



Least energy solutions of the generalized Hénon equation in reflectionally symmetric or point symmetric domains

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ABSTRACT

We study the generalized Hénon equation in reflectionally symmetric or point symmetric domains and prove that a least energy solution is neither reflectionally symmetric nor even. Moreover, we prove the existence of a positive solution with prescribed exact symmetry.

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1. Introduction

In this paper, we study the generalized Hénon equation in a symmetric domain

$$-\Delta u = h(x)u^p, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N with piecewise smooth boundary $\partial\Omega$, $1 < p < \infty$ when $N = 2$, $1 < p < (N + 2)/(N - 2)$ when $N \geq 3$, $h \in L^\infty(\Omega)$ and $h(x)$ may be positive or may change its sign. In this paper, we consider Ω and $h(x)$ which are reflectionally symmetric or point symmetric. We call Ω

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x_i -symmetric if it has a reflectional symmetry with respect to the hyperplane $x_i = 0$, i.e., $(x_1, \dots, x_N) \in \Omega$ implies

$$(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N) \in \Omega.$$

We call $h(x)$ x_i -symmetric if

$$h(x_1, \dots, x_N) = h(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N) \quad \text{in } \Omega.$$

We call Ω *point symmetric* if $x \in \Omega$ implies $-x \in \Omega$ and $h(x)$ *point symmetric* (or *even*) if $h(x) = h(-x)$ in Ω . In the same manner as in $h(x)$, we call a solution of (1.1) x_i -symmetric or even. Let I be a subset of the set $\{1, \dots, N\}$, where I may be empty. We call Ω , $h(x)$ and a solution of (1.1) *I-symmetric* if they are x_i -symmetric for all $i \in I$. We call a solution of (1.1) *exactly I-symmetric* if it is x_i -symmetric for all $i \in I$ and not x_j -symmetric for all $j \notin I$. When I is empty, we mean that an exactly I -symmetric solution has no x_i -symmetry for all i . Consider the weight function of the form $h(x) = g(|x|)^\lambda$ in a rectangle domain Ω . Then we have the next theorem, which is one of our main results.

Theorem 1.1. Let $\Omega := \prod_{i=1}^N (-L_i, L_i)$ be a rectangle with $L_i > 0$. Let $g(r)$ be a continuous function on $[0, T]$ which satisfies $0 \leq g(r) < g(T)$ for $0 \leq r < T$, where $T := (\sum_{i=1}^N L_i^2)^{1/2}$. Put $h(x) := g(|x|)^\lambda$ in (1.1). Then for λ large enough, (1.1) has an exactly I -symmetric positive solution for any subset I of the set $\{1, \dots, N\}$. Therefore there exist at least 2^N positive solutions.

The theorem above is applicable to many weight functions $h(x)$, e.g., $h(x) = |x|^\lambda$ (the original Hénon equation), $h(x) = e^{\lambda|x|}$, $(|x|/(1+|x|))^\lambda$, $(\sin(\pi|x|/(2T)))^\lambda$, etc. When Ω is a cube, we say that solutions u and v are *equivalent* if $u(gx) = v(x)$ with a certain orthogonal matrix g . For example, when $N = 5$, an exactly $\{1, 4, 5\}$ -symmetric solution is equivalent to an exactly $\{1, 2, 3\}$ -symmetric solution. Then we have the next corollary.

Corollary 1.2. Let Ω be a cube and let $h(x)$ be as in Theorem 1.1. Then for λ large enough, (1.1) has an exactly $\{1, \dots, n\}$ -symmetric positive solution for $n = 0, \dots, N$. Here we mean that the solution with $n = 0$ is not x_i -symmetric for all i . Therefore there exist at least $N + 1$ non-equivalent positive solutions.

The difficulty of Theorem 1.1 is to prove the exactness of symmetry, i.e., to prove that a solution is not x_j -symmetric for all $j \in I^c := \{1, \dots, N\} \setminus I$. It is not difficult to find an I -symmetric solution without the exactness. Indeed, we have the next lemma.

Lemma 1.3. Let I be a subset of $\{1, \dots, N\}$ and assume that Ω and $h(x)$ are I -symmetric and $h \in L^\infty(\Omega)$ with $h_+(x) \not\equiv 0$, where $h_+(x) := \max(h(x), 0)$. Then there exists an I -symmetric positive solution. If Ω and $h(x)$ are point symmetric with respect to the origin, then there exists an even positive solution.

Proof. We deal with the I -symmetry only, because the point symmetry can be treated in the same way as below. We define

$$L(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} h(x) (u_+)^{p+1} \right) dx, \\ H_0^1(\Omega, I) := \{u \in H_0^1(\Omega) : u(x) \text{ is } I\text{-symmetric}\}. \quad (1.2)$$

Here $H_0^1(\Omega)$ is a usual Sobolev space. Restricting $L(u)$ onto $H_0^1(\Omega, I)$ and applying the mountain pass lemma by Ambrosetti and Rabinowitz [1], one can obtain a critical point u of L in $H_0^1(\Omega, I)$, i.e., $L'(u)v = 0$ for $v \in H_0^1(\Omega, I)$, where L' denotes the Fréchet derivative of L . Then it becomes a critical

point in $H_0^1(\Omega)$, i.e., $L'(u)v = 0$ for all $v \in H_0^1(\Omega)$, because of the principle of symmetric criticality by Palais [17]. Thus we get an I -symmetric solution of the equation

$$-\Delta u = h(x)u_+^p \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.3)$$

Multiplying both sides by $u_-(x) := \min(u(x), 0)$ and integrating it over Ω , we have

$$\|\nabla u_-\|_2^2 = \int_{\Omega} hu_+^p u_- dx = 0,$$

which shows $u_-(x) = 0$, i.e., $u \geq 0$ in Ω . In the equation above, $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm and hereafter $\|u\|_q$ denotes the $L^q(\Omega)$ norm of u . A solution u belongs to $H_0^1(\Omega) \cap W_{loc}^{2,q}(\Omega) \cap L^\infty(\Omega)$ for all $q < \infty$, which will be proved in Lemma 3.1. If h is nonnegative in Ω , then the strong maximum principle ensures that $u > 0$. If $h(x)$ changes its sign, we put $C := \|hu^{p-1}\|_\infty$. Adding Cu to (1.3) with u_+ replaced by u , we get

$$(C - \Delta)u = (C + hu^{p-1})u \geq 0 \quad \text{in } \Omega,$$

which with the strong maximum principle shows that $u > 0$ in Ω . Thus u is an I -symmetric positive solution of (1.1). \square

We are looking for an I^c -asymmetric solution in the set of I -symmetric solutions. To this end, we use a *Rayleigh quotient*

$$R(u) := \left(\int_{\Omega} |\nabla u|^2 dx \right) / \left(\int_{\Omega} h(x)|u|^{p+1} dx \right)^{2/(p+1)},$$

with the definition domain

$$D(R) := \left\{ u \in H_0^1(\Omega) : \int_{\Omega} h(x)|u|^{p+1} dx > 0 \right\}.$$

In this paper, we deal with sign-changing weights $h(x)$ as well as positive weights. If $h(x) \leq 0$ in the whole Ω , there exists no positive solution by the maximum principle. Hence we always assume that $h_+(x) \not\equiv 0$. Then $D(R)$ is not empty and an open subset of $H_0^1(\Omega)$. We define the *least energy* R_0 by

$$R_0 := \inf\{R(u) : u \in D(R)\}.$$

Because of the Sobolev embedding theorem, the Rayleigh quotient R has a positive lower bound, and therefore R_0 is well defined and positive. We define the *Nehari manifold*

$$\mathcal{N} := \left\{ u \in H_0^1(\Omega) \setminus \{0\} : \int_{\Omega} (|\nabla u|^2 - h(x)|u|^{p+1}) dx = 0 \right\}.$$

It is obvious that $\mathcal{N} \subset D(R)$. Observe that for any $u \in D(R)$, there is a $\lambda > 0$ such that $\lambda u \in \mathcal{N}$. Furthermore, $R(\lambda u) = R(u)$ for any $\lambda > 0$. Hence we have

$$R_0 = \inf\{R(u) : u \in D(R)\} = \inf\{R(u) : u \in \mathcal{N}\}.$$

The infimum is achieved at a certain point $u \in \mathcal{N}$. Then u satisfies (1.1). For the proof, we refer the readers to [14] or [15]. We call u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = R_0$. It is well known that a least energy solution is positive or negative (see [14] or [15]). We choose a positive solution as a least energy solution because we replace u by $-u$, if necessary.

To explain our motivation, we consider the Hénon equation

$$-\Delta u = |x|^\lambda u^p, \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B, \quad (1.4)$$

where B is a unit ball in \mathbb{R}^N . The equation above was introduced by Hénon [11] to study spherically symmetric clusters of stars. Smets, Willem and Su [20] have proved that if λ is large enough, then a least energy solution of (1.4) is not radially symmetric. On the other hand, by using the mountain pass lemma in the radially symmetric function space, one can prove the existence of a radial positive solution. Therefore (1.4) has both a radial positive solution and a nonradial positive solution. There are many contributions which have studied the Hénon equation [2–9, 12, 18, 19].

On the other side, Moore and Nehari [16, pp. 32–33] have studied the two point boundary value problem of the ordinary differential equation

$$u''(t) + h(t)u^p = 0, \quad u > 0 \text{ in } (-1, 1), \quad u(-1) = u(1) = 0. \quad (1.5)$$

Here $h(t) = 0$ for $|t| < a$ and $h(t) = 1$ for $a < |t| < 1$. When $a(< 1)$ is sufficiently close to 1, they have constructed at least three positive solutions of (1.5): the first one is even, the second one $u(t)$ is non-even and the third one is the reflection $u(-t)$.

Inspired the results above, the author has studied (1.4) with $|x|^\lambda$ replaced by more general nonnegative radial functions $h(|x|)$ and has proved in [14] that if the ratio of the density of $h(|x|)$ in $|x| < a$ to that in $a < |x| < 1$ is small enough and a is sufficiently close to 1, then a least energy solution is not radially symmetric. See also [13] and [15].

