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J. Differential Equations 267 (2019) 4293–4319

Journal of Differential Equations

www.elsevier.com/locate/jde

Global in time solvability of the Navier-Stokes equations in the half-space ☆

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Received 19 September 2018; revised 1 April 2019
Available online 21 May 2019

Abstract

In this paper, we study the initial value problem of the Navier-Stokes equations in the half-space. Let a solenoidal initial velocity be given in the function space $\dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$ for $0 < \alpha < 2$, $1 < p, q < \infty$ with $\alpha + 1 = \frac{n}{p} + \frac{2}{q}$ and $\frac{2}{q} < 1 + \frac{n}{p}$. We prove the global in time existence of weak solution $u \in L^q(0, \infty; \dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+)) \cap L^{q_0}(0, \infty; L^{p_0}(\mathbb{R}^n_+))$ for some $p < p_0 < \infty$ and $q < q_0 < \infty$ with $\frac{n}{p_0} + \frac{2}{q_0} = 1$, when the given initial velocity has small norm in function space $\dot{B}^{-\frac{2}{q_0}}_{p_0q_0,0}(\mathbb{R}^n_+)$ ($\supset \dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$). The solution is unique in the class $L^{q_0}(0, \infty; L^{p_0}(\mathbb{R}^n_+))$. Pressure estimates are also given.

MSC: primary 35K61; secondary 76D07

Keywords: Stokes equations; Navier-Stokes equations; Homogeneous initial boundary value; Half-space

☆ Tongkeun Chang was supported by NRF-2017R1D1A1B03033427 and Bum Ja Jin was supported by NRF-2016R1D1A1B03934133.

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https://doi.org/10.1016/j.jde.2019.04.037
0022-0396/

1. Introduction

In this paper, we study the following nonstationary Navier–Stokes equations

$$\begin{aligned} u_t - \Delta u + \nabla p = -\mathrm{div}(u \otimes u), \qquad &\mathrm{div}\, u = 0 \text{ in } \mathbb{R}^n_+ \times (0,\infty), \\ u|_{t=0} = h, \qquad &u|_{x_n=0} = 0, \end{aligned} \tag{1.1}$$

where $u = (u_1, \cdots, u_n)$ and p are the unknown velocity and pressure, respectively, $h = (h_1, \cdots, h_n)$ is the given initial data.

Since the nonstationary Navier–Stokes equations are invariant under the scaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad h_\lambda = \lambda h(\lambda x),$$

it is important to study (1.1) in the so-called critical spaces, i.e., the function spaces with norms invariant under the scaling $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$.

There are a number of papers dealing with global well-posedness for (1.1) in critical spaces. Among them, R. Farwig, H. Sohr and W. Varnhorn [15] showed the unique existence of the Leray Hopf weak solution $u \in L^q(0,\infty; L^p(\Omega))$, $1 = \frac{3}{p} + \frac{2}{q}$ if the initial data $h \in L^2_\sigma(\Omega)$ and $\|e^{-At}h\|_{L^q(0,\infty;L^p(\Omega))}$ is small enough (here Ω is bounded domain in $\mathbb{R}^3$ with boundary of class $C^{2,1}$ and A is the Stokes operator). R. Farwig, Y. Giga and P. Hsu [13] showed the unique existence of the Leray Hopf weak solution $u \in L^q_\alpha(0,\infty; L^p(\Omega))$, $\frac{1}{2} > \alpha = \frac{1}{2}(1 - \frac{3}{p} - \frac{2}{q}) > 0$ if the initial data $h \in L^2_\sigma(\Omega)$ and $\|t^\alpha e^{-At}h\|_{L^q(0,\infty;L^p(\Omega))}$ is small enough, where $L^q_\alpha(0,\infty;X)$ is the weighted Bochner space with norm $\|\cdot\|_{L^q_\alpha(0,\infty;X)} = \|t^\alpha \cdot\|_{L^q(0,\infty;X)}$.

In [7], T. Chang and B. Jin showed the unique existence of solution u in $L^q_\alpha(0,\infty; L^p(\mathbb{R}^n_+))$, $\frac{1}{2} > \alpha = \frac{1}{2}(1 - \frac{n}{p} - \frac{2}{q}) > 0$ for h small in $\dot{B}^{-\frac{2}{q}}_{pq}(\mathbb{R}^n_+)$ (also see [8] for $\alpha = 0$).

