



Transverse instability of periodic and generalized solitary waves for a fifth-order KP model

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Abstract

We consider a fifth-order Kadomtsev–Petviashvili equation which arises as a two-dimensional model in the classical water-wave problem. This equation possesses a family of generalized line solitary waves which decay exponentially to periodic waves at infinity. We prove that these solitary waves are transversely spectrally unstable and that this instability is induced by the transverse instability of the periodic tails. We rely upon a detailed spectral analysis of some suitably chosen linear operators.

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1. Introduction

We consider a fifth-order Kadomtsev–Petviashvili (KP) equation

$$\partial_t \partial_x u = \partial_x^2 \left(\partial_x^4 u + \partial_x^2 u + \frac{1}{2} u^2 \right) + \partial_y^2 u, \quad (1.1)$$

in which the unknown u depends upon two space variables $(x, y) \in \mathbb{R}^2$ and time $t \in \mathbb{R}$. This equation arises as a two-dimensional model for capillary-gravity water waves in the regime of

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critical surface tension, when the Bond number is close to $1/3$ [15]. While the exact values of the coefficients in (1.1) are unimportant, their signs have been chosen corresponding to the case of Bond number less than $1/3$. We can also regard (1.1) as a two-dimensional version of the Kawahara equation

$$\partial_t u = \partial_x \left(\partial_x^4 u + \partial_x^2 u + \frac{1}{2} u^2 \right), \quad (1.2)$$

just as the KP equation is a two-dimensional version of the well-known Korteweg–de Vries (KdV) equation.

The Kawahara equation (1.2) possesses a family of traveling generalized solitary waves [9]. In contrast to the solitary waves of the KdV equation which tend exponentially to zero as $|x| \rightarrow \infty$, the generalized solitary waves of (1.2) decay exponentially to periodic waves as $|x| \rightarrow \infty$. The amplitude of these periodic waves may be exponentially small but not zero [16]. Our purpose is to study the transverse stability of these generalized solitary waves, i.e., their stability as solutions of the equation (1.1), for perturbations which depend upon both spatial variables x and y . While the stability of solitary waves for the KdV equation has been intensively studied, very little is known about the stability of the generalized solitary waves for (1.2). A key difference is that asymptotically these generalized solitary waves tend to periodic waves, not to a constant, and the stability of these periodic waves is not fully understood. We mention the result in [8] showing that these periodic waves are spectrally stable as solutions of (1.2) provided their amplitude is sufficiently small.

Our main result shows that the generalized solitary waves of (1.2) are transversely spectrally unstable. The starting point of our analysis is a formulation of the transverse instability problem in terms of the spectrum of a suitably chosen operator. In particular, this allows to distinguish between linear instability, due to point spectrum, and essential instability, due to essential spectrum. Next, the key step is the spectral analysis of the operator found for the asymptotic periodic waves. We prove that these periodic waves are transversely linearly unstable with respect to perturbations which are co-periodic (i.e., they have the same period as the periodic wave) in the longitudinal direction, and transversely essentially unstable with respect to perturbations which are localized in the longitudinal direction. Finally, a rather general perturbation argument allows to conclude to the essential instability of the generalized solitary waves.

It is interesting to compare these properties with known stability results for the KP equation. Recall that the KP equation comes in two flavors: KP-I, which is valid for strong surface tension (Bond number greater than $1/3$), and KP-II, which is valid for weak or zero surface tension (Bond number less than $1/3$) [15]. Both equations reduce to the KdV equation in one dimension, but display completely opposite transverse dynamics. While KP-I predicts transverse instability of both periodic and solitary waves, KP-II predicts stability [1,4,6,12–14,17,18,20]. The latter may at first sight seem to contradict our result, but note that the KP-II equation does not capture the small periodic tails of the generalized solitary waves (nor is it uniformly valid in the limit of critical surface tension).

Generalized line solitary waves with small amplitude are known to exist for the full capillary-gravity water-wave problem in the regime of weak surface tension [10,16]. When the surface tension is close to critical, these are to leading order described by the Kawahara equation (1.2). The results in this paper predict that these generalized solitary waves are transversely unstable and that this instability is due to that of the asymptotic periodic waves. This prediction will make the object of future work. We point out that the instability predictions based on the KP-I equation

for the regime of strong surface tension have been confirmed for the full water-wave problem for both periodic and solitary waves [2,7,19]. Also notice that in the regime of weak surface tension, transverse instability of solitary waves have been recently proved for a class of true solitary waves which appear in the modulational regime, which is different from the long-wave setting studied here [3].

Finally, the question of nonlinear transverse instability for these generalized solitary waves remains open. This question has recently been solved in the positive for line solitary waves of the KP-I equation [20], as well as the full water wave problem with strong surface tension [21]. Although it would seem natural to expect a similar result for the fifth-order KP equation, the proof is far from straightforward for generalized solitary waves, since the instability is due to essential spectrum (even considering solutions which are periodic in the transverse direction).

The paper is organized as follows. In Section 2 we recall the existence results for both periodic and generalized solitary waves of the Kawahara equation (1.2). The main results are presented in Section 3, and the proofs are given in Section 4.

2. Existence of line traveling waves

In this section we recall the existence results for both periodic and generalized solitary traveling waves of the equation (1.2).

We consider traveling waves moving with constant speed c . In a comoving frame, after replacing $x + ct$ by x , these traveling waves are stationary solutions of the equation

$$\partial_t u = \partial_x \left(\partial_x^4 u + \partial_x^2 u - cu + \frac{1}{2}u^2 \right), \quad (2.1)$$

and therefore satisfy the ODE

$$\partial_x^4 u + \partial_x^2 u - cu + \frac{1}{2}u^2 = C,$$

obtained after integrating the right hand side of (2.1) once with respect to x . As a remnant of the Galilean invariance of (1.2), we may set $C = 0$ and restrict to the solutions of the ODE

$$\partial_x^4 u + \partial_x^2 u - cu + \frac{1}{2}u^2 = 0. \quad (2.2)$$

2.1. Small periodic waves

The existence of small periodic solutions of (2.2) for small speeds c has been proved in [8] (see also [16, Chapters 4 and 7]). We recall in the next proposition the result from [8] which also gives a number of properties of these periodic solutions which are essential in our analysis.