Observing these results, we conjecture that if Ω and $h(x)$ are symmetric with respect to $x_1 = 0$ and if the density of $h(x)(\geq 0)$ is thick in a subdomain D of Ω far away from the hyperplane $x_1 = 0$, but thin in $\Omega \setminus D$, then a least energy solution has no reflectional symmetry with respect to x_1 . Therefore (1.1) has both a reflectionally symmetric solution and an asymmetric solution. Moreover, we conjecture that if Ω and $h(x)$ are point symmetric with respect to the origin and if $h(x)$ is large in a subdomain D of Ω far away from the origin and small in $\Omega \setminus D$, then a least energy solution is not even. The purpose of this paper is to prove these conjectures to be true.

To get our results, for a subset I of $\{1, \dots, N\}$, we define

$$\begin{aligned} D(R, I) &:= D(R) \cap H_0^1(\Omega, I), & \mathcal{N}(I) &:= \mathcal{N} \cap H_0^1(\Omega, I), \\ R_0(I) &:= \inf\{R(u) : u \in D(R, I)\} = \inf\{R(u) : u \in \mathcal{N}(I)\}, \end{aligned}$$

where $H_0^1(\Omega, I)$ has been defined by (1.2). Moreover, we define even function spaces

$$\begin{aligned} H_0^1(\Omega, e) &:= \{u \in H_0^1(\Omega) : u(x) = u(-x)\}, \\ D(R, e) &:= D(R) \cap H_0^1(\Omega, e), & \mathcal{N}(e) &:= \mathcal{N} \cap H_0^1(\Omega, e), \\ R_0(e) &:= \inf\{R(u) : u \in D(R, e)\} = \inf\{R(u) : u \in \mathcal{N}(e)\}. \end{aligned}$$

We call $R_0(I)$ the I -symmetric least energy and u an I -symmetric least energy solution if $u \in \mathcal{N}(I)$ and $R(u) = R_0(I)$. To avoid confusion, a usual least energy solution is called a *global least energy solution*. By replacing $R_0(I)$ and $\mathcal{N}(I)$ by $R_0(e)$ and $\mathcal{N}(e)$, respectively, we define the *even least energy* and an *even least energy solution*.

Our idea of the proof of Theorem 1.1 is to show that an I -symmetric least energy solution becomes exactly I -symmetric. Indeed, if an I -symmetric least energy solution has another x_j -symmetry with a

certain $j \notin I$, then we shall construct an I -symmetric function v whose energy $R(v)$ is less than the I -symmetric least energy. This is a contradiction, and hence our assertion is proved.

On the other hand, to find a non-even solution for a point symmetric domain, we show that a global least energy solution is not even. To this end, we compare the even least energy $R_0(e)$ with a global least energy R_0 . We shall show that $R_0 < R_0(e)$. This guarantees that a global least energy solution cannot be even. Before stating the main results, we prove that $D(R, I)$, $\mathcal{N}(I)$, $D(R, e)$ and $\mathcal{N}(e)$ are not empty in the next lemma.

Lemma 1.4. *Assume that $h \in L^\infty(\Omega)$ and $h_+(x) \not\equiv 0$. If Ω and h are I -symmetric, then $D(R, I)$ and $\mathcal{N}(I)$ are not empty. If Ω and h are point symmetric, then $D(R, e)$ and $\mathcal{N}(e)$ are not empty. In any case, $D(R)$ and \mathcal{N} are not empty.*

Proof. For $\delta > 0$, we put

$$D := \{x \in \Omega : h(x) > \delta \text{ and } \text{dist}(x, \partial\Omega) > \delta\},$$

$$\text{dist}(x, \partial\Omega) := \inf\{|x - y| : y \in \partial\Omega\}. \quad (1.6)$$

Since $h_+(x) \not\equiv 0$, we choose $\delta > 0$ so small that D has a positive Lebesgue measure. Define $u(x) = 1$ in D and $u(x) = 0$ in $\Omega \setminus D$. Then we see that

$$\int_{\Omega} h|u|^{p+1} dx \geq \delta \text{Vol}(D) > 0,$$

where $\text{Vol}(D)$ denotes the Lebesgue measure of D . Since D is a symmetric domain, u is symmetric. Here “symmetric” means I -symmetric or point symmetric. Since $u \in L^\infty(\Omega)$, there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that $u_n \rightarrow u$ strongly in $L^q(\Omega)$ for all $q < \infty$. Moreover we can assume that u_n is symmetric because the usual mollifier has symmetry. Indeed, we choose a radially symmetric function $J(|x|)$ in $C_0^\infty(\mathbb{R}^N)$ such that $J(x) \geq 0$ and

$$\text{supp } J \subset \{x : |x| < 1\}, \quad \int_{\mathbb{R}^N} J(x) dx = 1.$$

Here $\text{supp } J$ denotes the support of J . For $n \in \mathbb{N}$, we put $J_n(x) = n^N J(nx)$ and define

$$u_n(x) := \int_{\mathbb{R}^N} J_n(x - y) u(y) dy.$$

Then $u_n \in C_0^\infty(\Omega)$ for n large enough and it is symmetric and satisfies

$$\int_{\Omega} h|u_n|^{p+1} dx \rightarrow \int_{\Omega} h|u|^{p+1} dx > 0.$$

For n large enough, u_n belongs to $D(R, I)$ or $D(R, e)$, i.e., these sets are not empty. Note that for any $u \in D(R, I)$, there exists a $\lambda > 0$ such that $\lambda u \in \mathcal{N}(I)$. Thus $\mathcal{N}(I) \neq \emptyset$. In the same way, $\mathcal{N}(e)$ is not empty. Since $D(R, \sigma) \subset D(R)$ and $\mathcal{N}(\sigma) \subset \mathcal{N}$ with $\sigma = I$ or $\sigma = e$, the sets $D(R)$ and \mathcal{N} also are not empty. \square

This paper is organized in seven sections. In Section 2, we state the main results and give some examples of $h(x)$. In Section 3, we construct a function which has a lower energy than the symmetric least energy provided that a symmetric least energy solution satisfies a certain inequality. This inequality is fulfilled under our assumption on $h(x)$, which will be proved in Section 5 for the I -symmetry and in Section 6 for the point symmetry. To this end, we need an a priori $L^\infty(\Omega)$ estimate of a symmetric least energy solution. This will be given in Section 4. In Section 7, we prove the main results.

2. Main results

In this section, we state the main results and give some examples of $h(x)$. We first study the reflectional symmetry. Let $\Omega(s < x_1 < t)$ denote the set of points in Ω between two hyperplanes $x_1 = s$ and $x_1 = t$, i.e.,

$$\Omega(s < x_1 < t) := \{(x_1, x') \in \Omega : s < x_1 < t\}, \quad (2.1)$$

with $x = (x_1, x')$ and $x' = (x_2, \dots, x_N)$. We define $L > 0$ by

$$L := \sup\{x_1 : (x_1, x') \in \Omega\}. \quad (2.2)$$

If Ω is x_1 -symmetric, it lies between two parallel hyperplanes $x_1 = L$ and $x_1 = -L$. We fix $a \in (0, L)$ satisfying

$$\frac{8(L-a)}{a} < \frac{p-1}{p}, \quad (2.3)$$

that is, $8pL/(9p-1) < a < L$. We introduce two alternative assumptions (A) and (B) on $h(x)$: the former deals with a sign-changing weight and the latter with a positive one.

(A) $h(x) \leq 0$ in $\Omega(|x_1| < a)$ and $h_+(x) \not\equiv 0$ in $\Omega(a < |x_1| < L)$.

(B) $h(x) \geq 0$, $\not\equiv 0$ in Ω .

In case (B), we define for $N \geq 3$,

$$\xi(h) := \|h\|_{L^\infty(\Omega(|x_1| < a))}^{N+2-(N-2)p} \|h\|_{L^\infty(\Omega)}^{(N-2)(p-1)} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-4},$$

and for $N = 2$ with $r \in (0, 1)$,

$$\xi(h, r) := \|h\|_{L^\infty(\Omega(|x_1| < a))} \|h\|_{L^\infty(\Omega)}^{(1-r)/r} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-1/r}.$$

Here $\|\cdot\|_{L^q(\Omega)}$ denotes the $L^q(\Omega)$ norm and $\operatorname{dist}(x, \partial\Omega)$ is the distance function defined by (1.6). We always assume that Ω is a bounded domain whose boundary $\partial\Omega$ is piecewise smooth. The rectangle domain fulfills the condition above. We state the first main result.

Theorem 2.1. *Let Ω and h be x_1 -symmetric and $h \in L^\infty(\Omega)$. Fix $a \in (0, L)$ satisfying (2.3) and assume either (A) or (B). In case (B), we assume in addition that $\xi(h)$ is small enough for $N \geq 3$ and $\xi(h, r)$ is small enough with a certain $r \in (0, 1)$ for $N = 2$. Then a least energy solution of (1.1) is not x_1 -symmetric. Therefore (1.1) has both an x_1 -symmetric positive solution and an x_1 -asymmetric positive solution.*

In the theorem above, the assumption of $\xi(h)$ to be small enough means that the ratio of $h(x)$ in $|x_1| < a$ to that in $a < |x_1| < L$ is small enough. Indeed, consider a simple example $h(x) := m$ in $|x_1| < a$ and $h(x) := M$ in $a < |x_1| < L$ with $M > m > 0$. Then one finds that

$$C_1(m/M)^{N+2-(N-2)p} \leq \xi(h) \leq C_2(m/M)^{N+2-(N-2)p},$$

with $C_i > 0$ independent of m and M . Thus the smallness of $\xi(h)$ is equivalent to that of m/M .