R. Danchin and P. Zhang in [12] have studied global solvability of inhomogeneous Navier-Stokes equations in the half space with bounded density, and showed that if the initial velocity in $\dot{B}^{-1+\frac{n}{p}}_{pq}(\mathbb{R}^n_+)$ with $\frac{n}{3} < p < n$, $1 < q < \infty$ and $q \geq \frac{2p}{3p-n}$ is small and initial density in $L^\infty(\mathbb{R}^n_+)$ is close enough to the homogeneous fluid, then (1.1) has a unique solution satisfying

$$\begin{gathered} t^\alpha \big(u_t, D^2_x u, \nabla p\big) \in L^q(0,\infty; L^p(\mathbb{R}^n_+)),\ t^\beta \nabla u \in L^{q_2}(0,\infty; L^{p_2}(\mathbb{R}^n_+)), \\ t^\gamma u \in L^{q_3}(0,\infty; L^{p_3}(\mathbb{R}^n_+)) \end{gathered}$$

for some $\beta, \gamma > 0$, $1 < p_2, p_3, q_2, q_3 < \infty$ with $\alpha = \beta + \gamma$, $\frac{1}{p} = \frac{1}{p_2} + \frac{1}{p_3}$ and $\frac{1}{q} = \frac{1}{q_2} + \frac{1}{q_3}$.

The limiting case $q = \infty$ has been studied by M. Cannone, F. Planchon, and M. Schonbek [5] for $h \in L^3(\mathbb{R}^3_+)$, by H. Amann [3] for $h \in b^{-1+\frac{n}{p}}_{p,\infty}(\Omega)$, $p > \frac{n}{3}$, $p \neq n$, where Ω is a standard domain like $\mathbb{R}^3, \mathbb{R}^3_+$, exterior or bounded domain in $\mathbb{R}^3$, by M. Ri, P. Zhang and Z. Zhang [23] for $h \in b^0_{n\infty}(\Omega)$, where Ω is $\mathbb{R}^n$, $\mathbb{R}^n_+$ or bounded domain with smooth boundary, and $b^s_{p\infty}(\Omega)$ denotes the completion of the generalized Sobolev space $H^s_p(\Omega)$ in $B^s_{p\infty}(\Omega)$. In particular, in [5], the solution exists globally in time when $\|h\|_{\dot{B}^{-1+\frac{3}{p}}_{p\infty}(\mathbb{R}^3_+)}$, $p > 3$ ($L^3 \subset \dot{B}^{-1+\frac{3}{p}}_{p\infty}(\mathbb{R}^n_+)$) is small enough. See also [2,14,17,20–22,25] and the references therein for initial value problem of Navier-Stokes equations in the half space.

In this paper, we consider critical function spaces $\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$ for the space of initial data and $L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))$ for the solution space, respectively, where $0<\alpha<2$, $1<p,q<\infty$ with $\alpha+1=\frac{n}{p}+\frac{2}{q}$ and $\frac{2}{q}<1+\frac{n}{p}$.

In [10], the case $p=q$ has been considered, and the unique existence u in $\dot B^{\alpha,\frac{\alpha}{2}}_{pp}(\mathbb{R}^n_+\times(0,\infty))$ has been shown. Note that $\dot B^{\alpha,\frac{\alpha}{2}}_{pp}(\mathbb{R}^n_+\times(0,\infty))=L^p(0,\infty;\dot B^\alpha_{pp})\cap L^p(\mathbb{R}^n_+;\dot B^{\frac{\alpha}{2}}_{pp}(0,\infty))$.

Our study in this paper is motivated by the result in [5] and [12]. The following texts state our main results.

Theorem 1.1. *Let $0<\alpha<2$, $1<p,q<\infty$ with $\alpha+1=\frac{n}{p}+\frac{2}{q}$ and $\frac{2}{q}<1+\frac{n}{p}$. Assume that $h\in\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$ with $\operatorname{div}h=0$. Then, there is $\epsilon_*>0$ and $p<p_0<\infty$, $q<q_0<\infty$ with $\frac{n}{p_0}+\frac{2}{q_0}=1$ so that if $\|h\|_{\dot B^{-\frac{2}{q_0}}_{p_0q_0,0}(\mathbb{R}^n_+)}<\epsilon_*$, then (1.1) has a solution $u\in L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))\cap L^{q_0}(0,\infty;L^{p_0}(\mathbb{R}^n_+))$. The solution is unique in $L^{q_0}(0,\infty;L^{p_0}(\mathbb{R}^n_+))$.*

Note that $-\frac{2}{q_0}-\frac{n}{p_0}=-1=\alpha-\frac{2}{q}-\frac{n}{p}$, $p<p_0$ and $q<q_0$. Hence Besov embeddings $\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)\subset\dot B^{-\frac{2}{q_0}}_{p_0q_0,0}(\mathbb{R}^n_+)$ hold (see Theorem 6.5.1 in [4]).

