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Algebraic differential equations with functional coefficients concerning ζ and Γ

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Abstract

We prove that the Riemann zeta function and the Euler gamma function cannot satisfy a class of algebraic differential equations with functional coefficients that are connected to the zeros of the Riemann zeta function on the critical line.
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The famous, as yet unproven, Riemann hypothesis states that all the nontrivial zeros of the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ (analytically continued as a meromorphic function in $\mathbf{C}$) lie on the critical line $\mathbf{L} := \{z \in \mathbf{C} : \mathrm{Re} z = 1/2\}$. Although every effort to prove the Riemann hypothesis has failed, it is however well-known that ζ has an infinity of zeros on the critical line (see e.g. [15], p. 256). In this paper, we will prove that the Riemann zeta function ζ and the Euler gamma function Γ, which are connected by the well-known Riemann functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\frac{\pi z}{2} \Gamma(1-z)\zeta(1-z),$$

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cannot satisfy a class of algebraic differential equations with functional coefficients satisfying a growth condition connected to infinitely many zeros of ζ on the critical line $\mathbf{L}$.

In a profound theorem Hölder [4] proved that the Euler gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re} z > 0$$

(analytically continued as a meromorphic function in the complex plane $\mathbf{C}$) does not satisfy any nontrivial algebraic differential equation with polynomial coefficients, i.e., if $P(w_0, w_1, \cdots, w_n)$ is a nonzero polynomial in its arguments with coefficients being polynomials in $\mathbf{C}$, then

$$P(\Gamma, \Gamma' \cdots, \Gamma^{(n)})(z) \not\equiv 0$$

where $z \in \mathbf{C}$. In Problem 18 of his famous list of 23 problems, Hilbert raised the question whether the Riemann zeta function ζ and allied functions satisfy any nontrivial algebraic differential equations (see [3]). Hilbert himself stated that ζ does not satisfy any nontrivial algebraic differential equation, and the problem was solved in great generality by Ostrowski [12]. Since then, these results have been extended in many different directions (see [1,2,5,7–9,11,13,16], etc. for related results and references therein), for example by also considering more general functional coefficients in considered equations. In contrast to the functional relation of ζ and Γ by the above mentioned Riemann functional equation, one is naturally interested in knowing whether ζ and Γ are related by any nontrivial algebraic differential equation. Bank and Kaufman [1] proved that Γ cannot satisfy any nontrivial algebraic differential equation with coefficients in a class of meromorphic functions, which however grow more slowly than ζ (see the precise statement in [1] and [6] in terms of the Nevanlinna characteristic function, which will not be used in the present paper); thus this result cannot apply to algebraic differential equations whose coefficients involving ζ. Voronin [16] proved that ζ cannot satisfy a large class of differential equations; but the result does not include the case when coefficients of algebraic differential equations involve Γ. More recently, Markus [11] showed that the composition function $\zeta(\sin(2\pi z))$ does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of Γ and its derivatives, and conjectured that the Riemann zeta function ζ itself does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of Γ and its derivatives, or equivalently (see [11]), Γ does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of ζ and its derivatives. In the present paper we will show that ζ and Γ do not satisfy a class of algebraic differential equations with functional coefficients in a "wild" class $\mathcal{L}$ of complex functions (not necessarily meromorphic or even continuous) that are connected to the zeros of ζ on the critical line, which particularly contains the ring of polynomials. The main theorem is shown to be best possible in a certain sense and has a number of consequences, one of which particularly shows that $P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z)) \not\equiv 0$ in $\mathbf{C}$ for any nontrivial distinguished polynomial P whose coefficients can be allowed to be any polynomials of ζ over $\mathbf{C}$, over the ring of polynomials or, more generally, over the class $\mathcal{L}$ (see Corollary 3 below).

As mentioned above, the Riemann zeta function ζ has infinitely many zeros on the critical line $\mathbf{L}$. In fact, by the Riemann functional equation, ζ has infinitely many zeros on the half critical lines $\{z \in \mathbf{C} : \mathrm{Re} z = 1/2, \mathrm{Im} z > 0\}$ and $\{z \in \mathbf{C} : \mathrm{Re} z = 1/2, \mathrm{Im} z < 0\}$.