Corollary 2.2. *Let Ω be x_1 -symmetric and let h depend only on x_1 , which is defined by $h(x_1) = g(|x_1|)^\lambda$. Here $g \in C[0, L]$, $0 \leq g(t) < g(L)$ for $0 \leq t < L$ and L is defined by (2.2). If λ is large enough, a least energy solution of (1.1) is not x_1 -symmetric.*

The corollary above deals with $h(x_1)$ depending only on x_1 . In the next corollary, we consider a radially symmetric potential $h(|x|)$.

Corollary 2.3. *Let Ω be an x_1 -symmetric domain satisfying*

$$\sup\{|x|: x \in \Omega(|x_1| < \alpha)\} < \sup\{|x|: x \in \Omega\}, \quad (2.4)$$

with $\alpha := 8pL/(9p-1)$. Denote the right hand side of (2.4) by T . Let $g \in C[0, T]$ satisfy that $0 \leq g(t) < g(T)$ for $0 \leq t < T$. Define $h(x) := g(|x|)^\lambda$. If λ is large enough, a least energy solution of (1.1) is not x_1 -symmetric.

As examples of Ω satisfying (2.4), we have an ellipse

$$\Omega := \{(x_1, x_2): x_1^2/a^2 + x_2^2/b^2 < 1\},$$

with $0 < b < a$ or an N -dimensional rectangle, $\Omega := \prod_{i=1}^N (-L_i, L_i)$.

In order to study x_i -symmetry, we introduce some notation. When Ω is x_i -symmetric, we define

$$L_i := \sup\{x_i: (x_1, \dots, x_i, \dots, x_N) \in \Omega\}, \quad (2.5)$$

$$\Omega(|x_i| < a) := \{x \in \Omega: |x_i| < a\},$$

for $a > 0$, where x_i is the i -th coordinate of x .

Corollary 2.4. *Let Ω be a bounded domain which is x_i -symmetric for all $1 \leq i \leq N$ and satisfies*

$$\sup\{|x|: x \in \Omega(|x_i| < \alpha_i)\} < \sup\{|x|: x \in \Omega\}, \quad (2.6)$$

for all i with $\alpha_i := 8pL_i/(9p-1)$, where L_i has been defined by (2.5). Define $h(x) := g(|x|)^\lambda$, where $g(r)$ satisfies the same assumption as in Corollary 2.3. Then for λ large enough, an I -symmetric least energy solution is exactly I -symmetric for any subset I of $\{1, \dots, N\}$.

An N -dimensional rectangle $\Omega := \prod_{i=1}^N (-L_i, L_i)$ satisfies the assumption of Corollary 2.4. Hence Theorem 1.1 follows directly from Corollary 2.4.

Example 2.5. We give examples of $h(x)$ satisfying our assumption.

- (i) Let $a > 0$ satisfy (2.3). Define $h(x) := m$ for $|x_1| < a$ and $h(x) := 1$ for $a < |x_1| < L$. Here L is defined by (2.2) and m is a constant in $(-\infty, \varepsilon_0)$ with $\varepsilon_0 > 0$ small enough. If $m \leq 0$, then (A) is satisfied. If $0 < m < \varepsilon_0$ with $\varepsilon_0 > 0$ sufficiently small, then (B) is fulfilled and $\xi(h)$ is small enough.

- (ii) Let Ω be x_1 -symmetric and put $h(x_1) := (|x_1| - a)/(L - a)$, where a satisfies (2.3). Then (A) holds and hence a least energy solution is not x_1 -symmetric.
- (iii) Let $h(x) = |x|^\lambda$, $e^{\lambda|x|}$, $(|x|/(1 + |x|))^\lambda$, etc., and Ω be an N -dimensional rectangle. Then for λ large enough, (1.1) has an exactly I -symmetric positive solution for any subset I of $\{1, 2, \dots, N\}$ by Theorem 1.1.

Remark 2.6. We state Chern and Lin's result [6], in which they have studied the semilinear elliptic equation

$$-\Delta u = f(|x|, u), \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B. \quad (2.7)$$

Here B is the unit ball in \mathbb{R}^N and $f(|x|, u)$ is a superlinear and subcritical nonlinear term, which includes the Hénon equation $f(|x|, u) = |x|^\lambda u^p$. In [6], they have proved the theorem below.

Let u be a least energy solution of (2.7) and P_0 be a maximum point of u . Then the following conclusions hold.

- (i) If $P_0 = O$, where O denotes the origin, then u is radially symmetric.
- (ii) If $P_0 \neq O$, then u is axially symmetric with respect to $\overrightarrow{OP_0}$. Furthermore, on each sphere $S_r = \{x: |x| = r\}$ with $0 < r < 1$, $u(x)$ is increasing as the angle of \overrightarrow{Ox} and $\overrightarrow{OP_0}$ decreases.

As stated before, when $f(|x|, u) = |x|^\lambda u^p$ with $\lambda > 0$ large enough, a least energy solution of (2.7) is not radially symmetric. Then (ii) holds. After an orthogonal transformation, we assume $P_0 = (0, \dots, 0, p_0)$ with $0 < p_0 < 1$. Then the axial symmetry ensures the reflectional symmetry with respect to the hyperplane $x_i = 0$ with $1 \leq i \leq N - 1$. On the other hand, Theorem 2.1 says that a least energy solution is not reflectionally symmetric. However, our theorem does not contradict Chern and Lin's result. Indeed, for $h(x) = |x|^\lambda$ and $\Omega = B$, the assumption of Theorem 2.1 is not satisfied. To show it, we compute $\xi(h)$. Observing the definition of $\Omega(|x_1| < a)$ with $\Omega = B$ and $h(x) = |x|^\lambda$, we see that $\|h\|_{L^\infty(\Omega(|x_1| < a))} = \|h\|_{L^\infty(\Omega)} = 1$. Then it holds that

$$\xi(h) = \left(\int_{|x| < 1} |x|^\lambda (1 - |x|)^{p+1} dx \right)^{-4} \rightarrow \infty \text{ as } \lambda \rightarrow \infty.$$

Thus the assumption of Theorem 2.1 is not fulfilled.

As mentioned before, the assumption of $\xi(h)$ to be small implies that the ratio of $h(x)$ in $|x_1| < a$ to that in $a < |x_1| < L$ is small enough. Therefore $\Omega = B$ with $h(x) = |x|^\lambda$ does not satisfy the assumption of $\xi(h)$, but a rectangle $\Omega = (-L_1, L_1) \times (-L_2, L_2)$ does. In a rectangle domain, the density of $|x|^\lambda$ concentrates at 4 vertices $(\pm L_1, \pm L_2)$ as $\lambda \rightarrow \infty$, and hence the ratio of $|x|^\lambda$ in $|x_i| < a$ to that in $a < |x_i| < L_i$ with $i = 1, 2$ is small enough.

Now, we consider a point symmetry. We denote by $\Omega(s < |x| < t)$ the intersection of Ω and the annulus $s < |x| < t$, i.e.,

$$\Omega(s < |x| < t) := \{x \in \Omega: s < |x| < t\}. \quad (2.8)$$

We put

$$T := \sup\{|x|: x \in \Omega\}. \quad (2.9)$$

Fix $a \in (0, T)$ satisfying

$$\frac{8(T^N - a^N)}{Na^N} < \frac{p-1}{p}. \quad (2.10)$$

We introduce the assumption on $h(x)$:

(C) $h(x) \leq 0$ in $\Omega(|x| < a)$ and $h_+(x) \not\equiv 0$ in $\Omega(a < |x| < T)$.

We define $\eta(h)$ and $\eta(h, r)$ by $\xi(h)$ and $\xi(h, r)$ with $\Omega(|x_1| < a)$ replaced by $\Omega(|x| < a)$, i.e.,

$$\eta(h) := \|h\|_{L^\infty(\Omega(|x| < a))}^{N+2-(N-2)p} \|h\|_{L^\infty(\Omega)}^{(N-2)(p-1)} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-4},$$

$$\eta(h, r) := \|h\|_{L^\infty(\Omega(|x| < a))} \|h\|_{L^\infty(\Omega)}^{(1-r)/r} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-1/r}.$$

Theorem 2.7. *Let Ω and h be point symmetric and $h \in L^\infty(\Omega)$. Fix $a \in (0, T)$ satisfying (2.10) and assume either (B) or (C). In case (B), we assume in addition that $\eta(h)$ is small enough for $N \geq 3$ and $\eta(h, r)$ is small enough with a certain $r \in (0, 1)$ for $N = 2$. Then a least energy solution of (1.1) is not even. Therefore (1.1) has both an even positive solution and a non-even positive solution.*

Corollary 2.8. *Let Ω be point symmetric and define T by (2.9). Put $h(x) = g(|x|)^\lambda$, where $g(t)$ satisfies the same assumption as in Corollary 2.3. Then for λ large enough, a least energy solution of (1.1) is not even.*

3. A function with lower energy

In this section, we shall construct a function v which has a lower energy than the x_1 -symmetric least energy or the even least energy. We begin with the regularity of solutions. Let $C_{loc}^{1,\theta}(\Omega)$ denote the set of $C^1(\Omega)$ functions whose first derivatives are Hölder continuous with exponent θ in any compact subset of Ω . The boundary $\partial\Omega$ is said to satisfy the exterior cone condition if for every point $\xi \in \partial\Omega$ there exists a finite right cone K , with vertex ξ , satisfying $\bar{K} \cap \bar{\Omega} = \{\xi\}$.

Lemma 3.1. *Assume that $h \in L^\infty(\Omega)$ and Ω is a bounded domain. Let $u \in H_0^1(\Omega)$ satisfy (1.1) in the distribution sense. Then we have*

$$u \in H_0^1(\Omega) \cap W_{loc}^{2,q}(\Omega) \cap C_{loc}^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad (3.1)$$

for all $q < \infty$ and all $\theta \in (0, 1)$. If $\partial\Omega$ satisfies the exterior cone condition, then $u \in C(\bar{\Omega})$. If $h(x)$ is locally Hölder continuous in Ω , then u belongs to $C^2(\Omega)$.

