $|U_x|\leqslant c\delta\omega^2$ and using (3.3) and Lemma 3.1, we obtain

$$|J_2|\leqslant c\delta\int \big(\phi^2+\zeta^2\big)\omega^2\,dx,$$

$$|J_3|\leqslant c\int \big(|\zeta_x h_x|+|\phi\Theta_x h_x|+|\zeta_x h|\omega+|\phi\Theta_x h|\omega\big)\,dx \leqslant c(\delta+\eta)\int \big(\phi^2+\zeta^2\big)\omega^2\,dx + c\big(1+\eta^{-1}\big)\big\|(\phi_x,\zeta_x)\big\|^2, \quad \text{for any } \eta>0,$$

$$|J_4|\leqslant c\int \big(|\phi|+|\zeta|\big)\big(|\psi_x|^2+|U_x|^2\big)\,dx \leqslant c\varepsilon_1\|\psi_x\|^2+c\delta(1+t)^{-\frac{3}{2}},$$

where $c=c(\underline{m},\overline{m})>0$.

The estimate of J_1 is more subtle. Noticing that $R\zeta-p_+\phi=h-\gamma p_+\phi$ and $\phi_t=\psi_x$, we compute

$$
\begin{aligned}
-2J_1 &= 2\int v^{-1}\big(h^2-\gamma p_+ h\phi\big)\phi_t g^2\,dx = \int \big(2v^{-1}h^2g^2\phi_t - \gamma p_+ v^{-1}g^2\big(\phi^2\big)_t\big)\,dx \\
&= \left(\int v^{-1}hg^2\phi(2h-\gamma p_+\phi)\,dx\right)_t - 2\int v^{-1}hg\phi(2h-\gamma p_+\phi)g_t\,dx \\
&\quad + \int v^{-2}v_t g^2 h\phi(2h-\gamma p_+\phi)\,dx - \int v^{-1}g^2\phi(4h-\gamma p_+\phi)h_t\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
-2J_1 &= \left(\int v^{-1}hg^2\phi(2h-\gamma p_+\phi)\,dx\right)_t - 2\alpha^{-1}(\gamma-1)\int v^{-1}hg\phi(2h-\gamma p_+\phi)\omega_x\,dx \\
&\quad + \int v^{-2}u_x g^2\phi\big[h(2h-\gamma p_+\phi) + (\gamma-1)(4h-\gamma p_+\phi)(R\zeta - p_+\phi)\big]\,dx \\
&\quad + \kappa(\gamma-1)\int \frac{V\zeta_x - \phi\Theta_x}{Vv}\big[v^{-1}g^2\phi(4h-\gamma p_+\phi)\big]_x\,dx \\
&\quad - (\gamma-1)\int v^{-1}g^2\phi(4h-\gamma p_+\phi)G\,dx \\
&= \left(\int v^{-1}hg^2\phi(2h-\gamma p_+\phi)\,dx\right)_t + \sum_{i=1}^{4} J_1^i. \qquad (4.15)
\end{aligned}
$$

By using (4.13) and (3.3), we estimate J_1^i $(i=1,\dots,4)$ term by term:

$$
\begin{aligned}
\big|J_1^1\big| &\leqslant c\int |h||\phi|\big(|h|+|\phi|\big)|\omega_x|\,dx \leqslant c(1+t)^{-1}\int \big(|\phi|^3+|\zeta|^3\big)\,dx \\
&\leqslant c(1+t)^{-1}\big(\|\phi\|^{\frac{5}{2}}\|\phi_x\|^{\frac{1}{2}} + \|\zeta\|^{\frac{5}{2}}\|\zeta_x\|^{\frac{1}{2}}\big) \leqslant c(1+t)^{-\frac{4}{3}} + c\big\|(\phi_x,\zeta_x)\big\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\big|J_1^2\big| &\leqslant c\int \big(|U_x|+|\phi_x|\big)\big(|\phi|^3+|\zeta|^3\big)\,dx \\
&\leqslant c\big(\|\phi\|^2\|\phi_x\| + \|\zeta\|^2\|\zeta_x\|\big)\big(\|U_x\|+\|\psi_x\|\big) \leqslant c(1+t)^{-\frac{3}{2}} + c\big\|(\phi_x,\psi_x,\zeta_x)\big\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\big|J_1^3\big| &\leqslant c\int \big(|\zeta_x|+|\phi\Theta_x|\big)\big(|\omega|\big(|\phi|^2+|\zeta|^2\big) + \big(|\phi|+|\zeta|\big)\big(|\phi_x|+|\zeta_x|\big)\big)\,dx \\
&\leqslant c\int \big(|\zeta_x|+|\phi|(1+t)^{-\frac{1}{2}}\big)\big((1+t)^{-\frac{1}{2}}\big(|\phi|^2+|\zeta|^2\big) + \big(|\phi|+|\zeta|\big)\big(|\phi_x|+|\zeta_x|\big)\big)\,dx \\
&\leqslant c(1+t)^{-\frac{4}{3}} + c\big\|(\phi_x,\zeta_x)\big\|^2,
\end{aligned}
$$

$$
\big|J_1^4\big| \leqslant c\int |\phi|\big(|\phi|+|\zeta|\big)\big(|U_x|^2+|\psi_x|^2\big)\,dx \leqslant c(1+t)^{-\frac{3}{2}} + c\|\psi_x\|^2,
$$

where $c = c(\underline{m},\overline{m},M_1) > 0$,

$$
\left|\int v^{-1}hg^2\phi(2h-\gamma p_+\phi)\,dx\right| \leqslant c(\underline{m},\overline{m},M_1). \qquad (4.16)
$$

Using estimates J_i $(i=1,\dots,4)$ and (4.16), we obtain from (4.14)

$$\left|\int_0^t \langle h_t, hg^2\rangle\,ds\right| \leqslant c(\delta+\eta)\int_0^t\!\!\int \left(\phi^2+\zeta^2\right)\omega^2\,dx\,ds$$

$$+c+c\left(1+\eta^{-1}\right)\int_0^t \left\|(\phi_x,\psi_x,\zeta_x)\right\|^2 ds, \quad \text{for any } \eta>0. \tag{4.17}$$

Applying Proposition 3.1 with (4.17), we obtain (4.11).

