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Bifurcation analysis of a spruce budworm model with diffusion and physiological structures ☆

Xiaofeng Xu [a,b], Junjie Wei [a,*]

[a] *Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, PR China*
[b] *Department of Mathematics, Northeast Forestry University, Harbin, Heilongjiang 150040, PR China*

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Abstract

In this paper, the dynamics of a spruce budworm model with diffusion and physiological structures are investigated. The stability of steady state and the existence of Hopf bifurcation near positive steady state are investigated by analyzing the distribution of eigenvalues. The properties of Hopf bifurcation are determined by the normal form theory and center manifold reduction for partial functional differential equations. And global existence of periodic solutions is established by using the global Hopf bifurcation result of Wu. Finally, some numerical simulations are carried out to illustrate the analytical results.
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1. Introduction

The spruce budworm is one of the most destructive insects in North American forests, where spruce and balsam fir trees grow. Normally, the spruce budworm exists in low numbers in these forests, kept in check by the predators, primarily birds. However, outbreak of these insects occurs periodically (every 30–40 years lasting for about 10 years) causing billions of dollars loss to for-

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* Corresponding author.
E-mail address: weijj@hit.edu.cn (J. Wei).

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est industry [1–6]. Understanding the dynamics of spruce budworm population is very important to control the growth of budworm and protect spruce and fir forests.

Notice that, as far as the budworm dynamics are concerned, the forest variables may be treated as constants. Also since the birds do not feed exclusively on budworms, their numbers are for the most part independent of the budworm population. In 1979, Ludwig et al. [7] proposed the diffusing budworm population dynamics governed by the equation:

$$u_t = d\Delta u + ru\left(1 - \frac{u}{K}\right) - \frac{Bu^2}{A^2 + u^2}, \tag{1}$$

and investigated the dynamics of Eq. (1) in spatial one dimension. In 2013, Wang and Yeh [8] studied the steady-state problem of (1) and obtained the S-shaped bifurcation diagrams. For some other results about budworm population dynamics we refer to the papers and monographs [9–13].

In fact, the logistic model of budworm population with Holling type III predation function is widely accepted in the literature because of the existence of a stable periodic orbit. But both predation and carrying capacity are unlikely to be a primary cause of budworm population oscillation, see [5]. In 2008, Vaidya and Wu [14] derived a delay differential equation for the matured budworm population from a structured population model and by considering the inactive stage from egg to the second instar caterpillars (L_2) as the immature stage

$$\dot{u} = -Du(t) - \frac{\beta u^2(t)}{\gamma^2 + u^2(t)} + q_1 e^{-\tilde{d}\tau} u(t-\tau) e^{-\alpha_1 u(t-\tau)}. \tag{2}$$

They showed that the simulation results of Eq. (2) are in very good agreement with the real data from the Green River area of New Brunswick, Canada and discussed the role of the parameters on controlling budworm population. They also pointed out that spatial nonhomogeneity should be incorporated as additional factors in further research. So, in this paper, we consider the following diffusive budworm model with Neumann boundary conditions

$$\frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) - Du(x,t) - \frac{\beta u^2(x,t)}{\gamma^2 + u^2(x,t)} + q_1 e^{-\tilde{d}\tau} u(x,t-\tau) e^{-\alpha_1 u(x,t-\tau)}, \tag{3}$$

where $u(x,t)$ is the mature budworm density at location x and time t, $d_1 > 0$ is the diffusion coefficient, $D > 0$ is the average mortality rate of the mature budworms, $\beta > 0$ represents the predation rate of the birds, $\gamma > 0$ is the budworm population when the predation rate is at half of the maximum, τ is the maturation time delay, $\tilde{d} > 0$ is the average mortality rate of the immature budworms and $b(u) = q_1 u e^{-\alpha_1 u}$ is the birth function of budworms with $q_1, \alpha_1 > 0$.

We would like to mention that, when the stability and Hopf bifurcation at the positive steady state are considered, the difficulty resides in the presence of the dependent delay and the fact that some coefficients in the equations depend upon this delay. Consequently, the characteristic equation of the linearized system has delay-dependent coefficients. As mentioned by Beretta and Kuang [15], models with delay-dependent coefficients often exhibit very rich dynamics as compared to those with constant coefficients. In the analysis, we need to study a series of first degree transcendental polynomial with delay coefficients. The problem of determining the distribution of roots to such polynomials is very complex and there are very few studies on this topic (see [16–21], and references therein).

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The rest of the paper is organized as follows. In Section 2, we investigate the existence of the constant steady state. In Section 3, we analyze the distribution of the roots of the characteristic equation, and give various conditions on the stability of steady state and the existence of Hopf bifurcation. In Section 4, we establish the extended existence of bifurcation periodic solutions by using the global Hopf bifurcation result of Wu [22]. In Section 5, we carry out some numerical simulations to support the analytical results.

2. Existence analysis of the non-negative constant steady state

In this paper, we consider the non-dimensional model of Eq. (3) with Neumann boundary conditions by changing the variables as $\hat{u}(x,\hat{t}) = \frac{1}{\gamma}u(x,t)$, $\hat{t} = \frac{\beta t}{\gamma}$ and $\hat{\tau} = \frac{\beta\tau}{\gamma}$. After removing the hat, we obtain the following non-dimensional model:

$$\begin{cases} u_t = d\Delta u - r_1 u - \dfrac{u^2}{1+u^2} + be^{-r_2\tau}u_\tau e^{-\alpha u_\tau}, x \in \Omega, t > 0, \\ u_\nu = 0, \qquad x \in \partial\Omega, t > 0, \\ u = \eta(x,t) \geq 0, \qquad x \in \Omega, t \in [-\tau, 0], \end{cases} \tag{4}$$

where

$$d = \frac{\gamma d_1}{\beta}, \quad r_1 = \frac{\gamma D}{\beta}, \quad b = \frac{\gamma q_1}{\beta}, \quad r_2 = \frac{\gamma \tilde{d}}{\beta}, \quad \alpha = \gamma\alpha_1, \tag{5}$$

$u_\tau = u(x, t-\tau)$, $\Omega = (0, l\pi)$, ν is the outward unit normal vector on $\partial\Omega$ and $\eta(x,t)$ is Hölder continuous with $\eta(x,0) \in C^1(\bar{\Omega})$.

Obviously, $u = 0$ is always a steady state of Eq. (4). Let $u = u_0$ be a positive constant steady state of Eq. (4). Then u_0 satisfies

$$r_1 + \frac{u}{1+u^2} = be^{-r_2\tau}e^{-\alpha u}. \tag{6}$$

It is easy to see that, if $be^{-r_2\tau} \leq r_1$,

$$r_1 + \frac{u}{1+u^2} > r_1 \geq be^{-r_2\tau} > be^{-r_2\tau}e^{-\alpha u},$$

for any $u \in (0,\infty)$. Consequently, Eq. (4) has no positive constant steady state when $be^{-r_2\tau} \leq r_1$. Next, we consider the case $be^{-r_2\tau} > r_1$, i.e. $b > r_1$ and

$$0 \leq \tau < \tau_{\max} := \frac{1}{r_2}\ln\frac{b}{r_1}.$$

In this case, Eq. (4) maybe has 1, 2 or 3 positive constant steady states depending upon the value of parameters. However, if we assume that $\alpha > 1$, one can obtain that Eq. (4) has a unique positive constant steady state. In fact, we can multiply Eq. (6) by $1+u^2$ and denote

$$g_1(u) = r_1(1+u^2) + u, \qquad g_2(u,\tau) = be^{-r_2\tau}(1+u^2)e^{-\alpha u}. \tag{7}$$

Table 1
Spruce budworm population model parameters.

Symbol	Description	Value	Source
β	Max. budworms predated	105700 larvae/ha/yr	(Ludwig et al., 1979) (Ludwig et al., 1978)
γ	Related to predation function	69748 larvae	(Ludwig et al., 1979) (Ludwig et al., 1978)
τ	Maturation delay	0.75–2 yrs	(Fleming and Shoemaker, 1992) (Sheehan et al., 1989) (Royama, 1984)
$\tilde{d}$	Death rate of immature	0.95–2.53	(Sheehan et al., 1989) (Royama, 1984) Calculated
D	Death rate of mature	0.30	(Blais,1981) (Sheehan et al., 1989) Estimated
α_1	Related to birth function	0.00017	Estimated
q_1	Related to birth function	9.83×10^5	(Royama, 1984) Estimated, Calculated

Then u_0 is a positive root of Eq. (6) if and only if it is a positive root of $g_1(u) = g_2(u)$. Notice that

$$g_1'(u) = 2r_1u + 1, \qquad \frac{\partial g_2(u,\tau)}{\partial u} = be^{-r_2\tau}e^{-\alpha u}\big(2u - \alpha(1+u^2)\big),$$

and $g_1(0) = r_1$, $g_2(0,\tau) = be^{-r_2\tau}$. Clearly, for $u \in (0,\infty)$, we have $g_1'(u) > 0$ and $\frac{\partial g_2(u,\tau)}{\partial u} < 0$ if $\alpha > 1$. These imply that $\frac{\partial(g_1 - g_2)}{\partial u} > 0$ when $\alpha > 1$. Therefore, from $g_1(0) < g_2(0,\tau)$ and $g_1 - g_2 \to \infty$ as $u \to \infty$, the equation $g_1(u) = g_2(u,\tau)$ has a unique positive root.

Now we can state the following theorem on the existence of the nonnegative constant steady state.

Theorem 2.1.