Proof. By the standard bootstrap argument, we see that $u \in L^\infty(\Omega)$. Since $h \in L^\infty(\Omega)$, it holds that

$$-\Delta u = hu^p \in L^\infty(\Omega),$$

which with the interior elliptic regularity theorem shows that u belongs to $W_{loc}^{2,q}(\Omega)$ for all $q < \infty$. The Sobolev embedding theorem implies that $u \in C_{loc}^{1,\theta}(\Omega)$. If $\partial\Omega$ satisfies the exterior cone condition, then $u \in C(\bar{\Omega})$ (see [10, Theorem 9.30]). If $h(x)$ is locally Hölder continuous in Ω , then the elliptic regularity theorem ensures that u belongs to $C^2(\Omega)$. The proof is complete. \square

When Ω is a rectangle (which clearly satisfies the exterior cone condition), any solution belongs to $C(\bar{\Omega})$.

Lemma 3.2. Let u be a nontrivial solution of (1.1). Then it belongs to \mathcal{N} and hence to $D(R)$. Moreover it satisfies

$$R(u) = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{(p-1)/(p+1)} = \left(\int_{\Omega} h|u|^{p+1} dx \right)^{(p-1)/(p+1)}. \quad (3.2)$$

Proof. Multiplying (1.1) by u and integrating it over Ω , we have

$$0 < \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} h|u|^{p+1} dx,$$

which means that $u \in \mathcal{N} \subset D(R)$. This relation with the definition of $R(u)$ leads to (3.2) and the proof is complete. \square

Let us define a function v whose energy is less than the x_1 -symmetric least energy or the even least energy. Let u be an x_1 -symmetric positive solution or an even positive solution. Then we define

$$v(x_1, x') := (1 + \varepsilon x_1)u(x_1, x') \quad \text{for } \varepsilon > 0, \quad (3.3)$$

with $x = (x_1, x')$ and $x' = (x_2, \dots, x_N)$. Since $u \in D(R)$, v also belongs to $D(R)$ for $\varepsilon > 0$ small enough because

$$\int_{\Omega} h|v|^{p+1} dx \rightarrow \int_{\Omega} h|u|^{p+1} dx > 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Proposition 3.3. Let u be an x_1 -symmetric positive solution or an even positive solution when Ω and $h(x)$ are x_1 -symmetric or point symmetric, respectively. Define v by (3.3). If u satisfies

$$\int_{\Omega} u^2 dx < \frac{p-1}{p} \int_{\Omega} |\nabla u|^2 x_1^2 dx, \quad (3.4)$$

then $R(v) < R(u)$ for $\varepsilon > 0$ small enough.

Proof. First, denote the integral of $|\nabla u|^2$ on Ω by A . Then Lemma 3.2 means that

$$\begin{aligned} R(u) &= \left(\int_{\Omega} |\nabla u|^2 dx \right)^{(p-1)/(p+1)} \\ &= \left(\int_{\Omega} h|u|^{p+1} dx \right)^{(p-1)/(p+1)} = A^{(p-1)/(p+1)}. \end{aligned} \quad (3.5)$$

Next, multiplying (1.1) by ux_1^2 and integrating it over Ω , we have

$$\int_{\Omega} (|\nabla u|^2 x_1^2 + 2u_{x_1} u x_1) dx = \int_{\Omega} h u^{p+1} x_1^2 dx.$$

Integrating by parts yields

$$\int_{\Omega} (|\nabla u|^2 x_1^2 - u^2) dx = \int_{\Omega} hu^{p+1} x_1^2 dx.$$

Combining this relation with (3.4), we get

$$\int_{\Omega} |\nabla u|^2 x_1^2 dx < p \int_{\Omega} hu^{p+1} x_1^2 dx. \quad (3.6)$$

To estimate the Rayleigh quotient of v , we shall compute the integral of $|\nabla v|^2$. From an easy calculation, it follows that

$$|\nabla v|^2 = (1 + \varepsilon x_1)^2 |\nabla u|^2 + 2\varepsilon(1 + \varepsilon x_1)u_{x_1}u + \varepsilon^2 u^2. \quad (3.7)$$

Using the integration by parts, we see that

$$2\varepsilon \int_{\Omega} (1 + \varepsilon x_1)u_{x_1}u dx = -\varepsilon^2 \int_{\Omega} u^2 dx.$$

Then we integrate (3.7) to get

$$\int_{\Omega} |\nabla v|^2 dx = \int_{\Omega} (1 + 2\varepsilon x_1 + \varepsilon^2 x_1^2) |\nabla u|^2 dx. \quad (3.8)$$

We claim

$$\int_{\Omega} x_1 |\nabla u|^2 dx = 0. \quad (3.9)$$

We extend u to \mathbb{R}^N by setting $u = 0$ for $x \notin \Omega$. Then $u, v \in H^1(\mathbb{R}^N)$. Let u and Ω be x_1 -symmetric. Since $|\nabla u|^2$ is even in x_1 , we see that

$$\int_{-\infty}^{\infty} x_1 |\nabla u|^2 dx_1 = 0.$$

Integrating both sides with respect to $x' = (x_2, \dots, x_N)$, we have (3.9). If u is even and Ω is point symmetric, (3.9) clearly holds. Then (3.8) is reduced to

$$\int_{\Omega} |\nabla v|^2 dx = \int_{\Omega} (1 + \varepsilon^2 x_1^2) |\nabla u|^2 dx = A(1 + \varepsilon^2 B), \quad (3.10)$$

where A has been given by (3.5) and B is defined by

$$B := A^{-1} \int_{\Omega} x_1^2 |\nabla u|^2 dx.$$

We shall compute the denominator of $R(v)$. For any $t > -1$, there exists a $\delta \in (0, 1)$ by the Taylor theorem such that

$$(1+t)^{p+1} = 1 + (p+1)t + \frac{p(p+1)}{2}(1+\delta t)^{p-1}t^2.$$

Observing this identity, we define $\phi(x_1, \varepsilon)$ by the relation

$$(1 + \varepsilon x_1)^{p+1} = 1 + (p+1)\varepsilon x_1 + \frac{p(p+1)}{2}\varepsilon^2 x_1^2 (1 + \phi(x_1, \varepsilon)).$$

Then $\phi(x_1, \varepsilon)$ converges to zero uniformly on $|x_1| \leq L$ as $\varepsilon \rightarrow 0$. Here L is defined by (2.2). Using the expression above, we have

$$\begin{aligned} \int_{\Omega} h|v|^{p+1} dx &= \int_{\Omega} h(x)(1 + \varepsilon x_1)^{p+1} |u|^{p+1} dx \\ &= \int_{\Omega} h(x)|u|^{p+1} dx + \frac{p(p+1)}{2}\varepsilon^2 \int_{\Omega} h(x)|u|^{p+1} x_1^2 (1 + \phi) dx, \end{aligned}$$

where we have used the fact that the integral of $h(x)|u|^{p+1}x_1$ vanishes because of the same reason as in (3.9). We put

$$C_{\varepsilon} := A^{-1} \int_{\Omega} h|u|^{p+1} x_1^2 (1 + \phi(x_1, \varepsilon)) dx.$$

Using (3.5), we have

$$\int_{\Omega} h|v|^{p+1} dx = A(1 + (p(p+1)/2)\varepsilon^2 C_{\varepsilon}),$$

or equivalently

$$\left(\int_{\Omega} h|v|^{p+1} dx \right)^{-2/(p+1)} = A^{-2/(p+1)} (1 + (p(p+1)/2)\varepsilon^2 C_{\varepsilon})^{-2/(p+1)}.$$

Note that the integral of $hu^{p+1}x_1^2$ over Ω is positive because of (3.6). Thus we have

$$\lim_{\varepsilon \rightarrow 0} C_{\varepsilon} = A^{-1} \int_{\Omega} h|u|^{p+1} x_1^2 dx > 0. \quad (3.11)$$

By the mean value theorem, for $t > 0$ there is a $\delta \in (0, 1)$ such that

$$\begin{aligned} (1+t)^{-2/(p+1)} &= 1 - \frac{2}{p+1}(1+\delta t)^{-(p+3)/(p+1)}t \\ &\leq 1 - \frac{2}{p+1}(1+t)^{-(p+3)/(p+1)}t. \end{aligned}$$

Putting $t = (p(p+1)/2)\varepsilon^2 C_\varepsilon$, we have

$$\left(\int_{\Omega} h|v|^{p+1} dx \right)^{-2/(p+1)} \leq A^{-2/(p+1)} (1 - p\theta \varepsilon^2 C_\varepsilon), \quad (3.12)$$

where we have put

$$\theta := \left(1 + (p(p+1)/2)\varepsilon^2 C_\varepsilon \right)^{-(p+3)/(p+1)}.$$

Combining (3.10) with (3.12), we get

$$\begin{aligned} R(v) &= \left(\int_{\Omega} |\nabla v|^2 dx \right) \left(\int_{\Omega} h|v|^{p+1} dx \right)^{-2/(p+1)} \\ &\leq A(1 + \varepsilon^2 B) A^{-2/(p+1)} (1 - p\theta \varepsilon^2 C_\varepsilon) \\ &\leq R(u) \{ 1 + \varepsilon^2 (B - p\theta C_\varepsilon) \}, \end{aligned}$$

where we have used (3.5). By (3.6) and (3.11), we see that $B < p \lim_{\varepsilon \rightarrow 0} C_\varepsilon$. Since $\lim_{\varepsilon \rightarrow 0} \theta = 1$, it holds that $B - p\theta C_\varepsilon < 0$ for $\varepsilon > 0$ small enough, and hence $R(v) < R(u)$ for $\varepsilon > 0$ small. The proof is complete. \square

Proposition 3.3 remains valid with x_1 replaced by x_i . Let I be a proper subset of $\{1, \dots, N\}$ and choose an integer i in $\{1, 2, \dots, N\} \setminus I$. Put

$$v(x) := (1 + \varepsilon x_i)u(x). \quad (3.13)$$

If u is I -symmetric, so is v . Therefore we have

Corollary 3.4. *Let I be a proper subset of $\{1, \dots, N\}$ and let $i \notin I$. Let u be an $I \cup \{x_i\}$ -symmetric positive solution satisfying (3.4) with x_1 replaced by x_i . Define $v(x)$ by (3.13). Then v is I -symmetric and satisfies $R(v) < R(u)$ for $\varepsilon > 0$ small enough.*

4. A priori estimate

In this section, we deal with the case where $h(x)$ is nonnegative. Then we shall give an a priori estimate for the $L^\infty(\Omega)$ norm of I -symmetric least energy solutions or even least energy solutions. When I is the empty set, an I -symmetric least energy solution is equal to a global least energy solution.