Proposition 2.1. ([8, Theorem 1]) *There exist positive constants c_0 and a_0 such that, for any $c \in (-c_0, c_0)$, the equation (2.2) possesses a one-parameter family of even, periodic solutions $(\varphi_{a,c})_{a \in (-a_0, a_0)}$ of the form*

$$\varphi_{a,c}(x) = p_{a,c}(k_{a,c}x), \quad \forall x \in \mathbb{R},$$

with the following properties.

(i) The real-valued map $(a, c) \mapsto k_{a,c}$ is analytic on $(-a_0, a_0) \times (-c_0, c_0)$ and

$$k_{a,c} = k_0(c) + c\tilde{k}(a, c), \quad k_0(c) = \left(\frac{1 + \sqrt{1 + 4c}}{2} \right)^{1/2}, \quad \tilde{k}(a, c) = \sum_{n \geq 1} \tilde{k}_{2n}(c) a^{2n},$$

for any $(a, c) \in (-a_0, a_0) \times (-c_0, c_0)$, where $|\tilde{k}_{2n}(c)| \leq K_0/\rho_0^{2n}$, for any $n \geq 1$ and some positive constants K_0 and ρ_0 .

(ii) The map $(a, c) \mapsto p_{a,c}$ is analytic on $(-a_0, a_0) \times (-c_0, c_0)$ with values $H_{\text{per}}^6(0, 2\pi)$ and

$$p_{a,c}(z) = ac \cos(z) + c \sum_{\substack{n,m \geq 0, n+m \geq 2 \\ n-m \neq \pm 1}} \tilde{p}_{n,m}(c) e^{i(n-m)z} a^{n+m},$$

in which $\tilde{p}_{n,m}(c)$ are real numbers such that $\tilde{p}_{n,m}(c) = \tilde{p}_{m,n}(c)$ and $|\tilde{p}_{n,m}(c)| \leq C_0/\rho_0^{n+m}$, for any $c \in (-c_0, c_0)$ and some positive constant C_0 . (Here $H_{\text{per}}^6(0, 2\pi)$ is the space of 2π -periodic functions defined in Section 3.2 below.)

(iii) The Fourier coefficients $\hat{p}_q(a, c)$ of the 2π -periodic function $p_{a,c}$,

$$p_{a,c}(z) = \sum_{q \in \mathbb{Z}} \hat{p}_q(a, c) e^{iqz}, \quad \forall z \in \mathbb{R},$$

are real and satisfy $\hat{p}_0(a, c) = O(ca^2)$ and $\hat{p}_q(a, c) = O(c|a|^{|q|})$, for all $q \neq 0$, as $a \rightarrow 0$. Moreover, the map $a \mapsto \hat{p}_q(a, c)$ is even (resp. odd) for even (resp. odd) values of q , and in particular $p_{-a,c}(z) = p_{a,c}(z + \pi)$.

We collect below some properties of $k_{a,c}$ and $p_{a,c}$ which are needed in our proofs. First, a direct calculation allows to compute the expansions of $k_{a,c}$ and $p_{a,c}$, as $a \rightarrow 0$. Without writing explicitly the dependence upon c , for notational simplicity, we find

$$k_{a,c} = k_0 + \sum_{n \geq 1} k_{2n} a^{2n}, \quad k_0^2 = \frac{1 + \sqrt{1 + 4c}}{2}, \quad k_2(4k_0^3 - 2k_0) = -\frac{c}{4} + \frac{c^2}{8X_2}, \quad (2.3)$$

in which we used the notation

$$X_n = k_0^4 n^4 - k_0^2 n^2 - c, \quad \forall n \geq 2. \quad (2.4)$$

For $p_{a,c}$, we write

$$p_{a,c}(z) = ac \left(\cos(z) + \sum_{n \geq 1} p_n(z) a^n \right), \quad (2.5)$$

in which we find

$$p_1(z) = \frac{1}{4} - \frac{c}{4X_2} \cos(2z), \quad p_2(z) = \frac{c^2}{8X_2X_3} \cos(3z).$$

We point out that the Fourier coefficients ± 1 of the functions $p_n(z)$ are zero by construction.

Next, notice that $k_{a,0} = 1$, $p_{a,0} = 0$, and we claim that

$$\partial_c k_{a,c}^2|_{c=0} = 1 - q(a), \quad \partial_c p_{a,c}|_{c=0} = a \cos(z) + q(a), \quad q(a) = 1 - \sqrt{1 - \frac{1}{2}a^2}. \quad (2.6)$$

Indeed, recall that

$$k_{a,c}^4 \partial_z^4 p_{a,c} + k_{a,c}^2 \partial_z^2 p_{a,c} - c p_{a,c} + \frac{1}{2} p_{a,c}^2 = 0. \quad (2.7)$$

Differentiating this equality with respect to c and taking $c = 0$ we find

$$\left(\partial_z^4 + \partial_z^2 \right) \left(\partial_c p_{a,c}|_{c=0} \right) = 0,$$

so that $\partial_c p_{a,c}|_{c=0}$ belongs to the kernel of $\partial_z^4 + \partial_z^2$. Since $\partial_c p_{a,c}|_{c=0}$ is an even function, this implies that $\partial_c p_{a,c}|_{c=0}$ is a linear combination of $\cos(z)$ and 1, and taking into account the expansion in [Proposition 2.1](#) (ii), we obtain the second equality in [\(2.6\)](#). Next, we differentiate [\(2.7\)](#) twice with respect to c and take $c = 0$. This gives

$$\left(\partial_z^4 + \partial_z^2 \right) \left(\partial_c^2 p_{a,c}|_{c=0} \right) + 2a \left(\partial_c k_{a,c}^2|_{c=0} - 1 + q(a) \right) \cos(z) + q(a)^2 - 2q(a) + a^2 \cos^2(z) = 0,$$

and the solvability conditions for this equation imply the first and the third equalities in [\(2.6\)](#).

