ARTICLE IN PRESS YJDEQ:7631

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source

Johannes Lankeit

Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany

Received 18 July 2014; revised 1 October 2014

Abstract

We prove existence of global weak solutions to the chemotaxis system

$$u_t = \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2$$
$$v_t = \Delta v - v + u$$

under homogeneous Neumann boundary conditions in a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, for arbitrarily small values of $\mu > 0$.

Additionally, we show that in the three-dimensional setting, after some time, these solutions become classical solutions, provided that κ is not too large. In this case, we also consider their large-time behaviour: We prove decay if $\kappa \leq 0$ and the existence of an absorbing set if $\kappa > 0$ is sufficiently small.

MSC: primary 35K55; secondary 35B65, 35Q92, 92C17, 35B40

Keywords: Chemotaxis; Logistic source; Existence; Weak solutions; Eventual smoothness

E-mail address: johannes.lankeit@math.upb.de.

http://dx.doi.org/10.1016/j.jde.2014.10.016

Please cite this article in press as: J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.10.016

1. Introduction

Starting from the pioneering work of Keller and Segel [9], an extensive mathematical literature has grown on the Keller–Segel model and its variants, mathematical models describing chemotaxis, that is the tendency of (micro-)organisms to adapt the direction of their (otherwise random) movement toward increasing concentrations of a signalling substance. For a survey see [6] or [7,8].

If biological phenomena where chemotaxis plays a role are modelled not only on small timescales, often the growth of the population, whose density we will denote by u, must be taken into account. A prototypical choice to accomplish this is the addition of logistic growth terms $+\kappa u - \mu u^2$ in the evolution equation for u. Here $+\kappa u$, with $\kappa \in \mathbb{R}$ being the difference between birth rate and death rate of the population, is used to describe population growth, and the term $-\mu u^2$ models additional overcrowding effects. Negative values of κ can be used to include effects like spontaneous degradation into the model (e.g. in the case of a starving population) that – in contrast to the effects modelled by the quadratic term – take place also in regions with small population density. Unfortunately, it is unclear whether global classical solutions to the chemotaxis-system

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\ v_t &= \Delta v - v + u \\ \partial_\nu u|_{\partial\Omega} &= \partial_\nu v|_{\partial\Omega} = 0 \\ u(\cdot,0) &= u_0, \qquad v(\cdot,0) = v_0, \end{aligned} \tag{1}$$

where u_0, v_0 are given functions, exist in the smooth, bounded domain $\Omega \subset \mathbb{R}^n$ if $n \geq 3$ and $\mu > 0$ is small.

The parabolic–elliptic simplification (where v_t is replaced by 0) of (1) has been considered in [23], where – besides some study of asymptotic behaviour – it is shown that weak solutions exist for arbitrary $\mu > 0$ and that they are smooth and globally classical if $\mu > \frac{n-2}{n}$. In [24] the existence of (very) weak solutions is proven under more general conditions. Under additional assumptions, also the existence of a bounded absorbing set in $L^\infty(\Omega)$ is shown.

Turning to the parabolic–parabolic system, important findings are given in [26], which assert existence and uniqueness of global, smooth, bounded solutions to (1) under the condition that μ be large enough.

Additional results on existence of global solutions or even of an exponential attractor have been given in the two-dimensional case (see e.g. [16,17]). In this case, global solutions exist for arbitrary $\mu > 0$.

Not only the restriction to dimension 2, but also the inclusion of some kind of saturation effect in the chemotactic sensitivity [2], sublinear dependence of the chemotactic sensitivity on u [3] or even changing the second equation into one that models the consumption of the chemoattractant (as done in [20,22] for $\kappa = \mu = 0$) can make it possible to derive the global existence of solutions. The same can be accomplished by replacement of the secretion term $+u$ in the second equation of (1) by $+\frac{u}{(1+u)^{1-\beta}}$ with some $0 < \beta < \frac{9}{10}$, which enables the authors of [14] to show the existence of attractors in the corresponding dynamical system.

Please cite this article in press as: J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.10.016

On the other hand, the model

$$\begin{aligned} u_t &= \varepsilon \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\ 0 &= \Delta v - v + u \end{aligned} \tag{2}$$

has recently been shown to exhibit the following property [12]: If $\mu \in (0,1)$ and the (radially symmetric) initial datum u_0 is large in a certain $L^p(\Omega)$-space, there exists some finite time such that up to this time any given threshold will be surpassed by solutions to (2) for sufficiently small $\varepsilon > 0$. Although this demeanour may be interesting from an emergence-of-patterns point-of-view and although solutions become very large, it still is not the same as blow-up and, in fact, also occurs in case of bounded solutions, even in space-dimension 1 [29].

In [27] it is shown that in another related model,

$$u_t = \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^\alpha, \qquad v_t = \Delta v - m(t) + u, \qquad m(t) = \frac{1}{|\Omega|} \int_\Omega u,$$

blow-up may occur for space-dimension $n \geq 5$ and exponents $1 < \alpha < \frac{3}{2} + \frac{1}{2n-2}$.

Consequently, the supposition that any superlinear growth restriction already signifies the existence of a global, bounded solution does not stand unchallenged; and the question whether the above-mentioned results on the presence of global smooth solutions in similar situations find their analogue in the case of (1), the most prototypical chemotaxis system including logistic growth, is not clear at all.

In the present article, we therefore investigate the existence of solutions to (1). More precisely, we will construct weak solutions in the sense of Definition 5.1 below. We shall show that, in dimension 3 and under a smallness condition on κ, they become smooth after some time, which also excludes finite-time blow-up from then on. Note that this, however, does not provide any information on a small timescale.

To the aim sketched above we will then consider the approximate system

$$\begin{aligned} u_{\varepsilon t} &= \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \kappa u_\varepsilon - \mu u_\varepsilon^2 - \varepsilon u_\varepsilon^\theta \\ v_{\varepsilon t} &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon \\ \partial_\nu u_\varepsilon|_{\partial\Omega} &= \partial_\nu v_\varepsilon|_{\partial\Omega} = 0 \\ u_\varepsilon(\cdot,0) &= u_{0,\varepsilon}, \qquad v(\cdot,0) = v_{0,\varepsilon}, \end{aligned} \tag{3}$$

for $\theta > n+2$ with nonnegative initial values $u_{0,\varepsilon} \in C(\overline{\Omega})$ and $v_{0,\varepsilon} \in W^{1,n+1}(\Omega)$, where global classical solutions are quickly seen to exist, and derive estimates finally allowing for compactness arguments, which will provide the existence of a weak solution to (1) in Proposition 6.1 and Lemma 6.2.

We will employ the estimates from Section 4 to conclude that a solution must become small in an appropriate sense after some time. This, in turn, will be the starting point for an ODE comparison argument for the quantity $\int_\Omega u_\varepsilon^2(t) + \int_\Omega |\nabla v_\varepsilon(t)|^4$, whose thereby-obtained boundedness in conjunction with estimates on the Neumann heat semigroup results in eventual boundedness and hence in eventual smoothness of (u,v). We finally arrive at the following result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded convex domain and $u_0 \in L^2(\Omega)$, $v_0 \in W^{1,2}(\Omega)$ nonnegative. Let $\kappa \in \mathbb{R}$, $\mu > 0$.*

Then there is a nonnegative weak solution (u, v) (in the sense of Definition 5.1 below) to (1) with initial data u_0, v_0.

It can be approximated in the sense of a.e.-convergence by solutions of (3).

Furthermore, if $n = 3$, for any $\mu > 0$ there exists $\kappa_0 > 0$ such that if $\kappa < \kappa_0$, there is $T > 0$ such that u and v are a classical solution of (1) for $t > T$. Moreover, in this case, there are $C > 0$ and $\alpha > 0$ such that for any $t > T$

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}\times[t,t+1])} \le C.$$

Remark 1.2. Because we have adopted a weak concept of solution, it is conceivable that solutions to (1) are not unique. Investigation of this issue is beyond the scope of the present work and we state the following theorems only for solutions as provided by Theorem 1.1.

Besides the aforementioned results about attractors, little is known about asymptotic behaviour of solutions to models like (1). Recently, in [28] convergence to the positive homogeneous equilibrium was found for values of μ being sufficiently large as compared to the chemotactic sensitivity.

The richness of dynamics and pattern formation exhibited by chemotaxis models with growth [18,10] however indicates that any speculation about asymptotical behaviour, especially about convergence to homogeneous states, should be backed by rigorous examinations.

In the situation of (1), we can summarize the long-term behaviour as follows: If $\kappa \le 0$, solutions will converge to the trivial steady state – and any formation of interesting patterns has to take place on intermediary timescales.

Theorem 1.3. *Let $\mu > 0$, $\kappa \le 0$. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded convex domain and let (u, v) be the solution to (1) provided by Theorem 1.1. Then*

$$\big(u(t), v(t)\big) \to (0, 0) \quad \text{as } t \to \infty$$

in the sense of uniform convergence.

Remark 1.4. The same convergence result can be given for any classical solution of (1) for $\mu > 0$, $\kappa \le 0$ in $\Omega \subset \mathbb{R}^3$ as above. In this case, only minor adaptions of the proofs become necessary.

If κ is positive and sufficiently small, we can assert the existence of an absorbing set in the following sense:

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded convex domain. Then for any $\mu > 0$ there is $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0)$, there is $\alpha > 0$ and a bounded set $B_{\mu,\kappa} \subset (C^{2+\alpha}(\overline{\Omega}))^2$ such that for all $(u_0, v_0) \in L^2(\Omega) \times W^{1,2}(\Omega)$, the corresponding solution (u, v) as constructed in Theorem 1.1 admits the existence of $T > 0$ such that*

$$\big(u(t), v(t)\big) \in B_{\mu,\kappa} \quad \text{for all } t > T.$$

Please cite this article in press as: J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.10.016

Furthermore, for each fixed $\mu > 0$,

$$\operatorname{diam}_{L^\infty(\Omega)\times W^{1,\infty}(\Omega)}(B_{\mu,\kappa}) \to 0 \quad \text{as } \kappa \searrow 0.$$

Further steps in this direction may hopefully lead to an even more detailed insight, much in the spirit of [13,1], into the long-time behaviour of solutions to (1) in dimension 3 for small, positive μ.

Remark 1.6. In the calculations below, we will assume that $\mu > 0$ is a fixed number.

Throughout the article, we fix $\Omega \subset \mathbb{R}^n$ to be a convex bounded domain with smooth boundary and $u_0 \in L^2(\Omega)$, $v_0 \in W^{1,2}(\Omega)$ nonnegative. Also, let θ denote a number satisfying $\theta > n + 2$.

2. Existence of approximate solutions

The system (3) has a unique, global, classical solution. At a first glance, the source term $f(s) = \kappa s - \mu s^2 - \varepsilon s^\theta$ seems to satisfy the condition $f(s) \le a - \mu_0 s^2$ from Theorem 0.1 of [26], which would provide a global solution, but as μ_0 depends on a, this theorem is not applicable in the present case. Even tracing the dependence of μ_0 on a does not improve the situation.

We therefore use Lemma 1.1 of the same article, which asserts the local existence of a unique classical solution $(u_\varepsilon, v_\varepsilon)$ to (3) for initial data $u_{0\varepsilon} \in C(\overline{\Omega})$, $v_{0\varepsilon} \in W^{1,n+1}(\Omega)$. More specifically, it implies that this solution exists on a time interval $[0, T_{\max})$, $T_{\max} \in (0, \infty]$, and satisfies

$$\limsup_{t \nearrow T_{\max}} \left(\|u_\varepsilon(t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \right) = \infty$$

if $T_{\max} < \infty$. Hence, in order to show the global existence of this solution, it is sufficient to derive boundedness of u_ε, v_ε and ∇v_ε.

Our means of pursuing this aim will be

Proposition 2.1. *Let $q > n + 2$. Let (u, v) be a nonnegative classical solution of*

$$u_t = \Delta u - \nabla \cdot (u \nabla v) + f(u),$$
$$v_t = \Delta v - v + u$$

in $\Omega \times [0, T]$, $T > 0$, with homogeneous Neumann boundary conditions, for initial data $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \in L^\infty(\Omega)$ and some function f satisfying $f(s) \le C_0$ for all $s > 0$ with some $C_0 > 0$. Furthermore, assume that there exists $C > 0$ such that u satisfies

$$\left(\int_0^T \int_\Omega u^q \right)^{\frac{1}{q}} \le C.$$

Then u, v and ∇v are bounded in $\Omega \times [0, T]$.

Proof. Denote by C_1, C_2, C_3 the constants provided by Lemma 1.3 of [25] such that

$$\left\| \nabla e^{\tau\Delta} w \right\|_{L^\infty(\Omega)} \le C_1 \|\nabla w\|_{L^\infty(\Omega)} \tag{4}$$

for all $w \in W^{1,\infty}(\Omega)$ and

$$\left\|\nabla e^{\tau\Delta} w\right\|_{L^\infty(\Omega)} \le C_2\left(1+\tau^{-\frac{1}{2}-\frac{n}{2q}}\right)\|w\|_{L^q(\Omega)} \tag{5}$$

for all $w \in L^q(\Omega)$ as well as

$$\left\|e^{\tau\Delta}\nabla\cdot w\right\|_{L^\infty(\Omega)} \le C_3\left(1+\tau^{-\frac{1}{2}-\frac{n}{2q}}\right)\|w\|_{L^q(\Omega)} \tag{6}$$

for $w \in L^q(\Omega,\mathbb{R}^n)$. Here, $e^{\tau\Delta}\nabla\cdot$ denotes the extension of the corresponding operator on $(C_0^\infty(\Omega))^n$ to a continuous operator from $L^q(\Omega,\mathbb{R}^n)$ to $L^\infty(\Omega)$, see [25, Lemma 1.3]. Since $(-\frac{1}{2}-\frac{n}{2q})\cdot\frac{q}{q-1} = -\frac{1}{2}\frac{q+n}{q}\frac{q}{q-1} = -\frac{1}{2}(1+\frac{n+1}{q-1}) > -\frac{1}{2}(1+\frac{n+1}{(n+2)-1}) = -1$,

$$C_4 = \left(\int_0^T \left(1+(T-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)^{\frac{q}{q-1}}\mathrm{d}s\right)^{\frac{q-1}{q}} \tag{7}$$

is finite.