Theorem 1.2. *Let (p,q,α) and h satisfy the same conditions as in Theorem 1.1. Then, there is (p_1,q_1,β) with $1<p_1<p$, $1<q_1<q$ and $0<\beta<\alpha<\beta+1<2$ so that if $\alpha>\frac{1}{p}$, then the corresponding pressure p can be decomposed by $p=P_0+\sum_{j=1}^{n-1}D_{x_j}P_j+D_tp_1$ for some $p_1\in L^q(0,\infty;\dot B^{\alpha+1}_{pq}(\mathbb{R}^n_+))$, $P_j\in L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))$, $P_0\in L^{q_1}(0,\infty;\dot B^\beta_{p_1q}(\mathbb{R}^n_+))$.*

The explanation of function spaces and notations is placed in Section 2.

Remark 1.3.

(1) The authors in the papers [13] and [15] studied (1.1) when the initial data h is in space $\dot B^{-1+\frac{n}{p}}_{pq,0}(\mathbb{R}^n_+)$ with $(\frac{n}{p},\frac{2}{q})$ in I and the authors in the paper [12] studied (1.1) when $(\frac{n}{p},\frac{2}{q})$ is in II. In this paper, we study (1.1) when $(\frac{n}{p},\frac{2}{q})$ is in II $\cup$ III. (See Fig. 1.)
(2) Note that if $(\frac{n}{p},\frac{2}{q})$ is in I and III, then $-1+\frac{n}{p}<0$.
(3) If $(\frac{n}{p},\frac{2}{q})\in IV$, then Theorem 1.1 and Theorem 1.2 can be modified: For $0<\alpha<2$, $1<p,q<\infty$ with $\frac{n}{p}+\frac{2}{q}=\alpha+1$, there is $1<p_0,q_0<\infty$ with $\frac{n}{p_0}+\frac{2}{q}=1$ so that if $h\in\dot B^{\alpha-\frac{2}{q}}_{pq,0}\cap\dot B^{-\frac{2}{q_0}}_{p_0q_0}$, then there is $u\in L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))\cap L^{q_0}(0,\infty;L^{p_0}(\mathbb{R}^n_+))$. Without the condition $p<p_0$ and $q<q_0$ we cannot expect the embedding $\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)\subset\dot B^{-\frac{2}{q_0}}_{p_0q_0,0}(\mathbb{R}^n_+)$ (see section 5.1).

For the proof of Theorem 1.1, it is necessary to study the following initial value problem of the Stokes equations in $\mathbb{R}^n_+\times(0,\infty)$:

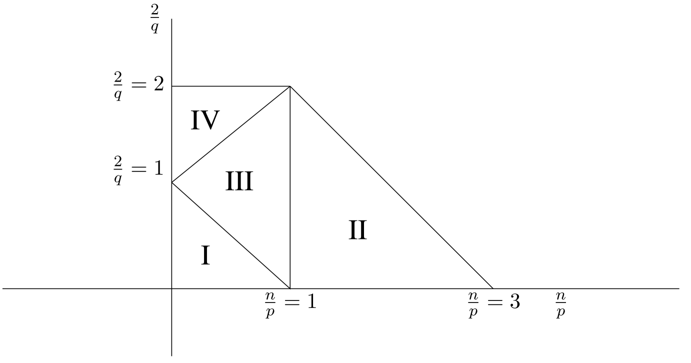

Fig. 1. Region of $(\frac{n}{p}, \frac{2}{q})$.

$$\begin{aligned} u_t - \Delta u + \nabla p = f, \qquad & \operatorname{div} u = 0 \text{ in } \mathbb{R}^n_+ \times (0,\infty), \\ u|_{t=0} = h, \qquad & u|_{x_n=0} = 0, \end{aligned} \tag{1.2}$$

where $f = \operatorname{div}\mathcal{F}$.