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Definition 1. Let $\mathcal{L}_\delta$ be the set of the zero function and all nonzero functions f from $\mathbf{C}$ to $\mathbf{C}\cup\infty$ satisfying the following property: there exist infinitely many zeros $z_n = 1/2 + iy_n$ of ζ on the critical line $\mathbf{L}$ such that $\{|f(z_n)|\}$ has a positive lower bound and

$$|f(z_n)|e^{-\delta|y_n|} = o(1) \tag{1}$$

as $n \to \infty$, where δ is a positive number.

The functions in the set $\mathcal{L}_\delta$ will serve as the coefficients in our equations below, which can be any complex functions satisfying the above growth condition, and are not even required to be continuous or meromorphic functions. We defined $\mathcal{L}_{\frac{\pi}{2}}$ in [9], where $P(z, \Gamma(z), \zeta(z)) \not\equiv 0$ (without derivatives involved) was considered; however $\mathcal{L}_{\frac{\pi}{2}}$ does not suffice our purpose in this paper anymore.

Example 1. Some important classes of functions belong to $\mathcal{L}_\delta$.

(i) It is easy to see that for any $\delta > 0$, $\mathcal{L}_\delta$ contains the ring of all polynomials in $\mathbf{C}$.
(ii) $\mathcal{L}_\delta$ may contain entire or meromorphic functions of finite or infinite order. For example, the function e^z, which is of order 1, belongs to $\mathcal{L}_\delta$ for any $\delta > 0$, since $|e^z| = e^{\frac{1}{2}}$ on the critical line. The entire function e^{e^z}, which is of infinite order, belongs to $\mathcal{L}_\delta$ for any $\delta > 0$, since $|e^{e^z}| = e^{e^{\frac{1}{2}}\cos(\mathrm{Im}\ z)}$ and thus $e^{-e^{\frac{1}{2}}} \le |e^{e^z}| \le e^{e^{\frac{1}{2}}}$ on the critical line.
(iii) The functions in $\mathcal{L}_\delta$ are not even required to be continuous or meromorphic, as mentioned above. For example, given any complex function f with $f(0) \neq 0, \infty$ (even not continuous), the composite $f(\zeta(z))$, if defined on $\mathbf{C}$, belongs to $\mathcal{L}_\delta$ for any $\delta > 0$, since for any zeros $z_n = 1/2 + iy_n$ of ζ with $y_n \to \infty$, $f(\zeta(z_n)) = f(0)$, which trivially satisfies (1) for any $\delta > 0$.

Definition 2. Let $I = (i_0, i_1, \cdots, i_n)$ be a multi-index with $|I| = i_0 + i_1 + \cdots + i_n$. A polynomial in the variables $u_0, u_1, \cdots, u_n$ with functional coefficients in a set $\mathcal{S}$ can be always written as

$$P(z, u_0, u_1, \cdots, u_n) = \sum_{I\in\Lambda} a_I(z) u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n},$$

where the coefficients a_I are functions in $\mathcal{S}$ and Λ is an index set. We call P a *distinguished polynomial* in $u_0, u_1, \cdots, u_n$ with coefficients in $\mathcal{S}$, or simply a *$\mathcal{S}$-distinguished polynomial*, if the index set Λ satisfies that $|I_i| \neq |I_j|$ for distinct indices I_i, I_j in Λ.

Note that any polynomial $P(z, u)$ in one argument u with function coefficients is a distinguished polynomial. Also, any polynomial $P(z, u, v)$ in two arguments u, v with function coefficients can be written as

$$P(z, u, v) = \sum_{k=0}^{m} P_k(z, u) v^k \tag{2}$$

where each $P_k(z, u)$ is a distinguished polynomial in one argument u.

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We will prove the following result:

Theorem 1. *Let*

$$P(z, u_0, u_1, \cdots, u_n, v) = \sum_{k=0}^{m} P_k(z, u_0, u_1, \cdots, u_n) v^k, \tag{3}$$

where P_k, not all identically zero, are $\mathcal{L}_\delta$-distinguished polynomials with $\delta < \frac{\pi}{2}$. Then

$$P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z), \zeta(z)) \not\equiv 0$$

in **C**.