Let $t \in [0,T]$. Employing (4) and (5) in the variations-of-constants formula for v, we obtain

$$\begin{aligned}
\left\|\nabla v(t)\right\|_{L^\infty(\Omega)} &\le \left\|\nabla e^{t(\Delta-1)}v_0\right\|_{L^\infty(\Omega)} + \int_0^t \left\|\nabla e^{(t-s)(\Delta-1)}u(s)\right\|_{L^\infty(\Omega)}\mathrm{d}s \\
&\le C_1\|\nabla v_0\|_{L^\infty(\Omega)} + C_2\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)\left\|u(s)\right\|_{L^q(\Omega)}\mathrm{d}s \\
&\le C_1\|\nabla v_0\|_{L^\infty(\Omega)} + C_2\left(\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)^{\frac{q}{q-1}}\mathrm{d}s\right)^{\frac{q-1}{q}} \\
&\quad\times\left(\int_0^t \|u(s)\|_{L^q(\Omega)}^q\mathrm{d}s\right)^{\frac{1}{q}} \\
&\le C_1\|\nabla v_0\|_{L^\infty(\Omega)} + C_2C_4C =: C_5.
\end{aligned} \tag{8}$$

We represent also u in terms of the semigroup, use the order-preserving property of the heat semigroup and estimate with the help of (6) to see that

$$\begin{aligned}
0 \le u(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot\left(u(s)\nabla v(s)\right)\mathrm{d}s + \int_0^t e^{(t-s)\Delta}f\left(u(s)\right)\mathrm{d}s \\
&\le \|u_0\|_{L^\infty(\Omega)} + C_3\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)\left\|u(s)\right\|_{L^q(\Omega)}\left\|\nabla v(s)\right\|_{L^\infty(\Omega)}\mathrm{d}s + TC_0.
\end{aligned}$$

Another application of Hölder's inequality, now in time, in combination with (8) and (7) gives

$$0 \le u(t) \le \|u_0\|_{L^\infty(\Omega)} + C_3C_5\left(\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \left(\int_0^t \|u(s)\|^q_{L^q(\Omega)}\right)^{\frac{1}{q}} + TC_0$$
$$\le \|u_0\|_{L^\infty(\Omega)} + C_3C_5C_4C + TC_0 =: C_6.$$

Boundedness of v on $\Omega \times [0,T]$ then is an easy consequence:

$$0 \le v(t) \le \left\|e^{t(\Delta-1)}v_0\right\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}u(s)\right\|_{L^\infty(\Omega)}\mathrm{ds} \le \|v_0\|_{L^\infty(\Omega)} + \int_0^t C_6\mathrm{ds}$$

for all $t \in [0,T]$. $\square$

For given nonnegative $u_0 \in L^2(\Omega)$, $v_0 \in W^{1,2}(\Omega)$ and $\varepsilon > 0$, we choose $u_{0,\varepsilon} \in C(\overline{\Omega})$, $v_{0,\varepsilon} \in W^{1,n+1}(\Omega)$ nonnegative such that

$$\|u_0 - u_{0,\varepsilon}\|_{L^2(\Omega)} \le \min\{\varepsilon, 1\}, \qquad \|v_0 - v_{0,\varepsilon}\|_{W^{1,2}(\Omega)} \le \min\{\varepsilon, 1\}. \tag{9}$$

From now on, by $(u_\varepsilon, v_\varepsilon)$ we denote the unique classical solution on $[0, T_{\max})$ to (3) with initial data $u_{0,\varepsilon}$ and $v_{0,\varepsilon}$. Proposition 2.1 in conjunction with the next two lemmata and Lemma 1.1 of [26] will show that, indeed, $T_{\max} = \infty$.

Note that, by (9), in the following lemmata estimates in terms of $u_{0,\varepsilon}$ or $v_{0,\varepsilon}$ can be made ε-independent by retreating to the corresponding integral of u_0 or v_0 plus 1.

3. Estimates

In this section we present estimates for different quantities involving u_ε and v_ε respectively, which can be obtained more or less directly from (3) together with ODE comparison arguments. In the following, denote

$$\kappa_+ := \max\{\kappa, 0\}.$$

Lemma 3.1. *For any $\varepsilon > 0$, the function u_ε satisfies*

$$\int_\Omega u_\varepsilon(t) \le \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}$$

for $t > 0$. Furthermore,

$$\limsup_{t\to\infty} \int_\Omega u_\varepsilon(t) \le \frac{\kappa_+|\Omega|}{\mu},$$

uniformly in $\varepsilon > 0$.

Proof. By Hölder's inequality, $(\int_\Omega u_\varepsilon)^2 \le (\int_\Omega u_\varepsilon^2)|\Omega|$. Hence, integration of the first equation of (3) yields

$$\begin{aligned}\left(\int_\Omega u_\varepsilon\right)_t = \int_\Omega u_{\varepsilon t} &\le 0 - 0 + \kappa_+ \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^2 - \varepsilon \int_\Omega u_\varepsilon^\theta \\ &\le \kappa_+ \int_\Omega u_\varepsilon - \frac{\mu}{|\Omega|}\left(\int_\Omega u_\varepsilon\right)^2. \end{aligned} \tag{10}$$

The claim can be seen by solving the logistic ODE. □

Lemma 3.2. *Let $\kappa > 0$, let $T > 0$. Then there exists $C > 0$ such that for all $\varepsilon > 0$*

$$\int_0^T \int_\Omega u_\varepsilon^2 + \frac{\varepsilon}{\mu} \int_0^T \int_\Omega u_\varepsilon^\theta \le \frac{\kappa_+}{\mu} \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} T + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} \le C.$$

Proof. The estimate

$$\begin{aligned}\int_0^T \int_\Omega u_\varepsilon^2 + \frac{\varepsilon}{\mu} \int_0^T \int_\Omega u_\varepsilon^\theta &\le \frac{\kappa_+}{\mu} \int_0^T \int_\Omega u_\varepsilon + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} - \frac{1}{\mu}\int_\Omega u_\varepsilon(T) \\ &\le \frac{\kappa_+}{\mu} \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} T + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}\end{aligned}$$

results from (10) after time-integration. □

Also for the second component of the solution some basic estimates are available:

Lemma 3.3. *Let $\kappa \in \mathbb{R}$, $\varepsilon > 0$. The inequality*

$$\int_\Omega v_\varepsilon(t) \le \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}, \int_\Omega v_{0,\varepsilon}\right\}$$

holds as well as

$$\int_\Omega v_\varepsilon^2(t) + \int_0^t \int_\Omega v_\varepsilon^2 \le \frac{\kappa_+}{\mu} \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} t + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon} + \int_\Omega v_{0,\varepsilon}^2$$

for all $t > 0$.

Proof. Integrating the second equation of (3) gives, by Lemma 3.1,

$$\frac{d}{dt}\int_\Omega v_\varepsilon(t) = \int_\Omega v_{\varepsilon t}(t) = \int_\Omega \Delta v_\varepsilon(t) - \int_\Omega v_\varepsilon(t) + \int_\Omega u_\varepsilon(t)$$
$$\le -\int_\Omega v_\varepsilon(t) + \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}$$

for $t > 0$, an ODI for $\int_\Omega v_\varepsilon$, whose solution directly shows

$$\int_\Omega v_\varepsilon(t) \le \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} + e^{-t}\int_\Omega v_{0,\varepsilon} \tag{11}$$

and hence the first part of the assertion.

As to the second part, we derive an ODI for $\frac{1}{2}\int_\Omega v_\varepsilon^2$ in quite the same way: For $t > 0$, by Young's inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2(t) = \int_\Omega v_\varepsilon(t)v_{\varepsilon t}(t) = \int_\Omega v_\varepsilon(t)\Delta v_\varepsilon(t) - \int_\Omega v_\varepsilon^2(t) + \int_\Omega u_\varepsilon(t)v_\varepsilon(t)$$
$$\le -\int_\Omega \left|\nabla v_\varepsilon(t)\right|^2 - \int_\Omega v_\varepsilon^2(t) + \frac{1}{2}\int_\Omega u_\varepsilon^2(t) + \frac{1}{2}\int_\Omega v_\varepsilon^2(t)$$
$$\le -\frac{1}{2}\int_\Omega v_\varepsilon^2(t) + \frac{1}{2}\int_\Omega u_\varepsilon^2(t).$$

Integrating this with respect to the time variable, so that we can use the bound from Lemma 3.2 on u_ε^2, we obtain

$$\int_\Omega v_\varepsilon^2(t) - \int_\Omega v_{0,\varepsilon}^2 \le -\int_0^t\int_\Omega v_\varepsilon^2 + \int_0^t\int_\Omega u_\varepsilon^2$$
$$\le -\int_0^t\int_\Omega v_\varepsilon^2 + \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}t + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}$$

for any $t > 0$ and the claim follows. □

The next lemma gives estimates on the derivatives of v.

Lemma 3.4. *Let $\kappa \in \mathbb{R}$, $\varepsilon > 0$. The solutions of (3) satisfy, for all $t > 0$,*

$$\left[\int_\Omega \left|\nabla v_\varepsilon(t)\right|^2 + \frac{1}{\mu}\int_\Omega u_\varepsilon(t)\right] \le \max\left\{\int |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+ + 1}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}\right\}$$

and

$$\int_0^t \int_\Omega \left|\Delta v_\varepsilon(t)\right|^2 \le \frac{\kappa_+}{\mu} \max\left\{ \int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu} \right\} t + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu} \int_\Omega u_{0,\varepsilon}.$$

Proof. Integration by parts and Young's inequality result in

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{\mu} \int_\Omega u_\varepsilon \right] \\
&\le -2 \int_\Omega |\Delta v_\varepsilon|^2 + 2 \int_\Omega \Delta v_\varepsilon v_\varepsilon - 2 \int_\Omega \Delta v_\varepsilon u_\varepsilon + \frac{\kappa_+}{\mu} \int_\Omega u_\varepsilon - \int_\Omega u_\varepsilon^2 - \frac{\varepsilon}{\mu} \int_\Omega u_\varepsilon^\theta \\
&\le -2 \int_\Omega |\Delta v_\varepsilon|^2 - 2 \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\Delta v_\varepsilon|^2 + \int_\Omega u_\varepsilon^2 - \int_\Omega u_\varepsilon^2 + \frac{\kappa_+}{\mu} \int_\Omega u_\varepsilon \\
&\le -\int_\Omega |\Delta v_\varepsilon|^2 - \int_\Omega |\nabla v_\varepsilon|^2 - \frac{1}{\mu} \int_\Omega u_\varepsilon + \frac{\kappa_+ + 1}{\mu} \int_\Omega u_\varepsilon
\end{aligned} \tag{12}$$

on $(0, \infty)$. From this, we can conclude by Lemma 3.1

$$\frac{d}{dt}\left[\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{\mu} \int_\Omega u_\varepsilon \right] \le -\left[\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{\mu} \int_\Omega u_\varepsilon \right] + \frac{\kappa_+ + 1}{\mu} \max\left\{ \int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu} \right\}$$

on $(0, \infty)$ and hence the claim follows by comparison with the solution of $y' = -y + \text{const}$. Re-sorting the terms in (12) moreover gives

$$\int_\Omega \left|\Delta v_\varepsilon(t)\right|^2 \le -\int_\Omega \left|\nabla v_\varepsilon(t)\right|^2 + \frac{\kappa_+}{\mu} \int_\Omega u_\varepsilon(t) - \frac{d}{dt}\left[\int_\Omega \left|\nabla v_\varepsilon(t)\right|^2 + \frac{1}{\mu} \int_\Omega u_\varepsilon(t) \right]$$

for $t > 0$, and therefore

$$\begin{aligned}
\int_0^t \int_\Omega |\Delta v_\varepsilon|^2 &\le \frac{\kappa_+}{\mu} \max\left\{ \int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu} \right\} t - \left[\int_\Omega \left|\nabla v_\varepsilon(t)\right|^2 + \frac{1}{\mu} \int_\Omega u_\varepsilon(t) \right] \\
&\quad + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu} \int_\Omega u_{0,\varepsilon} \\
&\le \frac{\kappa_+}{\mu} \max\left\{ \int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu} \right\} t + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu} \int_\Omega u_{0,\varepsilon}. \qquad \square
\end{aligned}$$

The bounds that have been derived so far can be combined to yield

Lemma 3.5. *Let $\kappa \in \mathbb{R}$. For any $T > 0$, there exists a constant $C = C(T, \mu, \kappa_+, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{W^{1,2}(\Omega)})$ such that for all $\varepsilon > 0$*

$$\frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \mu\int_0^T\int_\Omega u_\varepsilon^2\log(1+u_\varepsilon) + \varepsilon\int_0^T\int_\Omega u_\varepsilon^\theta\log(1+u_\varepsilon) \le C.$$

In particular: The families $\{u_\varepsilon^2\}_{\varepsilon\in(0,1)}$ *and* $\{\varepsilon u_\varepsilon^\theta\}_{\varepsilon\in(0,1)}$ *are equi-integrable over* $\Omega\times(0,T)$.