2.2. Generalized solitary waves

The existence of generalized solitary waves is a consequence of the result in [\[16, Chapter 7, Theorem 7.1.18\]](#) for general four-dimensional reversible ODEs in presence of a $0^2(i\omega)$ resonance. For completeness, we give the proof of the following proposition in [Appendix A](#).

Proposition 2.2. *There exist positive constants a_1 and M_1 such that for any $0 < \ell < \pi$ and $0 < \lambda < 1$, there exist $c_2(\ell) > 0$ and $a_2(\ell) > 0$ such that for all $c \in (0, c_2(\ell)]$ and $|a| \in [a_2(\ell)ce^{-\ell/\sqrt{c}}, a_1]$, the equation [\(2.2\)](#) possesses an even solution*

$$u_{a,c}(x) = h_{a,c}(x) + \varphi_{a,c}(x + \tau_{a,c} \tanh(\sqrt{c}x/2)), \quad (2.8)$$

with the following properties:

- (i) $|\partial_x^j h_{a,c}(x)| \leq M_1 c e^{-\lambda \sqrt{c}|x|}$ for $j = 0, 1, 2, 3$ and all $x \in \mathbb{R}$;
- (ii) $\varphi_{a,c}$ is the periodic solution in [Proposition 2.1](#);
- (iii) the asymptotic phase shift is such that $\tau_{a,c} = O(1)$, as $(a, c) \rightarrow (0, 0)$.

3. Transverse instability: main results

In this section, we state the main instability results for both periodic and generalized solitary waves. We give the proofs of these results in Section 4.

3.1. Formulation of the transverse instability problem

Assume that u_* is a one-dimensional solution of (2.1), for instance a periodic wave (as in Proposition 2.1) or a generalized solitary waves (as in Proposition 2.2). Consider the linearized equation

$$\partial_t \partial_x u = \partial_x^2 \left(\partial_x^4 u + \partial_x^2 u - cu + u_* u \right) + \partial_y^2 u, \quad (3.1)$$

and set

$$\mathcal{A}_* = \partial_x^2 \left(\partial_x^4 + \partial_x^2 - c + u_* \right).$$

Roughly speaking, the wave u_* is called transversely unstable if the equation (3.1) possesses solutions of the form

$$u(t, x, y) = e^{\lambda t} v(x, y),$$

for some $\operatorname{Re} \lambda > 0$ and v a time-independent function which belongs to the set of the allowed perturbations. Since u_* does not depend upon the transverse spatial variable y , the operator $\mathcal{A}_* + \partial_y^2$ in the right hand side of (3.1) has y -independent coefficients, so that using Fourier transform in y we can reformulate the instability statement and say the u_* is transversely unstable if the linearized equation

$$\partial_t \partial_x u = \mathcal{A}_* u - \omega^2 u,$$

has solutions of the form

$$u(t, x) = e^{\lambda t} v(x),$$

for some $\operatorname{Re} \lambda > 0$, $\omega \in \mathbb{R}$, and v in some space H of functions depending upon the longitudinal spatial variable x , only. In this setting, perturbations are bounded in the transverse variable y and determined by the choice of H in the longitudinal variable x (e.g., localized if $H = L^2(\mathbb{R})$ or periodic if $H = L^2(0, L)$).

We can now reformulate the transverse instability problem and say that u_* is *transversely spectrally unstable* if the linear operator $\lambda \partial_x - \mathcal{A}_* + \omega^2$, is not invertible in H for some $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ and $\omega \in \mathbb{R}$. The particular form of this operator allows to further say that u_* is transversely spectrally unstable if the spectrum of the linear operator $\lambda \partial_x - \mathcal{A}_*$ contains a negative value $-\omega^2$ for some $\operatorname{Re} \lambda > 0$. Notice that if $-\omega^2$ is an isolated eigenvalue of $\lambda \partial_x - \mathcal{A}_*$ then this definition implies transverse linear instability. If $-\omega^2$ belongs to the essential spectrum of $\lambda \partial_x - \mathcal{A}_*$,

$$\sigma_{\text{ess}}(\lambda \partial_x - \mathcal{A}_*) = \{ \nu \in \mathbb{C} ; \lambda \partial_x - \mathcal{A}_* - \nu \text{ is not Fredholm with index } 0 \}, \quad (3.2)$$

we may say that u_* is transversely essentially unstable. Our main results show that the periodic waves $\varphi_{a,c}$ are transversely unstable with respect to co-periodic longitudinal perturbations ([Theorem 1](#)), and that both the periodic waves $\varphi_{a,c}$ and the generalized solitary waves $u_{a,c}$ are transversely essentially unstable with respect to localized longitudinal perturbations ([Theorem 2](#) and [Theorem 3](#), respectively).