Taking $m = 0$ in Theorem 1, we obtain the following result for the Euler Γ-function:

Corollary 1. $P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z)) \not\equiv 0$ *in* **C** *for any nontrivial $\mathcal{L}_\delta$-distinguished polynomial $P(z, u_0, u_1, \cdots, u_n)$ with $\delta < \frac{\pi}{2}$.*

Corollary 1 can be though as a new property of Γ in the direction of the Hölder theorem. As noted above, the coefficients could be continuous or discontinuous functions, meromorphic or non-meromorphic functions, and of finite or infinite order if meromorphic. Such coefficients do not seem to have been investigated before, although related problems for equations with continuous functions were considered (see e.g. [16]). In the extension of the Hölder theorem of [1] mentioned above, the coefficients are contained in the set of meromorphic functions of order ≤ 1.

Remark 1. Theorem 1 and Corollary 1 turn out to be best possible in the sense that $\delta < \frac{\pi}{2}$ is sharp and the condition "distinguished" on the polynomials P_k cannot be dropped, as seen from the following examples.

(i) First, consider the polynomial $P(z, u_0, u_1) = \frac{1}{\Gamma'(z)} u_1 - 1$, which is a polynomial in u_1 whose coefficient is $\frac{1}{\Gamma'}$. Then $P(z, \Gamma(z), \Gamma'(z)) = \frac{1}{\Gamma'(z)} \Gamma'(z) - 1 \equiv 0$. One can check that the coefficient $\frac{1}{\Gamma'}$ belongs to $\mathcal{L}_\delta$ for any $\delta \geq \frac{\pi}{2}$. In fact, let $z_n = 1/2 + iy_n$ be any zeros of ζ with $y_n \to \infty$. Note that $\Gamma(z_n) = (1 + o(1))\sqrt{2\pi} e^{-\pi |y_n|/2}$ and $\frac{\Gamma'}{\Gamma}(z_n) = (1 + o(1) \log z_n$ (cf. the proof of Theorem 1 below). Thus, we see that for any $\delta \geq \pi/2$, $\frac{1}{\Gamma'(z_n)} \to \infty$ and $\frac{1}{\Gamma'(z_n)} e^{-\delta |y_n|} \to 0$, which shows that $\frac{1}{\Gamma'(z)}$ belongs to $\mathcal{L}_\delta$ for any $\delta \geq \frac{\pi}{2}$. Thus, the condition $\delta < \frac{\pi}{2}$ in Corollary 1 and thus Theorem 1 is sharp and cannot be improved.

(ii) Next, consider the equation $P(z, u_0, u_1) = \frac{\Gamma'}{\Gamma} u_0 - u_1 = 0$, which is not a distinguished polynomial. It is clear that $P(z, \Gamma, \Gamma') \equiv 0$. Note that $\frac{\Gamma'}{\Gamma}(z_n) = (1 + o(1) \log z_n$ for any zeros $z_n = 1/2 + iy_n$ of ζ with $y_n \to \infty$ (cf. (i) above). Thus, the coefficient $\frac{\Gamma'}{\Gamma}$ is in $\mathcal{L}_\delta$ for any $\delta > 0$. Hence, the condition "distinguished" in Corollary 1 and thus Theorem 1 cannot be dropped.

We give several more consequences. In Corollary 1, taking the coefficients to be composite functions $f(\zeta(z))$ with $f(0) \neq 0, \infty$, in view of Example 1(iii) we have the following

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Corollary 2. *$P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z)) \not\equiv 0$ in $\mathbf{C}$ for any nontrivial distinguished polynomial $P(z, u_0, u_1, \cdots, u_n)$ whose coefficients are composite functions $f(\zeta(z))$ defined on $\mathbf{C}$ with f being a complex function and $f(0) \neq 0, \infty$.*