Proof. Let $T>0$. Testing the first equation of (3) with $\log(1+u_\varepsilon)$ and integrating by parts gives

$$\begin{aligned}\int_\Omega u_{\varepsilon t}\log(1+u_\varepsilon) \le &-\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_\Omega \frac{u_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_\Omega u_\varepsilon\log(1+u_\varepsilon)\\ &-\mu\int_\Omega u_\varepsilon^2\log(1+u_\varepsilon) - \varepsilon\int_\Omega u_\varepsilon^\theta\log(1+u_\varepsilon),\end{aligned}$$

which, using $((1+u_\varepsilon)\log(1+u_\varepsilon)-u_\varepsilon)_t = u_{\varepsilon t}\log(1+u_\varepsilon)$, can be turned into

$$\begin{aligned}&\mu\int_\Omega u_\varepsilon^2\log(1+u_\varepsilon) + \varepsilon\int_\Omega u_\varepsilon^\theta\log(1+u_\varepsilon)\\ &\le -\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_\Omega \frac{u_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_\Omega u_\varepsilon\log(1+u_\varepsilon)\\ &\quad -\int_\Omega \big[(1+u_\varepsilon)\log(1+u_\varepsilon)-u_\varepsilon\big]_t.\end{aligned}$$

Integration in time hence yields

$$\begin{aligned}I &:= \mu\int_0^T\int_\Omega u_\varepsilon^2\log(1+u_\varepsilon) + \varepsilon\int_0^T\int_\Omega u_\varepsilon^\theta\log(1+u_\varepsilon)\\ &\le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_0^T\int_\Omega \frac{u_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_0^T\int_\Omega u_\varepsilon\log(1+u_\varepsilon)\\ &\quad -\int_\Omega \big(\big(1+u_\varepsilon(T)\big)\log\big(1+u_\varepsilon(T)\big)-u_\varepsilon(T)\big) + \int_\Omega \big((1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon})-u_{0,\varepsilon}\big)\\ &\le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \int_0^T\int_\Omega \frac{u_\varepsilon\nabla v_\varepsilon\cdot\nabla u_\varepsilon}{1+u_\varepsilon} + \kappa_+\int_0^T\int_\Omega u_\varepsilon\log(1+u_\varepsilon)\\ &\quad +\int_\Omega u_\varepsilon(T) + \int_\Omega (1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon}).\end{aligned} \tag{13}$$

We integrate the second term by parts:

$$\int_0^T\int_\Omega \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon}{1+u_\varepsilon} = -\int_0^T\int_\Omega \frac{u_\varepsilon^2}{1+u_\varepsilon}\Delta v_\varepsilon - \int_0^T\int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\left(\frac{u_\varepsilon}{1+u_\varepsilon}\right)$$
$$= -\int_0^T\int_\Omega \frac{u_\varepsilon^2}{1+u_\varepsilon}\Delta v_\varepsilon - \int_0^T\int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon \frac{1+u_\varepsilon-u_\varepsilon}{(1+u_\varepsilon)^2}.$$

Inserting this into (13) then results in

$$I \le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} - \int_0^T\int_\Omega \frac{u_\varepsilon^2}{1+u_\varepsilon}\Delta v_\varepsilon - \int_0^T\int_\Omega \frac{u_\varepsilon \nabla v_\varepsilon}{(1+u_\varepsilon)^{\frac{3}{2}}} \cdot \frac{\nabla u_\varepsilon}{(1+u_\varepsilon)^{\frac{1}{2}}}$$
$$+\kappa_+ \int_0^T\int_\Omega u_\varepsilon \log(1+u_\varepsilon) + \int_\Omega u_\varepsilon(T) + \int_\Omega (1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon}),$$

where application of the trivial inequality $\frac{u_\varepsilon}{(1+u_\varepsilon)^{\frac{3}{2}}} \le 1$ gives rise to

$$I \le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} - \int_0^T\int_\Omega \frac{u_\varepsilon}{1+u_\varepsilon} u_\varepsilon \Delta v_\varepsilon + \int_0^T\int_\Omega |\nabla v_\varepsilon| \frac{|\nabla u_\varepsilon|}{\sqrt{1+u_\varepsilon}}$$
$$+\kappa_+ \int_0^T\int_\Omega u_\varepsilon \log(1+u_\varepsilon) + \int_\Omega u_\varepsilon(T) + \int_\Omega (1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon}).$$

Estimating $\frac{u_\varepsilon}{1+u_\varepsilon} \le 1$, $\log(1+u_\varepsilon) \le u_\varepsilon$ and employing Young's inequality shows

$$I \le -\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \frac{1}{2}\int_0^T\int_\Omega u_\varepsilon^2 + \frac{1}{2}\int_0^T\int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_0^T\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon}$$
$$+\kappa_+ \int_0^T\int_\Omega u_\varepsilon^2 + \int_\Omega u_\varepsilon(T) + \int_\Omega (1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon})$$
$$= -\frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \left(\kappa_+ + \frac{1}{2}\right)\int_0^T\int_\Omega u_\varepsilon^2 + \int_\Omega u_\varepsilon(T) + \frac{1}{2}\int_0^T\int_\Omega |\Delta v_\varepsilon|^2 + \frac{1}{2}\int_0^T\int_\Omega |\nabla v_\varepsilon|^2$$
$$+\int_\Omega (1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon}).$$

And if we compile the bounds provided by Lemmata 3.2, 3.4 and 3.1, we arrive at

$$
\begin{aligned}
&I + \frac{1}{2}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} \\
&\le \left(\kappa_+ + \frac{1}{2}\right)\frac{1}{\mu}\left(\kappa_+ \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}T + \int_\Omega u_{0,\varepsilon}\right) + \max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\} \\
&\quad + \frac{1}{2}\left(\frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}T + \int_\Omega |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}\right) \\
&\quad + \frac{1}{2}T\max\left\{\int |\nabla v_{0,\varepsilon}|^2 + \frac{1}{\mu}\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+ + 1}{\mu}\max\left\{\int_\Omega u_{0,\varepsilon}, \frac{\kappa_+|\Omega|}{\mu}\right\}\right\} \\
&\quad + \int_\Omega (1+u_{0,\varepsilon})\log(1+u_{0,\varepsilon}) =: C. \qquad \square
\end{aligned}
$$

From the bound on $\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon}$ we can extract information on the behaviour of the spatial gradient of u.

Lemma 3.6. *Let $\kappa \in \mathbb{R}$. For all $T > 0$ there is $C > 0$ such that for all $\varepsilon > 0$*

$$\|u_\varepsilon\|_{L^{\frac{4}{3}}((0,T),W^{1,\frac{4}{3}}(\Omega))} \le C.$$

Proof. Denote by C_1 the constant provided by Lemma 3.2 and by C_2 that of Lemma 3.5. Then, by Hölder's and Young's inequalities,

$$
\begin{aligned}
\|u_\varepsilon\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((0,T),W^{1,\frac{4}{3}}(\Omega))} &= \int_0^T \|u_\varepsilon\|^{\frac{4}{3}}_{W^{1,\frac{4}{3}}(\Omega)} = \int_0^T\left(\int_\Omega u_\varepsilon^{\frac{4}{3}} + \int_\Omega |\nabla u_\varepsilon|^{\frac{4}{3}}\right) \\
&\le \left(\int_0^T\int_\Omega u_\varepsilon^2\right)^{\frac{2}{3}}\left(|\Omega|T\right)^{\frac{1}{3}} + \int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^{\frac{4}{3}}}{(1+u_\varepsilon)^{\frac{2}{3}}}(1+u_\varepsilon)^{\frac{2}{3}} \\
&\le C_1^{\frac{2}{3}}\left(|\Omega|T\right)^{\frac{1}{3}} + \frac{2}{3}\int_0^T\int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} + \frac{1}{3}\int_0^T\int_\Omega (1+u_\varepsilon)^2 \\
&\le C_1^{\frac{2}{3}}\left(|\Omega|T\right)^{\frac{1}{3}} + \frac{4}{3}C_2 + \frac{2}{3}T|\Omega| + \frac{2}{3}C_1 =: C. \qquad \square
\end{aligned}
$$

In order to gain convergence results from Aubin–Lions-type lemmas, we need some information on the time derivative. The following lemma provides this kind of information.

Lemma 3.7. *Let $\kappa \in \mathbb{R}$ and $T > 0$. Then there is $C > 0$ such that for all $\varepsilon > 0$*

$$\|u_{\varepsilon t}\|_{L^1((0,T),(W^{2,\infty}(\Omega))^*)} \le C.$$

Proof. Definition of the norm and integration by parts in (3) lead us to

$$\int_0^T \sup_{\|\varphi\|_{W^{2,\infty}(\Omega)}\le 1} \left|\int_\Omega u_{\varepsilon t}\varphi\right|$$
$$\le \int_0^T \sup_{\|\varphi\|_{W^{2,\infty}(\Omega)}\le 1} \left(\left|\int_\Omega u_\varepsilon \Delta\varphi\right| + \left|\int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi\right| + \left|\kappa \int_\Omega u_\varepsilon\varphi\right| + \mu\left|\int_\Omega u_\varepsilon^2\varphi\right| + \varepsilon\left|\int_\Omega u_\varepsilon^\theta\varphi\right|\right),$$

where we can use $\|\varphi\|_{W^{2,\infty}(\Omega)} \le 1$ and Young's inequality to see

$$\|u_{\varepsilon t}\|_{L^1((0,T),(W^{2,\infty}(\Omega))^*)} \le \int_0^T \left(\int_\Omega u_\varepsilon + \frac{1}{2}\int_\Omega u_\varepsilon^2 + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + |\kappa|\int_\Omega u_\varepsilon + \mu\int_\Omega u_\varepsilon^2 + \varepsilon\int_\Omega u_\varepsilon^\theta\right)$$

and infer boundedness of this norm, independent of ε, from Lemmata 3.1, 3.2 and 3.4. □

The space in which the spatial gradient is known to be bounded can be improved if a bound on u is assumed.

Lemma 3.8. *Let $\kappa \in \mathbb{R}$ and let $[T_1, T_2]$ be an interval such that there exists a constant M satisfying*

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le M$$

for all $t \in [T_1, T_2]$ and $\varepsilon > 0$. Then there is $C > 0$ such that for all $\varepsilon > 0$

$$\|\nabla u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))} \le C.$$

Proof. By Lemma 3.5 we can find $\widetilde{C} > 0$ such that

$$\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{1+u_\varepsilon} \le \widetilde{C}$$

for all $\varepsilon > 0$, ergo, setting $C = (1+M)\widetilde{C}$,

$$\int_{T_1}^{T_2}\int_\Omega |\nabla u_\varepsilon|^2 \le \int_{T_1}^{T_2}\int_\Omega \frac{1+M}{1+u_\varepsilon}|\nabla u_\varepsilon|^2 \le (1+M)\widetilde{C} = C. \qquad \square$$

Under similar conditions, also the time derivative is bounded in a better space.

Lemma 3.9. *Let $\kappa \in \mathbb{R}$ and let $[T_1, T_2]$ be an interval such that there exists a constant M satisfying*

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \le M$$

for all $t \in [T_1, T_2]$ and $\varepsilon > 0$. Then there is $C > 0$ such that for all $\varepsilon > 0$

$$\|u_{\varepsilon t}\|_{L^2((T_1,T_2);(W^{1,2}(\Omega))^*)} \le C.$$

Proof. Let φ be an element of $L^2((T_1, T_2); W^{1,2}(\Omega))$ with norm 1.

Let $\widetilde{C}$ be the bound on $\|\nabla u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))}$ provided by Lemma 3.8. Then

$$\begin{aligned}
\left|\int_{T_1}^{T_2}\int_\Omega u_{\varepsilon t}\varphi\right| &\le \left|\int_{T_1}^{T_2}\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\right| + \left|\int_{T_1}^{T_2}\int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi\right| + \left|\int_{T_1}^{T_2}\int_\Omega \left(\kappa u_\varepsilon - \mu u_\varepsilon^2 - \varepsilon u_\varepsilon^\theta\right)\varphi\right| \\
&\le \|\nabla u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))}\|\nabla\varphi\|_{L^2((T_1,T_2);L^2(\Omega))} \\
&\quad + \left(\sup_{t\in[T_1,T_2]}\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)}\right)\|u_\varepsilon\|_{L^2((T_1,T_2);L^2(\Omega))}\|\nabla\varphi\|_{L^2((T_1,T_2);L^2(\Omega))} \\
&\quad + \sqrt{(T_2-T_1)|\Omega|}\left\|\kappa u_\varepsilon + \mu u_\varepsilon^2 + u_\varepsilon^\theta\right\|_{L^\infty(\Omega\times(T_1,T_2))}\|\varphi\|_{L^2((T_1,T_2);L^2(\Omega))} \\
&\le \widetilde{C} + M\sqrt{(T_2-T_1)|\Omega|}M + \sqrt{(T_2-T_1)|\Omega|}\left(|\kappa|M + \mu M^2 + M^\theta\right) =: C
\end{aligned}$$

and hence boundedness of $u_{\varepsilon t}$ in $(L^2((T_1, T_2); W^{1,2}(\Omega)))^*$ follows. □

4. Preservation of smallness

In the last two lemmata, we have seen that boundedness can provide bounds also for derivatives. It will as well be important in establishing regularization effects. Therefore, in this section we will derive this boundedness and to this aim proceed as follows: At first we will prepare some estimates on $y_\varepsilon(t) := \int_\Omega u_\varepsilon^2(t) + \int_\Omega |\nabla v_\varepsilon|^4$. These will establish that y_ε satisfies a differential inequality with a polynomial right hand side; we will show that this polynomial has a positive root and y_ε eventually undermatches its value. Finally, we will use the bounds just gained to improve them to L^∞-bounds for the solutions under consideration.