3.2. Periodic waves

Consider the small periodic waves $\varphi_{a,c}$ constructed in [Proposition 2.1](#), and the linear operator

$$\lambda \partial_x - \mathcal{A}_{a,c}, \quad \mathcal{A}_{a,c} = \partial_x^2 \left(\partial_x^4 + \partial_x^2 - c + \varphi_{a,c} \right).$$

Since the period $2\pi/k_{a,c}$ of $\varphi_{a,c}$ depends upon a and c , it is convenient to rescale x and λ by taking

$$z = k_{a,c}x, \quad \lambda = k_{a,c}\Lambda,$$

and work with the rescaled operator

$$\Lambda \partial_z - \mathcal{B}_{a,c}, \quad \mathcal{B}_{a,c} = \partial_z^2 (k_{a,c}^4 \partial_z^4 + k_{a,c}^2 \partial_z^2 - c + p_{a,c}),$$

which has 2π -periodic coefficients.

First, for co-periodic perturbations we take $H = L^2(0, 2\pi)$ and as domain of definition for $\mathcal{B}_{a,c}$ the subspace $H_{\text{per}}^6(0, 2\pi)$ consisting of 2π -periodic functions,

$$H_{\text{per}}^j(0, 2\pi) = \{f \in H_{\text{loc}}^j(\mathbb{R}) ; f(z + 2\pi) = f(z), \forall z \in \mathbb{R}\},$$

for $j \geq 1$. Then $\mathcal{B}_{a,c}$ is closed in H , and our main result is the following theorem which is proved in [Section 4.1](#).

Theorem 1. *There exist positive constants c_3 and a_3 such that for any $c \in (-c_3, c_3)$ and $a \in (-a_3, a_3)$, there exists $\Lambda_{a,c} > 0$ such that for any $\Lambda \in (0, \Lambda_{a,c})$ the linear operator $\Lambda \partial_z - \mathcal{B}_{a,c}$ acting in $L^2(0, 2\pi)$ with domain $H_{\text{per}}^6(0, 2\pi)$ has a simple negative eigenvalue. Consequently, the periodic wave $\varphi_{a,c}$ is transversely linearly unstable with respect to co-periodic longitudinal perturbations.*

Next, for localized perturbations we take $H = L^2(\mathbb{R})$ and as domain of definition of $\mathcal{B}_{a,c}$, and also $\Lambda \partial_z - \mathcal{B}_{a,c}$, the subspace $H^6(\mathbb{R})$. The spectral analysis in this space is based on a Bloch-wave (or Floquet in this case) decomposition, which shows that the spectrum of $\Lambda \partial_z - \mathcal{B}_{a,c}$ in $L^2(\mathbb{R})$ is the union of the spectra of the operators

$$\Lambda(\partial_z + i\gamma) - \mathcal{B}_{a,c,\gamma}, \quad \mathcal{B}_{a,c,\gamma} = (\partial_z + i\gamma)^2 \left(k_{a,c}^4 (\partial_z + i\gamma)^4 + k_{a,c}^2 (\partial_z + i\gamma)^2 - c + p_{a,c} \right),$$

acting in $L^2(0, 2\pi)$ with domain $H_{\text{per}}^6(0, 2\pi)$, for $\gamma \in (-1/2, 1/2]$ (e.g., see [\[5,11\]](#)). Then the transverse spectral instability of $\varphi_{a,c}$ with respect to localized perturbations is an immediate consequence of [Theorem 1](#). Moreover, from [\[5,11\]](#) we deduce that the spectrum is purely essential

spectrum, in the sense of definition (3.2), so that the instability is essential. Summarizing, we have the result below.

Theorem 2. *For any a , c , and Λ as in Theorem 1, the linear operator $\Lambda \partial_z - \mathcal{B}_{a,c}$ acting in $L^2(\mathbb{R})$ with domain $H^6(\mathbb{R})$ has negative essential spectrum. Consequently, the periodic wave $\varphi_{a,c}$ is transversely essentially unstable with respect to localized longitudinal perturbations.*

3.3. Generalized solitary waves

Consider the generalized solitary waves found in Proposition 2.2 and the linear operator

$$\lambda \partial_x - \mathcal{C}_{a,c}, \quad \mathcal{C}_{a,c} = \partial_x^2 \left(\partial_x^4 + \partial_x^2 - c + u_{a,c} \right),$$

acting in $L^2(\mathbb{R})$ with domain $H^6(\mathbb{R})$ (localized perturbations). The key observation in the spectral analysis of $\lambda \partial_x - \mathcal{C}_{a,c}$ is that it is a relatively compact perturbation of the asymptotic operator

$$\lambda \partial_x - \mathcal{C}_{a,c}^\infty = \begin{cases} \lambda \partial_x - \mathcal{A}_{a,c}^+, & \text{for } x > 0 \\ \lambda \partial_x - \mathcal{A}_{a,c}^-, & \text{for } x < 0 \end{cases}, \quad \mathcal{A}_{a,c}^\pm = \partial_x^2 \left(\partial_x^4 + \partial_x^2 - c + \varphi_{a,c}(\cdot \pm \tau_{a,c}) \right).$$

Since the essential spectrum is stable under relatively compact perturbations, the generalized solitary wave $\varphi_{a,c}$ is transversely unstable provided the asymptotic operator has negative essential spectrum. The latter property is a consequence of Theorem 1, just as Theorem 2. This leads to the following result, which is proved in Section 4.2

Theorem 3. *For any a , c , and Λ as in Proposition 2.2 and Theorem 1, the linear operator $k_{a,c} \Lambda \partial_x - \mathcal{C}_{a,c}$ acting in $L^2(\mathbb{R})$ with domain $H^6(\mathbb{R})$ has negative essential spectrum. Consequently, the generalized solitary wave $u_{a,c}$ is transversely essentially unstable.*

4. Proofs

4.1. Proof of Theorem 1

We claim that it is enough to prove that the operator $\mathcal{B}_{a,c}$ has a simple positive eigenvalue, or equivalently, that the operator $-\mathcal{B}_{a,c}$ has a simple negative eigenvalue, for sufficiently small a and c . Indeed, notice that the operator $\Lambda \partial_z - \mathcal{B}_{a,c}$ is a small relatively bounded perturbation of $-\mathcal{B}_{a,c}$, for any sufficiently small $\Lambda \in \mathbb{C}$. If $-\mathcal{B}_{a,c}$ has a simple negative eigenvalue, then a standard perturbation argument implies that $\Lambda \partial_z - \mathcal{B}_{a,c}$ has a simple eigenvalue in some open disk centered on the real axis and contained in the open left half complex plane. For real values Λ , the operator $\Lambda \partial_z - \mathcal{B}_{a,c}$ is real, so its spectrum is symmetric with respect to the real axis. Consequently, the simple eigenvalue above is necessarily real and negative, which proves the claim.