Next, we prove (4.12). Denoting by $f(x,t)=\int_{-\infty}^x \omega^2(y,t)\,dy$, we have

$$\left\|f(\cdot,t)\right\|_{L_\infty} \leqslant c(1+t)^{-\frac{1}{2}}, \qquad \left\|f_t(\cdot,t)\right\|_{L_\infty} \leqslant c(1+t)^{-\frac{3}{2}}. \tag{4.18}$$

Rewriting $(4.4)_2$ as

$$\psi_t+\left(\frac{R\zeta-p_+\phi}{v}\right)_x=\mu\left(\frac{\phi_x}{v}\right)_x+F, \quad F=-U_t+\mu\left(v^{-1}U_x\right)_x,$$

and multiplying it by $(R\zeta-p_+\phi)vf$, we have

$$\begin{aligned}
&\frac{1}{2}\int (R\zeta-p_+\phi)^2\omega^2\,dx\\
&=\int \psi_t(R\zeta-p_+\phi)vf\,dx-\int v^{-1}(R\zeta-p_+\phi)^2 v_x f\,dx\\
&\quad+\mu\int v^{-1}\psi_x\left((R\zeta-p_+\phi)vf\right)_x dx-\int F(R\zeta-p_+\phi)^2 vf\,dx\\
&=\left(\int \psi(R\zeta-p_+\phi)vf\,dx\right)_t-\int \psi(R\zeta-p_+\phi)_t vf\,dx-\int \psi(R\zeta-p_+\phi)v_t f\,dx\\
&\quad-\int \psi(R\zeta-p_+\phi)vf_t\,dx-\int v^{-1}(R\zeta-p_+\phi)^2 v_x f\,dx\\
&\quad+\mu\int v^{-1}\psi_x\left((R\zeta-p_+\phi)vf\right)_x dx-\int F(R\zeta-p_+\phi)^2 vf\,dx\\
&=\left(\int \psi(R\zeta-p_+\phi)vf\,dx\right)_t+\sum_{i=5}^{10} J_i.
\end{aligned} \tag{4.19}$$

We estimate J_i, $i=5,\dots,10$, as follows: Using (4.13) and $\phi_t=\psi_x$, we have

$$\begin{aligned}
J_5&=-(\gamma-1)\int \psi vf\left(\frac{R}{\gamma-1}\zeta+p_+\phi\right)_t dx+\gamma p_+\int \psi vf\,dx\\
&=(\gamma-1)\int \psi f(R\zeta-p_+\phi)(U_x+\psi_x)\,dx+\kappa(\gamma-1)\int \frac{V\zeta_x-\phi\Theta_x}{V(V+\phi)}(\psi vf)_x\,dx\\
&\quad-(\gamma-1)\int \psi vfG\,dx+\frac{\gamma p_+}{2}\int vf\left(\psi^2\right)_x=\sum_{i=1}^{4} J_5^i.
\end{aligned} \tag{4.20}$$

It is easy to show that

$$\begin{aligned}\left|J_5^1\right|+|J_6| &\leqslant c(1+t)^{-\frac{1}{2}}\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}\big(\|\psi\|+\|\zeta\|\big)\big(\|U_x\|+\|\psi_x\|\big)\\ &\leqslant c\big((1+t)^{-\frac{5}{4}}\|\psi_x\|^{\frac{1}{2}}+(1+t)^{-\frac{1}{2}}\|\psi_x\|^{\frac{3}{2}}\big)\leqslant c\|\psi_x\|^2+c(1+t)^{-\frac{5}{3}},\end{aligned}$$

$$\begin{aligned}\left|J_5^2\right| &\leqslant c\int\big(|\zeta_x|+|\phi|(1+t)^{-\frac{1}{2}}\big)|\psi_x vf+\psi v_x f+\psi v f_x|\,dx\\ &\leqslant c\big(\|\zeta_x\|+(1+t)^{-\frac{1}{2}}\|\phi\|\big)\big(\big(\|\phi_x\|+\|\zeta_x\|\big)(1+t)^{-\frac{1}{2}}+\|\psi\|(1+t)^{-1}\big)\\ &\leqslant c\big\|(\phi_x,\psi_x,\zeta_x)\big\|^2+c(1+t)^{-\frac{3}{2}},\end{aligned}$$

$$\begin{aligned}\left|J_5^3\right| &\leqslant c(1+t)^{-\frac{1}{2}}\|\psi\|_{L_\infty}\big(\|\psi_x\|^2+(1+t)^{-\frac{3}{2}}\big)\leqslant c\|\psi_x\|^2+c\varepsilon_0(1+t)^{-2},\\ \left|J_5^4\right| &= -\frac{\gamma p_+}{2}\int v\omega^2\psi^2\,dx-\frac{\gamma R}{2}\int f\psi^2\Theta_x\,dx-\frac{\gamma p_+}{2}\int f\psi^2\phi_x\,dx\\ &\leqslant -\frac{\gamma p_+}{2}\int v\omega^2\psi^2\,dx+c\delta\int v\omega^2\psi^2\,dx+c(1+t)^{-\frac{1}{2}}\|\psi\|^{\frac{3}{2}}\|\psi_x\|^{\frac{1}{2}}\|\phi_x\|\\ &\leqslant -\frac{\gamma p_+}{4}\int v\omega^2\psi^2\,dx+c\big\|(\phi_x,\psi_x)\big\|^2+c(1+t)^{-2},\end{aligned}$$

where $c=c(\underline{m},\overline{m},M_1)>0$. On the other hand, we get

$$|J_7|\leqslant c(1+t)^{-\frac{3}{2}}\int|\psi|\big(|\phi|+|\zeta|\big)\,dx\leqslant c(1+t)^{-\frac{3}{2}},$$

$$\begin{aligned}|J_8| &= \left|\int v^{-1}(R\zeta-p_+\phi)^2\Theta_x f\,dx+\int v^{-1}(R\zeta-p_+\phi)^2\phi_x f\,dx\right|\\ &\leqslant c\delta\int\big(\phi^2+\zeta^2\big)\omega^2\,dx+c\big\|(\phi_x,\zeta_x)\big\|^2+c(1+t)^{-2},\end{aligned}$$

$$|J_9|\leqslant c\big\|(\phi_x,\psi_x,\zeta_x)\big\|^2+c(1+t)^{-\frac{3}{2}},$$

$$|J_{10}|\leqslant c\|F\|_{L_1}(1+t)^{-\frac{1}{2}}\big(\|\phi_x\|_{L_\infty}+\|\zeta_x\|_{L_\infty}\big)\leqslant c\delta(1+t)^{-\frac{3}{2}}+c\delta\big\|(\phi_x,\zeta_x)\big\|^2$$

and

$$\left|\int\psi(R\zeta-p_+\phi)vf\,dx\right|\leqslant c(\underline{m},\overline{m},M_1). \tag{4.21}$$

Integrating (4.19) over $(0,t)$, together with all the estimates of J_i, $i=5,\dots,10$, and (4.21), yields (4.12).