At first we state the following easy consequence of Poincaré's inequality.

Lemma 4.1. *If we denote $\overline{w} = \frac{1}{|\Omega|}\int_\Omega w$, then*

$$\int_\Omega w^2 \le C_P \int_\Omega |\nabla w|^2 + |\Omega|\overline{w}^2,$$

for all $w \in W^{1,2}(\Omega)$, where C_P is the Poincaré-constant of Ω, defined by $\int_\Omega (w - \overline{w})^2 \le C_P \int_\Omega |\nabla w|^2$ for $w \in W^{1,2}(\Omega)$.

Proof. As announced, this is a direct consequence of Poincaré's inequality:

$$C_P \int_\Omega |\nabla w|^2 \geq \int_\Omega (w-\overline{w})^2 = \int_\Omega w^2 - 2\int_\Omega w\overline{w} + \int_\Omega \overline{w}^2 = \int_\Omega w^2 - 2\overline{w}|\Omega|\overline{w} + |\Omega|\overline{w}^2$$
$$= \int_\Omega w^2 - |\Omega|\overline{w}^2. \qquad \square$$

Another elementary but useful identity is the following:

Lemma 4.2. *Let $w \in C^3(\Omega)$. Then*

$$\Delta|\nabla w|^2 = 2\nabla w \nabla \Delta w + 2\left|D^2 w\right|^2.$$

In the proof of Lemma 4.4 we will also make use of the well-known Gagliardo–Nirenberg inequality:

Lemma 4.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $p, q, r, s \geq 1$, $j, m \in \mathbb{N}_0$ and $\alpha \in [\frac{j}{m}, 1]$ satisfying $\frac{1}{p} = \frac{j}{m} + (\frac{1}{r} - \frac{m}{n})\alpha + \frac{1-\alpha}{q}$. Then there are positive constants C_1 and C_2 such that for all functions $w \in L^q(\Omega)$ with $\nabla w \in L^r(\Omega)$, $w \in L^s(\Omega)$,*

$$\left\|D^j w\right\|_{L^p(\Omega)} \leq C_1 \left\|D^m w\right\|^\alpha_{L^r(\Omega)} \|w\|^{1-\alpha}_{L^q(\Omega)} + C_2 \|w\|_{L^s(\Omega)}.$$

Proof. See [15, p. 126]. $\square$

We are aiming for an estimate for $\int_\Omega |\nabla v_\varepsilon|^4$. During the calculations we therefore will have to get rid of integrals of $|\nabla v_\varepsilon|^6$. The Gagliardo–Nirenberg inequality enables us to replace them by more convenient terms.

Lemma 4.4. *Let $n = 3$. For any $a > 0$ there is $C(a) > 0$ such that, for any $\kappa \in \mathbb{R}$, $\varepsilon > 0$,*

$$\int_\Omega |\nabla v_\varepsilon|^6 \leq a \int_\Omega \left|\nabla|\nabla v_\varepsilon|^2\right|^2 + C(a)\left[\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right].$$

Proof. For given $j = 0$, $m = 1$, Ω, $p = 3$, $r = q = 2$, $s = 2$, the Gagliardo–Nirenberg inequality (Lemma 4.3) provides constants C_1 and C_2 such that for $w \in L^2(\Omega)$ and with $\alpha = \frac{1}{2}$ the inequality

$$\|w\|^3_{L^3(\Omega)} \leq 8C_1^3 \|\nabla w\|^{\frac{3}{2}}_{L^2(\Omega)} \|w\|^{\frac{3}{2}}_{L^2(\Omega)} + 8C_2^3 \|w\|^3_{L^2(\Omega)},$$

holds true (where we at the same time have used $(x+y)^3 < 8(x^3+y^3)$). Applied to $w = |\nabla v_\varepsilon|^2$ this means

$$\int_\Omega |\nabla v_\varepsilon|^6 \le 8C_1^3\Big(\int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2\Big)^{\frac34}\Big(\int_\Omega |\nabla v_\varepsilon|^4\Big)^{\frac34} + 8C_2^3\Big(\int_\Omega |\nabla v_\varepsilon|^4\Big)^{\frac32}.$$

With $p=\frac43$, $q=4$, corresponding to $a>0$ Young's inequality provides $\widetilde{C}(a)>0$ such that

$$\int_\Omega |\nabla v_\varepsilon|^6 \le a\int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2 + \widetilde{C}(a)\Big(\int_\Omega |\nabla v_\varepsilon|^4\Big)^3 + 8C_2^3\Big(\int_\Omega |\nabla v_\varepsilon|^4\Big)^{\frac32}$$

and the claim results with the choice of $C(a)=\max\{8C_2^3, \widetilde{C}(a)\}$. □

With the help of Lemma 4.4, we separate u_ε, ∇u_ε and ∇v_ε in one of the terms arising from differentiation of $\int_\Omega u_\varepsilon^2$.

Lemma 4.5. *Let $n=3$. Corresponding to $\mu>0$ there exists $C>0$ such that for any $\kappa\in\mathbb{R}$ and $\varepsilon>0$ the estimate*

$$\begin{aligned}\int_\Omega u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon \le \frac14\int_\Omega |\nabla u_\varepsilon|^2 + \mu\int_\Omega u_\varepsilon^3 + \frac12\int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2 \\ + C\bigg(\Big(\int_\Omega |\nabla v_\varepsilon|^4\Big)^3 + \Big(\int_\Omega |\nabla v_\varepsilon|^4\Big)^{\frac32}\bigg)\end{aligned}$$

holds.

Proof. Double application of Young's inequality yields a constant $\widetilde{C}>0$ such that

$$\int_\Omega u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon \le \frac14\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^2|\nabla v_\varepsilon|^2 \le \frac14\int_\Omega |\nabla u_\varepsilon|^2 + \mu\int_\Omega u_\varepsilon^3 + \widetilde{C}\int_\Omega |\nabla v_\varepsilon|^6.$$

Using Lemma 4.4 with $a=\frac12$ to estimate $\int_\Omega |\nabla v_\varepsilon|^6$ this produces the assertion, with the choice $C=\widetilde{C}C(\frac12)$. □

The term $\int_\Omega |\nabla|\nabla v_\varepsilon|^2|^2$, known to us from Lemma 4.4, arises from the following estimate with the "right" sign.

Lemma 4.6. *Let $\kappa\in\mathbb{R}$, let $q\ge 1$. Then*

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} \le -q(q-1)\int_\Omega |\nabla v_\varepsilon|^{2q-4}\big|\nabla|\nabla v_\varepsilon|^2\big|^2 - 2q\int_\Omega |\nabla v_\varepsilon|^{2q} + 2q\int_\Omega |\nabla v_\varepsilon|^{2q-1}|\nabla u_\varepsilon|.$$

Proof. Evaluating the derivative and inserting the second equation of (3) gives

$$\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} = 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon\cdot\nabla\Delta v_\varepsilon - 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon\cdot\nabla v_\varepsilon + 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon\cdot\nabla u_\varepsilon.$$

Here, Lemma 4.2 and integration by parts eventuate

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^{2q} &= q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\Delta|\nabla v_\varepsilon|^2 - 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\left|D^2 v_\varepsilon\right|^2 \\ &\quad - 2q\int_\Omega |\nabla v_\varepsilon|^{2q} + 2q\int_\Omega |\nabla v_\varepsilon|^{2q-2}\nabla v_\varepsilon\cdot\nabla u_\varepsilon \\ &\le -q(q-1)\int_\Omega |\nabla v_\varepsilon|^{2q-4}\left|\nabla|\nabla v_\varepsilon|^2\right|^2 - 2q\int_\Omega |\nabla v_\varepsilon|^{2q} + 2q\int_\Omega |\nabla v_\varepsilon|^{2q-1}|\nabla u_\varepsilon|.\end{aligned}$$

In this step we used convexity of Ω to estimate the boundary integral

$$\int_{\partial\Omega} |\nabla v_\varepsilon|^{2q-2}\nabla\left(|\nabla v_\varepsilon|^2\right)\cdot\nu \le 0$$

due to the fact that in convex domains $\partial_\nu|\nabla v_\varepsilon|^2|_{\partial\Omega} \le 0$ follows from $\partial_\nu v|_{\partial\Omega} = 0$, confer [21, Lemma 3.2]. □

The other summand arising in the calculation of $y_\varepsilon'(t)$ can be estimated as follows:

Lemma 4.7. *For any $\kappa \in \mathbb{R}$, $\varepsilon > 0$,*

$$\frac{d}{dt}\int_\Omega u_\varepsilon^2 \le -2\int_\Omega |\nabla u_\varepsilon|^2 + 2\int_\Omega u_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon + 2\kappa\int_\Omega u_\varepsilon^2 - 2\mu\int_\Omega u_\varepsilon^3.$$

Proof. This results from integration by parts and estimation of the negative last term in

$$2\int_\Omega u_\varepsilon u_{\varepsilon t} \le 2\int_\Omega u_\varepsilon\Delta u_\varepsilon - 2\int_\Omega u_\varepsilon\nabla\cdot(u_\varepsilon\nabla v_\varepsilon) + 2\kappa\int_\Omega u_\varepsilon^2 - 2\mu\int_\Omega u_\varepsilon^3 - 2\varepsilon\int_\Omega u_\varepsilon^\theta. \qquad \square$$

We put the estimates that we have found so far to their use and state

Proposition 4.8. *Let $n = 3$ and $\mu > 0$. There is a constant $A > 0$ such that for all $\varepsilon > 0$, for all $\nu > 0$, $\eta \in (0, 4]$ and $\widehat{\kappa} > 0$ the following holds: If $\kappa \in \mathbb{R}$ satisfies $\kappa < \widehat{\kappa}$ and $2\kappa + \eta \le \frac{1}{C_P}$, where C_P is the Poincaré-constant associated with Ω, then the quantity*

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon^2(t) + \int_\Omega \left|\nabla v_\varepsilon(t)\right|^4 \tag{14}$$

satisfies the differential inequality

$$y_\varepsilon'(t) \le \nu - \eta y(t) + A\left(1 + \frac{1}{4\nu}\right) y^3(t) + \frac{4\widehat{\kappa}^2|\Omega|}{C_P \mu^2} =: p\big(y_\varepsilon(t)\big)$$

for all $t > T_0$ with some $T_0 = T_0(\mu, \kappa, \widehat{\kappa}) > 0$ depending on $\mu, \kappa, \widehat{\kappa}$ only.

Proof. With the aid of Lemma 3.1, fix $T_0 > 0$ such that

$$\int_\Omega u_\varepsilon(\tau) < \frac{2\widehat{\kappa}|\Omega|}{\mu} \quad \text{for all } \tau > T_0,\ \varepsilon > 0. \tag{15}$$

By Lemma 4.7 and Lemma 4.6 with $q = 2$, we have

$$\begin{aligned} y_\varepsilon'(t) &= \frac{d}{dt}\left(\int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^4\right) \\ &\le -2\int_\Omega |\nabla u_\varepsilon|^2 + 2\int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon + 2\kappa \int_\Omega u_\varepsilon^2 - 2\mu \int_\Omega u_\varepsilon^3 \\ &\quad - 2\int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2 - 4\int_\Omega |\nabla v_\varepsilon|^4 + 4\int_\Omega |\nabla v_\varepsilon|^3 |\nabla u_\varepsilon|. \end{aligned}$$

By application of Lemma 4.5 to the second and Young's inequality and Lemma 4.4 to the last term, this becomes

$$\begin{aligned} y_\varepsilon'(t) &\le -2\int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + 2\mu \int_\Omega u_\varepsilon^3 + \int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2 \\ &\quad + 2C\left(\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right) \\ &\quad + 2\kappa \int_\Omega u_\varepsilon^2 - 2\mu \int_\Omega u_\varepsilon^3 - 2\int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2 - 4\int_\Omega |\nabla v_\varepsilon|^4 + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\quad + 8\left(\frac{1}{8}\int_\Omega \big|\nabla|\nabla v_\varepsilon|^2\big|^2 + C\left(\frac{1}{8}\right)\left(\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}}\right)\right) \\ &\le 2\kappa \int_\Omega u_\varepsilon^2 + A\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + A^{\frac{1}{2}}\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{3}{2}} - 4\int_\Omega |\nabla v_\varepsilon|^4 - \int_\Omega |\nabla u_\varepsilon|^2, \end{aligned}$$

where we denoted $A^{\frac{1}{2}} = \max\{2C + 8C(\frac{1}{8}), 1\} \le A$, C being the constant from Lemma 4.5 and $C(\frac{1}{8})$ taken from Lemma 4.4.

Another application of Young's inequality with $\nu > 0$ – so as to remove the unsolicited exponent $\frac{3}{2}$ – and sorting other terms, in order that the term $-\eta y$ appears, leave us with

$$y_\varepsilon'(t) \le (2\kappa+\eta)\int_\Omega u_\varepsilon^2 + A\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 + \nu + \frac{A}{4\nu}\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 - \eta\int_\Omega u_\varepsilon^2$$
$$-4\int_\Omega |\nabla v_\varepsilon|^4 - \int_\Omega |\nabla u_\varepsilon|^2,$$

where we apply Lemma 4.1 to the last summand and use that by (15) $\bar{u}_\varepsilon(t) = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(t) \le \frac{2\widehat{\kappa}}{\mu}$ for $t > T_0$ to arrive at

$$\begin{aligned} y_\varepsilon'(t) &\le \left((2\kappa+\eta) - \frac{1}{C_P}\right)\int_\Omega u_\varepsilon^2 + A\left(1+\frac{1}{4\nu}\right)\left(\int_\Omega |\nabla v_\varepsilon|^4\right)^3 \\ &\quad + \nu - \eta\left(\int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^4\right) + \frac{|\Omega|}{C_P}\bar{u}_\varepsilon^2 \\ &\le \nu - \eta y_\varepsilon(t) + A\left(1+\frac{1}{4\nu}\right)\big(y_\varepsilon(t)\big)^3 + \frac{4|\Omega|\widehat{\kappa}^2}{C_P\mu^2} \end{aligned}$$

as long as $(2\kappa+\eta)C_P \le 1$ and $\eta \in (0,4]$. □

The function y_ε satisfies a differential inequality with polynomial right hand side. This information is not very useful in obtaining boundedness if not accompanied by the statement that comparison with a stationary solution to the differential equation might be possible, i.e. that there is a root of the polynomial. Such is provided by the following lemma.