For small a and c , the operator $\mathcal{B}_{a,c}$ is a small relatively bounded perturbation of the operator

$$\mathcal{B}_{0,0} = \partial_z^2 \left(\partial_z^4 + \partial_z^2 \right),$$

which has constant coefficients. We can compute the spectrum of $\mathcal{B}_{0,0}$ using Fourier series, and find

$$\sigma(\mathcal{B}_{0,0}) = \{-n^2(n^4 - n^2), n \in \mathbb{Z}\},$$

where 0 is a semi-simple triple eigenvalue and all other eigenvalues are negative. Then a standard perturbation argument shows that there exists a neighborhood V of 0 in the complex plane and a positive constant m such that $V \subset \{v \in \mathbb{C}; |\operatorname{Re} v| < m/2\}$ and for sufficiently small a and c , the spectrum of $\mathcal{B}_{a,c}$ decomposes as

$$\sigma(\mathcal{B}_{a,c}) = \sigma_1(\mathcal{B}_{a,c}) \cup \sigma_2(\mathcal{B}_{a,c}), \quad \sigma_1(\mathcal{B}_{a,c}) \subset V, \quad \sigma_2(\mathcal{B}_{a,c}) \subset \{v \in \mathbb{C}; \operatorname{Re} v < -m\},$$

and $\sigma_1(\mathcal{B}_{a,c})$ contains precisely three eigenvalues, not necessarily distinct, counted with multiplicities. We show that one of these eigenvalues is positive when $a \neq 0$ and that the other two eigenvalues are equal to 0.

For $a = 0$, the operator $\mathcal{B}_{0,c}$ has constant coefficients and using Fourier series, again, we can compute its spectrum,

$$\sigma(\mathcal{B}_{0,c}) = \{-n^2(k_0^2 n^4 - k_0^2 n^2 - c), n \in \mathbb{Z}\},$$

where k_0 is the constant in the expansion (2.3) of $k_{a,c}$. In particular, 0 is a triple eigenvalue of $\mathcal{B}_{0,c}$ with associated eigenfunctions 1, $\cos(z)$, and $\sin(z)$.

For $a \neq 0$, we write

$$\mathcal{B}_{a,c} = \partial_z^2 \mathcal{L}_{a,c}, \quad \mathcal{L}_{a,c} = k_{a,c}^4 \partial_z^4 + k_{a,c}^2 \partial_z^2 - c + p_{a,c}.$$

The spectrum of $\partial_z^2 \mathcal{L}_{a,c}$ has been studied in [8]. According to [8, Remark 3.5 (ii)], the kernel of $\partial_z^2 \mathcal{L}_{a,c}$ is two-dimensional, spanned by the odd function $\partial_z p_{a,c}$ and an even function $\xi_{a,c}^e$ which is equal to 1 when $a = 0$. Since the kernel of $\mathcal{B}_{a,c}$ contains the kernel of $\partial_z^2 \mathcal{L}_{a,c}$, 0 is at least a double eigenvalue of $\mathcal{B}_{a,c}$ with two associated eigenfunctions $\xi_{a,c}^o$ and $\xi_{a,c}^e$ which are smooth continuations, for small a , of the vectors $\sin(z)$ and 1, and are odd and even functions, respectively.

In order to compute the third eigenvalue in $\sigma_1(\mathcal{B}_{a,c})$, we consider a basis for the associated three-dimensional spectral subspace which is a smooth continuation of the basis $\{1, \cos(z), \sin(z)\}$ found for $a = 0$. Since $\mathcal{B}_{a,c}$ leaves invariant the subspaces consisting of even and odd functions, two vectors in this basis are even functions and the third one is an odd function. Clearly, the two eigenfunctions $\xi_{a,c}^o$ and $\xi_{a,c}^e$ above belong to this basis, and a third vector is an even function that we denote by $\psi_{a,c}$. Since $\xi_{a,c}^e = 1 + O(|a|)$ and $\psi_{a,c} = \cos(z) + O(|a|)$, upon replacing $\psi_{a,c}$ by a linear combination of $\psi_{a,c}$ and $\xi_{a,c}^e$, we can always choose $\psi_{a,c}$ to be orthogonal to 1. Then, writing

$$\mathcal{B}_{a,c} \psi_{a,c} = \nu_{a,c} \psi_{a,c} + \mu_{a,c} \xi_{a,c}^e,$$

for small a , and taking the scalar product with 1 we conclude that $\mu_{a,c} = 0$. This implies that we can determine $\psi_{a,c}$ and the third eigenvalue $\nu_{a,c}$ by solving the eigenvalue problem

$$\mathcal{B}_{a,c} \psi_{a,c} = \nu_{a,c} \psi_{a,c}, \tag{4.1}$$

in which $\nu_{a,c}$ and $\psi_{a,c}$ depend smoothly upon a and c , and

$$v_{a,c} = O(|a|), \quad \psi_{a,c} = \cos(z) + O(|a|).$$

To complete the proof, we show that $v_{a,c}$ is positive.