The proof of Lemma 4.2 is completed. □

Next, we estimate the L_2-norm of the perturbation (ϕ_x,ψ_x,ζ_x).

Lemma 4.3. *Suppose $(\phi,\psi,\zeta)\in X([0,T])$ satisfies (4.2). Then it holds for $t\in[0,T]$,*

$$\big\|\phi_x(t)\big\|^2+\int_0^t\big\|\phi_x(s)\big\|^2\,ds\leqslant c\big(\delta+\big\|(\psi_0,\zeta_0)\big\|^2+\|\phi_0\|_1^2\big),$$

where $c=c(\underline{m},\overline{m},M_1)>0$.

Proof. We rewrite $(4.1)_2$ as

$$\mu\left(\frac{\phi_x}{v}\right)_t + p\frac{\phi_x}{v} = \psi_t + \frac{R\zeta_x}{v} - \frac{R\zeta - p_+\phi}{v^2}V_x - \mu\left(\frac{V_x}{v}\right)_t + U_t, \tag{4.22}$$

where we used $(\frac{u_x}{v})_x = (\frac{\phi_x}{v})_t + (\frac{V_x}{v})_t$ due to $v_t = u_x$.

Multiplying (4.19) by ϕ_x/v and using

$$\left(\frac{\phi_x}{v}\right)_t = \frac{\psi_{xx}}{v} - \frac{\phi_x\psi_x + \phi_x U_x}{v^2},$$

we have

$$\begin{aligned}\left(\frac{\mu}{2}\frac{\phi_x^2}{v^2} - \psi\frac{\phi_x}{v}\right)_t + p\frac{\phi_x^2}{v^2} &= -\left(\frac{\psi\psi_x}{v}\right)_x + \left(\frac{\psi_x^2}{v} + \frac{R\zeta_x\phi_x}{v^2}\right) - \left(\frac{\psi\psi_x V_x}{v^2} + \frac{R\zeta - p_+\phi}{v^3}V_x\phi_x\right)\\ &\quad + \left(\frac{\psi\phi_x U_x}{v^2} - \mu\left(\frac{V_x}{v}\right)_t - U_t\right)\frac{\phi_x}{v}\\ &= -\left(\frac{\psi\psi_x}{v}\right)_x + \sum_{i=1}^{3} I_i. \end{aligned} \tag{4.23}$$

We estimate I_i, $i = 1, \dots, 3$, as follows:

$$\int |I_1|\,dx \leqslant \eta\|\phi_x\|^2 + c\left(1+\eta^{-1}\right)\left\|(\psi_x,\zeta_x)\right\|^2, \quad \text{for any } \eta > 0,$$

$$\begin{aligned}\int |I_2|\,dx &\leqslant c\int \left(|\psi\psi_x V_x| + \left(|\phi| + |\zeta|\right)|V_x\phi_x|\right)dx\\ &\leqslant c\delta\left\|(\phi_x,\psi_x)\right\|^2 + c\delta\int \left(\phi^2+\psi^2+\zeta^2\right)\omega^2\,dx,\end{aligned}$$

$$\begin{aligned}\int |I_3|\,dx &\leqslant c\int \left(|\psi\phi_x U_x| + \left|\left(\frac{V_x}{v}\right)_t - U_t\right||\phi_x|\right)dx\\ &\leqslant c\delta\int \left((1+t)^{-\frac{1}{2}}\omega + |\psi_x|\right)|\phi_x|\,dx \leqslant c\delta\left\|(\phi_x,\psi_x)\right\|^2 + c\delta(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Using estimates I_i, $i = 1, \dots, 3$, we obtain from (4.23)

$$\begin{aligned}&\left\|\phi_x(t)\right\|^2 + \int_0^t \left\|\phi_x(s)\right\|^2 ds\\ &\quad\leqslant c\left\|(\psi_0,\phi_{0x})\right\|^2 + c\delta + c\left(\left\|\psi(t)\right\|^2 + \int_0^t \left\|(\psi_x,\zeta_x)(s)\right\|^2 ds\right) + c\delta\int_0^t\!\!\int \left|(\phi,\psi,\zeta)\right|^2\omega^2\,dx\,ds.\end{aligned} \tag{4.24}$$

Lemma 4.3 follows directly from (4.24), Lemmas 4.1 and 4.2. □

Remark 4.1. From Lemmas 4.1–4.3 and (2.8), we obtain $(4.3)_1$.

Lemma 4.4. *Suppose $(\phi, \psi, \zeta) \in X([0, T])$ satisfies (4.2). Then it holds for $t \in [0, T]$,*

$$\big\|(\psi_x, \zeta_x)(t)\big\|^2 + \int_0^t \big\|(\psi_{xx}, \zeta_{xx})(s)\big\|^2 \, ds \leqslant c\big(1 + \big\|(\phi_0, \psi_0, \zeta_0)\big\|_1^2\big).$$

Proof. Rewrite $(4.4)_2$ as

$$\psi_t - \mu \frac{\psi_{xx}}{v} = -\left(\frac{R\zeta - p_+\phi}{v}\right)_x - \mu \frac{\psi_x v_x}{v^2} + F. \tag{4.25}$$

Multiplying (4.23) by $-\phi_{xx}$ and integrating the resulting system with respect to x, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\psi_x(t)\big\|^2 + \mu \int \frac{\psi_{xx}^2}{v} \, dx \\
&\quad = \int \left(\frac{R\zeta - p_+\phi}{v}\right)_x \psi_{xx} \, dx + \mu \int \frac{\psi_x v_x}{v^2} \psi_{xx} \, dx - \int F \psi_{xx} \, dx \equiv \sum_{i=4}^{6} \int I_i \, dx; \qquad (4.26)
\end{aligned}$$

$$\int |I_4| \, dx \leqslant \frac{\mu}{6} \int \frac{\psi_{xx}^2}{v} \, dx + c\big\|(\phi_x, \zeta_x)\big\|^2 + c\delta \int \big(\phi^2 + \zeta^2\big)\omega^2 \, dx,$$

$$\begin{aligned}
\int |I_5| \, dx &\leqslant c \int |\psi_x|\big(|\phi_x| + |U_x|\big)|\psi_{xx}| \, dx \\
&\leqslant c\|\phi_x\| \|\psi_x\|^{\frac{1}{2}} \|\psi_{xx}\|^{\frac{3}{2}} + c\delta\|\psi_{xx}\|^2 + c\delta(1+t)^{-1}\|\psi_x\|^2 \\
&\leqslant \frac{\mu}{6} \int \frac{\psi_{xx}^2}{v} \, dx + c\delta(1+t)^{-1}\|\psi_x\|^2 + c\|\phi_x\|^4\|\psi_x\|^2,
\end{aligned}$$

$$\int |I_6| \, dx \leqslant \frac{\mu}{6} \int \frac{\psi_{xx}^2}{v} \, dx + c\delta\big\|(\phi_x, \zeta_x)\big\|^2 + c\delta(1+t)^{-\frac{5}{4}}.$$