If $P(z, u_0, u_1, \cdots, u_n)$ is a nontrivial distinguished polynomial in its arguments $u_0, u_1, \cdots, u_n$ whose coefficients are polynomials in ζ, then it is easy to check that it can be written as a polynomial in the form (3) of Theorem 1, i.e., $P = \sum_{k=0}^{m} P_k(z, u_0, u_1, \cdots, u_n)\zeta^k$, where P_k are some distinguished polynomials in $u_0, u_1, \cdots, u_n$. Thus, Theorem 1 also implies the following

Corollary 3. *$P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z)) \not\equiv 0$ in $\mathbf{C}$ for any nontrivial distinguished polynomial $P(z, u_0, u_1, \cdots, u_n)$ whose coefficients are polynomials of ζ over $\mathbf{C}$, over the ring of polynomials or, more generally, over $\mathcal{L}_\delta$ with $\delta < \frac{\pi}{2}$.*

If $P(z, u, v)$ is a nontrivial polynomial in u, v with functional coefficients, P can then be written into the form (2) with $P_k(z, u)$ being distinguished polynomial in one argument u. Thus, Theorem 1 immediately yields the algebraical independence ζ and $\Gamma^{(n)}$, $n \geq 0$ ($\Gamma^{(0)} = \Gamma$) over $\mathcal{L}_\delta$ ($\delta < \frac{\pi}{2}$) and, particularly, over the ring of polynomials:

Corollary 4. *ζ and $\Gamma^{(n)}$ ($n \geq 0$) are algebraically independent over $\mathcal{L}_\delta$ with $\delta < \frac{\pi}{2}$ and thus over the ring of polynomials, i.e., $P(z, \Gamma^{(n)}(z), \zeta(z)) \not\equiv 0$ in $\mathbf{C}$ for any nontrivial polynomial $P(z, u, v)$ whose coefficients are functions in $\mathcal{L}_\delta$ with $\delta < \frac{\pi}{2}$.*

We refer to [10] and [9] for the algebraic independence of ζ and Γ over the ring of polynomials in $\mathbf{C}$. For this particular case, the coefficients may be improved to $\mathcal{L}_{\frac{\pi}{2}}$ (cf. [9]).

Proof of Theorem 1. We may write $P(z, u_0, u_1, \cdots, u_n, v)$ to the following form:

$$\begin{aligned}
&P(z, u_0, u_1, \cdots, u_n, v)\\
&= v^m \sum_{I\in\Lambda_m} a_{m,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} + v^{m-1} \sum_{I\in\Lambda_{m-1}} a_{m-1,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} + \cdots\\
&+ v \sum_{I\in\Lambda_1} a_{1,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} + \sum_{I\in\lambda_0} a_{0,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n},
\end{aligned}$$

where m is the highest power of v in the polynomial P and Λ_j's are index sets, and the coefficients $a_{i,I}$ are either identically zero in $\mathbf{C}$ or nonzero functions in $\mathcal{L}_\delta$. By the assumption of the theorem, the coefficients $a_{i,I}$ are not all identically zero in $\mathbf{C}$.

Suppose that $\Gamma, \Gamma', \cdots, \Gamma^{(n)}, \zeta$ satisfy $P(z, u_0, u_1, \cdots, u_n, v) = 0$ in $\mathbf{C}$, i.e.,

$$\begin{aligned}
&P(z, \Gamma, \Gamma', \cdots, \Gamma^{(n)}, \zeta)\\
&= \zeta^m \sum_{I\in\Lambda_m} a_{m,I} \Gamma^{i_0} (\Gamma')^{i_1} \cdots (\Gamma^{(n)})^{i_n} + \zeta^{m-1} \sum_{I\in\Lambda_{m-1}} a_{m-1,I} \Gamma^{i_0} (\Gamma')^{i_1} \cdots (\Gamma^{(n)})^{i_n}\\
&+ \cdots + \zeta \sum_{I\in\Lambda_1} a_{1,I} \Gamma^{i_0} (\Gamma')^{i_1} \cdots (\Gamma^{(n)})^{i_n} + \sum_{I\in\Lambda_0} a_{0,I} \Gamma^{i_0} (\Gamma')^{i_1} \cdots (\Gamma^{(n)})^{i_n}\\
&\equiv 0 \qquad (4)
\end{aligned}$$

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We will prove that all the coefficients $a_{i,I}$ in (4) are identically zero in $\mathbf{C}$ for all possible i, I. This, of course, contradicts to the assumption of the theorem.