In [16], M. Giga, Y. Giga and H. Sohr showed that if $f \in L^q(0,T;\hat{D}(A_p^{-\alpha}))$ and $h=0$ then the solution u of Stokes equations (1.2) satisfies that for $0<\alpha<1$,

$$\int_0^T \Big(\|(\frac{d}{dt})^{1-\alpha} u(t)\|^q_{L^p(\Omega)} + \|A_p^{1-\alpha} u(t)\|^q_{L^p(\Omega)} \Big) dt \le c(p,q,\Omega,\alpha) \int_0^T \|A_p^{-\alpha} f(t)\|^q_{L^p(\Omega)} dt,$$

where A_p is Stokes operator in Ω for standard domain Ω such as bounded domain, exterior domain or half space, and $\hat{D}(T)$ is the completion of $D(T)$ in the homogeneous norm $\|T\cdot\|$. In particular, if $f = \operatorname{div}\mathcal{F}$ with $\mathcal{F} \in L^q(0,T;L^p_\sigma(\Omega))$ then

$$\int_0^T \Big(\|(\frac{d}{dt})^{\frac{1}{2}} u(t)\|^q_{L^p(\Omega)} + \|\nabla u(t)\|^q_{L^p(\Omega)} \Big) dt \le c(p,q,\Omega) \int_0^T \|\mathcal{F}(t)\|^q_{L^p(\Omega)} dt.$$

Estimates for the pressure were, however, not given in [16].

H. Koch and V. A. Solonnikov [18] showed the unique local in time existence of solution $u \in L^q(0,T;L^q(\mathbb{R}^3_+))$ of (1.2) when $f = \operatorname{div}\mathcal{F}$, $\mathcal{F} \in L^q(0,T;L^q(\mathbb{R}^3_+))$ and $h=0$. They also showed that the corresponding pressure p is decomposed by $p = p_1 + \frac{\partial P}{\partial t}$, where p_1 and P satisfy $\|p_1\|_{L^q(0,T;L^q(\mathbb{R}^3_+))} + \|P\|_{L^q(0,T;W^2_q(\mathbb{R}^3_+))} \le c\|\mathcal{F}\|_{L^q(0,T;L^q(\mathbb{R}^3_+))}$. See also [17,19,20,25] and the references therein.

The following theorem states our result on the unique solvability of the Stokes equations (1.2).

Theorem 1.4. *Let $1<p,q<\infty$ and $0\le\alpha\le 2$. Let $h \in \dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$ with $\operatorname{div} h = 0$ and $\mathcal{F} \in L^{q_1}(0,\infty,\dot{B}^{\beta}_{p_1q,0}(\mathbb{R}^n_+))$ for (p_1,q_1,β) satisfying $1<p_1\le p$, $1<q_1\le q$, $0<\beta\le\alpha\le\beta+1\le 2$ and $0=\alpha-\beta-1+n(\frac{1}{p_1}-\frac{1}{p})+\frac{2}{q_1}-\frac{2}{q}$. Then there is a solution u of (1.2) with*

$$\|u\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} \le c\big(\|h\|_{\dot B^{\alpha-2/q}_{pq,0}(\mathbb{R}^n_+)} + \|\mathcal{F}\|_{L^{q_1}(0,\infty,\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))}\big). \tag{1.3}$$

In addition, if $\alpha > \frac{1}{p}$ *then the corresponding pressure* p *can be decomposed by* $p = D_t p_1 + \sum_{j=1}^{n-1} D_{x_j} P_j + P_0$, $p_1 \in L^q(0,\infty;\dot B^{\alpha+1}_{pq}(\mathbb{R}^n_+))$, $P_j \in L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))$ *and* $P_0 \in L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))$ *with*

$$\|p_1\|_{L^q(0,\infty;\dot B^{\alpha+1}_{pq}(\mathbb{R}^n_+))} + \sum_{j=1}^{n-1}\|P_j\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} + \|P_0\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))} \le c\big(\|h\|_{\dot B^{\alpha-2/q}_{pq,0}(\mathbb{R}^n_+)} + \|\mathcal{F}\|_{L^{q_1}(0,\infty,\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))}\big). \tag{1.4}$$

We organize this paper as follows. In Section 2, we introduce the function spaces, definition of the weak solutions of Stokes equations and Navier-Stokes equations. In Section 3, the various estimates of operators related with Newtonian kernel and Gaussian kernel are given. In Section 4, we complete the proof of Theorem 1.4. In Section 5, we give the proof of Theorem 1.1 and Theorem 1.2 applying the estimates in Theorem 1.4 to the approximate solutions.