Lemma 4.1. *Suppose that $h(x) \geq 0$, $\neq 0$ in Ω . Let Ω and $h(x)$ be I -symmetric or point symmetric and u be an I -symmetric least energy solution or even least energy solution, respectively, where I may be empty. Then*

$$R(u) \leq \text{Vol}(\Omega) \left(\int_{\Omega} h(x) \text{dist}(x, \partial\Omega)^{p+1} dx \right)^{-2/(p+1)}.$$

Here $\text{Vol}(\Omega)$ denotes the \mathbb{R}^N -Lebesgue measure of Ω .

Proof. We put $\phi(x) := \text{dist}(x, \partial\Omega)$. Then ϕ is I -symmetric or even and belongs to $D(R, I)$ or $D(R, e)$. Since $|\nabla\phi(x)| \leq 1$ a.e. in Ω , we see that

$$\begin{aligned} R(\phi) &= \left(\int_{\Omega} |\nabla\phi|^2 dx \right) \left(\int_{\Omega} h(x) |\phi|^{p+1} dx \right)^{-2/(p+1)} \\ &\leq \text{Vol}(\Omega) \left(\int_{\Omega} h(x) \text{dist}(x, \partial\Omega)^{p+1} dx \right)^{-2/(p+1)}. \end{aligned}$$

Since u is an I -symmetric least energy solution or even least energy solution, it holds that $R(u) \leq R(\phi)$ and the proof is complete. \square

We give an a priori $L^\infty(\Omega)$ estimate of an I -symmetric least energy solution or an even least energy solution in the next lemma.

Lemma 4.2. *Let u and h be as in Lemma 4.1. Then for $N \geq 3$, there exists a constant $C > 0$ independent of u and h such that*

$$\|u\|_\infty \leq C \|h\|_\infty^\mu \left(\int_{\Omega} h(x) \text{dist}(x, \partial\Omega)^{p+1} dx \right)^\nu,$$

where $\|\cdot\|_\infty$ denotes the $L^\infty(\Omega)$ norm and μ and ν are given by

$$\mu := \frac{N-2}{N+2-(N-2)p}, \quad \nu := \frac{-4}{(p-1)(N+2-(N-2)p)}.$$

When $N = 2$, for any $r \in (0, 1)$ there exists a $C_r > 0$ independent of u and h such that

$$\|u\|_\infty \leq C_r \|h\|_\infty^{(1-r)/r(p-1)} \left(\int_{\Omega} h(x) \text{dist}(x, \partial\Omega)^{p+1} dx \right)^{-1/r(p-1)}.$$

Proof. We use a bootstrap argument. Recall (3.1). Let $\alpha \geq 1$. Multiplying (1.1) by u^α and integrating it over Ω , we have

$$\alpha \int_{\Omega} |\nabla u|^2 u^{\alpha-1} dx = \int_{\Omega} h u^{p+\alpha} dx,$$

which is rewritten as

$$\int_{\Omega} |\nabla(u^{(\alpha+1)/2})|^2 dx = \frac{(\alpha+1)^2}{4\alpha} \int_{\Omega} h u^{p+\alpha} dx \leq (\alpha+1) \int_{\Omega} h u^{p+\alpha} dx,$$

where we have used $\alpha \geq 1$. Let $q = 2N/(N-2)$ if $N \geq 3$ and $2 < q < \infty$ if $N = 2$. Using the Sobolev embedding $\|v\|_q^2 \leq C \|\nabla v\|_2^2$ with $v = u^{(\alpha+1)/2}$, we get

$$\|u^{(\alpha+1)/2}\|_q^2 \leq (\alpha+1) C \|h u^{p-1}\|_\infty \|u\|_{\alpha+1}^{\alpha+1},$$

or equivalently

$$\|u\|_{(\alpha+1)q/2} \leq (\alpha+1)^{1/(\alpha+1)} K^{1/(\alpha+1)} \|u\|_{\alpha+1},$$

where we have put $K := C \|hu^{p-1}\|_\infty$. We define $p_n := (q/2)^{n-1} p_1$, where p_1 will be determined later on. Putting $\alpha = p_n - 1$, we get

$$\|u\|_{p_{n+1}} \leq p_n^{1/p_n} K^{1/p_n} \|u\|_{p_n}.$$

Multiplying both sides over $n = 1, 2, \dots, m-1$, we have

$$\|u\|_{p_m} \leq \left(\prod_{n=1}^{m-1} p_n^{1/p_n} \right) \left(\prod_{n=1}^{m-1} K^{1/p_n} \right) \|u\|_{p_1}.$$

Let $m \rightarrow \infty$ and observe that $\prod_{n=1}^{m-1} p_n^{1/p_n}$ converges to a finite limit and compute

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \frac{q}{p_1(q-2)}.$$

Then we have

$$\begin{aligned} \|u\|_\infty &\leq CK^{q/p_1(q-2)} \|u\|_{p_1} \\ &\leq C \|h\|_\infty^{q/p_1(q-2)} \|u\|_\infty^{q(p-1)/p_1(q-2)} \|u\|_{p_1}. \end{aligned} \quad (4.1)$$

Hereafter C denotes various positive constants independent of h and u .

Let $N \geq 3$. We choose $p_1 = q = 2N/(N-2)$. Then (4.1) is reduced to

$$\|u\|_\infty \leq C \|h\|_\infty^{(N-2)/((N+2)-(N-2)p)} \|u\|_{2N/(N-2)}^{4/((N+2)-(N-2)p)}.$$

Using the Sobolev inequality with (3.2) and Lemma 4.1, we get

$$\begin{aligned} \|u\|_{2N/(N-2)} &\leq C \|\nabla u\|_2 = CR(u)^{(p+1)/(2(p-1))} \\ &\leq C \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-1/(p-1)}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \|u\|_\infty &\leq C \|h\|_\infty^{(N-2)/(N+2-(N-2)p)} \\ &\quad \times \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-4/(p-1)((N+2)-(N-2)p)}. \end{aligned}$$

Let $N = 2$ and $r \in (0, 1)$. We choose $q > 2$ satisfying $(p-1)q/(q-2) > 1$ and then define p_1 by the relation

$$\frac{(p-1)q}{p_1(q-2)} = 1 - r.$$

Since $(p-1)q/(q-2) > 1$, it holds that $p_1 > 1$. Then (4.1) is rewritten as

$$\|u\|_{\infty}^r \leq C \|h\|_{\infty}^{q/p_1(q-2)} \|u\|_{p_1}.$$

Since $N = 2$ and $p_1 > 1$, the embedding $\|u\|_{p_1} \leq C \|\nabla u\|_2$ holds. The same method as in the case $N \geq 3$ is valid for getting the conclusion. The proof is complete. \square

5. x_1 -symmetric least energy solution

In this section, we investigate the properties of x_1 -symmetric least energy solutions. Recall the notation $R_0(I)$ defined in Introduction. Then $R_0(\{1\})$ denotes the x_1 -symmetric least energy. Because of Proposition 3.3, to prove the asymmetry of a global least energy solution, it is enough to show that an x_1 -symmetric least energy solution satisfies (3.4). Indeed, Proposition 3.3 under (3.4) ensures that $R_0 \leq R(v) < R_0(\{1\})$. Therefore a global least energy solution cannot be x_1 -symmetric. To prove (3.4), we investigate the properties of x_1 -symmetric least energy solutions.

Let u be a positive solution of (1.1). Put $u = 0$ outside of Ω . Then $u \in H^1(\mathbb{R}^N)$. For $x_1 \in \mathbb{R}$, we define

$$\begin{aligned} \Omega(x_1) &:= \{x' \in \mathbb{R}^{N-1} : (x_1, x') \in \Omega\}, \\ w(x_1) &:= \int_{\mathbb{R}^{N-1}} u(x_1, x')^2 dx' = \int_{\Omega(x_1)} u(x_1, x')^2 dx'. \end{aligned} \quad (5.1)$$

Proposition 5.1. *Let Ω be an x_1 -symmetric bounded domain with piecewise smooth boundary. Let u be an x_1 -symmetric positive solution. Define w by (5.1). Then w is in $W^{1,1}(\mathbb{R}) \cap C^1(-L, L)$, even and $w(x_1) = 0$ for $|x_1| \geq L$, where L has been defined by (2.2). Moreover it satisfies*

$$w_{x_1}(x_1) = 2 \int_{\mathbb{R}^{N-1}} u_{x_1} u dx' = 2 \int_{\Omega(x_1)} u_{x_1} u dx', \quad (5.2)$$

$$w_{x_1}(t) - w_{x_1}(s) = 2 \int_{\Omega(s < x_1 < t)} (|\nabla u|^2 - hu^{p+1}) dx, \quad (5.3)$$

for $-L < s < t < L$, where $w_{x_1}(x_1)$ denotes the derivative of $w(x_1)$ and $\Omega(s < x_1 < t)$ is given by (2.1).