(i) *If $be^{-r_2\tau} \le r_1$, then $u = 0$ is the only biologically meaningful constant steady state of Eq. (4);*
(ii) *If $be^{-r_2\tau} > r_1$ and $\alpha > 1$, then, in addition to the trivial steady state, there exists a unique positive constant steady state of Eq. (4), which satisfies Eq. (6).*

Remark 2.2. We would like to mention that the assumption $\alpha > 1$, which ensures the uniqueness of positive constant steady state, is biologically reasonable. For convenience, we copy the table of spruce budworm model parameters here, which is given by Vaidya and Wu [14]. From Table 1 and (5), one can easily obtain that $\alpha \approx 11.8572 \gg 1$.

3. Stability and Hopf bifurcation analysis

The existence and uniqueness of the solution of systems Eq. (4) can be obtained by using results introduced in Wu [22]. In this section, we investigate the stability of steady state and existence of Hopf bifurcation.

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3.1. Stability of steady state $u = 0$

In order to prove the global stability of $u = 0$, we first give the following lemmas.

Lemma 3.1. *Every solution $u(x,t)$ of Eq. (4) satisfies $u(x,t) \geq 0$ for $x \in \Omega$ and $t > 0$. Furthermore, if $\eta(x,0) \not\equiv 0$, then every solution $u(x,t)$ of Eq. (4) satisfies $u(x,t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$.*

Proof. Let $v(x,t)$ be the unique solution of

$$\begin{cases} v_t = d\Delta v - r_1 v - \dfrac{v^2}{1+v^2}, & x \in \Omega, t > 0, \\ v_\nu = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = \eta(x,0), & x \in \Omega. \end{cases} \tag{8}$$

Note that $u(x,t-\tau) = \eta(x,t-\tau) \geq 0$ for $t \in [0,\tau]$. Then comparison principle implies that $u(x,t) \geq v(x,t) \geq 0$ for $x \in \Omega$ and $t \in [0,\tau]$. That is $u(x,t) \geq 0$ for $x \in \Omega$ and $t \in [0,\tau]$, and one can obtain $u(x,t) \geq v(x,t) \geq 0$ for $x \in \Omega$ and $t \in [0,2\tau]$ in the same way. Hence, by using the mathematical induction, we have $u(x,t) \geq v(x,t) \geq 0$ for $x \in \Omega$ and $t > 0$. Moreover, the strong maximum principle implies that $u(x,t) \geq v(x,t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$ if $\eta(x,0) \not\equiv 0$. The proof is complete. □

Lemma 3.2. *([23, Theorem 3.1]) Assume that $d, \delta, \tau > 0$, $f \in C^1\big([0,\infty),[0,\infty)\big)$ and $f(0) = 0$. If $\sup_{y\in(0,\infty)} f'(y) < \delta$, then every solution $u(x,t)$ of*

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - \delta u(x,t) + f\big(u(x,t-\tau)\big), \tag{9}$$

with Neumann boundary conditions and non-negative initial conditions, converges to zero (uniformly in x) as $t \to \infty$ if and only if $f(y) < \delta y$ for all $y > 0$.

Applying Lemmas 3.1 and 3.2, we have the following result about global attractivity of $u = 0$.

Lemma 3.3. *If $be^{-r_2\tau} < r_1$, then the solution $u(x,t)$ of Eq. (4) converges to $u = 0$ (uniformly in x) as $t \to \infty$.*

Proof. In order to apply Lemma 3.2, we denote

$$g(y) = be^{-r_2\tau} y e^{-\alpha y}. \tag{10}$$

Clearly, $g \in C^1\big([0,\infty),[0,\infty)\big)$ and $g(0) = 0$. Let $v(x,t)$ be the unique solution of

$$\begin{cases} v_t = d\Delta v - r_1 v + be^{-r_2\tau} v_\tau e^{-\alpha v_\tau}, x \in \Omega, t > 0, \\ v_\nu = 0, \quad x \in \partial\Omega, t > 0, \\ v = \eta(x,t) \geq 0, \quad x \in \Omega, t \in [-\tau, 0]. \end{cases} \tag{11}$$

Then comparison principle implies that $u(x,t) \le v(x,t)$ for $x \in \Omega, t > 0$. From (10), we have

$$g'(y) = be^{-r_2\tau}e^{-\alpha y}(1-\alpha y),$$
$$g''(y) = be^{-r_2\tau}e^{-\alpha y}(\alpha^2 y - 2\alpha).$$

Thus, one has $g''(y) < 0$ when $y \in (0, \frac{2}{\alpha})$ and $g''(y) > 0$ when $y \in (\frac{2}{\alpha}, \infty)$. These lead to $g'(y)$ is monotone decreasing when $y \in (0, \frac{2}{\alpha})$ and monotone increasing when $y \in (\frac{2}{\alpha}, \infty)$. Also note that $g'(0) = be^{-r_2\tau}$ and $g'(\infty) = 0$. Therefore, if $be^{-r_2\tau} < r_1$, we have

$$\sup_{y\in(0,\infty)} g'(y) = be^{-r_2\tau} < r_1,$$

and

$$g(y) = be^{-r_2\tau}ye^{-\alpha y} < be^{-r_2\tau}y < r_1 y,$$

for all $y > 0$. From Lemma 3.1 and Lemma 3.2, every solution $v(x,t)$ of Eq. (11) converges to $u = 0$ (uniformly in x) as $t \to \infty$, so does the solution $u(x,t)$ of Eq. (4). The proof is complete. □

Moreover, define the real-valued Sobolev space

$$X = \{u \in H^2(0, l\pi) | u_x = 0, x = 0, l\pi\}, \tag{12}$$

and the abstract space

$$\mathcal{C} = C([-\tau, 0], X). \tag{13}$$

Then, the linearization of Eq. (4) at $u = 0$ can be rewritten as an abstract differential equation in the phase space $\mathcal{C}$,

$$\dot{U}(t) = d\Delta U(t) + L(U_t), \tag{14}$$

where $U(t) = u(x,t)$, $U_t(\theta) = U(t+\theta)$ and

$$L(\phi) = -r_1\phi(0) + be^{-r_2\tau}\phi(-\tau).$$

From Wu [22], the corresponding characteristic equation of Eq. (14) is equivalent to

$$\lambda + \frac{dn^2}{l^2} + r_1 - be^{-r_2\tau}e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{15}$$

It is well known that all roots of Eq. (15) have negative real parts if and only if $be^{-r_2\tau} < \frac{dn^2}{l^2} + r_1$ for all $n \in \mathbb{N}_0$ (see [24] or [25]). Thus, $be^{-r_2\tau} < r_1$ implies all roots of Eq. (15) have negative real parts and, for Eq. (4), $u = 0$ is locally asymptotically stable. Meanwhile, $be^{-r_2\tau} > r_1$ leads to Eq. (15) has at least one root with positive real part and, for Eq. (4), $u = 0$ is unstable.

Summarizing the discussion above and applying Lemma 3.3, we arrive at the following preliminary result.

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Theorem 3.4. *For Eq. (4), $u = 0$ is global asymptotically stable if $be^{-r_2\tau} < r_1$ and unstable if $be^{-r_2\tau} > r_1$.*

Theorem 3.4 shows that $be^{-r_2\tau} = r_1$ is a critical situation for the stability of $u = 0$. Clearly, $\lambda = 0$ is a simple root of Eq. (15) if $be^{-r_2\tau} = r_1$. The following result is to describe the stability of $u = 0$ when $be^{-r_2\tau} = r_1$.

Theorem 3.5. *If $be^{-r_2\tau} = r_1$, then for Eq. (4), $u = 0$ is unstable.*

Proof. From Theorem 3.4 we know that all roots of Eq. (15) with $be^{-r_2\tau} = r_1$, except $\lambda = 0$, have negative real parts. In order to investigate the stability of $u = 0$ for Eq. (4), we employ the center manifold theory and normal form method. Here, we shall use the method of computing normal forms for PFDEs introduced by Faria [26].

Following the same algorithms as those in [26], let $\Lambda = \{0\}$ and $B = 0$. Clearly, the non-resonance conditions relative to Λ are satisfied. Therefore, there exists a 1-dimensional ODE which governs the dynamics of Eq. (4) near the origin (see [27]).

Firstly, Eq. (4) can be written in $\mathcal{C} = C([-\tau, 0], X)$ of the form:

$$\dot{U}(t) = d\Delta U(t) + L(U_t) + F(U_t), \tag{16}$$

where

$$L(\phi) = -r_1\phi(0) + be^{-r_2\tau}\phi(-\tau),$$

and

$$F(\phi) = -2\phi^2(0) - 2\alpha be^{-r_2\tau}\phi^2(-\tau) + 3\alpha^3 be^{-r_2\tau}\phi^3(-\tau) + \mathcal{O}(4),$$

for any $\phi \in \mathcal{C}$.

Choosing

$$\eta(\theta) = \begin{cases} -be^{-r_2\tau}, & \theta = -\tau, \\ 0, & -\tau < \theta < 0, \\ -\left(\frac{dn^2}{l^2} + r_1\right), & \theta = 0, \end{cases}$$

we obtain

$$-\frac{dn^2}{l^2}\phi(0) + L(\phi) = \int_{-\tau}^{0} d\eta(\theta)\phi(\theta), \quad \phi(\theta) \in \mathcal{C}.$$

Using the adjoint theory to decompose $\mathcal{C}$ by Λ, we can obtain $\mathcal{C} = P \bigoplus Q$, where $P = span\{\Phi(\theta)\}$ with $\Phi(\theta) = 1$ being the center space for

$$\dot{U}(t) = d\Delta U(t) + L(U_t).$$

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Choose a basis Ψ for the adjoint space P^* such that $(\Psi, \Phi) = 1$, where $(\cdot, \cdot)$ is the bilinear form on $\mathcal{C}^* \times \mathcal{C}$ defined by

$$(\psi(s), \phi(\theta)) = \psi(0)\phi(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi.$$

Thus, $\psi(s) = \frac{1}{1+r_1\tau}$.