2. Notations, function spaces and definitions of weak solutions

We denote by x' and $x = (x', x_n)$ the points of the spaces $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$, respectively. The multiple derivatives are denoted by $D^k_x D^m_t = \frac{\partial^{|k|}}{\partial x^k}\frac{\partial^m}{\partial t}$ for multi-index k and nonnegative integers m. Throughout this paper we denote by c various generic constants.

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we denote $\dot H^s_p(\mathbb{R}^n)$ and $\dot B^s_{pq}(\mathbb{R}^n)$ the generalized homogeneous Sobolev spaces (space of Bessel potentials) and the homogeneous Besov spaces in $\mathbb{R}^n$, respectively (see [4,27] for the definition of function spaces). Denote by $\dot H^s_p(\mathbb{R}^n_+)$ and $\dot B^s_{pq}(\mathbb{R}^n_+)$ the restrictions of $\dot H^s_p(\mathbb{R}^n)$ and $\dot B^s_{pq}(\mathbb{R}^n)$, respectively, with norms

$$\|f\|_{\dot H^s_p(\mathbb{R}^n_+)} = \inf\{\|F\|_{\dot H^s_p(\mathbb{R}^n)} \mid F|_{\mathbb{R}^n_+} = f,\ F \in \dot H^s_p(\mathbb{R}^n)\},$$
$$\|f\|_{\dot B^s_{pq}(\mathbb{R}^n_+)} = \inf\{\|F\|_{\dot B^s_{pq}(\mathbb{R}^n)} \mid F|_{\mathbb{R}^n_+} = f,\ F \in \dot B^s_{pq}(\mathbb{R}^n)\}.$$

For a non-negative integer k, $\dot H^k_p(\mathbb{R}^n_+) = \{f \mid \sum_{|l|=k}\|D^l f\|_{L^p(\mathbb{R}^n_+)} < \infty\}$. In particular, $\dot H^0_p(\mathbb{R}^n_+) = L^p(\mathbb{R}^n_+)$.

For $s \in \mathbb{R}$, we denote by $\dot B^s_{pq}(\mathbb{R}^n_+)$, $1 \le p, q \le \infty$ the usual homogeneous Besov space in $\mathbb{R}^n_+$ and denote by

$$\dot H^s_{p,0}(\mathbb{R}^n_+) = \{f \in \dot H^s_p(\mathbb{R}^n_+) \mid \tilde f \in \dot H^s_p(\mathbb{R}^n)\},$$
$$\dot B^s_{pq,0}(\mathbb{R}^n_+) = \{f \in \dot B^s_{pq}(\mathbb{R}^n_+) \mid \tilde f \in \dot B^s_{pq}(\mathbb{R}^n)\},$$

where $\tilde f$ is the zero extension of f over $\mathbb{R}^n$. Note that $\|\tilde f\|_{\dot H^s_p(\mathbb{R}^n)} \le c\|f\|_{\dot H^s_{p,0}(\mathbb{R}^n_+)}$ and $\|\tilde f\|_{\dot B^s_{pq}(\mathbb{R}^n)} \le c\|f\|_{\dot B^s_{pq,0}(\mathbb{R}^n_+)}$.

Note that for $s \ge 0$, $\dot B^{-s}_{pq,0}(\mathbb{R}^n_+)$, $1 < p, q \le \infty$ is the dual space of $\dot B^s_{p'q'}(\mathbb{R}^n_+)$, that is, $\dot B^{-s}_{pq,0}(\mathbb{R}^n_+) = (\dot B^s_{p'q'}(\mathbb{R}^n_+))^*$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

For the Banach space X, we denote by $L^q(0,\infty;X)$, $1\le q<\infty$ the usual Bochner space with norm

$$\|f\|_{L^q(0,\infty;X)} := \Big(\int_0^\infty \|f(t)\|_X^q dt\Big)^{\frac{1}{q}}.$$