Let us first show that all the coefficients $a_{0,I}$ in the last sum of (4) are identically zero. Suppose that the index set Λ_0 contains t indices $I_1, I_2, \cdots, I_t$. List all these indices according to the increasing order, i.e., $|I_1| < |I_2| < \cdots < |I_t|$. We can do this by the assumption that the polynomials P_k in the theorem are distinguished. We thus can write the last sum in (4) into

$$\sum_{j=1}^{t} a_{0,I_j} \Gamma^{i_0} (\Gamma')^{i_1} \cdots (\Gamma^{(n)})^{i_n}. \tag{5}$$

We next prove that the first coefficient a_{0,I_1} in (5) is identically zero in $\mathbf{C}$. Suppose that $a_{0,I_1} \not\equiv 0$ in $\mathbf{C}$. We will derive a contradiction. By the definition of $\mathcal{L}_\delta$, there exist infinitely many zeros $z_l = 1/2 + iy_l$ $(l = 1, 2, 3, \cdots)$ of ζ on the critical line $\mathbf{L}$ such that $\{|a_{0,I_1}(z_l)|\}$ has a positive lower bound and

$$|a_{0,I_1}(z_l)| e^{-\delta|y_l|} = o(1). \tag{6}$$

By taking a subsequence if necessary, we may assume that $|y_l| \to \infty$ as $l \to \infty$.

Note by (4) that for all l,

$$P(z_l, \Gamma(z_l), \Gamma'(z_l), \cdots, \Gamma^{(l)}(z_l), \zeta(z_l)) = 0,$$

and $\zeta(z_l) = 0$. We obtain, from (4) and (5), that

$$\Big(\sum_{j=1}^{t} a_{0,I_j} \Gamma^{i_0} (\Gamma')^{i_1} \cdots (\Gamma^{(n)})^{i_n}\Big)(z_l) = 0.$$

Dividing the both sides of the above equality by $\Gamma^{|I_1|}$, we have that

$$\Big(\sum_{j=1}^{t} a_{0,I_j} \Big(\frac{\Gamma'}{\Gamma}\Big)^{i_1} \cdots \Big(\frac{\Gamma^{(n)}}{\Gamma}\Big)^{i_n} \Gamma^{|I_j|-|I_1|}\Big)(z_l) = 0. \tag{7}$$

We will make use of the following known results (see e.g. p. 151 in [14]):

$$\Gamma(1/2 + iy) = (1 + o(1))\sqrt{2\pi} e^{-\pi|y|/2}$$

and

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + \int_0^\infty \frac{[u] - u + \frac{1}{2}}{u + z} du$$

with the logarithm having its principle value. The last equality implies, by a Lebesgue's convergence theorem and an elementary argument, that

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$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - \int_0^\infty \frac{[u] - u + \frac{1}{2}}{(u+z)^2} du \tag{8}$$

and then that

$$\frac{\Gamma'(z)}{\Gamma(z)} = (1 + o(1)) \log z \tag{9}$$

uniformly for all $z \in \mathbf{C} \setminus \{z : |\arg z - \pi| \le \epsilon\}$ and for any small $\epsilon > 0$. We assert that

$$\frac{\Gamma''(z)}{\Gamma(z)} = (1 + o(1))(\log z)^2. \tag{10}$$

To see this, note that

$$\frac{\Gamma''(z)}{\Gamma(z)} = (\frac{\Gamma'(z)}{\Gamma(z)})' + (\frac{\Gamma'(z)}{\Gamma(z)})^2,$$

and that from (8),

$$(\frac{\Gamma'(z)}{\Gamma(z)})' = \frac{1}{z} + \frac{1}{2z^2} + 2\int_0^\infty \frac{[u] - u + \frac{1}{2}}{(u+z)^3} dz.$$

This together with (9) implies (10). Continuing this argument, we obtain that for any positive integer q,

$$\Gamma^{(q)}(z) = (1 + o(1))(\log z)^q \Gamma(z)$$

uniformly for all $z \in \mathbf{C} \setminus \{z : |\arg z - \pi| \le \epsilon\}$.