Taking the enlarged phase space

$$BC = \left\{\phi : [-\tau, 0] \to X, \phi \text{ is continuous on } [-\tau, 0) \text{ and } \lim_{\theta\to 0} \phi(\theta) \text{ exists}\right\},$$

we obtain the abstract differential equation with the form

$$\frac{d}{dt} U_t = AU_t + X_0 F(U_t), \tag{17}$$

where

$$A\phi = \dot{\phi}(\theta) + X_0[d\Delta\phi(0) + L(\phi) - \dot{\phi}(0)],$$

and $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} 0, & -\tau \le \theta < 0, \\ 1, & \theta = 0. \end{cases}$$

Consider the projection

$$\pi : BC \mapsto P, \quad \pi(\phi + X_0\xi) = \Phi[(\Psi, \langle\phi(\cdot), \beta_n\rangle)\beta_n + \Psi(0)\langle\xi, \beta_n\rangle\beta_n],$$

where

$$\beta_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}\cos\frac{nx}{l}, & n \in \mathbb{N}. \end{cases}$$

This leads to the decomposition $BC = P \bigoplus \mathrm{Ker}\pi$. Then using the decomposition

$$U_t = \Phi z(t)\beta_n + y(t),$$

where $z(t) = (\Psi, \langle U_t(\cdot), \beta_n\rangle) \in \mathbb{R}$, $y(t) \in \mathcal{Q}^1$, we decompose Eq. (17) as

$$\begin{aligned} \dot{z} &= Bz + \Psi(0)\langle F(\Phi z\beta_n + y), \beta_n\rangle, \\ \frac{d}{dt} y &= A_{\mathcal{Q}^1} y + (I - \pi)X_0 F(\Phi z\beta_n + y). \end{aligned} \tag{18}$$

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Note that, when $be^{-r_2\tau} = r_1$, $\lambda = 0$ is a root of Eq. (15) if and only if $n = 0$. Therefore, the flow on the manifold is given by the following 1-dimensional ODE

$$\dot{z} = -\frac{2 + 2\alpha b e^{-r_2\tau}}{1 + r_1\tau} z^2 + \mathcal{O}(3). \tag{19}$$

Clearly, when $be^{-r_2\tau} = r_1$, the zero solution of Eq. (19) is unstable, so is the zero solution of Eq. (4). The proof is completed. □

3.2. *Stability of steady state* $u = u_0$

In this subsection, we are going to investigate the stability of positive steady state u_0 of Eq. (4) and show that it can be destabilized via Hopf bifurcation. The time delay τ will be used as a bifurcation parameter. Throughout this subsection, we always assume that $be^{-r_2\tau} > r_1$ and $\alpha > 1$, which can ensure the existence and uniqueness of positive constant steady state u_0. We would like to emphasize that $be^{-r_2\tau} > r_1$ is equivalent to $b > r_1$ and $\tau \in [0, \tau_{\max})$.

Similarly, we can obtain the linearization of Eq. (4) at $u = u_0$ in the phase space $\mathcal{C}$,

$$\dot{U}(t) = d\Delta U(t) + L(U_t), \tag{20}$$

where

$$L(\phi) = -\Big(r_1 + \frac{2u_0}{(1+u_0^2)^2}\Big)\phi(0) + be^{-r_2\tau}e^{-\alpha u_0}(1 - \alpha u_0)\phi(-\tau).$$

Thus, the corresponding characteristic equation of Eq. (20) is equivalent to

$$\lambda + \frac{dn^2}{l^2} + r_1 + \frac{2u_0}{(1+u_0^2)^2} + be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0. \tag{21}$$

Note that some of these coefficients (including u_0) depend on the time delay τ. We rewrite Eq. (21) in the general form

$$P_n(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0, \tag{22}$$

where

$$\begin{aligned} P_n(\lambda, \tau) &= \lambda + \frac{dn^2}{l^2} + r_1 + \frac{2u_0}{(1+u_0^2)^2}, \\ Q(\lambda, \tau) &= be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1). \end{aligned} \tag{23}$$

In order to apply the geometrical criterion due to Beretta and Kuang [15], we need to verify the following properties for $n \in \mathbb{N}_0$ and $\tau \in [0, \tau_{\max})$.

(i) $P_n(0, \tau) + Q(0, \tau) \neq 0$;
(ii) $P_n(i\omega, \tau) + Q(i\omega, \tau) \neq 0$;
(iii) $\limsup\{\left|\frac{Q(\lambda,\tau)}{P_n(\lambda,\tau)}\right|; |\lambda| \to \infty, \mathrm{Re}\lambda \geq 0\} < 1$;

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(iv) $F_n(\omega, \tau) := |P_n(i\omega, \tau)|^2 + |Q(i\omega, \tau)|^2$ has a finite number of zeros;
(v) Each positive root $\omega_n(\tau)$ of $F_n(\omega, \tau) = 0$ is continuous and differentiable in τ whenever it exists.

Given the fact that

$$\begin{aligned} P_n(0, \tau) + Q(0, \tau) &= \frac{dn^2}{l^2} + r_1 + \frac{2u_0}{(1+u_0^2)^2} + be^{-r_2\tau} e^{-\alpha u_0}(\alpha u_0 - 1) \\ &= \left[\frac{u}{1+u^2}\big(g_1(u) - g_2(u)\big)\right]_u\bigg|_{u=u_0} + \frac{dn^2}{l^2} \\ &= \frac{u_0}{1+u_0^2}\big(g_1'(u_0) - g_2'(u_0)\big) + \frac{dn^2}{l^2} \\ &> 0, \end{aligned}$$

and

$$P_n(i\omega, \tau) + Q(i\omega, \tau) = i\omega + P_n(0, \tau) + Q(0, \tau),$$

(i) and (ii) are satisfied.

From (23), we know that

$$\lim_{|\lambda| \to \infty} \left| \frac{Q(\lambda, \tau)}{P_n(\lambda, \tau)} \right| = 0.$$

Therefore (iii) follows.

Let F_n be defined as in (iv) with the following expression

$$F_n(\omega, \tau) := \omega^2 + \left(\frac{dn^2}{l^2} + r_1 + \frac{2u_0}{(1+u_0^2)^2}\right)^2 - b^2 e^{-2r_2\tau} e^{-2\alpha u_0}(\alpha u_0 - 1)^2.$$

It is obvious that property (iv) is satisfied, and by the Implicit Function Theorem, (v) is also satisfied.

From Theorem 2.1 we have know that Eq. (4) has a unique positive constant steady state. We denote the positive constant steady state as $u_0 = u_0(\tau)$ since u_0 satisfies Eq. (6) which depends on τ. Then we have the following result.

Lemma 3.6. *If $b > r_1$ and $\alpha > 1$, then $u_0(\tau)$ is a strictly decreasing function on* $[0, \tau_{\max})$.

Proof. From the discussion of Theorem 2.1, we have know that $u_0(\tau)$ satisfies $g_1(u) = g_2(u, \tau)$. Assume that, for $i = 1, 2$, $\tau_i \in [0, \tau_{\max})$, u_i is the unique positive root of $g_1(u) = g_2(u, \tau_i)$ and $\tau_1 < \tau_2$. Then we only need to verify $u_1 > u_2$. In fact, if $u_1 \le u_2$, by the definition of g_2 in (7), we have

$$g_2(u_1, \tau_2) < g_2(u_1, \tau_1) = g_1(u_1) \le g_1(u_2) = g_2(u_2, \tau_2) \le g_2(u_1, \tau_2).$$

This is a contradiction and the proof is complete. □

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From Eq. (6), we know that $u_{\max} := u_0(0)$ satisfies

$$r_1 + \frac{u}{1+u^2} = be^{-\alpha u}.$$

Denote

$$\begin{aligned} T_n(\tau) &= \frac{dn^2}{l^2} + r_1 + \frac{2u_0}{(1+u_0^2)^2}, \\ h(\tau) &= be^{-r_2\tau}e^{-\alpha u_0}(\alpha u_0 - 1). \end{aligned} \tag{24}$$

Then, when $\tau = 0$, the roots of Eq. (21) satisfy

$$\begin{aligned} \lambda &= -T_n(0) - h(0) \\ &= -\Big[\frac{u_{\max}}{1+u_{\max}^2}\big(g_1'(u_{\max}) - g_2'(u_{\max})\big) + \frac{dn^2}{l^2}\Big] \\ &< 0. \end{aligned}$$

That is, all roots of Eq. (21) have negative real part when $\tau = 0$.

Now, let $\lambda = i\omega(\omega > 0)$ be a root of Eq. (21). Substituting it into Eq. (21) and separating the real and imaginary parts, we have

$$T_n(\tau) = -h(\tau)\cos\omega\tau, \quad \omega = h(\tau)\sin\omega\tau. \tag{25}$$

This leads to

$$\omega_n = \sqrt{h^2(\tau) - T_n^2(\tau)}. \tag{26}$$

Clearly, ω_n make sense if and only if $|h(\tau)| > T_n(\tau)$. Note that

$$T_n(\tau) + h(\tau) = P_n(0,\tau) + Q(0,\tau) > 0. \tag{27}$$

Thus, $|h(\tau)| > T_n(\tau)$ is equivalent to $h(\tau) > T_n(\tau)$.