Due to the estimates I_i $(i = 4, \ldots, 6)$, we have from (4.26)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\psi_x(t)\big\|^2 + \frac{\mu}{2} \int \frac{\psi_{xx}^2}{v} \, dx \\
&\quad \leqslant c\big\|(\phi_x, \psi_x, \zeta_x)(t)\big\|^2 + c\delta \int \big(\phi^2 + \zeta^2\big)\omega^2 \, dx + c\delta(1+t)^{-\frac{5}{4}} + c\big\|\phi_x(t)\big\|^4\big\|\psi_x(t)\big\|^2,
\end{aligned}$$

and integrating it with respect to t yields

$$\begin{aligned}
\big\|\psi_x(t)\big\|^2 + \int_0^t \big\|\psi_{xx}(\tau)\big\|^2 \, d\tau &\leqslant c\big\|\psi_x(0)\big\|^2 + c\int_0^t \big\|(\phi_x, \psi_x, \zeta_x)(\tau)\big\|^2 \, d\tau + c\delta \\
&\quad + c\delta \int_0^t\!\!\int \big(\phi^2 + \zeta^2\big)\omega^2 \, dx \, d\tau + c \max_{0 \leqslant t \leqslant T} \big\|\phi_x(t)\big\|^4 \int_0^t \big\|\psi_x(\tau)\big\|^2 \, d\tau.
\end{aligned} \tag{4.27}$$

Now, we estimate L_2-norm of ζ_x. Rewriting $(4.1)_3$ as

$$\frac{R}{\gamma-1}\zeta_t-\kappa\frac{\zeta_{xx}}{v}=-(pu_x-PU_x)-\kappa\frac{\zeta_x v_x}{v^2}-\kappa\left(\frac{\phi\Theta_x}{vV}\right)_x+G$$

and multiplying it by $-\zeta_{xx}$, we have

$$\begin{aligned}\frac{R}{2(\gamma-1)}\frac{d}{dt}\left\|\zeta_x(t)\right\|^2+\kappa\int\frac{\zeta_{xx}^2}{v}\,dx&=\int\left[\frac{R\theta}{v}\psi_x+RU_x\left(\frac{\theta}{v}-\frac{\Theta}{V}\right)\right]\zeta_{xx}\,dx+\kappa\int\frac{\zeta_x v_x}{v^2}\zeta_{xx}\,dx\\&\quad+\kappa\int\left(\frac{\phi\Theta_x}{vV}\right)_x\zeta_{xx}\,dx-\int G\zeta_{xx}\,dx\equiv\sum_{i=7}^{10}\int I_i\,dx;\end{aligned}\tag{4.28}$$

$$\int|I_7|\,dx\leqslant\frac{\kappa}{8}\int\frac{\zeta_{xx}^2}{v}\,dx+c\|\psi_x\|^2+c\delta\int\left(\phi^2+\zeta^2\right)\omega^2\,dx,$$

$$\begin{aligned}\int|I_8|\,dx&\leqslant c\|\phi_x\|\|\zeta_x\|^{\frac{1}{2}}\|\zeta_{xx}\|^{\frac{3}{2}}+c\delta\|\zeta_{xx}\|^2+c\delta(1+t)^{-1}\|\zeta_x\|^2\\&\leqslant\frac{\kappa}{8}\int\frac{\zeta_{xx}^2}{v}\,dx+c\delta(1+t)^{-1}\|\zeta_x\|^2+c\|\phi_x\|^4\|\zeta_x\|^2,\end{aligned}$$

$$\begin{aligned}\int|I_9|\,dx&\leqslant\frac{\kappa}{8}\int\frac{\zeta_{xx}^2}{v}\,dx+c\left[\|\Theta_{xx}\|^2+\|\Theta_x\|_{L_\infty}^2\|\phi_x\|^2+\|\Theta_x\|_{L_\infty}^2\left(\|\phi_x\|^2+\|V_x\|^2\right)\right]\\&\leqslant\frac{\kappa}{8}\int\frac{\psi_{xx}^2}{v}\,dx+c\delta(1+t)^{-1}\|\phi_x\|^2+c\delta(1+t)^{-\frac{3}{2}},\end{aligned}$$

$$\int|I_{10}|\,dx\leqslant\frac{\kappa}{8}\int\frac{\zeta_{xx}^2}{v}\,dx+c\delta(1+t)^{-1}\|\phi_x\|^2+c\delta(1+t)^{-\frac{5}{4}}.$$

Due to the estimates I_i $(i=7,\dots,10)$, we have from (4.28)

$$\begin{aligned}&\left\|\zeta_x(t)\right\|^2+\int_0^t\left\|\zeta_{xx}(\tau)\right\|^2d\tau\\&\quad\leqslant c\left\|\psi_x(0)\right\|^2+c\int_0^t\left\|(\phi_x,\psi_x,\zeta_x)(\tau)\right\|^2d\tau+c\delta\\&\qquad+c\delta\int_0^t\!\!\int\left(\phi^2+\zeta^2\right)\omega^2\,dx\,d\tau+c\max_{0\leqslant t\leqslant T}\left\|\phi_x(t)\right\|^4\int_0^t\left\|(\psi_x,\zeta_x)(\tau)\right\|^2d\tau.\end{aligned}\tag{4.29}$$

Lemma 4.4 follows from (4.27), (4.29), $(4.3)_1$ and Lemmas 4.1–4.3. $\square$

Remark 4.2. From Lemmas 4.1–4.4 and (2.8), we obtain $(4.3)_2$ which will close the proof of Proposition 4.1 together with $(4.3)_1$.