Proof. Recall (3.1). It is obvious that $w \in L^1(\mathbb{R})$, $w(x_1)$ is even and $w(x_1) = 0$ for $|x_1| \geq L$. Choose a sequence $u_n \in C_0^\infty(\Omega)$ which converges to u in $H_0^1(\Omega)$. Put $u_n = 0$ outside of Ω and define

$$w_n(x_1) := \int_{\mathbb{R}^{N-1}} u_n(x_1, x')^2 dx' = \int_{\Omega(x_1)} u_n(x_1, x')^2 dx'.$$

Then w_n is in $C_0^\infty(\mathbb{R})$ and converges to w in $L^1(\mathbb{R})$. Therefore, to prove $w \in W^{1,1}(\mathbb{R})$, it is enough to show that w_n is a Cauchy sequence in $W^{1,1}(\mathbb{R})$. We use the Schwarz inequality to get

$$\begin{aligned} |w_{nx_1}(x_1) - w_{mx_1}(x_1)| &\leq 2 \int_{\mathbb{R}^{N-1}} |u_{nx_1} - u_{mx_1}| |u_n| dx' + 2 \int_{\mathbb{R}^{N-1}} |u_{mx_1}| |u_n - u_m| dx' \\ &\leq 2 \|u_n\|_{L^2(\mathbb{R}^{N-1})} \|u_{nx_1} - u_{mx_1}\|_{L^2(\mathbb{R}^{N-1})} \\ &\quad + 2 \|u_n - u_m\|_{L^2(\mathbb{R}^{N-1})} \|u_{mx_1}\|_{L^2(\mathbb{R}^{N-1})}. \end{aligned}$$

Integrating it with respect to x_1 and using the Schwarz inequality again, we have

$$\int_{-\infty}^{\infty} |w_{nx_1} - w_{mx_1}| dx_1 \leq 2 \|u_n\|_{L^2(\Omega)} \|u_n - u_m\|_{H_0^1(\Omega)} + 2 \|u_n - u_m\|_{L^2(\Omega)} \|u_m\|_{H_0^1(\Omega)},$$

which converges to 0 as $n, m \rightarrow \infty$. Hence w_n is a Cauchy sequence in $W^{1,1}(\mathbb{R})$ and therefore $w \in W^{1,1}(\mathbb{R})$. Then $w(x_1)$ is absolutely continuous and so it is differentiable a.e. in \mathbb{R} and satisfies (5.2).

Multiplying (1.1) by u and integrating it over $\Omega(s < x_1 < t)$ with $-L < s < t < L$, we have

$$\int_{\Omega(s < x_1 < t)} (|\nabla u|^2 - hu^{p+1}) dx = \int_{\partial\Omega(s < x_1 < t)} \frac{\partial u}{\partial n} u d\sigma, \quad (5.4)$$

where $d\sigma$ is the standard measure on $\partial\Omega(s < x_1 < t)$. Note that $\partial u/\partial n$ is well defined a.e. on $\partial\Omega$ because $\partial\Omega$ is supposed to be piecewise smooth. We divide $\partial\Omega(s < x_1 < t)$ into

$$\partial\Omega(s < x_1 < t) = S \cup \Omega(s) \cup \Omega(t), \quad S := \partial\Omega \cap ((s, t) \times \mathbb{R}^{N-1}).$$

On S , u vanishes. Since $\partial/\partial n$ denotes the outward normal derivative, we observe that

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x_1} \quad \text{on } \Omega(t), \quad \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x_1} \quad \text{on } \Omega(s).$$

Since u has the C^1 regularity in the interior of Ω , $\partial u/\partial x_1$ is well defined. Hence by (5.2), we find that

$$\int_{\partial\Omega(s < x_1 < t)} \frac{\partial u}{\partial n} u d\sigma = \int_{\Omega(t)} \frac{\partial u}{\partial x_1} u dx' - \int_{\Omega(s)} \frac{\partial u}{\partial x_1} u dx' = (1/2)w_{x_1}(t) - (1/2)w_{x_1}(s).$$

Substituting the inequality above into (5.4), we see that (5.3) holds a.e. $s, t \in (-L, L)$. Since the right hand side of (5.3) is continuous in t , so is $w_{x_1}(t)$. Thus w is in $C^1(-L, L)$ and (5.3) holds for all $s, t \in (-L, L)$. \square

Lemma 5.2. Let $u(x)$ and $w(x_1)$ be as in Proposition 5.1. Let a satisfy (2.3). Then we have

$$\max_{a \leq x_1 \leq L} w(x_1) \leq \frac{4(L-a)}{La} \int_{\Omega(a < x_1 < L)} |u_{x_1}|^2 x_1^2 dx. \quad (5.5)$$

Proof. Let b be a maximum point of $w(x_1)$ on $[a, L]$. Using the Schwarz inequality and (5.2), we get

$$\begin{aligned} w(b) &= \left| \int_b^L w_{x_1}(x_1) dx_1 \right| \\ &\leq \left(\int_b^L w_{x_1}(x_1)^2 x_1^2 dx_1 \right)^{1/2} \left(\int_b^L x_1^{-2} dx_1 \right)^{1/2} \\ &\leq ((L-b)/Lb)^{1/2} \left(\int_b^L \left(2 \int_{\Omega(x_1)} u_{x_1} u dx' \right)^2 x_1^2 dx_1 \right)^{1/2}. \end{aligned} \quad (5.6)$$

By the Schwarz inequality again, for $b \leq x_1 \leq L$ we see that

$$\left(\int_{\Omega(x_1)} u_{x_1} u \, dx' \right)^2 \leq \int_{\Omega(x_1)} u^2 \, dx' \int_{\Omega(x_1)} u_{x_1}^2 \, dx' \leq w(b) \int_{\Omega(x_1)} u_{x_1}^2 \, dx'.$$

Then (5.6) leads to

$$w(b) \leq 2((L-b)/Lb)^{1/2} w(b)^{1/2} \left(\int_b^L \left(\int_{\Omega(x_1)} u_{x_1}^2 x_1^2 \, dx' \right) dx_1 \right)^{1/2},$$

or equivalently

$$w(b) \leq \frac{4(L-b)}{Lb} \int_{\Omega(b < x_1 < L)} u_{x_1}^2 x_1^2 \, dx.$$

Since $a \leq b$, the inequality above proves (5.5). \square

The next lemma gives a sufficient condition for an x_1 -symmetric least energy solution to satisfy (3.4). Indeed, (5.7) in the next lemma implies (3.4) because of (2.3).

Lemma 5.3. *Let $u(x)$ and $w(x_1)$ be as in Proposition 5.1. Suppose that the maximum of $w(x_1)$ on $[0, L]$ is achieved at a point in $[a, L]$. Then it holds that*

$$\int_{\Omega} u^2 \, dx \leq 4a^{-1}(L-a) \int_{\Omega} |\nabla u|^2 x_1^2 \, dx. \quad (5.7)$$

Proof. Let $b \in [a, L]$ be a maximum point of $w(x_1)$. Recall that w is even. Using Lemma 5.2, we obtain

$$\begin{aligned} \int_{\Omega} u^2 \, dx &= \int_{-L}^L w(x_1) \, dx_1 \leq 2Lw(b) \\ &\leq 8a^{-1}(L-a) \int_{\Omega(a < x_1 < L)} |u_{x_1}|^2 x_1^2 \, dx \\ &\leq 4a^{-1}(L-a) \int_{\Omega} |\nabla u|^2 x_1^2 \, dx. \quad \square \end{aligned}$$

Lemma 5.4. *Let $u(x)$ and $w(x_1)$ be as in Proposition 5.1. Assume that $h(x) \leq 0$ in $\Omega(|x_1| < a)$. Then $w(x_1)$ is convex in $(-a, a)$ and its maximum on $[0, L]$ is achieved at a point in $[a, L]$. Therefore (5.7) holds.*

Proof. Since $h(x) \leq 0$ in $\Omega(|x_1| < a)$, (5.3) means

$$w_{x_1}(t) - w_{x_1}(s) \geq 2 \int_{\Omega(s < x_1 < t)} |\nabla u|^2 \, dx \geq 0 \quad \text{for } -a < s < t < a.$$

Thus w_{x_1} is increasing, i.e., w is convex in $(-a, a)$. Since $w(x_1)$ is even, it holds that $w'(0) = 0$ and $w'(x_1) \geq 0$ in $[0, a]$. Consequently, the maximum of w on $[0, L]$ is achieved at a point in $[a, L]$. \square

The lemma above ensures that an x_1 -symmetric least energy solution satisfies (3.4) in case (A). In the next lemma, we deal with case (B) and consider the case where the maximum point of $w(x_1)$ is in $[0, a)$. Recall $\xi(h)$ and $\xi(h, r)$, which have been defined before Theorem 2.1.

Lemma 5.5. Assume that $h \geq 0, \neq 0$ in Ω . Let u be an x_1 -symmetric positive solution and an I -symmetric least energy solution with a certain I , where I may be empty. (As stated before, when $I = \emptyset$, an I -symmetric least energy solution means a global least energy solution.) Suppose that the maximum of $w(x_1)$ on $[0, L]$ is achieved at a point $b \in [0, a)$. Then there exist $\varepsilon_0, \varepsilon(r) > 0$ such that if $\xi(h) < \varepsilon_0$ for $N \geq 3$ or if $\xi(h, r) < \varepsilon(r)$ for $N = 2$ with a certain $0 < r < 1$, then

$$\int_{\Omega} u(x)^2 dx \leq 8a^{-1}(L-a) \int_{\Omega} |\nabla u|^2 x_1^2 dx.$$

Here ε_0 and $\varepsilon(r)$ do not depend on $b, h(x)$ and $u(x)$.