It is well known that a stability change at $u = u_0$ can only happen when there are characteristic roots crossing the imaginary axis to the right. Then, from the discuss above, we can arrive at the following result.

Theorem 3.7. *Assume that $\alpha > 1$ and denote $b_0 = e\big(r_1 + \frac{\alpha}{1+\alpha^2}\big)$.*

(i) *If $r_1 < b \le b_0$, then for Eq. (4), $u = u_0$ is locally asymptotically stable for any $\tau \in [0, \tau_{\max})$;*
(ii) *If $b > b_0$, there exists a $\hat{\tau} \in (0, \tau_{\max})$ such that $g_1(\frac{1}{\alpha}) = g_2(\frac{1}{\alpha}, \hat{\tau})$ and for Eq. (4), $u = u_0$ is locally asymptotically stable for any $\tau \in [\hat{\tau}, \tau_{\max})$.*

Proof. Assume that $b > r_1$ and $\alpha > 1$ hold. Then we can obtain

$$b \le b_0 \Leftrightarrow g_2\Big(\frac{1}{\alpha}, 0\Big) \le g_1\Big(\frac{1}{\alpha}\Big) \Leftrightarrow u_{\max} \le \frac{1}{\alpha}.$$

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This implies that $h(\tau) \le 0 < T_n(\tau)$ for all $n \in \mathbb{N}_0$ and $\tau \in [0, \tau_{\max})$ if $b \le b_0$. The proof of (i) is complete.

If $b > b_0$, we have $u_{\max} > \frac{1}{\alpha}$. Let $u_0(\hat{\tau}) = \frac{1}{\alpha}$, then, from Eq. (6), we have

$$\hat{\tau} = \frac{1}{r_2} \ln \frac{b}{b_0}. \tag{28}$$

By using the monotonicity of $u_0(\tau)$, one has $u_0(\tau) \le \frac{1}{\alpha}$ for all $\tau \in [\hat{\tau}, \tau_{\max})$. It follows that $h(\tau) \le 0 < T_n(\tau)$ for all $n \in \mathbb{N}_0$ and $\tau \in [\hat{\tau}, \tau_{\max})$.

Moreover, when $\tau = \hat{\tau}$, the roots of Eq. (21) satisfy

$$\lambda = -\Big(\frac{dn^2}{l^2} + r_1 + \frac{2\alpha^3}{(1+\alpha^2)^2}\Big) < 0.$$

The proof of (ii) is complete. □

From Theorem 3.7, we know that Hopf bifurcation near u_0 can only possibly happen when $b > b_0$, $\tau \in [0, \hat{\tau})$ and $\alpha > 1$. Set

$$I_n = \big\{\tau | \tau \in [0, \hat{\tau}), \ \ \text{satisfies} \ \ h(\tau) > T_n(\tau)\big\}.$$

Assume that I_n is nonempty. Then for $\tau \in I_n$, there exists a unique $\omega_n = \omega_n(\tau) > 0$, which satisfies Eq. (26), such that $F_n(\omega_n, \tau) = 0$. Let $\theta_n(\tau) \in [0, 2\pi]$ be defined for $\tau \in I_n$ by

$$\sin\theta_n(\tau) = \frac{\omega_n(\tau)}{h(\tau)}, \quad \cos\theta_n(\tau) = -\frac{T_n(\tau)}{h(\tau)}. \tag{29}$$

Then we have $\theta_n(\tau) \in [\frac{\pi}{2}, \pi]$ by the fact that $\tau \in [0, \hat{\tau})$ implies $\sin\theta_n(\tau) > 0$ and $\cos\theta_n(\tau) < 0$. From the above definitions, it follows that $\theta_n(\tau)$ is well and uniquely defined for all $\tau \in I_n$.

One can check that $i\omega_n(\tau^*)$ ($\omega_n(\tau^*) > 0$) is a purely imaginary root of Eq. (21) if and only if τ^* is a root of the function S_n^m, defined by

$$S_n^m(\tau) = \tau - \frac{\theta_n(\tau) + 2m\pi}{\omega_n(\tau)}, \quad \tau \in I_n, \ \text{with} \ \ n, m \in \mathbb{N}_0. \tag{30}$$

Obviously, $S_n^m(0^+) < 0$ if $0 \in \partial I_n$. Observing that when τ is close to the border of I_n, which is not equal to zero, $\omega_n(\tau) \to 0$ as well as $\sin\theta_n(\tau) \to 0$ and $\cos\theta_n(\tau) \to -1$ imply $\theta_n(\tau) \to \pi$, therefore, $S_n^m(\tau) \to -\infty$ for any $m \in \mathbb{N}_0$.

The following is the result introduced by Beretta and Kuang [15].

Lemma 3.8. *For a fixed $n_0 \in \mathbb{N}_0$, assume that the function $S_{n_0}^m(\tau)$ has a simple positive root $\tau^* \in I_{n_0}$ for some $m \in \mathbb{N}_0$, then a pair of simple purely imaginary roots $\pm i\omega_{n_0}(\tau^*)$ of Eq. (21) exists at $\tau = \tau^*$ and*

$$\begin{aligned} &\operatorname{Sign}\Big\{\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\Big|_{\lambda = i\omega_{n_0}(\tau^*)}\Big\} \\ &= \operatorname{Sign}\Big\{\frac{\partial F_{n_0}}{\partial \omega_{n_0}}(\omega_{n_0}(\tau^*), \tau^*)\Big\} \times \operatorname{Sign}\Big\{\frac{dS_{n_0}^m(\tau)}{d\tau}\Big|_{\tau=\tau^*}\Big\}. \end{aligned} \tag{31}$$

Since

$$\frac{\partial F_{n_0}}{\partial \omega_{n_0}}(\omega_{n_0}, \tau) = 2\omega_{n_0},$$

condition Eq. (31) is equivalent to

$$\delta(\tau^*) = \text{Sign}\left\{ \frac{d\text{Re}\lambda(\tau)}{d\tau}\bigg|_{\lambda=i\omega_{n_0}(\tau^*)} \right\} = \text{Sign}\left\{ \frac{dS_{n_0}^m(\tau)}{d\tau}\bigg|_{\tau=\tau^*} \right\}.$$

Therefore, this pair of simple conjugate purely imaginary roots crosses the imaginary axis from left to right if $\delta(\tau^*) = 1$ *and from right to left if* $\delta(\tau^*) = -1$.

Remark 3.9. It can be easily observed that $I_{n+1} \subset I_n$, $S_n^m(\tau) > S_n^{m+1}(\tau)$ for all $\tau \in I_n$ and $S_n^m(\tau) > S_{n+1}^m(\tau)$ for all $\tau \in I_{n+1}$. Thus, if $S_{n_0}^{m_0}$ has no zero in I_{n_0}, then for any $n \ge n_0$, $m \ge m_0$, $\tau \in I_n$, we have $S_n^m(\tau) \le S_{n_0}^{m_0}(\tau)$ and S_n^m have no zeros in I_n.

Remark 3.10. For $n, m \in \mathbb{N}_0$, denote the set of the zeros of S_n^m by

$$J_n^m = \{\tau_n^m | \tau_n^m \in I_n, S_n^m(\tau_n^m) = 0\}.$$

In what follows, we always assume $\frac{dS_n^m(\tau)}{d\tau}(\tau_n^m) \neq 0$ and $J_{n_1}^{m_1} \cap J_{n_2}^{m_2} = \emptyset$ for $n_1 > n_2$ and $m_1 < m_2$. Rearrange these roots in the set

$$J = \bigcup_{n,m\in\mathbb{N}_0} J_n^m = \{\tau_0,\ \tau_1, \cdots \tau_k\}, \quad \text{with } \tau_i < \tau_{i+1}, 0 \le i \le k-1.$$

Applying Corollary 2.4 in [16], we can draw the conclusion: If $b > b_0$ and $\alpha > 1$, then all roots of Eq. (21) have negative real parts when $\tau \in [0, \tau_0) \cup (\tau_k, \hat{\tau}]$ and at least a pair of roots has positive real parts when $\tau \in (\tau_0, \tau_k)$. Furthermore, all other roots of Eq. (21), except a pair of purely imaginary roots, have negative real parts when $\tau = \tau_0$ or $\tau = \tau_k$.

Now we can state the following theorem on the existence of a Hopf bifurcation at the positive steady state.

Theorem 3.11. *Assume that* $b > b_0$ *and* $\alpha > 1$.

(i) *If either* I_0 *is empty or the function* S_0^0 *has no positive zero in* $I_0(\neq \emptyset)$*, then for all* $\tau \in [0, \hat{\tau})$*, the steady state* u_0 *of Eq. (4) is locally asymptotically stable;*
(ii) *If* $J \neq \emptyset$*, then the steady state* u_0 *of Eq. (4) is locally asymptotically stable for* $\tau \in [0, \tau_0) \cup (\tau_k, \hat{\tau}]$ *and unstable for* $\tau \in (\tau_0, \tau_k)$ *with a Hopf bifurcation occurring at* u_0 *when* $\tau = \tau_i \in J$.