For $1<q<\infty$ and $0<\theta<1$, we denote by $(X,Y)_{\theta,q}$ and $[X,Y]_\theta$ the real interpolation and complex interpolation, respectively, of the Banach space X and Y. In particular, for $0<\theta<1$, $\alpha,\alpha_1,\alpha_2\in\mathbb{R}$ and $1<p_1,p_2,q_1,q_2,p,q,r<\infty$,

$$[\dot H^{\alpha_1}_{p_1}(\mathbb{R}^n_+),\dot H^{\alpha_2}_{p_2}(\mathbb{R}^n_+)]_\theta=\dot H^{\alpha}_{p}(\mathbb{R}^n_+),\qquad (\dot H^{\alpha_1}_{p}(\mathbb{R}^n_+),\dot H^{\alpha_2}_{p}(\mathbb{R}^n_+))_{\theta,r}=\dot B^{\alpha}_{pr}(\mathbb{R}^n_+),\tag{2.1}$$

$$[\dot B^{\alpha_1}_{p_1q_1}(\mathbb{R}^n_+),\dot B^{\alpha_2}_{p_2q_2}(\mathbb{R}^n_+)]_\theta=\dot B^{\alpha}_{pq}(\mathbb{R}^n_+),\qquad (\dot B^{\alpha_1}_{pq}(\mathbb{R}^n_+),\dot B^{\alpha_2}_{pq}(\mathbb{R}^n_+))_{\theta,r}=\dot B^{\alpha}_{pr}(\mathbb{R}^n_+),\tag{2.2}$$

$$[L^{q_1}(0,\infty;X),L^{q_2}(0,\infty;Y)]_\theta=L^q(0,\infty;[X,Y]_\theta),\tag{2.3}$$

$$(L^{q_1}(0,\infty;X),L^{q_2}(0,\infty;Y))_{\theta,q}=L^q(0,\infty;(X,Y)_{\theta,q}),\tag{2.4}$$

when $\alpha=\theta\alpha_1+(1-\theta)\alpha_2$, $\frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2}$ and $\frac{1}{q}=\frac{\theta}{q_1}+\frac{1-\theta}{q_2}$. See Theorem 6.4.5, Theorem 5.1.2 and Theorem 5.6.2 in [4].

Definition 2.1 *(Weak solution of the Stokes equations).* Let $1<p,q<\infty$ and $0\le\alpha\le 2$. Let $h,\mathcal{F}$ satisfy the same hypotheses as in Theorem 1.4. A vector field $u\in L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))$ is called a weak solution of the Stokes equations (1.2) if the following conditions are satisfied:

$$-\int_0^\infty\int_{\mathbb{R}^n_+} u\cdot\Delta\Phi dxdt=\int_0^\infty\int_{\mathbb{R}^n_+}\big(u\cdot\Phi_t-\mathcal{F}:\nabla\Phi\big)dxdt+\int_{\mathbb{R}^n_+} h(x)\cdot\Phi(x,0)dx$$

for each $\Phi\in C^\infty_0(\mathbb{R}^n_+\times[0,\infty))$ with $\mathrm{div}_x\Phi=0$. In addition, for each $\Psi\in C^1_c(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} u(x,t)\cdot\nabla\Psi(x)dx=0\quad \text{for all}\quad 0<t<\infty.\tag{2.5}$$

Definition 2.2 *(Weak solution to the Navier-Stokes equations).* Let $1<p,q<\infty$ and $0\le\alpha\le 2$ with $\alpha+1=\frac{n}{p}+\frac{2}{q}$. Let h satisfy the same hypothesis as in Theorem 1.1. A vector field $u\in L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))$ is called a weak solution of the Navier-Stokes equations (1.1) if the following variational formulations are satisfied:

$$-\int_0^\infty\int_{\mathbb{R}^n_+} u\cdot\Delta\Phi dxdt=\int_0^\infty\int_{\mathbb{R}^n_+}\big(u\cdot\Phi_t+(u\otimes u):\nabla\Phi\big)dxdt+\int_{\mathbb{R}^n_+} h(x)\cdot\Phi(x,0)dx\tag{2.6}$$

for each $\Phi\in C^\infty_0(\mathbb{R}^n_+\times[0,\infty))$ with $\mathrm{div}_x\Phi=0$. In addition, for each $\Psi\in C^1_c(\mathbb{R}^n_+)$, u satisfies (2.5).

Remark 2.3. If $0 < \alpha < \frac{2}{q}$, then the term $\int_{\mathbb{R}^n_+} h(x) \cdot \Phi(x,0)dx$ should be replaced by $< h, \Phi(\cdot,0) >$, where $< \cdot, \cdot >$ is the duality pairing between $\dot{B}^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)$ and $\dot{B}^{-\alpha+\frac{2}{q}}_{p'q'}(\mathbb{R}^n_+)$.

3. Preliminary estimates

3.1. Trace theorem

the following lemma is well known trace theorem. The proof is contained in appendix A.

Lemma 3.1. *Let* $1 < p, q < \infty$.