5. Proof of Theorem 2.2

Since we want to get L_∞-global stability result, the techniques in Section 4 do not apply any longer. To overcome this difficulty, we introduce function $X_{\hat{m},M}(t_1,t_2;m_\theta)$ by

$$X_{\hat{m},M}(t_1,t_2;m_\theta)=\left\{(\phi,\psi,\zeta)\in X\big([t_1,t_2]\big)\;\middle|\;0<\frac{1}{4}m_\theta^{-1}\leqslant\theta(x,t)\leqslant 4m_\theta,\right.$$
$$\left.0<\frac{1}{2}\hat{m}^{-1}\leqslant v(x,t)\leqslant 2\hat{m},\ \sup_{[t_1,t_2]}\big\|(\phi,\psi,\varphi)(t)\big\|_1\leqslant M\right\}\tag{5.1}$$

for t_1, t_2 $(0\leqslant t_1<t_2<\infty)$ and $\hat{m}$, M $(0<\hat{m}^{-1}<\hat{m}<\infty,\ 0<M<\infty)$, where

$$\varphi=s(v,\theta)-s(V,\Theta).$$

By Proposition 3.2 and the assumptions listed in Theorem 2.2, we know that the Cauchy problem (1.1), (1.2) admits a unique solution $(\phi,\psi,\zeta)(t,x)\in X_{m_v,M}(0,t_0;m_\theta)$ for some sufficiently small positive constant $t_0>0$ only depending on $\|(\phi,\psi,\zeta)(0)\|_1^2$ with $M=2\|(\phi,\psi,\varphi)(0)\|_1$.

To prove Theorem 2.2, we need the following a priori estimate:

Proposition 5.1 *(A priori estimate). Assume that the conditions of Theorem 2.2 hold. Then there exists a positive constant δ_0 such that if $\gamma<1+\delta_0$ and $(\phi,\psi,\zeta)\in X_{\hat{m},M}(0,t_0;m_\theta)$ is the solution of (3.2) for some positive $T>0$, then there exist positive constants $c_3(m_v,m_\theta)$ and $c_4(m_v,m_\theta)$ such that the following hold*

$$\begin{cases}0<c_3(m_v,m_\theta)^{-1}\leqslant v(t,x)\leqslant c_3(m_v,m_\theta),\\ 0<\dfrac{1}{2}m_\theta^{-1}\leqslant\theta(t,x)\leqslant 2m_\theta\end{cases}\tag{5.2}$$

and

$$\big\|(\phi,\varphi,\zeta)(t)\big\|_1^2+\int_0^t\big\{\big\|(\phi_x,\varphi_x)\big\|^2+\|\zeta_x\|_1^2\big\}\,d\tau\leqslant c_4(m_v,m_\theta)\big(1+\big\|(\phi_0,\psi_0,\varphi_0)\big\|_1^2\big).\tag{5.3}$$

Proposition 1 is proved by a series of lemmas. First, we have

Lemma 5.1. *It follows that*

$$\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\zeta}{\sqrt{\delta}}\right)(t)\right\|^2+\int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}},\frac{\zeta_x}{\sqrt{v}}\right)(\tau)\right\|^2d\tau$$
$$\leqslant c(m_v,m_\theta)\left\{\big\|(\phi_0,\psi_0,\sqrt{\delta}\varphi_0)\big\|^2+\delta^{\frac{1}{2}}c(\hat{m},M)\left(1+\int_0^t\big\|\phi_x(\tau)\big\|^2d\tau\right)\right\},\tag{5.4}$$

where $\delta=\gamma-1$.

Proof. We use (4.4) and (4.5). Noticing that

$$c_1(m_v)c_1(\hat{m})\phi^2 \leqslant \Phi\left(\frac{v}{V}\right) + \Phi\left(\frac{V}{v}\right) \leqslant c_2(m_v)c_2(\hat{m})\phi^2,$$

$$c_1(m_\theta)\zeta^2 \leqslant \Phi\left(\frac{\theta}{\Theta}\right) + \Phi\left(\frac{\Theta}{\theta}\right) \leqslant c_1(m_\theta)\zeta^2,$$

we get from (4.5)

$$|Q| \leqslant \frac{\mu\Theta}{4v\theta}\psi_x^2 + \frac{\kappa\Theta}{4v\theta^2}\zeta_x^2 + c(\hat{m}, m_v, m_\theta)\left[|U_x|\left(\phi^2 + \frac{\zeta^2}{\delta}\right) + \Theta_x^2\big(\phi^2 + \zeta^2\big)\right]. \tag{5.5}$$

From (3.11), we have

$$\begin{aligned}\int\left(|R_1\psi| + \left|\frac{R_2\zeta}{\theta}\right|\right)dx &\leqslant c(m_\theta)\big(\|R_1\|_{L_1} + \|R_2\|_{L_1}\big)\big(\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}} + \|\zeta\|^{\frac{1}{2}}\|\zeta_x\|^{\frac{1}{2}}\big)\\ &\leqslant \frac{\mu\Theta}{4v\theta}\psi_x^2 + \frac{\kappa\Theta}{4v\theta^2}\zeta_x^2 + c(\hat{m}, m_\theta, M)\delta(1+t)^{-\frac{4}{3}}.\end{aligned} \tag{5.6}$$

Using Lemma 3.2, (5.5) and (5.6), we get from (4.5)

$$\begin{aligned}&\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)}, \psi, \frac{\zeta}{\sqrt{\delta}}\right)(t)\right\|^2 + \int_0^t \left\|\left(\frac{\psi_x}{\sqrt{v}}, \frac{\zeta_x}{\sqrt{v}}\right)(\tau)\right\|^2 d\tau\\ &\quad\leqslant c(m_v, m_\theta)\big\|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\big\|^2 + c(\hat{m}, m_\theta, M)\delta\\ &\qquad + c(\hat{m}, m_v, m_\theta)\delta\int_0^t\!\!\int (1+\tau)^{-1}\left(\phi^2 + \frac{\zeta^2}{\delta}\right)\exp\left(-\frac{\hat{c}x^2}{2\delta(1+\tau)}\right)dx\,d\tau,\end{aligned} \tag{5.7}$$

where $\hat{c}$ is the positive constant in Lemma 3.2. Notice that in (5.7), we used the fact that

$$\|\zeta_0\|_k \leqslant c(m_v, m_\theta)(\gamma - 1)\big\|(\phi_0, \varphi_0)\big\|_k, \quad k = 0, 1.$$

Lemma 5.1 follows directly from (5.7) and the following Lemma 5.2. □

Lemma 5.2. *The following estimate holds*

$$\int_0^t\!\!\int\left(\phi^2 + \psi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\,dx\,d\tau \leqslant c\delta^{-\frac{1}{2}}\left(1 + \int_0^t \big\|(\phi_x, \psi_x, \zeta_x)\big\|^2\,d\tau\right), \tag{5.8}$$

where ω is the function in (2.3) *by choosing $\alpha = \hat{c}/2$ and $c = c(\hat{m}, m_v, m_\theta, M)$.*