Theorem 3.11 gives some sufficient conditions to ensure that Eq. (4) undergoes a Hopf bifurcation at u_0. Next, under the conditions of Theorem 3.11(ii), we shall use the center manifold and normal form theories presented by Wu [22] and Faria [26] to study the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions from u_0. As the details are given in the Appendix, we summarize the results in the following theorem.

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Theorem 3.12. *Assume that the conditions ensuring that Hopf bifurcation at u_0 occurs in Theorem 3.11(ii) are fulfilled. Then the periodic solutions bifurcated from u_0 are asymptotically stable (unstable) on the center manifold if* $\mathrm{Re}(c_1(0)) < 0$ (> 0). *In particular, if $b > b_0$ and $\alpha > 1$, then the bifurcating periodic solutions at the bifurcation value $\tau = \tau_0, \tau_k$ is stable (unstable) if* $\mathrm{Re}(c_1(0)) < 0$ (> 0).

Here $c_1(0)$ is derived in the Appendix. In Section 5, we shall give a example to illustrate the above results.

4. Global Hopf bifurcation analysis

In this section, we study the global continuation of periodic solutions bifurcating for Eq. (4) by using global Hopf bifurcation theorem given by Wu [22]. Assume that $b > b_0$, $\alpha > 1$ and $J \neq \emptyset$. Then from Theorem 3.11(ii) we know that Hopf bifurcation occurs at u_0 and nontrivial periodic solutions exist when τ is near $\tau_n^m \in J$. Denote

$$J_+^m = \{\tau | \tau \in I_n, S_n^m(\tau) = 0, \frac{dS_n^m(\tau)}{d\tau} > 0, n \in \mathbb{N}_0\},\ m \in \mathbb{N}_0,$$
$$J_-^m = \{\tau | \tau \in I_n, S_n^m(\tau) = 0, \frac{dS_n^m(\tau)}{d\tau} < 0, n \in \mathbb{N}_0\},\ m \in \mathbb{N}_0,$$

and $J^m = J_+^m \cup J_-^m$. Assume further that $J - J^0 \neq \emptyset$. In this section, we will investigate the global continuation of periodic solutions bifurcated from the point (u_0, τ_n^m) $(\tau_n^m \in J - J^0)$ as the bifurcation parameter τ varies.

Lemma 4.1. *Assume that $b > b_0$ and $\alpha > 1$. Then system (4) has no nontrivial periodic solution of period τ.*

Proof. From Lemma 3.3, we know that system (4) has no nontrivial periodic solution of period $\tau > \tau_{\max}$. Let $u(x,t)$ be a periodic solution to Eq. (4) of period $\tau \in [0, \tau_{\max}]$. Then it is a periodic solution to the following system

$$\begin{cases} u_t = d\Delta u - r_1 u - \dfrac{u^2}{1+u^2} + be^{-r_2\tau} ue^{-\alpha u}, x \in \Omega, t > 0, \\ u_\nu = 0, \qquad x \in \partial\Omega, t > 0, \\ u(x,0) = \eta(x,0) \geq 0, \qquad x \in \Omega. \end{cases} \tag{32}$$

The corresponding ordinary differential equation of Eq. (32) is

$$\begin{cases} v_t = -r_1 v - \dfrac{v^2}{1+v^2} + be^{-r_2\tau} ve^{-\alpha v}, \\ v = v_0. \end{cases} \tag{33}$$

Now we can distinguish two cases.

Case (i): $\tau \in [0, \tau_{\max})$.

From Theorem 2.1(ii), Eq. (33) has the unique positive equilibrium u_0 and $v_t > 0$ for $0 < v < u_0$ and $v_t < 0$ for $v > u_0$. Therefore, $\lim_{t\to\infty} v(t, v_0) = u_0$ for any $v_0 > 0$. If $\eta(x, 0) \not\equiv 0$, then strong maximum principle implies that $u(x, t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$. Hence, for a fixed $\epsilon > 0$, we have $u(x, \epsilon) > 0$ for $x \in \bar{\Omega}$. Let $w(x, t) = u(x, t + \epsilon)$ and

$$m = \min_{x\in\bar{\Omega}} u(x, \epsilon), \quad M = \max_{x\in\bar{\Omega}} u(x, \epsilon).$$

Then $w(x, t)$ satisfies

$$\begin{cases} w_t = d\Delta w - r_1 w - \dfrac{w^2}{1+w^2} + be^{-r_2\tau} we^{-\alpha w}, & x \in \Omega, t > 0, \\ w_\nu = 0, \quad x \in \partial\Omega, t > 0, \\ w(x, 0) = u(x, \epsilon), \quad x \in \Omega, \end{cases} \tag{34}$$

and $z(t, M)$ and $z(t, m)$ are upper and lower solutions of Eq. (34). Notice that

$$\lim_{t\to\infty} v(t, M) = \lim_{t\to\infty} v(t, m) = u_0.$$

Which implies $\lim_{t\to\infty} u(x, t) = \lim_{t\to\infty} w(x, t) = u_0$. Moreover, if $\eta(x, 0) \equiv 0$, then we have $u(x, t) \equiv 0$. Therefore, system (4) has no nontrivial periodic solution of period $\tau \in [0, \tau_{\max})$.

Case (ii): $\tau = \tau_{\max}$.

From Theorem 2.1(i), Eq. (33) has the unique negative equilibrium $v = 0$ and $v_t < 0$ for any $v > 0$. It follows that $\lim_{t\to\infty} v(t, v_0) = 0$ for any $v_0 \geq 0$. Using the similar method, we can obtain that every solution $u(x, t)$ of Eq. (32) converges to $u = 0$ (uniformly in x) as $t \to \infty$. Therefore, system (4) has no nontrivial periodic solution of period $\tau = \tau_{\max}$. The proof is complete. □

Throughout this section, we closely follow the notations in [22]. To state the global Hopf bifurcation theorem, we define that

(i) $E = C(S^1, X)$ is a real isometric Banach representation of the group $G = S^1 := \{z \in \mathbb{C} : |z| = 1\}$;
(ii) Let $E^G := \{x \in E : gx = x$ for all $g \in G\}$. Then $E^G = X$, and E has an isotypical direct sum decomposition $E = E^G \bigoplus_{k=1}^{\infty} E_k$ where $E_k = \{e^{ikt}x : x \in X\}$ for $k \geq 1$.

Then from [22], Eq. (4) can be casted into an integral equation which is continuously differentiable, completely continuous and G-invariant.

Assume that $b > b_0$ and $\alpha > 1$. From Lemma 3.3 and the proof of Lemma 4.1 we know that Eq. (4) has no positive nonconstant steady state. Thus, u_0 is the unique positive steady state solution of Eq. (4) when $\tau \in [0, \tau_{\max})$. From $P_n(0, \tau) + Q(0, \tau) > 0$ for any $\tau \in [0, \tau_{\max})$ and $n \in \mathbb{N}_0$, we know that 0 is not an eigenvalue of Eq. (21), hence the assumption H(1) in [22, Sect.6.5] is satisfied. When $\tau = \tau_n^m$, Eq. (21) has a unique pair of purely imaginary eigenvalues

$\pm i\omega_n(\tau_n^m)$, hence the assumption H(2) in [22, Sect.6.5] is satisfied. We choose sufficiently small $\varepsilon_0, \varsigma_0 > 0$, and define the local steady state manifold

$$M = \{(u_0, \tau, \omega) : |\tau - \tau_n^m| < \varepsilon_0, |\omega - \omega_n(\tau_n^m)| < \varsigma_0\} \subset E^G \times \mathbb{R} \times \mathbb{R}_+.$$

Then for

$$(\tau, \omega) \in [\tau_n^m - \varepsilon_0, \tau_n^m + \varepsilon_0] \times [\omega_n(\tau_n^m) - \varsigma_0, \omega_n(\tau_n^m) + \varsigma_0],$$

$\pm i\omega_n(\tau_n^m)$ is an eigenvalue of Eq. (21) if and only if $\tau = \tau_n^m$ and $\omega = \omega_n(\tau_n^m)$. From [22, Lemma 6.5.3], we conclude that $(u_0, \tau_n^m, \omega_n(\tau_n^m))$ is an isolated singular point in M.

Let $\mu_k(u_0, \tau_n^m, \omega_n(\tau_n^m))(k = 1, 2, \cdots)$ be the generalized crossing number defined in [22, Sect.6.5]. Then from Lemma 3.8, if $\lambda(\tau) = \alpha(\tau) \pm i\beta(\tau)$ are the eigenvalues of Eq. (21) satisfying $\lambda(\tau_n^m) = \pm i\omega_n(\tau_n^m)$, then $\frac{dS_n^m(\tau)}{d\tau} > 0(< 0)$ implies that $\mu_1(u_0, \tau_n^m, \omega_n(\tau_n^m)) = 1(-1)$. Hence one obtains the local topological Hopf bifurcation for Eq. (4) at $\tau = \tau_n^m$.