Proof. Since $w_{x_1}(b) = 0$, we substitute $s = b$ in (5.3) to get

$$-w_{x_1}(t) = 2 \int_{\Omega(b < x_1 < t)} (hu^{p+1} - |\nabla u|^2) dx \leq 2 \int_{\Omega(b < x_1 < t)} hu^{p+1} dx.$$

Integrating both sides over (b, a) with respect to t , we find that

$$0 \leq w(b) - w(a) \leq 2 \int_b^a \left(\int_{\Omega(b < x_1 < t)} hu^{p+1} dx \right) dt.$$

For $t \in [b, a]$, we compute

$$\begin{aligned} \int_{\Omega(b < x_1 < t)} hu^{p+1} dx &\leq \|h\|_{L^\infty(\Omega(0 < x_1 < a))} \|u\|_{L^\infty(\Omega(0 < x_1 < a))}^{p-1} \int_b^t \left(\int_{\mathbb{R}^{N-1}} u(x_1, x')^2 dx' \right) dx_1 \\ &\leq (t-b) \|h\|_{L^\infty(\Omega(0 < x_1 < a))} \|u\|_{L^\infty(\Omega)}^{p-1} w(b). \end{aligned}$$

Hence it follows that

$$w(b) - w(a) \leq a^2 \|h\|_{L^\infty(\Omega(0 < x_1 < a))} \|u\|_{L^\infty(\Omega)}^{p-1} w(b). \quad (5.8)$$

We shall show that

$$a^2 \|h\|_{L^\infty(\Omega(0 < x_1 < a))} \|u\|_{L^\infty(\Omega)}^{p-1} \leq 1/2, \quad (5.9)$$

if $\xi(h)$ and $\xi(h, r)$ are small enough. Let $N \geq 3$. By Lemma 4.2, we have

$$\begin{aligned}
& a^2 \|h\|_{L^\infty(\Omega(0 < x_1 < a))} \|u\|_{L^\infty(\Omega)}^{p-1} \\
& \leq a^2 C \|h\|_{L^\infty(\Omega(|x_1| < a))} \|h\|_{L^\infty(\Omega)}^{\mu(p-1)} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{\nu(p-1)} \\
& = a^2 C \xi(h)^{1/(N+2-(N-2)p)},
\end{aligned}$$

where μ and ν have been defined in Lemma 4.2 and $C > 0$ is independent of b , $h(x)$ and $u(x)$. If $\xi(h)$ is small enough, then (5.9) holds.

Let $N = 2$. Lemma 4.2 implies that

$$\begin{aligned}
& a^2 \|h\|_{L^\infty(\Omega(0 < x_1 < a))} \|u\|_{L^\infty(\Omega)}^{p-1} \\
& \leq a^2 \|h\|_{L^\infty(\Omega(|x_1| < a))} C_r^{p-1} \|h\|_{L^\infty(\Omega)}^{(1-r)/r} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-1/r} \\
& = a^2 C_r^{p-1} \xi(h, r) \leq 1/2,
\end{aligned}$$

provided that $\xi(h, r)$ is small enough. Thus (5.9) holds. By (5.8) and (5.9), we see that $w(b) \leq 2w(a)$. By Lemma 5.2, we obtain

$$\begin{aligned}
\int_{\Omega} u(x)^2 dx &= 2 \int_0^L w(x_1) dx_1 \leq 2Lw(b) \leq 4Lw(a) \\
&\leq 16a^{-1}(L-a) \int_{\Omega(a < x_1 < L)} u_{x_1}^2 x_1^2 dx \\
&\leq 8a^{-1}(L-a) \int_{\Omega} |\nabla u|^2 x_1^2 dx. \quad \square
\end{aligned}$$

6. Even least energy solution

In this section, we investigate the properties of even least energy solutions. Many lemmas in this section can be proved in the same way as in Section 5. We use the polar coordinates $x = r\sigma$, $r = |x|$ and $\sigma = x/|x| \in S^{N-1}$. Here S^{N-1} denotes the unit sphere in \mathbb{R}^N . For $r > 0$, we define

$$\Omega(r) := \{\sigma \in S^{N-1} : r\sigma \in \Omega\}.$$

Note that in Section 5, $\Omega(r)$ denotes the projection on \mathbb{R}^{N-1} of the intersection of Ω with the hyperplane $x_1 = r$, however in this section it denotes the projection on S^{N-1} of the intersection of Ω with the sphere $|x| = r$. Put $u = 0$ outside of Ω . Instead of $w(x_1)$, we introduce an auxiliary function

$$W(r) := \int_{S^{N-1}} u(r, \sigma)^2 d\sigma = \int_{\Omega(r)} u(r, \sigma)^2 d\sigma. \quad (6.1)$$

Recall T defined by (2.9). Since $dx = r^{N-1} dr d\sigma$, we have

$$\int_{\Omega} u^2 dx = \int_0^T W(r) r^{N-1} dr.$$

In what follows, the derivative of $W(r)$ is denoted by $W_r(r)$ and $\Omega(s < |x| < t)$ is given by (2.8). We define the weighted Sobolev space

$$W^{1,1}(0, \infty; r^{N-1}) := \left\{ w: \int_0^\infty (|w_r(r)| + |w(r)|) r^{N-1} dr < \infty \right\}.$$

Any function in the space above becomes absolutely continuous in any compact subinterval of $(0, \infty)$.

Proposition 6.1. *Let Ω be a point symmetric bounded domain with piecewise smooth boundary. Let u be an even positive solution. Define W by (6.1). Then $W \in W^{1,1}(0, \infty; r^{N-1}) \cap C^1(0, T)$, $W(r) = 0$ for $r \geq T$ and it satisfies*

$$W_r(r) = 2 \int_{S^{N-1}} u_r u d\sigma = 2 \int_{\Omega(r)} u_r u d\sigma, \quad (6.2)$$

$$t^{N-1} W_r(t) - s^{N-1} W_r(s) = 2 \int_{\Omega(s < |x| < t)} (|\nabla u|^2 - hu^{p+1}) dx, \quad (6.3)$$

for $0 < s < t < T$.

Proof. Along the lines of the proof of Proposition 5.1, we can prove that $W \in W^{1,1}(0, \infty; r^{N-1})$, $W(r) = 0$ for $r \geq T$ and (6.2) holds. Instead of (5.4), we have

$$\int_{\Omega(s < |x| < t)} (|\nabla u|^2 - hu^{p+1}) dx = \int_{\partial\Omega(s < |x| < t)} \frac{\partial u}{\partial n} u d\tau,$$

where $d\tau$ is the measure on $\partial\Omega(s < |x| < t)$. This surface is decomposed into

$$\partial\Omega(s < |x| < t) = S(t) \cup S(s) \cup S_0,$$

where

$$S(r) := \{x \in \Omega: |x| = r\}, \quad S_0 := \partial\Omega \cap \{x: s < |x| < t\}.$$

Note that $u = 0$ on S_0 and $d\tau = r^{N-1} d\sigma$ on $S(r)$, where $d\sigma$ is the measure on S^{N-1} . Then we have

$$\begin{aligned} \int_{\partial\Omega(s < |x| < t)} \frac{\partial u}{\partial n} u d\tau &= \int_{S(t)} \frac{\partial u}{\partial n} u d\tau + \int_{S(s)} \frac{\partial u}{\partial n} u d\tau \\ &= t^{N-1} \int_{S^{N-1}} \frac{\partial u}{\partial r} u d\sigma - s^{N-1} \int_{S^{N-1}} \frac{\partial u}{\partial r} u d\sigma \\ &= t^{N-1} W_r(t)/2 - s^{N-1} W_r(s)/2. \end{aligned}$$

This completes the proof. \square

Lemma 6.2. Let $u(x)$ and $W(r)$ be as in Proposition 6.1. Let a satisfy (2.10). After exchanging the x_1 -axis with a certain x_i -axis, we have

$$\max_{a \leq r \leq T} W(r) \leq \frac{4(T^N - a^N)}{T^N a^N} \int_{\Omega(a < |x| < T)} |u_r|^2 x_1^2 dx. \quad (6.4)$$

Proof. Let us choose x_i in the following way. Observe the trivial identity

$$\int_{\Omega(a < |x| < T)} u_r^2 r^2 dx = \sum_{i=1}^N \int_{\Omega(a < |x| < T)} u_r^2 x_i^2 dx.$$

Then there exists an $i \in \{1, \dots, N\}$ such that

$$\frac{1}{N} \int_{\Omega(a < |x| < T)} u_r^2 r^2 dx \leq \int_{\Omega(a < |x| < T)} u_r^2 x_i^2 dx.$$

We exchange the x_1 -axis with x_i -axis to get

$$\int_{\Omega(a < |x| < T)} u_r^2 r^2 dx \leq N \int_{\Omega(a < |x| < T)} u_r^2 x_1^2 dx. \quad (6.5)$$

Let b be a maximum point of $W(r)$ on $[a, T]$. Using the Schwarz inequality and (6.2), we get

$$\begin{aligned} W(b) &= \left| \int_b^T W_r(r) dr \right| \\ &\leq \left(\int_b^T W_r(r)^2 r^{N+1} dr \right)^{1/2} \left(\int_b^T r^{-N-1} dr \right)^{1/2} \\ &\leq ((T^N - b^N)/Nb^N T^N)^{1/2} \left(\int_b^T \left(2 \int_{\Omega(r)} u_r u d\sigma \right)^2 r^{N+1} dr \right)^{1/2}. \end{aligned} \quad (6.6)$$

By the Schwarz inequality again, for $b \leq r \leq T$ we see that

$$\left(\int_{\Omega(r)} u_r u d\sigma \right)^2 \leq \int_{\Omega(r)} u^2 d\sigma \int_{\Omega(r)} u_r^2 d\sigma \leq W(b) \int_{\Omega(r)} u_r^2 d\sigma.$$

Then (6.6) leads to

$$W(b) \leq \frac{4(T^N - b^N)}{NT^N b^N} \int_{\Omega(b < |x| < T)} u_r^2 r^2 dx,$$

where we have used $r^{N-1} dr d\sigma = dx$. Since $a \leq b$, the inequality above with (6.5) proves (6.4). \square

Hereafter by Lemma 6.2, we choose the x_1 -axis, which depends on a solution u .

Lemma 6.3. *Let $u(x)$ and $W(r)$ be as in Proposition 6.1. Suppose that the maximum of $W(r)$ on $[0, T]$ is achieved at a point in $[a, T)$. Then it holds that*

$$\int_{\Omega} u^2 dx \leq 4N^{-1}a^{-N}(T^N - a^N) \int_{\Omega} |\nabla u|^2 x_1^2 dx. \quad (6.7)$$

Proof. Let $b \in [a, T)$ be a maximum point of $W(r)$. Then we see that

$$\int_{\Omega} u^2 dx = \int_0^T W(r)r^{N-1} dr \leq (T^N/N)W(b),$$

which with Lemma 6.2 proves the lemma. \square

We deal with case (C) in the next lemma.