Lemma 4.9. *For any $\mu > 0$ there exists $\nu_0 > 0$ such that for all $\nu \in (0,\nu_0]$ there are $\widetilde{\kappa} > 0$, $\eta \in (0,4]$ such that the polynomial*

$$p(x) = \nu - \eta x + A\left(1+\frac{1}{4\nu}\right)x^3 + \frac{4\widehat{\kappa}^2|\Omega|}{C_P\mu^2}$$

defined in Proposition 4.8 has a positive root for $\widehat{\kappa} = \widetilde{\kappa}$.

Furthermore, for each $\widehat{\kappa} \in [0,\widetilde{\kappa}]$ it has a largest positive root $\delta_\nu(\widehat{\kappa})$ as well, satisfying

$$\delta_\nu(\widetilde{\kappa}) \le \delta_\nu(\widehat{\kappa}) \le \sqrt{\frac{4}{A(1+\frac{1}{4\nu})}}.$$

Proof. Because $p(x)$ is increasing in $\widehat{\kappa}$, $\delta_\nu(\widetilde{\kappa}) \le \delta_\nu(\widehat{\kappa})$ for $\widehat{\kappa} \in [0,\widetilde{\kappa}]$ is obvious.

Choose $\nu_0 > 0$ such that

$$\nu_0^2 + \frac{\nu_0}{4} < \min\left\{\frac{4}{27AC_P^3}, \frac{256}{27A}\right\}$$

and let $\nu \in (0, \nu_0]$. Then the inequality

$$\left(\nu + \frac{4|\Omega|}{\mu^2 C_P}\widehat{\kappa}^2\right)^2 < \frac{4\min\{(\frac{1}{C_P} - 2\widehat{\kappa})^3, 4^3\}}{27A(1+\frac{1}{4\nu})} \tag{16}$$

is satisfied with $\widehat{\kappa} = 0$. Let $\widetilde{\kappa} \in (0, \frac{1}{2C_P})$ be such that (16) is still satisfied for $\widehat{\kappa} = \widetilde{\kappa}$. This is possible due to continuity of the expressions in $\widehat{\kappa}$. Additionally, let $\eta = \min\{4, \frac{1}{C_P} - 2\widetilde{\kappa}\}$.

Consequently, the inequality

$$\left(\nu + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2\right)^2 < \frac{4\eta^3}{27A(1+\frac{1}{4\nu})}, \quad \text{that is } \nu - \frac{2}{3}\eta\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2 < 0 \tag{17}$$

holds. Observe that p attains a local minimum at

$$x_m = \sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} > 0,$$

where

$$\begin{aligned} p(x_m) &= \nu - \eta\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + A\left(1+\frac{1}{4\nu}\right)\frac{\eta}{3A(1+\frac{1}{4\nu})}\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2 \\ &= \nu - \frac{2}{3}\eta\sqrt{\frac{\eta}{3A(1+\frac{1}{4\nu})}} + \frac{4|\Omega|}{C_P\mu^2}\widetilde{\kappa}^2 \end{aligned}$$

is negative by (17) and therefore p has a root in (x_m, ∞). For any $\widehat{\kappa} \in [0, \widetilde{\kappa}]$ this root is smaller than $\sqrt{\frac{4}{A(1+\frac{4}{\nu})}}$, because for $x > \sqrt{\frac{4}{A(1+\frac{4}{\nu})}} > \sqrt{\frac{\eta}{A(1+\frac{4}{\nu})}}$ we have

$$p(x) > A\left(1+\frac{1}{4\nu}\right)x^3 - \eta x \geq 0. \qquad \square$$

We use this root for a comparison argument:

Proposition 4.10. *Let $n = 3$ and $\mu > 0$, let ν_0, η, $\widetilde{\kappa}$ and $\delta_\nu(\widetilde{\kappa})$ for some $\nu \in (0, \nu_0]$ be as in Lemma 4.9. Then for any $0 \leq \widehat{\kappa} \leq \widetilde{\kappa}$, $\delta_\nu(\widehat{\kappa}) > \delta_\nu(\widetilde{\kappa}) > 0$ is such that for every $\kappa \leq \widehat{\kappa}$ every $\varepsilon > 0$ has the following property: If y_ε from (14) satisfies*

$$y_\varepsilon(T) \leq \delta_\nu(\kappa)$$

for some $T > T_0$ (with $T_0 = T_0(\mu, \kappa, \widehat{\kappa})$ from Proposition 4.8), then $y_\varepsilon(t) \leq \delta_\nu(\kappa)$ for all $t > T$.

Proof. Choose as $\delta = \delta_\nu(\widehat{\kappa})$ the largest root of p from Lemma 4.9 and observe that according to Proposition 4.8 and the assumption on T

$$y_\varepsilon'(t) \le p\big(y(t)\big) \quad \text{for all } t > T \quad \text{and} \quad y_\varepsilon(T) \le \delta.$$

The comparison principle for ordinary differential equations therefore shows by means of comparison with $\bar{y} \equiv \delta$ that $y_\varepsilon(t) \le \delta$ for all $t > T$ as well. □

4.1. Eventual boundedness of y_ε

Proposition 4.10 asserts that y_ε stays small, should it ever fall below a certain value. We still have to ensure that the condition actually occurs.

Proposition 4.11. *Let $n = 3$. Let $\nu \in (0, \nu_0]$ with ν_0 as in Lemma 4.9. Then there exists $\kappa_0 \in (0, \frac{1}{8})$ such that for any $\kappa < \widehat{\kappa} \in (0, \kappa_0]$ there is $t_0 > 0$ such that for all $\tau > t_0$, for all $\varepsilon > 0$*

$$\int_\Omega u_\varepsilon^2(\tau) + \left|\nabla v_\varepsilon(\tau)\right|^4 < \delta_\nu(\widehat{\kappa})$$

where $\delta_\nu(\widehat{\kappa}) > 0$ is the positive root of p given by Lemma 4.9.

Furthermore, $\widehat{\kappa}$ satisfies

$$\widehat{\kappa} \le \sqrt{\frac{\delta_\nu(\widehat{\kappa})\mu^2}{(4 + 8C_\Omega)|\Omega|}}, \tag{18}$$

where C_Ω is a constant depending on the domain Ω only.

Proof. Due to the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$, there is $C_\Omega > 0$ such that

$$\int_\Omega |\nabla w|^4 \le C_\Omega \int_\Omega \left(w^2 + |\Delta w|^2\right) \tag{19}$$

for all $w \in W^{2,2}(\Omega)$.

Let ν be as given in the statement of the proposition, let $\widetilde{\kappa} > 0$ be as provided by Lemma 4.9 and let $\delta = \delta_\nu(\widetilde{\kappa})$. Choose

$$0 < \kappa_0 < \min\left\{\widetilde{\kappa}, \sqrt{\frac{\delta\mu^2}{(4 + 8C_\Omega)|\Omega|}}, \frac{1}{8}\right\} \tag{20}$$

and let $\widehat{\kappa} \in (0, \kappa_0]$ and $\kappa < \widehat{\kappa}$. (This already ensures (18) as well as the applicability of Proposition 4.10.)

Let $T_0 = T_0(\mu, \kappa, \widehat{\kappa})$ be as provided by Proposition 4.8 and let $t > T_0$. Note that as a result of (15) this entails

$$\int_\Omega u_\varepsilon(t) < \frac{2\widehat{\kappa}|\Omega|}{\mu}. \tag{21}$$

Furthermore denote

$$C_0 = \max\left\{1 + \int |\nabla v_0|^2 + \frac{1}{\mu}\int_\Omega u_0 + \frac{1}{\mu}, \frac{\widehat{\kappa}+1}{\mu}\max\left\{1 + \int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\}\right\} \tag{22}$$

and choose $T > 0$ so large that

$$\frac{1}{T}\left(\frac{2\widehat{\kappa}|\Omega|}{\mu^2} + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2} + C_\Omega\frac{\widehat{\kappa}}{\mu}\max\left\{1 + \int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\}t \right.$$
$$\left. + C_\Omega\frac{1}{\mu}\int_\Omega u_0 + \frac{C_\Omega}{\mu} + C_\Omega\int_\Omega v_0^2 + C_\Omega + 2C_\Omega C_0 + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2}\right) < \frac{\delta}{2}. \tag{23}$$

Combining (19) with Lemmata 3.2, 3.3 and 3.4 gives

$$\begin{aligned}
\int_t^{t+T}\int_\Omega \left(u_\varepsilon^2 + |\nabla v_\varepsilon|^4\right) &\le \int_t^{t+T}\int_\Omega u_\varepsilon^2 + C_\Omega\int_t^{t+T}\int_\Omega v_\varepsilon^2 + C_\Omega\int_t^{t+T}\int_\Omega |\Delta v_\varepsilon|^2 \\
&\le \frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_\varepsilon(t), \frac{\kappa_+|\Omega|}{\mu}\right\}T + \frac{1}{\mu}\int_\Omega u_\varepsilon(t) \\
&\quad + C_\Omega\frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_\varepsilon(t), \frac{\kappa_+|\Omega|}{\mu}\right\}T + C_\Omega\frac{1}{\mu}\int_\Omega u_\varepsilon(t) + C_\Omega\int_\Omega v_\varepsilon^2(t) \\
&\quad + C_\Omega\frac{\kappa_+}{\mu}\max\left\{\int_\Omega u_\varepsilon(t), \frac{\kappa_+|\Omega|}{\mu}\right\}T + C_\Omega\int_\Omega |\nabla v_\varepsilon(t)|^2 + \frac{C_\Omega}{\mu}\int_\Omega u(t).
\end{aligned}$$

Due to (21), upon another application of Lemmata 3.3 and 3.4 and taking into account that $\kappa_+ \le \widehat{\kappa}$, this reduces to

$$\begin{aligned}
\int_t^{t+T}\int_\Omega \left(u_\varepsilon^2 + |\nabla v_\varepsilon|^4\right) &\le \frac{\widehat{\kappa}}{\mu}\frac{2\widehat{\kappa}|\Omega|}{\mu}T + \frac{2\widehat{\kappa}|\Omega|}{\mu^2} \\
&\quad + C_\Omega\frac{\widehat{\kappa}}{\mu}\frac{2\widehat{\kappa}|\Omega|}{\mu}T + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2} + C_\Omega\frac{\widehat{\kappa}}{\mu}\max\left\{1 + \int_\Omega u_0, \frac{\widehat{\kappa}|\Omega|}{\mu}\right\}t \\
&\quad + C_\Omega\frac{1}{\mu}\left(1 + \int_\Omega u_0\right) + C_\Omega + C_\Omega\int_\Omega v_0^2 + C_\Omega\frac{\widehat{\kappa}}{\mu}\frac{2\widehat{\kappa}|\Omega|}{\mu}T \\
&\quad + 2C_\Omega C_0 + \frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2},
\end{aligned}$$

where C_0 is as defined in (22). Therefore,

$$\frac{1}{T}\int_t^{t+T}\int_\Omega \left(u_\varepsilon^2+|\nabla v_\varepsilon|^4\right) \le \frac{(2+4C_\Omega)|\Omega|}{\mu^2}\widehat{\kappa}^2$$
$$+\frac{1}{T}\left(\frac{2\widehat{\kappa}|\Omega|}{\mu^2}+\frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2}+C_\Omega\frac{\widehat{\kappa}}{\mu}\max\left\{1+\int_\Omega u_0,\frac{\widehat{\kappa}|\Omega|}{\mu}\right\}t\right.$$
$$\left.+C_\Omega\frac{1}{\mu}\int_\Omega u_0+\frac{C_\Omega}{\mu}+C_\Omega\int_\Omega v_0^2+C_\Omega+2C_\Omega C_0+\frac{2C_\Omega\widehat{\kappa}|\Omega|}{\mu^2}\right).$$

Our choice of κ_0 and T in (20) and (23), respectively, now entails

$$\frac{1}{T}\int_t^{t+T}\int_\Omega \left(u_\varepsilon^2+|\nabla v_\varepsilon|^4\right) \le \delta(\widetilde{\kappa}).$$

Accordingly, for at least one $t_0 \in (t, t+T)$

$$\int_\Omega \left(u_\varepsilon^2(t_0)+\left|\nabla v_\varepsilon(t_0)\right|^4\right) \le \delta(\widetilde{\kappa})$$

holds as well. Due to $\delta_\nu(\widetilde{\kappa}) \le \delta_\nu(\kappa_0) \le \delta_\nu(\widehat{\kappa})$ for $0 < \widehat{\kappa} < \kappa_0$, the claimed inequality for larger times τ is a direct consequence of Proposition 4.10. □

4.2. *Eventual boundedness of* $(u_\varepsilon, v_\varepsilon)$ *in* $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$

The next step is to refine these bounds on y_ε to bounds on u_ε, v_ε and ∇v_ε. $L^p - L^q$ estimates for the heat semigroup will be the cornerstone of this procedure.