Next we consider the global nature of the Hopf bifurcation. Let

$$S = \mathrm{Cl}\{(z, \tau, \omega) \in E \times \mathbb{R} \times \mathbb{R}_+ : u(\cdot, t) = z(\cdot, \omega t) \text{ is a nontrivial } \frac{2\pi}{\omega} \text{ periodic solution of Eq. (4)}\}.$$

Then from the local bifurcation theorem, $(u_0, \tau_n^m, \omega_n(\tau_n^m)) \in S$. We also define the complete steady state manifold:

$$M^* = \{(u_0, \tau) : \tau \in \mathbb{R}\} \subset E^G \times \mathbb{R}.$$

Let $\mathfrak{C}_n^m(u_0, \tau_n^m, \omega_n\tau_n^m)$ be the connected component of S, for which $(u_0, \tau_n^m, \omega_n\tau_n^m)$ belongs to and denote $\mathrm{Proj}_\tau \mathfrak{C}_n^m(u_0, \tau_n^m, \omega_n\tau_n^m)$ its projection on τ component. Then we can state the global Hopf bifurcation theorem given by Wu:

Lemma 4.2. *[22, Theorem 6.5.5] For each connected component $\mathfrak{C}_n^m$, at least one of the following holds:*

(i) $\mathfrak{C}_n^m$ *is unbounded, i.e.,*

$$\sup\{\max_{t\in\mathbb{R}} |z(t)| + |\tau| + \omega + \omega^{-1} : (z, \tau, \omega) \in \mathfrak{C}_n^m\} = \infty;$$

(ii) $\mathfrak{C}_n^m \cap M^* \times \mathbb{R}_+$ *is finite and for all $k \geq 1$, one has the equality*

$$\sum_{(u_0, \tau_n^m, \omega_n\tau_n^m) \in \mathfrak{C}_n^m \cap M^* \times \mathbb{R}_+} \mu_k(u_0, \tau_n^m, \omega_n(\tau_n^m)) = 0.$$

Lemma 4.3. *All periodic solutions of Eq. (4) are uniformly bounded.*

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Proof. Let $u(x,t)$ be a periodic solution of Eq. (4) and

$$u(x_0,t_0)=\max_{x\in\bar{\Omega},t>0} u(x,t).$$

Then we have

$$\frac{\partial u(x,t)}{\partial t}|_{(x_0,t_0)}=0,\quad \frac{\partial^2 u(x,t)}{\partial x^2}|_{(x_0,t_0)}\le 0.$$

Which implies that

$$r_1u(x_0,t_0)+\frac{u^2(x_0,t_0)}{1+u^2(x_0,t_0)}\le be^{-r_2\tau}u(x_0,t_0-1)e^{-\alpha u(x_0,t_0-1)}.$$

Clearly, it follows that $u(x_0,t_0)\le \frac{b}{\alpha r_1 e}$. The proof is complete. □

Lemma 4.4. *Suppose that $(z,\tau,\omega)\in\mathfrak{C}_n^m$ for $n,m\in\mathbb{N}_0$ and let $u(x,t)=z(x,\omega t)$ be a $\frac{2\pi}{\omega}$-periodic solution of Eq. (4) with delay τ. Then $u(x,t)>0$ for $x\in\bar{\Omega}$ and $t>0$.*

Proof. For each fixed $\tau\in\mathrm{Proj}_\tau\mathfrak{C}_n^m(u_0,\tau_n^m,\omega_n\tau_n^m)$, let $z(x,t,\tau)$ be the corresponding nontrivial periodic solution. Since $z(x,t,\tau)\to u_0$ uniformly for $x\in\bar{\Omega}$ and $t>0$ when $\tau\to\tau_n^m$, then $\inf_{x\in\bar{\Omega},\ t>0} z(x,t,\tau)$ is continuous with respect to τ and $z(x,t,\tau)>0$ when τ sufficiently reaches τ_n^m. Suppose that there exists a $\tau^*\in\mathrm{Proj}_\tau\mathfrak{C}_n^m(u_0,\tau_n^m,\omega_n\tau_n^m)$ such that $\inf_{x\in\bar{\Omega},\ t>0} z(x,t,\tau^*)=0$. Then there exists x^* and t^* satisfied $z(x^*,t^*,\tau^*)=0$. If there exists a x^{**} such that $z(x^{**},t^*,\tau^*)>0$, then $z(x,t,\tau^*)>0$ for $x\in\bar{\Omega}$ and $t>t^*$, which contradicts with $z(x^*,t^*,\tau^*)=0$. If $z(x,t^*,\tau^*)=0$ for $x\in\bar{\Omega}$, then we have $z(x,t,\tau^*)\equiv 0$, which contradicts with the fact that $z(x,t,\tau^*)$ is a nontrivial periodic solution of Eq. (4). The proof is complete. □

Up to now, we have prepared sufficiently to state the following global Hopf bifurcation results.

Theorem 4.5. *Assume that $b>b_0$, $\alpha>1$ and $J-J^0\neq\emptyset$. For any $\tau^*\in J_\pm^m$, $m\in\mathbb{N}$, there exists a $\tau_*\in J_\mp^m$ such that system (4) has at least one positive periodic orbit when τ varies between τ^* and τ_*.*

Proof. Without loss of generality, we assume that $\tau^*\in J_n^m\cap J_+^m$ for $n\in\mathbb{N}_0$, $m\in\mathbb{N}$ and still denote τ^* by τ_n^m. Notice that

$$2m\pi<\omega_n\tau_n^m<2(m+1)\pi,\quad m\in\mathbb{N}.$$

It follows that

$$\frac{1}{m+1}<\frac{2\pi}{\omega_n\tau_n^m}<\frac{1}{m},\quad m\in\mathbb{N}.$$

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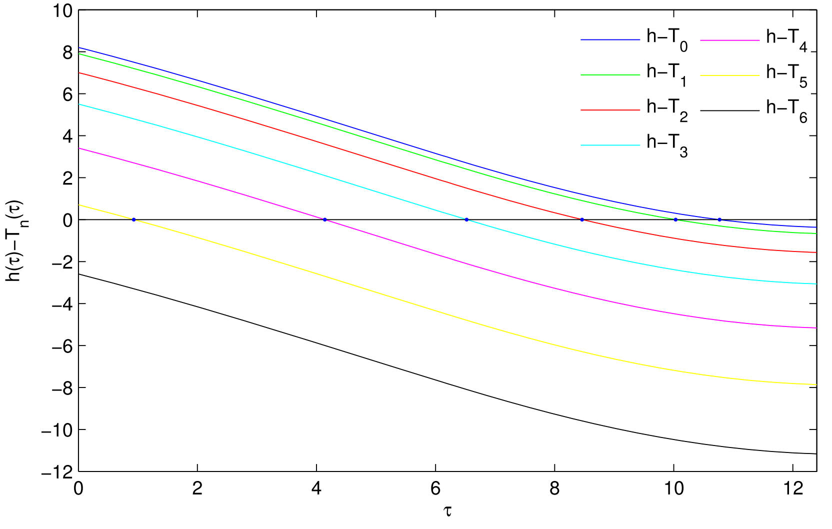

Fig. 1. The graphs of $h - T_n$ on $[0, \hat{\tau})$.

Assume that $(z, \tau, \omega) \in \mathfrak{C}_n^m(u_0, \tau_n^m, \omega_n \tau_n^m)$ for $m \in \mathbb{N}$. Applying Lemma 4.1, one has that $\frac{\tau}{m+1} < \frac{2\pi}{\omega} < \frac{\tau}{m}$. From Lemma 4.3, we have that the projection of $\mathfrak{C}_n^m(u_0, \tau_n^m, \omega_n \tau_n^m)$ onto the z-space is bounded. Meanwhile, Lemma 4.1 and Lemma 3.3 lead to system (4) has no nontrivial periodic solution when $\tau = 0$ or $\tau > \tau_{\max}$. Consequently, we can obtain $\mathfrak{C}_n^m$ is finite and

$$\sum_{(u_0, \tau_n^m, \omega_n \tau_n^m) \in \mathfrak{C}_n^m \cap M^* \times \mathbb{R}_+} \mu_1(u_0, \tau_n^m, \omega_n(\tau_n^m)) = 0.$$

This completes the proof. □

5. Simulations

According to Table 1 and the variable substitution at the beginning of Section 2, we choose a set of parameter values of Eq. (4):

$$r_1 = 0.1980, \; r_2 = 1.1, \; b = 648649.8, \; \alpha = 11.8572, \; d = 0.3, \; l = 1. \tag{35}$$

Under this parameter set, one can easily see that $b > b_0 \approx 0.7658$ and $\alpha > 1$. By calculation, we obtain that

$$\hat{\tau} \approx 12.4086 \text{ and } \tau_{\max} \approx 13.64.$$

In order to gain the set I_n, we picture the graphs of $h - T_n$ in Fig. 1. Note that $\tau \in I_n$ is equivalent to $h(\tau) - T_n(\tau) > 0$. Hence, we have

$$I_0 = [0, 10.77), \quad I_1 = [0, 10.03), \quad I_2 = [0, 8.46), \quad I_3 = [0, 6.52),$$
$$I_4 = [0, 4.14), \quad I_5 = [0, 0.93), \quad I_n = \emptyset, n \geq 6.$$

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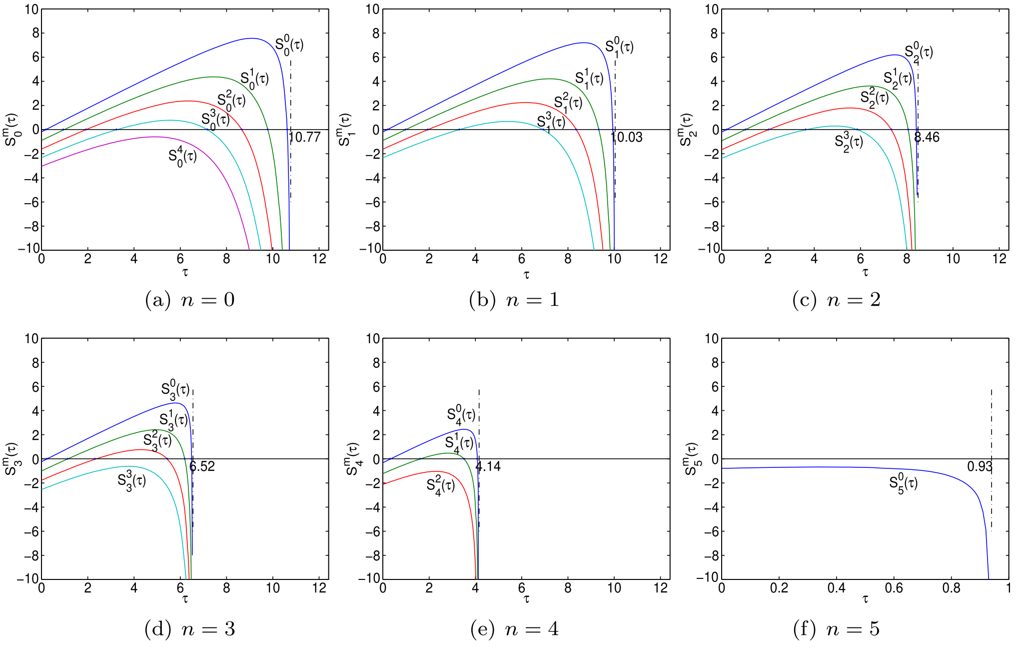

Fig. 2. The graphs of $S_n^m(\tau)$ on I_n.