(1) *If* $f \in \dot{B}^\alpha_{pq}(\mathbb{R}^n_+)$ *for* $\alpha > \frac{1}{p}$, *then* $f|_{x_n=0} \in \dot{B}^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1})$ *with*

$$\|f|_{x_n=0}\|_{\dot{B}^{\alpha-1/p}_{pp}(\mathbb{R}^{n-1})} \le c\|f\|_{\dot{H}^\alpha_p(\mathbb{R}^n_+)}, \quad \|f|_{x_n=0}\|_{\dot{B}^{\alpha-1/p}_{pq}(\mathbb{R}^{n-1})} \le c\|f\|_{\dot{B}^\alpha_{pq}(\mathbb{R}^n_+)}.$$

(2) *If* $f \in L^p(\mathbb{R}^n_+)$ *and* $\operatorname{div} f = 0$ *in* $\mathbb{R}^n_+$, *then* $f_n|_{x_n=0} \in \dot{B}^{-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1})$ *with*

$$\|f_n|_{x_n=0}\|_{\dot{B}^{-1/p}_{pp}(\mathbb{R}^{n-1})} \le c\|f\|_{L^p(\mathbb{R}^n_+)}.$$

3.2. Newtonial potential

The fundamental solution of the Laplace equation in $\mathbb{R}^n$ is denoted by

$$N(x) = \begin{cases} \frac{1}{\omega_n(2-n)|x|^{n-2}} & \text{if } n \ge 3, \\ \frac{1}{2\pi} \ln|x| & \text{if } n = 2, \end{cases}$$

ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

We define Nf by

$$Nf(x) = \int_{\mathbb{R}^{n-1}} N(x'-y', x_n) f(y')dy'. \tag{3.1}$$

Observe that $D_{x_n} Nf$ is Poisson operator of Laplace equation in $\mathbb{R}^n_+$ and $D_{x_i} Nf = D_{x_n} N R'_i f$ for $i \neq n$, where $R' = (R_1, \cdots, R_{n-1})$ is the $n-1$ dimensional Riesz operator. Poisson operator is bounded from $\dot{B}^{\alpha-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1})$ to $\dot{H}^\alpha_p(\mathbb{R}^n_+), \alpha \ge 0$ and R' is bounded from $\dot{B}^s_{pq}(\mathbb{R}^{n-1})$ to $\dot{B}^s_{pq}(\mathbb{R}^{n-1})$, $s \in \mathbb{R}$ (see [26]). Hence the following estimates hold.

Lemma 3.2. *Let* $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then*

$$\begin{aligned} \|\nabla_x Nf\|_{\dot{H}^\alpha_p(\mathbb{R}^n_+)} &\le c\|f\|_{\dot{B}^{\alpha-1/p}_{pp}(\mathbb{R}^{n-1})} \quad \alpha \ge 0, \\ \|\nabla_x Nf\|_{\dot{B}^\alpha_{pq}(\mathbb{R}^n_+)} &\le c\|f\|_{\dot{B}^{\alpha-1/p}_{pq}(\mathbb{R}^{n-1})} \quad \alpha > 0. \end{aligned} \tag{3.2}$$

According to Calderón-Zygmund inequality

$$\| \int_{\mathbb{R}^n} \nabla_x^2 N(\cdot - y) f(y) dy \|_{L^p(\mathbb{R}^n_+)} \le c \|f\|_{L^p(\mathbb{R}^n_+)} \quad \text{for} \quad 1 < p < \infty.$$

Using Lemma 3.2, (1) of Lemma 3.1 and Calderón-Zygmund inequality the following estimates also hold.

Lemma 3.3. *For* $\alpha \ge 0$ *and* $1 < p < \infty$.

$$\|\nabla_x^2 \int_{\mathbb{R}^n_+} N(\cdot - y) f(y) dy \|_{\dot H^\alpha_p(\mathbb{R}^n_+)} \le c \|f\|_{\dot H^\alpha_p(\mathbb{R}^n_+)}.$$

3.3. Gaussian kernel

The fundamental solution of the heat equation in $\mathbb{R}^n$ is denoted by

$$\Gamma(x,t) = \begin{cases} \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

Define $\Gamma_t * f(x) = \int_{\mathbb{R}^n} \Gamma(x-y,t) f(y) dy$.