Proposition 4.12. *Let $n = 3$. Then there exists a function $K: [0,\infty) \to [0,\infty)$ satisfying $\lim_{\delta\to 0} K(\delta) = 0$ with the following properties:*

Assume, $\nu \in (0, \nu_0)$, $\kappa < \kappa_0$ with ν_0, κ_0 as in Lemma 4.9 and Proposition 4.11 respectively. Choose $\widehat{\kappa} \in (\kappa_+, \kappa_0]$ and let $\delta_\nu(\widehat{\kappa})$ be as given by Proposition 4.11. Then there are $T_* > 0$ and $C > 0$ such that for all $t > T_*$ and for all $\varepsilon > 0$

$$\left\|u_\varepsilon(t)\right\|_{L^\infty(\Omega)} + \left\|\nabla v_\varepsilon(t)\right\|_{L^\infty(\Omega)} < K\left(\delta_\nu(\widehat{\kappa})\right).$$

Furthermore, corresponding to any $\|u_0\|_{L^2(\Omega)}$, $\|v_0\|_{L^2(\Omega)}$, there is $C > 0$ such that for all $t > T_*$, $\varepsilon > 0$

$$\left\|v_\varepsilon(t)\right\|_{L^\infty(\Omega)} \le Ce^{-(t-T_*)} + K\left(\delta_\nu(\widehat{\kappa})\right).$$

Proof. Let $\nu_0, \nu, \kappa \le \kappa_+ < \widehat{\kappa} < \kappa_0$ and $\delta := \delta_\nu(\widehat{\kappa})$ be as indicated in the statement of the proposition. Let $T_* - 2$ be the number from Proposition 4.11, let $t_0 \ge T_* - 2$ and let us first show boundedness of u_ε on $[t_0+1, t_0+2]$. Define

$$M := \sup_{\tau\in(t_0,t_0+2]} \left\|(\tau-t_0)^{\frac{3}{4}}u_\varepsilon(\tau)\right\|_{L^\infty(\Omega)}.$$

Let $p \in (3,4)$. By the choice of T_* and δ and Proposition 4.11, $\int_\Omega u^2(t) + \int_\Omega |\nabla v_\varepsilon(t)|^4 \le \delta$ for $t > T_*$. Together with Hölder's inequality this implies

$$\begin{aligned}
&\left\|u_\varepsilon(s)\nabla v_\varepsilon(s)\right\|_{L^p(\Omega)} \\
&\quad\le \left\|u_\varepsilon(s)\right\|_{L^{\frac{4p}{4-p}}(\Omega)} \left\|\nabla v_\varepsilon(s)\right\|_{L^4(\Omega)} \le \delta^{\frac{1}{4}}\left(\int_\Omega u_\varepsilon^{\frac{4p}{4-p}}(s)\right)^{\frac{4-p}{4p}} \\
&\quad\le \delta^{\frac{1}{4}}\left(\int_\Omega u_\varepsilon^2(s)\cdot u_\varepsilon^{\frac{4p}{4-p}-2}(s)\right)^{\frac{4-p}{4p}} \le \delta^{\frac{1}{4}}\left(\int_\Omega u_\varepsilon^2(s)\right)^{\frac{4-p}{4p}}\left(\sup_\Omega u_\varepsilon^{\frac{4p}{4-p}-2}(s)\right)^{\frac{4-p}{4p}} \\
&\quad\le \delta^{\frac{1}{4}+\frac{4-p}{4p}} \sup_\Omega\left(u_\varepsilon^{1-\frac{4-p}{2p}}(s)\right) \le \delta^{\frac{1}{p}}(s-t_0)^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}} \sup_\Omega\left((s-t_0)^{\frac{3}{4}}u_\varepsilon(s)\right)^{1-\frac{4-p}{2p}}
\end{aligned} \tag{24}$$

for $s \in (t_0, t_0+2]$. Triangle inequality and $L^p - L^q$-estimates [25, Lemma 1.3] give a constant $C_1 > 0$ such that

$$\left\|e^{\tau\Delta}u_\varepsilon(t_0)\right\|_{L^\infty(\Omega)} \le C_1\left(1+\tau^{-\frac{3}{4}}\right)\left\|u_\varepsilon(t_0)-\bar{u}_\varepsilon(t_0)\right\|_{L^2(\Omega)} + \left\|\bar{u}_\varepsilon(t_0)\right\|_{L^\infty(\Omega)},$$

where $\|\bar{u}_\varepsilon(t_0)\|_{L^\infty(\Omega)} = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(t_0) \le |\Omega|^{-\frac{1}{2}}\|u_\varepsilon(t_0)\|_{L^2(\Omega)} \le |\Omega|^{-\frac{1}{2}}\sqrt{\delta}$ and thus $\|\bar{u}_\varepsilon(t_0)\|_{L^2(\Omega)} \le \sqrt{\delta}$ lead to

$$\left\|e^{\tau\Delta}u_\varepsilon(t_0)\right\|_{L^\infty(\Omega)} \le C_1\left(1+\tau^{-\frac{3}{4}}\right)2\sqrt{\delta} + |\Omega|^{-\frac{1}{2}}\sqrt{\delta}. \tag{25}$$

Again, by semigroup representation and the fact that the heat semigroup is order-preserving,

$$\begin{aligned}
0 &\le \tau^{\frac{3}{4}}u_\varepsilon(t_0+\tau) \\
&\le \tau^{\frac{3}{4}}e^{\tau\Delta}u_\varepsilon(t_0) - \tau^{\frac{3}{4}}\int_0^\tau e^{(\tau-s)\Delta}\nabla\cdot\left(u_\varepsilon(t_0+s)\nabla v_\varepsilon(t_0+s)\right)\mathrm{d}s \\
&\quad + \tau^{\frac{3}{4}}\int_0^\tau e^{(\tau-s)\Delta}\left(\kappa_+ u_\varepsilon(t_0+s) - \mu u_\varepsilon^2(t_0+s)\right)\mathrm{d}s \\
&\le \tau^{\frac{3}{4}}\left\|e^{(\tau-s)\Delta}u_\varepsilon(t_0)\right\|_{L^\infty(\Omega)} + \tau^{\frac{3}{4}}\int_0^\tau \left\|e^{(\tau-s)\Delta}\nabla\cdot\left(u_\varepsilon(t_0+s)\nabla v_\varepsilon(t_0+s)\right)\right\|_{L^\infty(\Omega)}\mathrm{d}s \\
&\quad + \tau^{\frac{3}{4}}\int_0^\tau \kappa_+ M s^{-\frac{3}{4}}\mathrm{d}s.
\end{aligned}$$

Together with $L^p - L^q$-estimates, (24) and (25) this entails for $\tau \in [0,2]$ and some $C_2 > 0$ from [25, Lemma 1.3]

$$\begin{aligned}
\left\|\tau^{\frac{3}{4}} u_\varepsilon(t_0+\tau)\right\|_{L^\infty(\Omega)} &\le \tau^{\frac{3}{4}} C_1\left(1+\tau^{-\frac{3}{4}}\right)2\sqrt{\delta} + \tau^{\frac{3}{4}}|\Omega|^{-\frac{1}{2}}\sqrt{\delta} + 8\kappa_+ M \\
&\quad + 2\int_0^\tau \left(1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}}\right)\left\|u_\varepsilon(t_0+s)\nabla v_\varepsilon(t_0+s)\right\|_{L^p(\Omega)}\mathrm{ds} \\
&\le 2C_1\left(1+\tau^{\frac{3}{4}}\right)\sqrt{\delta} + \tau^{\frac{3}{4}}|\Omega|^{-\frac{1}{2}}\sqrt{\delta} + 8\kappa_0 M \\
&\quad + C_2\int_0^\tau \left(1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}}\right)\delta^{\frac{1}{p}} s^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}}\left\|s^{\frac{3}{4}}u_\varepsilon(t_0+s)\right\|_{L^\infty(\Omega)}^{1-\frac{4-p}{2p}}\mathrm{ds} \\
&\le 2C_1\left(1+\tau^{\frac{3}{4}}\right)\sqrt{\delta} + \tau^{\frac{3}{4}}|\Omega|^{-\frac{1}{2}}\sqrt{\delta} + 8\kappa_0 M \\
&\quad + C_2\int_0^\tau \left(1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}}\right)\delta^{\frac{1}{p}} s^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}} M^{1-\frac{4-p}{2p}}\mathrm{ds}.
\end{aligned}$$

As $\int_0^2 (1+(\tau-s)^{-\frac{1}{2}-\frac{3}{2p}})s^{-\frac{3}{4}+\frac{3}{4}\frac{4-p}{2p}}\mathrm{ds}$ is finite and $\frac{1}{1-8\kappa_0} > 0$, taking the supremum over $\tau \in [0,2]$, we infer

$$M \le C_3\sqrt{\delta} + C_4\delta^{\frac{1}{p}} M^{1-\frac{4-p}{2p}}$$

with obvious choices of the constants $C_3, C_4 > 0$.

Therefore

$$M \le D(\delta) := \sup\left\{\xi \in [0,\infty) : \xi - C_4\delta^{\frac{1}{p}}\xi^{1-\frac{4-p}{2p}} \le C_3\sqrt{\delta}\right\} < \infty.$$

Note that $D(\delta)$ tends to 0 as δ becomes small.

For $t \in [t_0, t_0+2]$

$$(t-t_0)^{\frac{3}{4}}\left\|u_\varepsilon(t)\right\|_{L^\infty(\Omega)} < D(\delta),$$

meaning that for $t \in [t_0+1, t_0+2]$

$$\left\|u_\varepsilon(t)\right\|_{L^\infty(\Omega)} < D(t-t_0)^{-\frac{3}{4}} \le D(\delta).$$

$D(\delta)$ is independent of the choice of $t_0 > T_* - 2$, therefore we can conclude

$$\left\|u_\varepsilon(t)\right\|_{L^\infty(\Omega)} \le D(\delta) \tag{26}$$

for any $t > T_* - 1$.

Boundedness of $\{\nabla v(\tau)\}_{\tau > T_*}$ in $L^\infty(\Omega)$ can be achieved from the following estimates: Let $t_0 = T_* - 1$ and denote $t = \tau - t_0$. Then [25, Lemma 1.3] provides $C_5 > 0$ such that

$$\begin{aligned}\|\nabla v_\varepsilon(\tau)\|_{L^\infty(\Omega)} &\le \|\nabla e^{t(\Delta-1)}v_\varepsilon(t_0)\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)}u_\varepsilon(t_0+s)\|_{L^\infty(\Omega)}\mathrm{ds}\\ &\le C_5 t^{-\frac{7}{8}}\|\nabla v_\varepsilon(t_0)\|_{L^4(\Omega)} + C_5\int_0^t \big(1+(t-s)^{-\frac{1}{2}}\big)e^{-(t-s)}\|u_\varepsilon(t_0+s)\|_{L^\infty(\Omega)}\mathrm{ds}\\ &\le C_5\delta^{\frac{1}{4}}t^{-\frac{7}{8}} + D(\delta)C_5\int_0^\infty \big(1+\sigma^{-\frac{1}{2}}\big)e^{-\sigma}d\sigma \qquad (27)\end{aligned}$$

is bounded on $[T_*,\infty)$. By similar reasoning together with Lemma 3.3, we obtain bounds on $\|v_\varepsilon(\tau)\|_{L^\infty(\Omega)}$. In preparation for these estimates, let $t_* > t_0$ and let us note that by (11) and Lemma 3.3

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t_*) &\le \frac{1}{|\Omega|}\Big(\int_\Omega v_\varepsilon(t_0)\Big)e^{-(t_*-t_0)} + \frac{\kappa_+}{\mu} + \frac{1}{|\Omega|}\int_\Omega u_\varepsilon(t_0)\\ &\le \Big(\frac{\kappa_+}{\mu}+C_6\Big)e^{-(t_*-t_0)} + \frac{\kappa_+}{\mu} + \frac{1}{|\Omega|^{\frac{1}{2}}}\|u_\varepsilon(t_0)\|_{L^2(\Omega)}\\ &\le C_6 e^{-(t_*-t_0)} + 2\frac{\widehat{\kappa}}{\mu} + \frac{\delta^{\frac{1}{2}}}{|\Omega|^{\frac{1}{2}}}\\ &\le C_6 e^{-(t_*-t_0)} + C_7\delta^{\frac{1}{2}},\end{aligned}$$

where C_6 depends on $\|u_0\|_{L^1(\Omega)}$ and $\|v_0\|_{L^1(\Omega)}$ (and $|\Omega|$) only, and where we have applied (18) in the last step, so that $C_7 = (1+\frac{1}{\sqrt{(4+8C_\Omega)}})\frac{1}{\sqrt{|\Omega|}}$ with C_Ω as in (18).