Proof. After getting the following two estimates

$$\iint_0^t \big(R\zeta + (\gamma-1)p_+\phi\big)^2 \omega^2\,dx\,d\tau$$
$$\leqslant c\delta^{\frac{3}{2}} \iint_0^t \left(\phi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\,dx\,ds + c\delta^{\frac{1}{2}}\left(1 + \int_0^t \big\|(\phi_x,\psi_x,\zeta_x)\big\|^2\,d\tau\right), \tag{5.9}$$

$$\iint_0^t \big((R\zeta - p_+\phi)^2 + \psi^2\big)\omega^2\,dx\,d\tau$$
$$\leqslant c\delta^{\frac{3}{2}} \iint_0^t \left(\phi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\,dx\,ds + c\delta^{\frac{1}{2}}\left(1 + \int_0^t \big\|(\phi_x,\psi_x,\zeta_x)\big\|^2\,d\tau\right), \tag{5.10}$$

adding two estimates and using

$$\big(R\zeta + (\gamma-1)p_+\phi\big)^2 + (R\zeta - p_+\phi)^2 \geqslant \delta\left(\frac{(R\zeta)^2}{\delta} + 2(p_+\phi)^2\right)$$

yields (5.8), where $c = c(\hat{m}, m_v, m_\theta, M)$.

We first prove (5.9). Noticing that

$$|U_x| \leqslant c\delta\omega^2, \qquad h = R\zeta + (\gamma-1)p_+\phi = O(1)\delta^{\frac{1}{2}}\left(\phi + \frac{\zeta}{\sqrt{\delta}}\right), \tag{5.11}$$

we estimate the right terms J_i $(i = 1,\dots,4)$ of (4.14). Using Lemma 3.2, (3.3) and (5.11), we obtain

$$|J_2| \leqslant c(\hat{m})\delta^2 \int \left(\phi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\,dx,$$

$$|J_3| \leqslant c(\hat{m}, m_v) \int \Big(\delta\big(|\zeta_x h_x| + |\phi\Theta_x h_x|\big) + \delta^{\frac{1}{2}}\omega\big(|\zeta_x h| + |\phi\Theta_x h|\big)\Big)\,dx$$
$$\leqslant c(\hat{m}, m_v)\delta^{\frac{3}{2}} \int \left(\phi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\,dx + c(\hat{m}, m_v)\delta^{\frac{1}{2}}\big\|(\phi_x,\zeta_x)\big\|^2,$$

$$|J_4| \leqslant c(\hat{m}, M)\delta \int \big(|\phi| + |\zeta|\big)\big(|\psi_x|^2 + |U_x|^2\big)\,dx \leqslant c(\hat{m}, M)\delta\|\psi_x\|^2 + c(\hat{m}, M)\delta(1+t)^{-\frac{3}{2}}.$$

To estimate J_1, we use (4.15). Noticing that (3.3), (5.11) and $|\omega_x| \leqslant c(1+t)^{-1}\delta^{-\frac{1}{2}}$, we have

$$\big|J_1^1\big| \leqslant c(\hat{m})\delta(1+t)^{-1} \int |h||\phi|\big(|\phi| + |h|\big)\,dx \leqslant c(\hat{m})\delta(1+t)^{-1} \int \big(|\phi|^3 + |\zeta|^3\big)\,dx$$
$$\leqslant c(\hat{m}, M)\delta(1+t)^{-\frac{4}{3}} + c(\hat{m}, M)\delta\big\|(\phi_x,\zeta_x)\big\|^2,$$

$$\begin{aligned}\left|J_1^2\right| &\leqslant c(\hat{m})\delta \int \big(|U_x| + |\psi_x|\big)\big(|\phi|^3 + |\zeta|^3\big)\,dx \\ &\leqslant c(\hat{m})\delta\big(\|\phi\|^2\|\phi_x\| + \|\zeta\|^2\|\zeta_x\|\big)\big(\|U_x\| + \|\psi_x\|\big) \\ &\leqslant c(\hat{m}, M)\delta(1+t)^{-\frac{3}{2}} + c(\hat{m}, M)\delta\big\|(\phi_x, \psi_x, \zeta_x)\big\|^2,\end{aligned}$$

$$\begin{aligned}\left|J_1^3\right| &\leqslant c(\hat{m}, m_v)\delta \int \big(|\zeta_x| + |\phi||\Theta_x|\big)\big|\big(v^{-1}g^2\big)_x\big|\big(|\phi|^2 + |\zeta|^2\big)\,dx \\ &\quad + c(\hat{m}, m_v)\delta \int \big(|\zeta_x| + |\phi||\Theta_x|\big)\big|g^2\big|\big|\big(\phi(2h - \gamma p_+\phi)\big)_x\big|\,dx \\ &\leqslant c(\hat{m}, m_v)\delta^{\frac{3}{2}} \int \big(|\zeta_x| + |\phi|(1+t)^{-\frac{1}{2}}\big)\big(|\phi_x| + (1+t)^{-\frac{1}{2}}\big)\big(|\phi|^2 + |\zeta|^2\big)\,dx \\ &\quad + c(\hat{m}, m_v)\delta^{\frac{3}{2}} \int \big(|\zeta_x| + |\phi|(1+t)^{-\frac{1}{2}}\big)\big(|\phi| + |\zeta|\big)\big(|\phi_x| + |\zeta_x|\big)\,dx \\ &\leqslant c(\hat{m}, m_v, M)\delta^{\frac{3}{2}}\big\{\|\zeta_x\|\big(\|\phi_x\| + \|\zeta_x\| + (1+t)^{-1}\big) + (1+t)^{-1}\big(\|\phi_x\|^{\frac{1}{2}} + \|\zeta_x\|^{\frac{1}{2}}\big)\big\} \\ &\quad + c(\hat{m}, m_v, M)\delta^{\frac{3}{2}}(1+t)^{-\frac{1}{2}}\big(\|\phi_x\|^{\frac{3}{2}} + \|\zeta_x\|^{\frac{3}{2}}\big) \\ &\leqslant c(\hat{m}, m_v, M)\delta^{\frac{3}{2}}\big(\big\|(\phi_x, \zeta_x)\big\|^2 + (1+t)^{-\frac{4}{3}}\big),\end{aligned}$$

$$\left|J_1^4\right| \leqslant c(\hat{m})\delta^2 \int |\phi|\big(|\phi| + |\zeta|\big)\big(|U_x|^2 + |\psi_x|^2\big)\,dx \leqslant c(\hat{m}, M)\delta^2(1+t)^{-\frac{3}{2}} + c(\hat{m})\delta^2\|\psi_x\|^2.$$