For each $j \ge 2$ in (7), we then deduce that

$$\begin{aligned}
&|\big(a_{0,I_j}(\frac{\Gamma'}{\Gamma})^{i_1} \cdots (\frac{\Gamma^{(n)}}{\Gamma})^{i_n} \Gamma^{|I_j|-|I_1|}\big)(z_l)| \\
&\le |a_{0,I_j}(z_l)||1 + o(1)||\log z_l|^{i_1+2i_2+\cdots+ni_n} (\sqrt{2\pi})^{|I_j|-|I_1|} e^{-(|I_j|-|I_1|)\pi|y_l|/2} \\
&\le |a_{0,I_j}(z_l)||1 + o(1)||\log z_l|^{i_1+2i_2+\cdots+ni_n} (\sqrt{2\pi})^{|I_j|-|I_1|)} e^{-\pi|y_l|/2} \\
&= |a_{0,I_j}(z_l)|e^{-\delta|y_n|}|1 + o(1)||\log z_l)|^{i_1+2i_2+\cdots+ni_n} (\sqrt{2\pi})^{|I_j|-|I_1|} e^{(\delta-\frac{\pi}{2})|y_l|} \\
&\to 0
\end{aligned}$$

as $l \to \infty$, in view of (6) and the assumption that $\delta < \pi/2$. Taking $l \to \infty$ in (7) yields that

$$a_{0,I_1}(z_l)(1 + o(1))(\log z_l)^{|I_1|} \to 0,$$

which implies that $a_{0,I_1}(z_l) \to 0$. But, by the definition of $\mathcal{L}_\delta$, $\{a_{0,I_1}(z_l)\}$ has a positive lower bound, a contradiction. This shows that a_{0,I_1} is identically zero.

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Now that a_{0,I_1} is identically zero, the expression (5) becomes

$$\sum_{j=2}^{t} a_{0,I_j}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}.$$

This is of the same form as (5), except that j starts from 2 now. The exact same argument as above shows that a_{0,I_2} is identically zero. Repeat this argument, we obtain that all the coefficients a_{0,I_j} in (5), i.e., all the coefficients in the last sum of (4), are identically zero. Therefore, (4) can be rewritten into

$$\begin{aligned}&P(z,\gamma,\Gamma',\cdots,\Gamma^{(n)},\zeta)\\ &=\zeta^m\sum_{I\in\Lambda_m} a_{m,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}+\zeta^{m-1}\sum_{I\in\Lambda_{m-1}} a_{m-1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}\\ &+\cdots+\zeta\sum_{I\in\Lambda_0} a_{1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}\equiv 0\end{aligned}$$

or

$$\begin{aligned}&P(z,\gamma,\Gamma',\cdots,\Gamma^{(n)},\zeta)\\ &=\zeta^{m-1}\sum_{I\in\Lambda_m} a_{m,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}+\zeta^{m-2}\sum_{I\in\Lambda_{m-1}} a_{m-1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}\\ &+\cdots+\sum_{I\in\Lambda_1} a_{1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}\equiv 0\end{aligned}$$

which is of the same form as (4), except that the highest power of ζ is $m-1$ now. Thus, the above same argument can be applied to deduce that all the coefficients $a_{1,I}$, $I\in\Lambda_1$, in the last sum will be identically zero. This process can be repeated until we obtain that all the coefficients $a_{i,I}$ are identically zero for all possible i, I, a contradiction to the assumption of the theorem. This completes the proof of the theorem. □

To conclude the paper, we note that it would be natural to extend the above study to L-functions. One expects a Riemann hypothesis for L-functions in the Selberg class. However, it remains an open problem whether a general L-function in the Selberg class has infinitely many zeros on the critical line $\mathbf{L}=\{z\in\mathbf{C}:\mathrm{Re}z=1/2\}$. We refer to [13, Chapter 10] for related results on functional independence of L-functions.

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