First, recall that $p_{-a,c}(x) = p_{a,c}(x + \pi)$ which implies that $v_{-a,c} = v_{a,c}$, and in particular $v_{a,c} = O(a^2)$, as $a \rightarrow 0$. Next, we claim that $v_{a,c} = O(c^2)$, as $c \rightarrow 0$, so that $v_{a,c} = O(a^2 c^2)$. Since $\mathcal{B}_{a,0} = \mathcal{B}_{0,0}$, we have that

$$v_{a,0} = 0, \quad \psi_{a,0} = \cos(z). \quad (4.2)$$

Differentiating (4.1) with respect to c and taking $c = 0$ we find

$$\mathcal{B}_{0,0}(\partial_c \psi_{a,c}|_{c=0}) + \partial_z^2 \left(2\partial_c k_{a,c}^2|_{c=0} \partial_z^4 + \partial_c k_{a,c}^2|_{c=0} \partial_z^2 - 1 + \partial_c p_{a,c}|_{c=0} \right) \cos(z) = \partial_c v_{a,c}|_{c=0} \cos(z).$$

The solvability condition for this equation gives

$$\partial_c v_{a,c}|_{c=0} = -\partial_c k_{a,c}^2|_{c=0} + 1 - [\partial_c p_{a,c}|_{c=0} \cos(z)]_1,$$

in which the bracket $[u]_1$ represents the coefficient of $\cos(z)$ in the Fourier expansion of u . Taking into account the equalities (2.6), we conclude that $\partial_c v_{a,c}|_{c=0} = 0$ which proves the claim.

Next, consider the expansions

$$v_{a,c} = v_2 a^2 + O(a^4), \quad \psi_{a,c}(z) = \cos(z) + \psi_1(z)a + \psi_2(z)a^2 + O(a^3),$$

for small a . Inserting these expansions into (4.1), at order $O(1)$ we find the eigenvalue problem at $a = 0$, which holds, and at order $O(a)$ the equality

$$\partial_z^2 \mathcal{L}_0 \psi_1 + \partial_z^2 \mathcal{L}_1 \cos(z) = 0,$$

in which

$$\mathcal{L}_0 = k_0^4 \partial_z^4 + k_0^2 \partial_z^2 - c, \quad \mathcal{L}_1 = c \cos(z).$$

A direct calculation gives

$$\psi_1(z) = -\frac{c}{2X_2} \cos(2z),$$

with X_2 given by (2.4). At order $O(a^2)$ we obtain

$$\partial_z^2 \mathcal{L}_0 \psi_2 + \partial_z^2 \mathcal{L}_1 \psi_1 + \partial_z^2 \mathcal{L}_2 \cos(z) = v_2 \cos(z),$$

in which

$$\mathcal{L}_2 = c \left(\frac{1}{4} - \frac{c}{4X_2} \cos(2z) \right) + 2k_0 k_2 \left(2k_0^2 \partial_z^4 + \partial_z^2 \right),$$

and k_2 is given by (2.3). The solvability condition for this equation gives

$$v_2 = \frac{c^2}{4X_2} > 0$$

which together with the fact that $v_{a,c} = O(a^2c^2)$ implies that $v_{a,c} > 0$ and completes the proof of [Theorem 1](#).

4.2. Proof of [Theorem 3](#)

Consider a , c , and Λ such that the results in [Proposition 2.2](#) and [Theorem 1](#) hold, and set $\lambda = k_{a,c}\Lambda$.

First, we claim that the operator $\lambda\partial_x - \mathcal{C}_{a,c}$ is a relatively compact perturbation of the asymptotic operator $\lambda\partial_x - \mathcal{C}_{a,c}^\infty$, when both operators act in $L^2(\mathbb{R})$ with domains $H^6(\mathbb{R})$. Indeed, the difference

$$\mathcal{G}_{a,c} = (\lambda\partial_x - \mathcal{C}_{a,c}^\infty) - (\lambda\partial_x - \mathcal{C}_{a,c}) = \begin{cases} \partial_x^2(g_{a,c}^+), & \text{for } x > 0 \\ \partial_x^2(g_{a,c}^-), & \text{for } x < 0 \end{cases}$$

where

$$g_{a,c}^\pm(x) = u_{a,c}(x) - \varphi_{a,c}(x \pm \tau_{a,c})$$

defines a closed operator in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. Since $g_{a,c}^\pm$ is a smooth function on \mathbb{R}^\pm with $\lim_{x \rightarrow \pm\infty} \partial_x^j g_{a,c}^\pm(x) = 0$, $0 \leq j \leq 4$, by [Proposition 2.2](#), using the compact embedding of $H^4(I)$ into $L^2(I)$ for any bounded interval I and the continuity of $\mathcal{G}_{a,c}$ as an operator from $H^6(\mathbb{R}^\pm)$ to $H^4(\mathbb{R}^\pm)$, we conclude that for any bounded sequence $(f_n)_{n \geq 1} \subset H^6(\mathbb{R})$, the sequence $(\mathcal{G}_{a,c}f_n)_{n \geq 1} \subset L^2(\mathbb{R})$ contains a convergent subsequence. This implies that $\mathcal{G}_{a,c}$ is relatively compact with respect to $\lambda\partial_x - \mathcal{C}_{a,c}^\infty$ and proves the claim. As a consequence, the operators $\lambda\partial_x - \mathcal{C}_{a,c}^\infty$ and $\lambda\partial_x - \mathcal{C}_{a,c}$ have the same essential spectrum, so that it is enough to show that $\lambda\partial_x - \mathcal{C}_{a,c}^\infty$ has negative essential spectrum.