Lemma 1.3 of [25] yields $C_8 > 0$, which, in conjunction with Poincaré's inequality and (26), gives

$$\begin{aligned}\|v_\varepsilon(t_*+t)\|_{L^\infty(\Omega)} &\le \|e^{t(\Delta-1)}v_\varepsilon(t_*)\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}u_\varepsilon(t_*+s)\|_{L^\infty(\Omega)}\mathrm{ds}\\ &\le \Big\|e^{t(\Delta-1)}\Big(v_\varepsilon(t_*) - \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t_*)\Big)\Big\|_{L^\infty(\Omega)} + \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t_*) + tD(\delta)\\ &\le C_8\big(1+t^{-\frac{3}{4}}\big)\Big\|v_\varepsilon(t_*) - \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t_*)\Big\|_{L^2(\Omega)} + \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t_*) + 2D(\delta)\\ &\le C_8\big(1+t^{-\frac{3}{4}}\big)C_P\|\nabla v_\varepsilon(t_*)\|_{L^2(\Omega)} + \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t_*) + 2D(\delta)\\ &\le C_8\big(1+t^{-\frac{3}{4}}\big)C_P|\Omega|^{\frac{1}{4}}\delta^{\frac{1}{4}} + C_6 e^{-(t_*-t_0)}e^{2-t} + C_7\delta^{\frac{1}{2}} + 2D(\delta)\end{aligned}$$

for any $t \in (0,2]$ and therefore

$$\left\| v_\varepsilon(\tau) \right\|_{L^\infty(\Omega)} \le 2C_8 C_P |\Omega|^{\frac{1}{4}} \delta^{\frac{1}{4}} + C_7 \delta^{\frac{1}{2}} + 2D(\delta) + C_6 e e^{-(\tau-(t_0+1))} \tag{28}$$

for any $\tau > t_0 + 1 = T_*$. Collecting terms from (26), (27) and (28), we obtain a suitable definition of C and of $K(\delta)$ – and as $\delta^{\frac{1}{4}}$, $\delta^{\frac{1}{2}}$ and $D(\delta)$ tend to 0 as $\delta \searrow 0$, indeed, $\lim_{\delta \searrow 0} K(\delta) = 0$. □

5. Definition of solutions

Definition 5.1. A pair of functions $(u, v) \in L^2_{loc}([0,\infty); L^2(\Omega)) \times L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ is called **weak solution** of (1) for initial data $(u_0, v_0) \in L^2(\Omega) \times W^{1,2}(\Omega)$ if for all test functions $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ the following holds:

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(0) = \int_0^\infty \int_\Omega u\Delta\varphi - \int_0^\infty \int_\Omega u\nabla v \cdot \nabla\varphi + \kappa \int_0^\infty \int_\Omega u\varphi - \mu \int_0^\infty \int_\Omega u^2\varphi \tag{29}$$

and, for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$,

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi. \tag{30}$$

6. Convergence to a solution

Purpose of the estimates from Section 3 was to make the extraction of convergent sequences of approximate solutions $(u_\varepsilon, v_\varepsilon)$ possible. The following proposition lists, in which sense we have obtained convergence.

Proposition 6.1. *There exist $u \in L^2_{loc}([0,\infty), L^2(\Omega))$ and $v \in L^2_{loc}([0,\infty), W^{1,2}(\Omega))$ and a sequence $\varepsilon_j \searrow 0$ such that for any $T > 0$*

$$u_{\varepsilon_j} \to u \qquad \text{a.e. in } \Omega \times [0,T], \tag{31}$$

$$u_{\varepsilon_j} \to u \qquad \text{in } L^2\big(\Omega \times (0,T)\big), \tag{32}$$

$$\varepsilon_j u_{\varepsilon_j}^\theta \rightharpoonup 0 \qquad \text{in } L^1\big(\Omega \times (0,T)\big), \tag{33}$$

$$v_{\varepsilon_j} \rightharpoonup v \qquad \text{in } L^2\big((0,T); W^{1,2}(\Omega)\big), \tag{34}$$

$$v_{\varepsilon_j} \to v \qquad \text{in } L^2\big(\Omega \times (0,T)\big), \tag{35}$$

$$v_{\varepsilon_j} \to v \qquad \text{a.e. in } \Omega \times [0,T], \tag{36}$$

$$\Delta v_{\varepsilon_j} \rightharpoonup \Delta v \qquad \text{in } L^2\big(\Omega \times (0,T)\big), \tag{37}$$

$$v_{\varepsilon_j t} \rightharpoonup v_t \qquad \text{in } L^2\big(\Omega \times (0,T)\big), \tag{38}$$

$$u_{\varepsilon_j} \nabla v_{\varepsilon_j} \rightharpoonup u\nabla v \qquad \text{in } L^1\big(\Omega \times (0,T)\big). \tag{39}$$

Proof. Lemmata 3.6 and 3.7 show boundedness of $\{u_\varepsilon\}_\varepsilon$ in $L^{\frac{4}{3}}((0,T);W^{1,\frac{4}{3}}(\Omega))$ and of the derivatives $\{u_{\varepsilon t}\}_\varepsilon$ in $L^1((0,T);(W^{2,\infty}(\Omega))^*)$ so that by a variant of the Aubin–Lions lemma [4, Prop. 6], $\{u_\varepsilon\}_\varepsilon$ is relatively compact in $L^{\frac{4}{3}}(\Omega\times(0,T))$; in particular, there is a sequence $\varepsilon_j \searrow 0$ (of which we will, without relabelling, choose further subsequences in the following) such that $u_{\varepsilon_j}\to u$ almost everywhere in $\Omega\times(0,T)$ for some $u\in L^{\frac{4}{3}}(\Omega\times(0,T))$. Boundedness of $\{u_\varepsilon\}_\varepsilon$ in $L^2(\Omega\times(0,T))$ due to Lemma 3.2 yields a subsequence along which $u_{\varepsilon_j}\rightharpoonup u$ in $L^2(\Omega\times(0,T))$.

By Lemma 3.5, $\{u_\varepsilon^2\}_\varepsilon$ is equi-integrable and thus, according to [5, Thm. IV.8.9], weakly sequentially precompact in $L^1(\Omega\times(0,T))$. Along a subsequence, $u_{\varepsilon_j}^2\rightharpoonup u^2$ in $L^1(\Omega\times(0,T))$ and hence

$$\|u_{\varepsilon_j}\|^2_{L^2(\Omega\times(0,T))}=\int\limits_{\Omega\times(0,T)} u_{\varepsilon_j}^2\cdot 1\to\int\limits_{\Omega\times(0,T)} u^2\cdot 1=\|u\|^2_{L^2(\Omega\times(0,T))}.$$

The combination of $u_{\varepsilon_j}\rightharpoonup u$ in $L^2(\Omega\times(0,T))$ and $\|u_{\varepsilon_j}\|_{L^2(\Omega\times(0,T))}\to\|u\|_{L^2(\Omega\times(0,T))}$ shows that actually (32) holds.

Similarly, we see that $\{\varepsilon u_\varepsilon^\theta\}_\varepsilon$ is equi-integrable (Lemma 3.5) and hence is weakly convergent along a subsequence. Pointwise a.e. convergence of u_ε^θ to u^θ identifies the weak limit of $\varepsilon_j u_{\varepsilon_j}^\theta$ as 0, which is (33).

According to Lemmata 3.3 and 3.4, $\{v_\varepsilon\}_\varepsilon$ is bounded in $L^\infty((0,T);W^{1,2}(\Omega))\hookrightarrow L^2((0,T);W^{1,2}(\Omega))$ and a subsequence with (34) can be found.

Furthermore, $\{v_{\varepsilon t}\}_\varepsilon=\{\Delta v_\varepsilon-v_\varepsilon+u_\varepsilon\}_\varepsilon$ is bounded in $L^2((0,T);L^2(\Omega))$ due to Lemmata 3.4, 3.3, 3.2, and the Aubin–Lions lemma yields (35) as well as, along another subsequence, (36). At the same time, we can conclude (37) and (38).

The statement (39), finally, results from a combination of (32) and (34). □

From now on, by (u,v) we will denote the limit provided by Proposition 6.1. Of course, it would be desirable for (u,v) to be a solution to the original problem. That is the case.

Lemma 6.2. *(u,v) is a solution to (1) in the sense of Definition 5.1.*

Proof. Take φ as specified in Definition 5.1 and test the equations of (3) against it. The convergence results of Proposition 6.1 then produce (29) and (30). □

Remark 6.3. None of the arguments used for Proposition 6.1 and Lemma 6.2 depend on dimension n nor on the specific values of $\mu>0$, $\kappa\in\mathbb{R}$.

7. Eventual smoothness. Proof of Theorem 1.1

In the most important scenario of spatial dimension 3, we can show that these solutions are not only solutions in some weak sense, but possess the property of eventual smoothness: From some time on, they are classical solutions. Our preparations from Section 4 that have provided boundedness of $(u_\varepsilon,v_\varepsilon)$ constitute the first step.

The next proposition transfers these properties to (u,v).

Proposition 7.1. *Let $n = 3$ and assume, $\kappa < \kappa_0$ with κ_0 from Proposition 4.11. With T_* denoting the number from Proposition 4.12,*

$$u \in L^2_{loc}\big([T_*, \infty), W^{1,2}(\Omega)\big), \qquad u_t \in L^2_{loc}\big([T_*, \infty), \big(W^{1,2}(\Omega)\big)^*\big).$$

Furthermore $u, v, \nabla v \in L^\infty(\Omega \times [T_*, \infty))$.

Proof. On the interval $[T_*, \infty)$, from Proposition 4.12 we obtain boundedness of $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{|\nabla v_\varepsilon|\}_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega \times [T_*, \infty))$ and hence can choose a sequence $\varepsilon_j \searrow 0$ such that $u_\varepsilon, v_\varepsilon, \nabla v_\varepsilon$ are weak-*-convergent in this space. For $T > 0$, boundedness of $\{u_\varepsilon\}_\varepsilon$ and $\{u_{\varepsilon t}\}_\varepsilon$ in $L^2_{loc}([T_*, T_* + T], W^{1,2}(\Omega))$ and $L^2_{loc}([T_*, T_* + T], (W^{1,2}(\Omega))^*)$ respectively are guaranteed by Lemma 3.8 and 3.9 and the choice of a weakly convergent subsequence yields the assertion. □

Corollary 7.2. *Under the conditions of Proposition 7.1, $u \in C_{loc}([T_*, \infty), L^2(\Omega))$.*

Proof. For any $T > 0$, $u \in L^2([T_*, T_* + T], W^{1,2}(\Omega))$ and $u_t \in L^2([T_*, T_* + T], (W^{1,2}(\Omega))^*)$. By Proposition 23.23 of [30], u is L^2-continuous on $[T_*, T_* + T]$. □

Actually, u and v are even Hölder continuous.

Lemma 7.3. *Let $n = 3$. Assume, $\kappa < \kappa_0$ with κ_0 from Proposition 4.11 and let T_* be as in Proposition 4.12. There is $\alpha > 0$ such that $u, v \in C^{\alpha,\frac{\alpha}{2}}_{loc}(\overline{\Omega} \times [T_* + 1, \infty))$. Moreover, there is $C > 0$ such that for every $T > T_* + 1$,*

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[T,T+1])} + \|v\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[T,T+1])} \le C.$$

Proof. Let T_* be as in Proposition 4.12 and let $t \ge T_*$ such that $\|u(t)\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega\times(T_*,\infty))}$ and $\|v(t)\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega\times(T_*,\infty))}$, which is the case for almost every such t.

Definition 5.1, Corollary 7.2 and Proposition 7.1 enable us to interpret u as a local weak solution in the sense of [19] of the equation

$$\widetilde{u}_t - \nabla \cdot (\nabla\widetilde{u} - \widetilde{u}\nabla v) = \kappa u - \mu u^2, \tag{40}$$

for $\widetilde{u}$ on $[T_*, \infty)$.

Using boundedness of $\kappa u - \mu u^2$ and ∇v, an application of Theorem 1.3 of [19] ensures $u \in C^{\alpha',\frac{\alpha'}{2}}(\overline{\Omega} \times [T_* + \frac{1}{2}, \infty))$ for some $\alpha' > 0$.

Theorem 1.3 of [19] additionally asserts that the norm $\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[t+\frac{1}{2},t+2])}$ can be estimated by a constant C_u which depends on the $L^\infty(\Omega)$-norm of $u(t)$ and some "data" of the problem, a term condensing structural information on the equation (such as exponents) and certain L^r-norms of coefficients and the right-hand side in (40).

Important to note is that, due to Proposition 7.1, $u, v, \nabla v \in L^\infty(\Omega \times [T_*, \infty))$ and therefore the restrictions of these functions to $\overline{\Omega} \times [t, t + 2]$ are bounded in $L^\infty(\overline{\Omega} \times [t, t + 2])$ independently of $t > T_*$. Hence C_u can be chosen independently of t.

Similar to Corollary 7.2, from (35), (38) and (34), we infer $v \in L^2_{loc}((0,\infty), W^{1,2}(\Omega)) \cap C((0,\infty), L^2(\Omega))$ and boundedness of u, v on $[T_* + \frac{1}{2}, \infty)$ imply, again by Theorem 1.3 of [19] applied to the solution v of

$$\widetilde{v}_t - \nabla \cdot (\nabla \widetilde{v}) = u - v \tag{41}$$

for $\widetilde{v}$, that $v \in C^{\alpha'', \frac{\alpha''}{2}}(\overline{\Omega} \times [t+1, t+2])$ for some $\alpha'' > 0$ – and that

$$\|v\|_{C^{\alpha'', \frac{\alpha''}{2}}(\overline{\Omega} \times [t+1, t+2])} \le C_v,$$

with some constant C_v which can be chosen independently of t.

Letting $\alpha = \min\{\alpha', \alpha''\}$, deriving a suitable constant C from the values of C_u and C_v and taking the arbitrariness of t into account, the claim follows. □

Thanks to the regularity of u and v that we have gained so far, we can interpret u and v as generalized solutions in the sense of [11] of the homogeneous Neumann boundary value problem with initial value $u(T_* + 1)$, $v(T_* + 1)$ to (40) or (41). As the coefficients are bounded, these problems are known to be uniquely solvable [11, Thm. III.5.1]. Therefore we can use existence theorems for smoother solutions to establish higher regularity of u and v.

Theorem IV.5.3 of [11] asserts the existence of $C^{2+\alpha, 1+\frac{\alpha}{2}}$ solutions, albeit under stronger smoothness assumptions on the initial datum than we can guarantee so far. In order to nevertheless apply this theorem, let us, for $t_0 > 0$, $T > t > 0$, introduce a smooth monotone function $\chi_{t_0,t,T} \colon [t_0, t_0 + T] \to \mathbb{R}$ satisfying $\chi(t_0) = 0$ and $\chi \equiv 1$ on $[t_0 + t, t_0 + T]$ as well as $\|\chi_{t_0,t,T}\|_{C^1(t_0, t_0+T)} \le 1 + \frac{2}{t}$.