Accordingly, the pictures of S_n^m on I_n can be drawn clearly (see Fig. 2). From Fig. 2, we obtain the set of the zeros of S_n^m

$$\begin{aligned}
&J_0^0 = \{0.19, 10.62\},\ J_0^1 = \{0.94, 9.78\},\ J_0^2 = \{1.92, 8.66\},\ J_0^3 = \{3.24, 7.17\},\\
&J_1^0 = \{0.20, 9.93\},\ \ J_1^1 = \{0.98, 9.33\},\ J_1^2 = \{1.94, 8.35\},\ J_1^3 = \{3.30, 6.93\},\\
&J_2^0 = \{0.21, 8.44\},\ \ J_2^1 = \{1.03, 8.06\},\ J_2^2 = \{2.03, 7.34\},\ J_2^3 = \{3.63, 5.86\},\\
&J_3^0 = \{0.23, 6.48\},\ \ J_3^1 = \{1.13, 6.19\},\ J_3^2 = \{2.31, 5.44\},\ J_4^0 = \{0.31, 4.07\},\\
&J_4^1 = \{1.61, 3.51\},
\end{aligned}$$

and $\frac{dS_n^m(\tau)}{d\tau}(\tau_n^m) \neq 0$. Hence, we have

$$\tau_0 \approx 0.19 \ \text{ and } \ \tau_k \approx 10.62 \ \text{ with } \ k = 33.$$

Therefore, the unique positive steady state u_0 is asymptotically stable when $\tau \in [0, \tau_0) \cup (\tau_k, \tau_{\max})$, and unstable when $\tau \in (\tau_0, \tau_k)$, as well as Hopf bifurcation takes place when $\tau = \tau_i \in J$. The stability of u_0 is illustrated by Fig. 3(a) and (b). When $\tau > \tau_{\max}$, the positive steady state u_0 disappears, and by Theorem 3.4, the steady state $u = 0$ is globally asymptotically stable (see Fig. 3(c)).

By using the algorithm given in the Appendix, we can obtain $\mathrm{Re}c_1(0)$ corresponding to $\tau = \tau_0$ and $\tau = \tau_k$, as

$$\mathrm{Re}c_1^0(0) \approx -76.83 \ \text{ and } \ \mathrm{Re}c_1^k(0) \approx -60.43,$$

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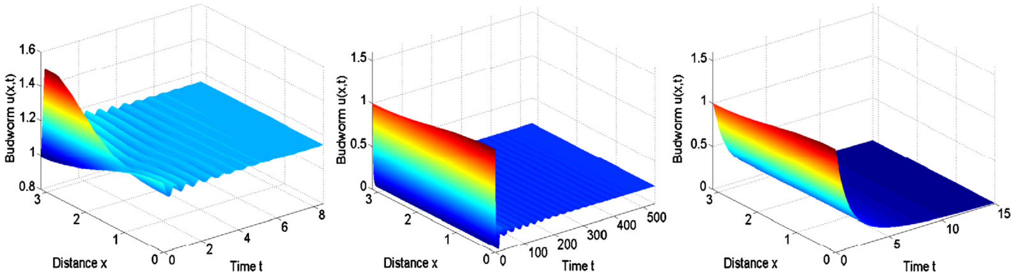
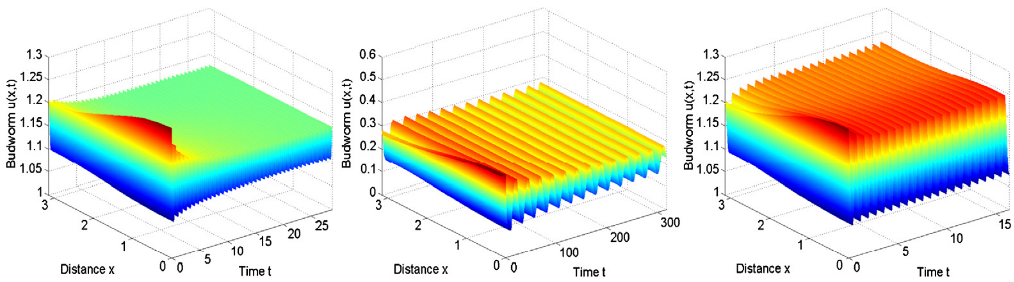


(a) $\tau = 0.18$ (b) $\tau = 10.8$ (c) $\tau = 15$

Fig. 3. The unique positive steady state u_0 of system (4) is asymptotically stable when $\tau \in [0, \tau_0) \cup (\tau_k, \tau_{\max})$ and the steady state $u = 0$ is globally asymptotically stable when $\tau > \tau_{\max}$, where $\eta(x, t) = 1.1 + 0.1 \cos x$ and $\tau = 0.18 < \tau_0 \approx 0.19$ in (a), $\tau = 10.8 > \tau_k \approx 10.62$ in (b) and $\tau = 15 > \tau_{\max} \approx 13.64$ in (c).



(a) $\tau = 0.195$ (b) $\tau = 10.6$ (c) $\tau = 0.205$

Fig. 4. For system (4), the bifurcating periodic solutions are asymptotically stable when $\tau > \tau_0$ and $\tau < \tau_k$ and is close to τ_0 and τ_k, respectively; and system (4) has a spatially inhomogeneous periodic solution when $\tau > \tau_1$ and is close to τ_1, where $\eta(x, t) = u_0 + 0.1 + 0.04 \cos x$ and $\tau = 0.195 > \tau_0 \approx 0.19$ in (a), $\tau = 10.6 < \tau_k \approx 10.62$ in (b) and $\tau = 0.205 > \tau_1 \approx 0.20$ in (c).

respectively. It follows that the direction of Hopf bifurcation is forward at τ_0 and backward at τ_k; The bifurcating periodic solutions at $\tau = \tau_0$ and $\tau = \tau_k$ are all stable. These are illustrated in Fig. 4 (a) and (b). Moreover, by using the similar method, we can obtain that the direction of Hopf bifurcation is forward at τ_1. This implies that system (4) has a spatially inhomogeneous periodic solutions when $\tau > \tau_1$ and is close to τ_1 (see Fig. 4(c)). From the global Hopf bifurcation result Theorem 4.5, we know that, when $\tau \in (\tau_5, \tau_{32}) \backslash J$, system (4) has at least one positive periodic orbit, where $\tau_5 \approx 0.94$ and $\tau_{32} \approx 9.78$. To verify the extended existence of bifurcating periodic solutions, we choose $\tau = 1.75$ and $\tau = 2.75$ far away from the bifurcation point. The corresponding numerical simulation results are shown in Fig. 5.

Remark 5.1. From Table 1 and the above-mentioned variable substitution, the time delay τ in the model given by Vaidya and Wu belongs to the interval [0.75, 2] is equivalent to the time delay τ in system (4) belongs to the interval [1.1366, 3.0309]. So we can conclude that the model given by Vaidya and Wu with diffusion has at least one positive periodic orbit when the parameters are choose as which in Table 1. This can be used to explain the phenomenon of periodic outbreaks of spruce budworm.

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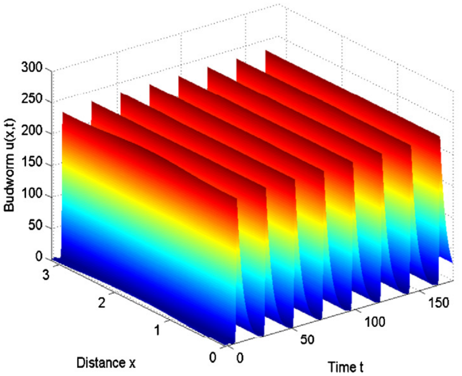
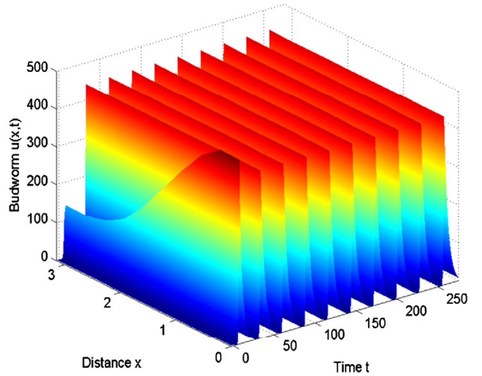


(a) $\tau = 1.75$, $\eta(x,t) = u_0 + 1 + x^2(x-\pi)^2$



(b) $\tau = 2.75$, $\eta(x,t) = u_0 + 1 + \cos x$

Fig. 5. System (4) has a positive periodic solution when $\tau = 1.75$ and $\tau = 2.75$.