On the other hand, we have

$$\left|\int v^{-1}hg^2\phi(2h - \gamma p_+\phi)\,dx\right| \leqslant c(\hat{m})\delta \int |h||\phi|\big(|h| + |\phi|\big)\,dx \leqslant c(\hat{m}, M)\delta. \tag{5.12}$$

Using estimates J_i $(i = 1, \dots, 4)$ and (5.12), we obtain from (4.14)

$$\frac{1}{\gamma - 1}\int_0^t \langle h_t, hg^2\rangle\,d\tau \leqslant c\delta^{\frac{3}{2}} \int_0^t\!\!\int \left(\phi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\,dx\,ds + c\delta^{\frac{1}{2}} \int_0^t \big\|(\phi_x, \psi_x, \zeta_x)\big\|^2\,ds + c\delta, \tag{5.13}$$

where $c = c(\hat{m}, m_v, M)$. Applying Proposition 3.1 to (5.13), we obtain (5.9).

Next, we prove (5.10). Denoting by $f(x,t) = \int_{-\infty}^x \omega^2(y,t)\,dy$, we have

$$\big\|f(\cdot,t)\big\|_{L_\infty} \leqslant c\delta^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}, \qquad \big\|f_t(\cdot,t)\big\|_{L_\infty} \leqslant c\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}. \tag{5.14}$$

We estimate the right terms J_i, $i = 5, \dots, 10$, of (4.19) as follows: Using Lemma 3.2 and (5.14), we have from (4.20)

$$\begin{aligned}\left|J_5^1\right| + |J_6| &\leqslant \delta^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}\big(\|\psi\| + \|\zeta\|\big)\big(\|U_x\| + \|\psi_x\|\big) \\ &\leqslant c\delta^{\frac{1}{2}}\big(\|\psi_x\|^2 + (1+t)^{-\frac{5}{3}}\big),\end{aligned}$$

$$\begin{aligned}\left|J_5^2\right| &\leqslant C(\hat{m}, m_v)\delta \int \big(|\zeta_x| + |\phi|(1+t)^{-\frac{1}{2}}\big)|\psi_x v f + \psi v_x f + \psi v f_x|\, dx \\ &\leqslant c\delta\big(\big\|(\phi_x, \psi_x, \zeta_x)\big\|^2 + (1+t)^{-\frac{3}{2}}\big),\end{aligned}$$

$$\left|J_5^3\right| \leqslant c\delta^{\frac{3}{2}}(1+t)^{-\frac{1}{2}}\|\psi\|_{L_\infty}\big(\|\psi_x\|^2 + (1+t)^{-\frac{3}{2}}\big) \leqslant c\delta^{\frac{3}{2}}\|\psi_x\|^2 + c\delta^{\frac{3}{2}}(1+t)^{-2},$$

$$\begin{aligned}J_5^3 &= -\frac{\gamma p_+}{2}\int v\omega^2\psi^2\, dx - \frac{\gamma p_+}{2}\int f\psi^2\bar{v}_x\, dx - \frac{\gamma p_+}{2}\int f\psi^2\phi_x\, dx \\ &\leqslant -\frac{\gamma p_+}{4}\int v\omega^2\psi^2\, dx + C(\hat{m}, M)\delta^{\frac{1}{2}}\big(\big\|(\phi_x, \psi_x)\big\|^2 + (1+t)^{-2}\big),\end{aligned}$$

where $c = c(\hat{m}, m_v, M) > 0$. On the other hand, we get

$$|J_7| \leqslant c\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}\int |\psi|\big(|\phi| + |\zeta|\big)\, dx \leqslant c\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},$$

$$\begin{aligned}|J_8| &= \left|\int v^{-1}(R\zeta - p_+\phi)^2\Theta_x f\, dx + \int v^{-1}(R\zeta - p_+\phi)^2\phi_x f\, dx\right| \\ &\leqslant c\delta^{\frac{3}{2}}\int\left(\phi^2 + \frac{\zeta^2}{\delta}\right)\omega^2\, dx + c\delta^{\frac{1}{2}}\big\|(\phi_x, \zeta_x)\big\|^2 + c\delta^{\frac{1}{2}}(1+t)^{-2},\end{aligned}$$

$$|J_9| \leqslant c\delta^{\frac{1}{2}}\big\|(\phi_x, \psi_x, \zeta_x)\big\|^2 + c\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},$$

$$|J_{10}| \leqslant c\delta^{\frac{1}{2}}\|F\|_{L_1}(1+t)^{-\frac{1}{2}}\big(\|\phi_x\|_{L_\infty} + \|\zeta_x\|_{L_\infty}\big) \leqslant c\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}} + c\delta^{\frac{1}{2}}\big\|(\phi_x, \zeta_x)\big\|^2$$

and

$$\left|\int \psi(R\zeta - p_+\phi)vf\, dx\right| \leqslant c\delta^{\frac{1}{2}}, \tag{5.15}$$

where $c = c(\hat{m}, m_v, M) > 0$.

Integrating (4.19) over $(0, t)$, together with all the estimates of J_i, $i = 5, \ldots, 10$, and (5.15) yields (5.10). The proof of Lemma 5.2 is completed. □

Lemma 5.3. *There exists a small positive constant δ_0 such that if $\delta = \gamma - 1 \leqslant \delta_0$, then it follows that*

$$0 < c_3(m_v, m_\theta)^{-1} \leqslant v(t, x) \leqslant c_3(m_v, m_\theta) \tag{5.16}$$

and

$$\begin{aligned}&\left\|\left(\phi, \psi, \frac{\zeta}{\sqrt{\delta}}\right)(t)\right\|^2 + \big\|\phi_x(t)\big\|^2 + \int_0^t \big\|(\phi_x, \psi_x, \zeta_x)(\tau)\big\|^2\, d\tau \\ &\quad\leqslant c_4(m_v, m_\theta)\big(1 + \big\|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\big\|_1^2\big).\end{aligned} \tag{5.17}$$

Proof. We first prove

$$\left\|\frac{\phi_x}{v}(t)\right\|^2+\int_0^t\left\|\frac{\phi_x}{v^{\frac{3}{2}}}(s)\right\|^2 ds\leqslant c(m_v,m_\theta)\big(1+\big\|(\phi_0,\psi_0,\sqrt{\delta}\varphi_0)\big\|_1^2\big). \tag{5.18}$$