Proposition 7.4. *Let $n = 3$ and assume that $\kappa < \kappa_0$ with κ_0 from Proposition 4.11. Then there are $T^* > 0$ and $\alpha > 0$ such that $u, v \in C^{2+\alpha, 1+\frac{\alpha}{2}}_{loc}(\overline{\Omega} \times [T^*, \infty))$.*

Moreover, there exists $C > 0$ such that for all $t > T^*$

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [t, t+1])} \le C.$$

Proof. Let T_* be as in Proposition 4.12 and $T_0 > T_* + 1$. Let $\chi = \chi_{T_0, \frac{1}{2}, 2}$ as defined above and observe that $(\chi v)(T_0) = 0$, $\partial_\nu(\chi v)|_{\partial\Omega} = 0$ and $\widetilde{v} := \chi v$ satisfies

$$\widetilde{v}_t - \Delta \widetilde{v} = \chi_t v + \chi u - \chi v \quad \text{on } (T_0, T_0 + 2), \tag{42}$$

a parabolic PDE with smooth coefficients and Hölder continuous right-hand side (due to Lemma 7.3). Theorem IV.5.3 of [11] in conjunction with the above-mentioned uniqueness property makes χv an element of $C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [T_0, T_0 + 2])$ and therefore $v \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [T_0 + \frac{1}{2}, T_0 + 2])$, where, according to the aforementioned theorem, its norm can be estimated by the $C^{\alpha, \frac{\alpha}{2}}$-norm of the right-hand side in (42) and therefore independently of $T_0 > T_* + 1$, cf. Lemma 7.3.

For an analogous procedure concerning u let $\chi = \chi_{T_0 + \frac{1}{2}, \frac{1}{2}, \frac{3}{2}}$ and consider $\widetilde{u} = \chi u$, satisfying $\widetilde{u}(T_0 + \frac{1}{2}) = 0$, $\partial_\nu \widetilde{u}|_{\partial\Omega} = 0$ and solving

$$\widetilde{u}_t - \Delta \widetilde{u} - \nabla \widetilde{u} \nabla v - \widetilde{u} \Delta v = \chi_t u + \chi\left(\kappa u - \mu u^2\right),$$

where the coefficients are Hölder continuous as well as the right-hand side and, by the same argument as before, [11, Thm. IV.5.3] asserts $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [T_0+1, T_0+2])$ with a T_0-independent estimate on the norm. The claim follows upon the choice $T^* = T_0 + 1$ and due to the independence of the Hölder norm of T_0. □

After these preparations, the proof of our main result consists in nothing more than collating the right statements:

Proof of Theorem 1.1. Existence of a solution is given by Proposition 6.1 in combination with Lemma 6.2, eventual smoothness and bounds on the Hölder norms by Proposition 7.4. □

8. Asymptotic behaviour

Now that existence and smoothness of (u, v) have been ensured, let us concentrate on the long time behaviour of solutions.

8.1. The case $\kappa \le 0$. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $\{(u_{\varepsilon_j}, v_{\varepsilon_j})\}_{j\in\mathbb{N}}$ be a sequence of solutions to (3) approaching (u, v) in the sense of Proposition 6.1. Let $\vartheta > 0$.

From Proposition 4.12 we can infer $\delta_0 > 0$ such that $K(\delta)$ from Proposition 4.12 satisfies $K(\delta) < \frac{\vartheta}{3}$ for any $\delta \in [0, \delta_0)$.

Now apply Lemma 4.9 with $\nu \in (0, \nu_0]$ so small that $\sqrt{\frac{4}{A(1+\frac{1}{4\nu})}} < \delta_0$ and choose $\widetilde{\kappa} > 0$ and $\eta \in (0, 4]$ as provided thereupon. In particular, this implies $\delta_\nu(\widehat{\kappa}) \le \delta_0$ for any $\widehat{\kappa} \in (0, \widetilde{\kappa})$.

Let $\widehat{\kappa} \in (0, \widetilde{\kappa})$ and let $T_0 = T_0(\mu, 0, \widehat{\kappa})$ be as in Proposition 4.8. As $\kappa \le 0 < \widehat{\kappa}$, Proposition 4.12 implies that there is $T > 0$ such that, independent of $j \in \mathbb{N}$,

$$\|u_{\varepsilon_j}(t)\|_{L^\infty(\Omega)} + \|v_{\varepsilon_j}(t)\|_{W^{1,\infty}(\Omega)} \le 2K\big(\delta_\nu(\widehat{\kappa})\big) + Ce^{-(t-T)} \quad \text{for all } t > T, \tag{43}$$

where C is a constant depending on the norm of the initial data (u_0, v_0).

Choose $T_\vartheta > T$ in such a way that $Ce^{-(T_\vartheta - T)} < \frac{\vartheta}{3}$ and that u, v are continuous on $[T_\vartheta, \infty)$ by Theorem 1.1.

Our choice of δ_0 thus shows that, independent of $j \in \mathbb{N}$,

$$\|u_{\varepsilon_j}(t)\|_{L^\infty(\Omega)} + \|v_{\varepsilon_j}(t)\|_{W^{1,\infty}(\Omega)} \le 2\frac{\vartheta}{3} + \frac{\vartheta}{3} = \vartheta \quad \text{for all } t > T_\vartheta.$$

Almost everywhere convergence of $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u, v)$ (as stated by Proposition 6.1 in (31), (36)) and continuity of u and v hence imply that

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \le \vartheta \quad \text{for all } t > T_\vartheta. \tag{44}$$

In conclusion,

$$\big(u(t), v(t)\big) \to 0 \quad \text{as } t \to \infty$$

in the sense of uniform convergence on Ω. □

8.2. Asymptotics for positive κ. Proof of Theorem 1.5

Proof of Theorem 1.5. Under the condition of κ being sufficiently small, Theorem 1.1 shows that the solutions constructed above enter some bounded set $B_{\mu,\kappa} \subset (C^{2+\alpha}(\overline{\Omega}))^2$, where $\alpha > 0$ is chosen as in Proposition 7.4.

As to the statement about the diameter of $B_{\mu,\kappa}$ in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ as $\kappa \to 0$, we can proceed almost exactly as in the proof of Theorem 1.3: Let $\vartheta > 0$. From Proposition 4.12 we can infer $\delta_0 > 0$ such that $K(\delta)$ from Proposition 4.12 satisfies $K(\delta) < \frac{\vartheta}{3}$ for any $\delta \in [0, \delta_0)$. The application of Lemma 4.9 with $\nu \in (0, \nu_0]$ satisfying $\sqrt{\frac{4}{A(1+\frac{1}{4\nu})}} < \delta_0$ provides $\eta \in (0, 4]$ and $\widetilde{\kappa} > 0$. Let $\widehat{\kappa} \in (0, \widetilde{\kappa})$.

We will prove that $\operatorname{diam} B_{\mu,\kappa} \le 2\vartheta$ if $\kappa < \widehat{\kappa}$.

Assume that $\kappa < \widehat{\kappa}$ and let $T_0 = T_0(\mu, \kappa, \widehat{\kappa})$ be as in Proposition 4.8. As $\kappa < \widehat{\kappa}$, Proposition 4.12 implies that there is $T > 0$ such that, independent of $j \in \mathbb{N}$,

$$\left\|u_{\varepsilon_j}(t)\right\|_{L^\infty(\Omega)} + \left\|v_{\varepsilon_j}(t)\right\|_{W^{1,\infty}(\Omega)} \le 2K\left(\delta_\nu(\widehat{\kappa})\right) + Ce^{-(t-T)} \quad \text{for all } t > T, \tag{45}$$

where C is a constant depending on the norm of the initial data (u_0, v_0).

Choose $T_\vartheta > T$ in such a way that $Ce^{-(T_\vartheta - T)} < \frac{\vartheta}{3}$ and that u, v are continuously differentiable on $[T_\vartheta, \infty)$ by Theorem 1.1.

Our choice of δ_0 thus shows that, independent of $j \in \mathbb{N}$,

$$\left\|u_{\varepsilon_j}(t)\right\|_{L^\infty(\Omega)} + \left\|v_{\varepsilon_j}(t)\right\|_{W^{1,\infty}(\Omega)} \le 2\frac{\vartheta}{3} + \frac{\vartheta}{3} = \vartheta \quad \text{for all } t > T_\vartheta.$$

We make use of the almost everywhere convergence of $(u_{\varepsilon_j}, v_{\varepsilon_j}) \to (u, v)$ (as stated by Proposition 6.1 in (31), (36)) and the fact that ∇v_{ε_j} is essentially bounded by some constant $\widetilde{C}$ on $\Omega \times [T_\vartheta, \infty)$ uniformly in j, which allows us to extract a L^∞-weak*-convergent subsequence leading to $\|\nabla v\|_{L^\infty(\Omega)} \le \widetilde{C}$.

Together with the continuity of u, v and ∇v these convergence results hence imply that

$$\left\|u(t)\right\|_{L^\infty(\Omega)} + \left\|v(t)\right\|_{W^{1,\infty}(\Omega)} \le \vartheta \quad \text{for all } t > T_\vartheta.$$

In terms of $B_{\mu,\kappa}$ this means

$$B_{\mu,\kappa} \subset B_\vartheta^{L^\infty(\Omega)\times W^{1,\infty}(\Omega)}(0)$$

and hence $\operatorname{diam}(B_{\mu,\kappa}) \le 2\vartheta$ for sufficiently small $\kappa > 0$. □

References

[1] M. Aida, T. Tsujikawa, M. Efendiev, A. Yagi, M. Mimura, Lower estimate of the attractor dimension for a chemotaxis growth system, J. Lond. Math. Soc. 74 (2) (2006) 453–474.

[2] K. Baghaei, M. Hesaaraki, Global existence and boundedness of classical solutions for a chemotaxis model with logistic source, C. R. Math. 351 (16) (2013) 585–591.

[3] X. Cao, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with logistic source, J. Math. Anal. Appl. 412 (1) (2014) 181–188.

[4] X. Chen, A. Jüngel, J.-G. Liu, A note on Aubin–Lions–Dubinskii lemmas, Acta Appl. Math. 133 (1) (2014) 33–43.

[5] N. Dunford, J.T. Schwartz, Linear Operators. I. General Theory, with the assistance of W.G. Bade and R.G. Bartle, Pure Appl. Math., vol. 7, Interscience Publishers, Inc., Interscience Publishers, Ltd., New York, London, 1958.
[6] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (1–2) (2009) 183–217.
[7] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein. 105 (3) (2003) 103–165.
[8] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences. II, Jahresber. Deutsch. Math.-Verein. 106 (2) (2004) 51–69.
[9] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (3) (1970) 399–415.
[10] K. Kuto, K. Osaki, T. Sakurai, T. Tsujikawa, Spatial pattern formation in a chemotaxis-diffusion-growth model, Phys. D 241 (19) (2012) 1629–1639.
[11] O. Ladyzhenskaya, V. Solonnikov, N. Uraltseva, Linear and Quasi-Linear Equations of Parabolic Type, Transl. Math. Monogr., vol. 23, American Mathematical Society, Providence, RI, 1968, reprinted with corrections 1988.
[12] J. Lankeit, Chemotaxis can prevent thresholds on population density, arXiv:1403.1837, 2014.
[13] E. Nakaguchi, M. Efendiev, On a new dimension estimate of the global attractor for chemotaxis-growth systems, Osaka J. Math. 45 (2) (2008) 273–281.
[14] E. Nakaguchi, K. Osaki, Global solutions and exponential attractors of a parabolic–parabolic system for chemotaxis with subquadratic degradation, Discrete Contin. Dyn. Syst. Ser. B 18 (10) (2013) 2627–2646.
[15] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. 13 (1959) 115–162.
[16] K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, Nonlinear Anal. 51 (1) (2002) 119–144.
[17] K. Osaki, A. Yagi, Global existence for a chemotaxis-growth system in $\mathbb{R}^2$, Adv. Math. Sci. Appl. 12 (2) (2002) 587–606.
[18] K.J. Painter, T. Hillen, Spatio-temporal chaos in a chemotaxis model, Phys. D 240 (4–5) (2011) 363–375.
[19] M.M. Porzio, V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, J. Differential Equations 103 (1) (1993) 146–178.
[20] Y. Tao, Boundedness in a chemotaxis model with oxygen consumption by bacteria, J. Math. Anal. Appl. 381 (2) (2011) 521–529.
[21] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, J. Differential Equations 252 (1) (2012) 692–715.
[22] Y. Tao, M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, J. Differential Equations 252 (3) (2012) 2520–2543.
[23] J.I. Tello, M. Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations 32 (4–6) (2007) 849–877.
[24] M. Winkler, Chemotaxis with logistic source: very weak global solutions and their boundedness properties, J. Math. Anal. Appl. 348 (2) (2008) 708–729.
[25] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, J. Differential Equations 248 (12) (2010) 2889–2905.
[26] M. Winkler, Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source, Comm. Partial Differential Equations 35 (8) (2010) 1516–1537.
[27] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, J. Math. Anal. Appl. 384 (2) (2011) 261–272.
[28] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, J. Differential Equations 257 (4) (2014) 1056–1077.
[29] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities?, J. Nonlinear Sci. (2014) 1–47.
[30] E. Zeidler, Nonlinear Functional Analysis and Its Applications. II/A, Linear Monotone Operators, Springer-Verlag, New York, 1990, translated from the German by the author and Leo F. Boron.

Please cite this article in press as: J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations (2014), http://dx.doi.org/10.1016/j.jde.2014.10.016