According to [Theorem 1](#), there exists $v_* < 0$ and a nontrivial 2π -periodic smooth function u_* such that

$$(\Lambda\partial_z - \mathcal{B}_{a,c})u_* = v_*u_*.$$

We set $v_*(x) = u_*(k_{a,c}x + \tau_{a,c})$, which solves the eigenvalue problem

$$(\lambda\partial_x - \mathcal{A}_{a,c}^+)v_* = k_{a,c}^2 v_* v_*,$$

and consider a cut-off function $\phi \in C_0^\infty(\mathbb{R})$ such that

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [1, 2] \\ 0, & \text{if } x \in (-\infty, 0] \cup [3, \infty) \end{cases}.$$

We define the sequence

$$v_n(x) = v_*(x)\phi_n(x), \quad n \geq 1,$$

where ϕ_n is the smooth function defined by

$$\phi_n(x) = \begin{cases} \phi(x), & \text{if } x \in [0, 1] \\ 1, & \text{if } x \in [1, n+1] \\ \phi(x-n+1), & \text{if } x \in [n+1, n+2] \\ 0, & \text{if } x \in (-\infty, 0] \cup [n+2, \infty) \end{cases}.$$

Since v_* is a periodic function we have that $\|v_n\| \rightarrow \infty$ as $n \rightarrow \infty$, and for $\mathcal{N}_* = \lambda \partial_x - \mathcal{C}_{a,c}^\infty - k_{a,c}^2 v_*$ we find

$$\|\mathcal{N}_* v_n\|^2 = \int_0^1 |\mathcal{N}_*(v_*(x)\phi(x))|^2 dx + \int_{n+1}^{n+2} |\mathcal{N}_*(v_*(x)\phi(x-n+1))|^2 dx \leq C_*,$$

for any $n \geq 1$, and some positive constant C_* which does not depend on n . As a consequence, the operator \mathcal{N}_* is not Fredholm, which implies that $k_{a,c}^2 v_* < 0$ belongs to the essential spectrum of $\lambda \partial_x - \mathcal{C}_{a,c}^\infty$. This completes the proof of [Theorem 3](#).

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Appendix A. Existence of generalized solitary waves

In this appendix we show how the results in [Proposition 2.2](#) follow from the general result in [\[16, Chapter 7, Theorem 7.1.18\]](#). In particular, this will also allow to recover the results in [Proposition 2.1](#).

We start by writing the equation [\(2.2\)](#) as a first order system

$$\frac{dU}{dx} = \mathcal{V}(U, c), \tag{A.1}$$

in which

$$U = \begin{pmatrix} u \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathcal{V}(U, c) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ -u_2 + cu - \frac{1}{2}u^2 \end{pmatrix}.$$

Notice that the system [\(A.1\)](#) is reversible, i.e., the vector field \mathcal{V} anti-commutes with the reflection $S = \text{diag}(1, -1, 1, -1)$.

For any $c \in \mathbb{R}$, the system [\(A.1\)](#) possesses the equilibrium $U = 0$. By linearizing at $U = 0$ we find the Jacobian matrix

$$J_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -1 & 0 \end{pmatrix},$$

with eigenvalues ν satisfying

$$\nu^4 + \nu^2 - c = 0.$$

At $c = 0$, we find the double non-semi-simple eigenvalue 0 and the simple eigenvalues $\pm i$. We consider a basis $\{\varphi_0, \varphi_1, \varphi_+, \varphi_-\}$ consisting of eigenvectors and generalized eigenvectors of J_0 ,

$$\varphi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_+ = \begin{pmatrix} -1 \\ -i \\ 1 \\ i \end{pmatrix}, \quad \varphi_- = \begin{pmatrix} -1 \\ i \\ 1 \\ -i \end{pmatrix},$$

satisfying

$$J_0\varphi_0 = 0, \quad J_0\varphi_1 = \varphi_0, \quad J_0\varphi_+ = i\varphi_+, \quad J_0\varphi_- = -i\varphi_-, \quad S\varphi_0 = \varphi_0,$$

together with the dual basis

$$\varphi_0^* = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varphi_1^* = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \varphi_+^* = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}, \quad \varphi_-^* = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}.$$

Following [16, Chapter 7], we compute the scalar products

$$d_{10} = \langle D_{Uc}^2 \mathcal{V}(0, 0) \varphi_0, \varphi_1^* \rangle = 1, \quad d_{20} = \langle D_{UU}^2 \mathcal{V}(0, 0) [\varphi_0, \varphi_0], \varphi_1^* \rangle = -1,$$

which are both non-zero. This shows that we are in the presence of a $0^2(i)$ resonance and that the quadratic vector field is not degenerate.

The starting point in the construction of generalized solitary waves is a normal form transformation of (A.1), followed by a scaling transformation [16, Section 7.1.1]. First, there exists a close to identity polynomial change of coordinates, analytically depending upon c , and preserving reversibility,

$$U = \tilde{\alpha}\varphi_0 + \tilde{\beta}\varphi_1 + \tilde{A}(\operatorname{Im} \varphi_+) + \tilde{B}(\operatorname{Re} \varphi_+) + \Phi(\tilde{Y}, c),$$

in which $\tilde{Y} = (\tilde{\alpha}, \tilde{\beta}, \tilde{A}, \tilde{B})$ and Φ is a polynomial in \tilde{Y} with coefficients depending analytically upon c , such that the system (A.1) is equivalent in a neighborhood of the origin to

$$\frac{d\tilde{Y}}{dx} = \tilde{N}(\tilde{Y}, c) + \tilde{R}(\tilde{Y}, c),$$

where $\tilde{R}(\tilde{Y}, c) = O(|\tilde{Y}|^3)$ and \tilde{N} is the normal form of order 2,

$$\tilde{N}(\tilde{Y}, c) = \begin{pmatrix} \tilde{\beta} \\ d_1(c)c\tilde{\alpha} + d_2(c)\tilde{\alpha}^2 + d_3(c)(\tilde{A}^2 + \tilde{B}^2) \\ -\tilde{B}(1 + c\omega_1(c) + m(c)\tilde{\alpha}) \\ \tilde{A}(1 + c\omega_1(c) + m(c)\tilde{\alpha}) \end{pmatrix}.$$