Estimate the right terms I_i, $i=1,\dots,3$, of (4.23) as follows:

$$\int|I_1|\,dx\leqslant\eta\left\|\frac{\phi_x}{v^{\frac{3}{2}}}\right\|^2+c(m_v,m_\theta)\big(1+\eta^{-1}\big)\left\|\left(\frac{\psi_x}{\sqrt{v}},\frac{\zeta_x}{\sqrt{v}}\right)\right\|^2,\quad\text{for any }\eta>0,$$

$$\int|I_2|\,dx\leqslant c(m_v,m_\theta)\delta^{\frac{1}{2}}\left\|\left(\frac{\phi_x}{v^{\frac{3}{2}}},\frac{\psi_x}{\sqrt{v}}\right)\right\|^2+c(\hat{m},m_v,m_\theta)\delta^{\frac{1}{2}}\int\left|\left(\phi,\psi,\frac{\zeta}{\sqrt{\delta}}\right)\right|^2\omega^2\,dx,$$

$$\int|I_3|\,dx\leqslant c(m_v,m_\theta,M)\delta^{\frac{1}{2}}\left\|\left(\frac{\phi_x}{v^{\frac{3}{2}}},\frac{\psi_x}{\sqrt{v}}\right)\right\|^2+c(m_v,m_\theta)\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}.$$

Using estimates I_i, $i=1,\dots,3$, we obtain from (4.23)

$$\begin{aligned}&\left\|\frac{\phi_x}{v}(t)\right\|^2+\int_0^t\left\|\frac{\phi_x}{v^{\frac{3}{2}}}(s)\right\|^2 ds\\&\quad\leqslant c(m_v,m_\theta)\big(\delta^{\frac{1}{2}}+\big\|(\psi_0,\phi_{0x})\big\|^2\big)+c(m_v,m_\theta)\left(\big\|\psi(t)\big\|^2+\int_0^t\left\|\left(\frac{\psi_x}{\sqrt{v}},\frac{\zeta_x}{\sqrt{v}}\right)(s)\right\|^2ds\right)\\&\qquad+c(\hat{m},m_v,m_\theta,M)\delta^{\frac{1}{2}}\int_0^t\!\!\int\left|\left(\phi,\psi,\frac{\zeta}{\sqrt{\delta}}\right)\right|^2\omega^2\,dx\,ds.\end{aligned}\tag{5.19}$$

From (5.19) and Lemmas 5.1–5.2, we get (5.18).

To use Y. Kanel's method (cf. [13]) to the proof of (5.16), we need to estimate $\|\frac{\tilde{v}_x}{\tilde{v}}(t)\|^2$ where $\tilde{v}=v/V$. In fact since

$$\frac{\tilde{v}_x}{\tilde{v}}=\frac{\phi_x}{v}-\left(\frac{V_x}{v}-\frac{V_x}{V}\right),$$

we have

$$\left\|\frac{\tilde{v}_x}{\tilde{v}}(t)\right\|^2\leqslant2\left\|\frac{\phi_x}{v}(t)\right\|^2+c(\hat{m})c(m_v)\|V_x\|^2\leqslant2\left\|\frac{\phi_x}{v}(t)\right\|^2+c(\hat{m})c(m_v)\delta^{\frac{1}{2}}. \tag{5.20}$$

Therefore, by using (5.4), (5.18) and (5.20), we get

$$\begin{aligned}&\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)},\psi,\frac{\zeta}{\sqrt{\delta}},\frac{\phi_x}{v},\frac{\tilde{v}_x}{\tilde{v}}\right)(t)\right\|^2+\int_0^t\left\|\left(\frac{\phi_x}{v^{\frac{3}{2}}},\frac{\psi_x}{\sqrt{v}},\frac{\zeta_x}{\sqrt{v}}\right)(\tau)\right\|^2d\tau\\&\quad\leqslant c(m_v,m_\theta)\big(1+\big\|(\phi_0,\psi_0,\sqrt{\delta}\varphi_0)\big\|_1^2\big).\end{aligned}\tag{5.21}$$

To prove (5.16), let

$$\Psi(\tilde{v}) = \int_1^{\tilde{v}} \frac{\sqrt{\Phi(\eta)}}{\eta}\, d\eta, \quad \Phi(\eta) = \eta - \ln\eta - 1. \tag{5.22}$$

Since

$$\Psi(\tilde{v}) \to \begin{cases} -\infty & \text{as } \tilde{v} \to 0_+, \\ +\infty & \text{as } \tilde{v} \to +\infty, \end{cases} \tag{5.23}$$

and

$$\left|\Psi\big(\tilde{v}(x,t)\big)\right| = \left|\int_{-\infty}^{x} \frac{\partial}{\partial y}\Psi\big(\tilde{v}(y,t)\big)\,dy\right| \leqslant \frac{1}{2}\left\|\left(\sqrt{\Phi\left(\frac{v}{V}\right)}, \frac{\tilde{v}_x}{\tilde{v}}\right)(t)\right\|^2, \tag{5.24}$$

(5.16) follows from (5.21)–(5.24). From (5.16) and (5.21), it is easy to get (5.17).

The proof of Lemma 5.3 is completed. □

Lemma 5.4. *It follows that*

$$\left\|(\psi_x, \zeta_x)(t)\right\|^2 + \int_0^t \left\|(\psi_{xx}, \zeta_{xx})(\tau)\right\|^2 d\tau \leqslant c_5(m_v, m_\theta)\big(1 + \left\|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\right\|_1^2\big) \tag{5.25}$$

and

$$0 < \frac{1}{2} m_\theta^{-1} \leqslant \theta(t,x) \leqslant 2m_\theta. \tag{5.26}$$

Proof. The estimate (5.25) is given in the same way as Lemma 4.4 using Lemma 3.2 instead of Lemma 3.1. So, we will omit. From (5.17) and (5.25), we have

$$\left|\zeta(x,t)\right| \leqslant \sqrt{2}\left\|\zeta(t)\right\|^{\frac{1}{2}}\left\|\zeta_x(t)\right\|^{\frac{1}{2}} \leqslant \delta^{\frac{1}{4}} c(m_v, m_\theta)\big(1 + \left\|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\right\|_1^2\big). \tag{5.27}$$

Since

$$0 < m_\theta^{-1} \leqslant \Theta(t,x) \leqslant m_\theta \tag{5.28}$$

and δ is small, we get (5.26) from (5.27) and (5.28). □

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