Lemma 6.4. *Let $u(x)$ and $W(r)$ be as in Proposition 6.1. Assume that $h(x) \leq 0$ in Ω ($|x| < a$). Then the maximum of $W(r)$ on $[0, T]$ is achieved at a point in $[a, T)$. Therefore (6.7) holds.*

Proof. Let $h(x) \leq 0$ in Ω ($|x| < a$). Then (6.3) implies that

$$t^{N-1}W_r(t) \geq s^{N-1}W_r(s) \quad \text{for } 0 < s < t < a. \quad (6.8)$$

We claim that $W_r(r) \rightarrow 0$ as $r \rightarrow 0$. If $0 \notin \Omega$, then our claim is clear. Let $0 \in \Omega$. Since u is even, it holds that $\nabla u(0) = 0$. Then $|\partial u(r, \sigma)/\partial r| \leq |\nabla u(x)| \rightarrow 0$ as $|x| \rightarrow 0$. Therefore

$$W_r(r) = 2 \int_{S^{N-1}} u_r u d\sigma \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Letting $s \rightarrow 0$ in (6.8), we see that $W_r(t) \geq 0$ for $0 < t < a$. Thus $W(r)$ is nondecreasing in $(0, a)$ and its maximum is achieved in $[a, T)$. \square

Now, we deal with the case where the maximum point of $W(r)$ is in $[0, a)$ in the next lemma.

Lemma 6.5. *Assume that $h \geq 0$, $\neq 0$ in Ω and let u be an even least energy solution. Suppose that the maximum of $W(r)$ on $[0, T]$ is achieved at a point $b \in [0, a)$. Then there exist $\varepsilon_0, \varepsilon(r) > 0$ such that if $\eta(h) < \varepsilon_0$ for $N \geq 3$ or if $\eta(h, r) < \varepsilon(r)$ for $N = 2$ with a certain $0 < r < 1$, then*

$$\int_{\Omega} u(x)^2 dx \leq 8N^{-1}a^{-N}(T^N - a^N) \int_{\Omega} |\nabla u|^2 x_1^2 dx.$$

Here ε_0 and $\varepsilon(r)$ do not depend on $b, h(x)$ and $u(x)$.

Proof. We use the same method as in the proof of Lemma 5.5. Since $W_r(b) = 0$, we substitute $s = b$ in (6.3) to get

$$-t^{N-1}W_r(t) \leq 2 \int_{\Omega(b < |x| < t)} hu^{p+1} dx.$$

Dividing the both sides by t^{N-1} and integrating it over (b, a) with respect to t , we see that

$$0 \leq W(b) - W(a) \leq 2 \int_b^a t^{-N+1} \left(\int_{\Omega(b < |x| < t)} hu^{p+1} dx \right) dt.$$

For $t \in [b, a]$, we compute

$$\begin{aligned} \int_{\Omega(b < |x| < t)} hu^{p+1} dx &\leq \|h\|_{L^\infty(\Omega(|x| < a))} \|u\|_{L^\infty(\Omega)}^{p-1} \int_b^t \left(\int_{S^{N-1}} u(r, \sigma)^2 d\sigma \right) r^{N-1} dr \\ &\leq \|h\|_{L^\infty(\Omega(|x| < a))} \|u\|_{L^\infty(\Omega)}^{p-1} t^{N-1} (t-b) W(b). \end{aligned}$$

Therefore we obtain

$$W(b) - W(a) \leq a^2 \|h\|_{L^\infty(\Omega(|x| < a))} \|u\|_{L^\infty(\Omega)}^{p-1} W(b).$$

This inequality coincides with (5.8). Along the lines of the proof of Lemma 5.5, we find that $W(b) \leq 2W(a)$ if $\eta(h)$ or $\eta(h, r)$ is small enough. By Lemma 6.2, we obtain

$$\begin{aligned} \int_{\Omega} u(x)^2 dx &= \int_0^T W(r) r^{N-1} dr \leq (2T^N/N) W(a) \\ &\leq 8N^{-1} a^{-N} (T^N - a^N) \int_{\Omega} |\nabla u|^2 x_1^2 dx. \quad \square \end{aligned}$$

7. Proof of the main results

In this section, we prove the main results.

Proof of Theorem 2.1. Let u be an x_1 -symmetric least energy solution. As mentioned at the beginning of Section 5, it is enough to show that u satisfies (3.4). First, we assume (A). Then Lemma 5.4 with (2.3) gives us (3.4). Next, we assume (B) and suppose that $\xi(h)$ with $N \geq 3$ is small enough or $\xi(h, r)$ is small enough for $N = 2$ with a certain $r \in (0, 1)$. Let $b \in [0, L]$ be the maximum point of $w(x_1)$. If $a \leq b < L$, then Lemma 5.3 guarantees (3.4). If $0 \leq b < a$, then Lemma 5.5 with $I = \{1\}$ shows (3.4). The proof is complete. \square

Proof of Corollary 2.2. Since g is defined on $[0, L]$, $h(x)$ is well defined on $\overline{\Omega}$. Since h is nonnegative, condition (B) holds. We fix a satisfying (2.3). Let $N \geq 3$. It is enough to show that $\xi(h) \rightarrow 0$ as $\lambda \rightarrow \infty$. We put

$$m := \max_{0 \leq t \leq a} g(t), \quad M := g(L) = \max_{0 \leq t \leq L} g(t).$$

From assumption, it follows that $m < M$. Observe that

$$m^{N+2-(N-2)p} M^{(N-2)(p-1)} M^{-4} = (m/M)^{N+2-(N-2)p} < 1.$$

Then we choose $M_1 \in (0, M)$ slightly less than M such that

$$m^{N+2-(N-2)p} M^{(N-2)(p-1)} M_1^{-4} < 1. \quad (7.1)$$

Choose $c \in (a, L)$ slightly less than L such that $\min_{c \leq t \leq L} g(t) > M_1$. Using this inequality, we get

$$\begin{aligned} \int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx &\geq \int_{\Omega(c \leq |x_1| \leq L)} g(|x_1|)^{\lambda} \operatorname{dist}(x, \partial\Omega)^{p+1} dx \\ &\geq M_1^{\lambda} \int_{\Omega(c \leq |x_1| \leq L)} \operatorname{dist}(x, \partial\Omega)^{p+1} dx. \end{aligned}$$

Combining the inequality above with (7.1), we obtain

$$\begin{aligned} \xi(h) &= \|h\|_{L^{\infty}(\Omega(|x_1| < a))}^{N+2-(N-2)p} \|h\|_{L^{\infty}(\Omega)}^{(N-2)(p-1)} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-4} \\ &\leq C(m^{N+2-(N-2)p} M^{(N-2)(p-1)} M_1^{-4})^{\lambda} \rightarrow 0, \end{aligned}$$

as $\lambda \rightarrow \infty$, where $C > 0$ is independent of λ . Thus $\xi(h)$ is small enough when λ is sufficiently large. For $N = 2$ also, the method above works well. The proof is complete. \square

We shall show Corollary 2.4 only and omit the proofs of other theorems and corollaries. Indeed, Corollary 2.3 can be proved in the same way as in Corollaries 2.2 and 2.4. Theorem 2.7 and Corollary 2.8 are proved in the same method as in Theorem 2.1 and Corollary 2.2. Theorem 1.1 follows directly from Corollary 2.4. We conclude this paper by proving Corollary 2.4.

Proof of Corollary 2.4. By (2.6), we choose $a_i > 0$ slightly larger than α_i such that $8pL_i/(9p-1) < a_i < L_i$ and (2.6) holds with α_i replaced by a_i . We define $\xi_i(h)$ and $\xi_i(h, r)$ by $\xi(h)$ and $\xi(h, r)$ with $L^{\infty}(\Omega(|x_1| < a))$ replaced by $L^{\infty}(\Omega(|x_i| < a_i))$. We put

$$\begin{aligned} T_i &:= \sup\{|x|: x \in \Omega(|x_i| < a_i)\}, \quad T := \sup\{|x|: x \in \Omega\}, \\ m_i &:= \max_{0 \leq t \leq T_i} g(t), \quad M := g(T) = \max_{0 \leq t \leq T} g(t). \end{aligned}$$

Choose $M_1 \in (0, M)$ slightly less than M . Using the same argument as in the proof of Corollary 2.2, we obtain

$$\begin{aligned} \xi_i(h) &= \|h\|_{L^{\infty}(\Omega(|x_i| < a_i))}^{N+2-(N-2)p} \|h\|_{L^{\infty}(\Omega)}^{(N-2)(p-1)} \left(\int_{\Omega} h(x) \operatorname{dist}(x, \partial\Omega)^{p+1} dx \right)^{-4} \\ &\leq C(m_i^{N+2-(N-2)p} M^{(N-2)(p-1)} M_1^{-4})^{\lambda} \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty. \end{aligned}$$

Therefore (3.4) holds with x_1 replaced by x_i for all i when λ is large enough. By Theorem 2.1, a global least energy solution has no x_i -symmetry for all i .

Let us show that an I -symmetric least energy solution is exactly I -symmetric. If $I = \{1, 2, \dots, N\}$, our claim is clear. Let I be a proper subset of $\{1, \dots, N\}$. Suppose on the contrary that an I -symmetric least energy solution u has another x_i -symmetry with an $i \notin I$. Since $\xi_i(h)$ is small enough, by Lemmas 5.3 and 5.5, u satisfies (3.4) with x_1 replaced by x_i . Then Corollary 3.4 ensures that v defined

by (3.13) is I -symmetric and $R(v) < R(u)$. This contradicts the definition of the I -symmetric least energy. Consequently, an I -symmetric least energy solution must be exactly I -symmetric. The proof is complete. \square

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