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Appendix A. Properties of Hopf bifurcation

In this section, by applying the normal formal theory and the center manifold theorem of partial differential equations with delay [22,26], we study the direction of Hopf bifurcation and stability of the bifurcating periodic solutions from u_0 under the conditions of Theorem 3.11(ii).

By normalizing the delay τ by the time scaling $t \mapsto \frac{t}{\tau}$ and using the change of variables $u(x,t) = u(x,\tau t)$, system (4) is transformed into

$$\frac{\partial u(x,t)}{\partial t} = d\tau\Delta u(x,t) - \tau\Big(r_1 u(x,t) - \frac{u^2(x,t)}{1+u^2(x,t)} + be^{-r_2\tau}u(x,t-1)e^{-\alpha u(x,t-1)}\Big). \quad \text{(A.1)}$$

Without loss of generality, we denote the critical value τ_n^m at which Eq. (A.1) undergoes a Hopf bifurcation at u_0. Set $\tau = \tau_n^m + \mu$, then $\mu = 0$ is the Hopf bifurcation value of system (A.1). Let $U(t) = u(x,t) - u_0$, then in an abstract form in the space $\mathscr{C} = C([-1,0],X)$, Eq. (A.1) can be written in the form:

$$\dot{U}(t) = \tilde{d}\Delta U(t) + L_\mu(U_t) + F(\mu, U_t),$$

where $\tilde{d} = d(\tau_n^m + \mu)$ and $L_\mu : \mathscr{C} \to X$, $F : \mathscr{C} \to X$ are defined, respectively, by

$$L_\mu(\phi) = (\tau_n^m + \mu)\big(-T_0(\tau_n^m + \mu)\phi(0) - h(\tau_n^m + \mu)\phi(-1)\big),$$
$$F(\mu,\phi) = (\tau_n^m + \mu)\big(a_{20}\phi^2(0) + a_{02}\phi^2(-1) + a_{30}\phi^3(0) + a_{03}\phi^3(-1)\big) + \mathcal{O}(4),$$

where T_0 and h are defined by (24) and

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$$a_{20}=\frac{2(3u_0^2-1)}{(1+u_0^2)^3},\qquad a_{02}=be^{-r_2\tau}e^{-\alpha u_0}(\alpha^2u_0-2\alpha),$$
$$a_{30}=\frac{24u_0(1-u_0^2)}{(1+u_0^2)^4},\qquad a_{03}=be^{-r_2\tau}e^{-\alpha u_0}(3\alpha^2-\alpha^3u_0).$$

The linearized equation at the origin has the form

$$\dot{U}(t)=\tilde{d}\Delta U(t)+L_\mu(U_t). \tag{A.2}$$

By Riesz representation theorem, there exist a bounded variation function

$$\eta(\theta,\mu)=\begin{cases}(\tau_n^m+\mu)h(\tau_n^m+\mu), & \theta=-1,\\ 0, & -1<\theta<0,\\ -(\tau_n^m+\mu)\left(\frac{dn^2}{l^2}+T_0(\tau_n^m+\mu)\right), & \theta=0,\end{cases}$$

such that

$$-(\tau_n^m+\mu)\frac{dn^2}{l^2}\phi(0)+L_\mu(\phi)=\int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta),\quad \phi(\theta)\in C([-1,0],\mathbb{R}).$$

We have that the solution operator of Eq. (A.2) is a C_0 semigroup, and the infinitesimal generator A_μ is given by

$$A_\mu(\phi(\theta))=\begin{cases}\dot{\phi}(\theta), & \theta\in[-1,0),\\ \tilde{d}\Delta\phi(0)+L_\mu(\phi), & \theta=0,\end{cases} \tag{A.3}$$

and the domain dom (A_μ) of A_μ is

$$\text{dom }(A_\mu):=\{\phi\in\mathscr{C}:\dot{\phi}\in\mathscr{C},\phi(0)\in\text{dom }(\Delta),\dot{\phi}(0)=\tilde{d}\Delta\phi(0)+L_\mu(\phi)\}.$$

Hence, Eq. (A.1) can be rewritten as the abstract ODE in $\mathscr{C}$:

$$\dot{U}_t=A_\mu U_t+R(\mu,U_t),$$

where

$$R(\mu,U_t)(\phi)=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,U_t), & \theta=0.\end{cases}$$

Define the adjoint operators of A_0

$$A^*(\psi(s))=\begin{cases}-\dot{\psi}(s), & s\in(0,1],\\ \int_{-1}^0 d\eta(s,0)\psi(-s), & s=0,\end{cases}$$

under the bilinear form

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$$(\psi(s), \phi(\theta)) = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta, 0)\phi(\xi) d\xi,$$

where $\psi(s) \in C([0,1], \mathbb{R}^*)$ and $\phi(\theta) \in C([-1,0], \mathbb{R})$. It can be verified that $\pm i\omega_n\tau_n^m$ are the eigenvalues of A_0 and A^*, and $q(\theta) = e^{i\omega_n\tau_n^m\theta}$ and $q^*(s) = \bar{D}_n e^{i\omega_n\tau_n^m s}$ are eigenvectors of A_0 and A^* corresponding to the eigenvalue $i\omega_n\tau_n^m$ and $-i\omega_n\tau_n^m$, respectively, where $\omega_n = \omega_n(\tau_n^m)$ and

$$D_n = \left(1 + \tau_n^m T_n(\tau_n^m) + i\omega_n\tau_n^m\right)^{-1}.$$

By setting

$$f_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}\cos\frac{nx}{l}, & n \in \mathbb{N}, \end{cases}$$

and using a computation process similar to that in [22], we can obtain the coefficients which will be used in determining the important quantities:

$$g_{20} = \begin{cases} 0, & n \in \mathbb{N}, \\ 2D_n\tau_n^m(a_{20} + a_{02}e^{-2i\omega_n\tau_n^m}), & n = 0, \end{cases}$$

$$g_{11} = \begin{cases} 0, & n \in \mathbb{N}, \\ 2D_n\tau_n^m(a_{20} + a_{02}), & n = 0, \end{cases}$$

$$g_{02} = \begin{cases} 0, & n \in \mathbb{N}, \\ 2D_n\tau_n^m(a_{20} + a_{02}e^{2i\omega_n\tau_n^m}), & n = 0, \end{cases}$$

$$\begin{aligned} g_{21} = \frac{D_n\tau_n^m}{l\pi}\int_0^{l\pi}\Bigg[&\Big(a_{20}\big(W_{20}(0) + 2W_{11}(0)\big) + a_{02}\big(e^{i\omega_n\tau_n^m}W_{20}(-1) \\ &+ 2e^{-i\omega_n\tau_n^m}W_{11}(-1)\big)\Big)f_n^2 + (3a_{30} + 3a_{03}e^{-i\omega_n\tau_n^m})f_n^4\Bigg]dx, \end{aligned}$$

where

$$W_{20}(\theta) = \left(\frac{ig_{20}}{\omega_n\tau_n^m}e^{i\omega_n\tau_n^m\theta} + \frac{i\bar{g}_{02}}{3\omega_n\tau_n^m}e^{-i\omega_n\tau_n^m\theta}\right)f_n + E_1 e^{2i\omega_n\tau_n^m\theta},$$

$$W_{11}(\theta) = \left(\frac{-ig_{11}}{\omega_n\tau_n^m}e^{i\omega_n\tau_n^m\theta} + \frac{i\bar{g}_{11}}{\omega_n\tau_n^m}e^{-i\omega_n\tau_n^m\theta}\right)f_n + E_2,$$

and

$$E_1 = \frac{a_{20} + a_{02}e^{-2i\omega_n\tau_n^m}}{2i\omega_n + T_0(\tau_n^m) + h(\tau_n^m)e^{-2i\omega_n\tau_n^m}} + \frac{\left(a_{20} + a_{02}e^{-2i\omega_n\tau_n^m}\right)\cos\frac{2nx}{l}}{2i\omega_n + \frac{4dn^2}{l^2} + T_0(\tau_n^m) + h(\tau_n^m)e^{-2i\omega_n\tau_n^m}},$$

$$E_2 = \frac{a_{20} + a_{02}}{T_0(\tau_n^m) + h(\tau_n^m)} + \frac{(a_{20} + a_{02})\cos\frac{2nx}{l}}{\frac{4dn^2}{l^2} + T_0(\tau_n^m) + h(\tau_n^m)}.$$

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Consequently, g_{21} could be expressed explicitly.

Thus, we can compute the following values:

$$c_1(0) = \frac{i}{2\omega_n \tau_n^m}(g_{20}g_{11} - 2|g_{11}|^2) + \frac{1}{2}g_{21},$$
$$\mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\tau_n^m \delta(\tau_n^m)}, \quad \beta_2 = 2\mathrm{Re}(c_1(0)).$$

By the general Hopf bifurcation theory (see [28]), we know that μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the direction of the Hopf bifurcation is forward (backward), that is the bifurcating periodic solutions exist when $\mu_2 > 0(\mu_2 < 0)$; and β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions on the center manifold are orbitally stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$.

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