Here d_1, d_2, d_3, ω_1 and m are analytic functions of c and

$$d_1(0) = d_{10} = 1, \quad d_2(0) = d_{20} = -1.$$

Next, for $c > 0$, we introduce the scaling

$$x = c^{-1/2}y, \quad \tilde{\alpha} = \frac{3}{2}c\alpha, \quad \tilde{\beta} = \frac{3}{2}c^{3/2}\beta, \quad \tilde{A} = cA, \quad \tilde{B} = cB,$$

which leads to the system

$$\frac{dY}{dy} = N(Y, \sqrt{c}) + R(Y, \sqrt{c}), \quad (\text{A.2})$$

with

$$N(Y, \sqrt{c}) = \begin{pmatrix} \beta \\ \alpha - \frac{3}{2}\alpha^2 - \frac{2}{3}d_3(0)(A^2 + B^2) \\ -B\left(\frac{1}{\sqrt{c}} + \omega_1(0)\sqrt{c} + m(0)\sqrt{c}\alpha\right) \\ A\left(\frac{1}{\sqrt{c}} + \omega_1(0)\sqrt{c} + m(0)\sqrt{c}\alpha\right) \end{pmatrix},$$

and $R(Y, \sqrt{c})$ representing higher order terms.

The result in [16, Theorem 7.1.4] shows the existence of periodic orbits for (A.2). Transforming back to the equation (2.2) this result is precisely the one stated in Proposition 2.1, when restricting to positive values $c \in (0, c_0]$. Next, the result in [16, Theorem 7.1.18] shows the existence of reversible homoclinic orbits to small periodic orbits for the system (A.2), provided the size of the periodic orbits is larger than an exponentially small critical size. For the equation (2.2) this leads to the result in Proposition 2.2.

References

- [1] J.C. Alexander, R.L. Pego, R.L. Sachs, On the transverse instability of solitary waves in the Kadomtsev–Petviashvili equation, *Phys. Lett. A* 226 (1997) 187–192.
- [2] M.D. Groves, M. Haragus, S.M. Sun, Transverse instability of gravity-capillary line solitary water waves, *C. R. Acad. Sci. Paris Sér. I Math.* 333 (2001) 421–426.
- [3] M.D. Groves, S.M. Sun, E. Wahlén, A dimension-breaking phenomenon for water waves with weak surface tension, *Arch. Ration. Mech. Anal.* 220 (2016) 747–807.
- [4] S. Hakkaev, M. Stanislavova, A. Stefanov, Transverse instability for periodic waves of KP-I and Schrödinger equations, *Indiana Univ. Math. J.* 61 (2012) 461–492.
- [5] M. Haragus, Stability of periodic waves for the generalized BBM equation, *Rev. Roumaine Math. Pures Appl.* 53 (2008) 445–463.
- [6] M. Haragus, Transverse spectral stability of small periodic traveling waves for the KP equation, *Stud. Appl. Math.* 126 (2011) 157–185.

- [7] M. Haragus, Transverse dynamics of two-dimensional gravity-capillary periodic water waves, *J. Dynam. Differential Equations* 27 (2015) 683–703.
- [8] M. Haragus, E. Lombardi, A. Scheel, Spectral stability of wave trains in the Kawahara equation, *J. Math. Fluid Mech.* 8 (2006) 482–509.
- [9] J.K. Hunter, J. Scheurle, Existence of perturbed solitary wave solutions to a model equation for water waves, *Phys. D* 32 (1988) 253–268.
- [10] G. Iooss, K. Kirchgässner, Water waves for small surface tension: an approach via normal form, *Proc. Roy. Soc. Edinburgh Sect. A* 122 (1992) 267–299.
- [11] M.A. Johnson, Stability of small periodic waves in fractional KdV-type equations, *SIAM J. Math. Anal.* 45 (2013) 3168–3193.
- [12] M.A. Johnson, K. Zumbrun, Transverse instability of periodic traveling waves in the generalized Kadomtsev–Petviashvili equation, *SIAM J. Math. Anal.* 42 (2010) 2681–2702.
- [13] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.* 15 (1970) 539–541.
- [14] E.A. Kuznetsov, M.D. Spector, G.E. Falkovich, On the stability of nonlinear waves in integrable models, *Phys. D* 10 (1984) 379–386.
- [15] D. Lannes, *The Water Waves Problem. Mathematical Analysis and Asymptotics*, Mathematical Surveys and Monographs, vol. 188, American Mathematical Society, Providence, RI, 2013.
- [16] E. Lombardi, *Oscillatory Integrals and Phenomena Beyond All Algebraic Orders*, With applications to homoclinic orbits in reversible systems *Lecture Notes in Mathematics*, vol. 1741, Springer-Verlag, Berlin, 2000.
- [17] T. Mizumachi, Stability of line solitons for the KP-II equation in \mathbb{R}^2 , *Mem. Amer. Math. Soc.* 238 (1125) (2015).
- [18] T. Mizumachi, N. Tzvetkov, Stability of the line soliton of the KP-II equation under periodic transverse perturbations, *Math. Ann.* 352 (2012) 659–690.
- [19] R.L. Pego, S.M. Sun, On the transverse linear instability of solitary water waves with large surface tension, *Proc. Roy. Soc. Edinburgh Sect. A* 134 (2004) 733–752.
- [20] F. Rousset, N. Tzvetkov, Transverse nonlinear instability for two-dimensional dispersive models, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009) 477–496.
- [21] F. Rousset, N. Tzvetkov, Transverse instability of the line solitary water-waves, *Invent. Math.* 184 (2011) 257–388.