Lemma 3.4. *Let* $1 \le p, q < \infty$.

$$\|\Gamma_t * f\|_{L^q(0,\infty; \dot H^\alpha_p(\mathbb{R}^n))} \le c \|f\|_{\dot B^{\alpha-2/q}_{pq}(\mathbb{R}^n)} \quad \textit{for} \quad \alpha \ge 0, \tag{3.3}$$

$$\|\Gamma_t * f\|_{L^q(0,\infty; \dot B^\alpha_{pq}(\mathbb{R}^n))} \le c \|f\|_{\dot B^{\alpha-2/q}_{pq}(\mathbb{R}^n)} \quad \textit{for} \quad \alpha > 0. \tag{3.4}$$

The proof of the estimate (3.3) is given in Proposition 4.1 of [11]. Using real interpolation to (3.3), we obtain the estimate (3.4).

3.4. Hölder type inequality

The following Hölder type inequality is well-known result (see Lemma 2.2 in [6]). For $\beta > 0$, $\frac{1}{r_i} + \frac{1}{s_i} = \frac{1}{p}$, $i = 1, 2$,

$$\|f_1 f_2\|_{\dot B^\beta_{pq}(\mathbb{R}^n)} \le c \big(\|f_1\|_{\dot B^\beta_{s_1 q}(\mathbb{R}^n)} \|f_2\|_{L^{r_1}(\mathbb{R}^n)} + \|f_1\|_{L^{s_2}(\mathbb{R}^n)} \|f_2\|_{\dot B^\beta_{r_2 q}(\mathbb{R}^n)} \big). \tag{3.5}$$

Let g be a function defined in $\mathbb{R}^n_+$. Let $\tilde g$ be extension of g over $\mathbb{R}^n$ defined by Theorem 5.19 in [1]. Then, for $1 \le p, q \le \infty$ and $0 < \beta$, we have

$$\|\tilde g\|_{\dot B^\beta_{pq}(\mathbb{R}^n)} \le c \|g\|_{\dot B^\beta_{pq}(\mathbb{R}^n_+)}, \qquad \|\tilde g\|_{L^p(\mathbb{R}^n)} \le c \|g\|_{L^p(\mathbb{R}^n_+)}. \tag{3.6}$$

Using (3.5) and (3.6), the following estimates hold.

Lemma 3.5. *Let $0 < \beta$ and $1 \le p, q \le \infty$. Then, for $\frac{1}{r_i} + \frac{1}{s_i} = \frac{1}{p}$, $i = 1, 2$,*

$$\|f_1 f_2\|_{\dot B^\beta_{pq}(\mathbb{R}^n_+)} \le c\big(\|f_1\|_{\dot B^\beta_{s_1 q}(\mathbb{R}^n_+)}\|f_2\|_{L^{r_1}(\mathbb{R}^n_+)} + \|f_1\|_{L^{s_2}(\mathbb{R}^n_+)}\|f_2\|_{\dot B^\beta_{r_2 q}(\mathbb{R}^n_+)}\big).$$

3.5. Helmholtz projection

The Helmholtz projection $\mathbb{P}$ in the half-space $\mathbb{R}^n_+$ is given by

$$\mathbb{P}f = f - \nabla\mathbb{Q}f = f - \nabla\mathbb{Q}_1 f - \nabla\mathbb{Q}_2 f, \tag{3.7}$$

where $\mathbb{Q}_1 f$ and $\mathbb{Q}_2 f$ satisfy the following equations;

$$\Delta\mathbb{Q}_1 f = \mathrm{div}\, f, \qquad \mathbb{Q}_1 f|_{x_n=0} = 0$$

and

$$\Delta\mathbb{Q}_2 f = 0, \qquad D_{x_n}\mathbb{Q}_2 f|_{x_n=0} = \big(f_n - D_{x_n}\mathbb{Q}_1 f\big)|_{x_n=0}.$$

Note that $\mathbb{Q}_1 f$ and $\mathbb{Q}_2 f$ are represented by

$$\mathbb{Q}_1 f(x) = -\int_{\mathbb{R}^n_+} D_{y_i}\big(N(x-y) - N(x-y^*)\big) f_i(y)dy, \tag{3.8}$$

$$\mathbb{Q}_2 f(x) = \int_{\mathbb{R}^{n-1}} N(x'-y', x_n)\big(f_n(y',0) - D_{y_n}\mathbb{Q}_1 f(y',0)\big)dy'. \tag{3.9}$$

Note that $\mathrm{div}\mathbb{P}f = 0$ and $(\mathbb{P}f)_n|_{x_n=0} = 0$.

Lemma 3.6. *Let $f = div\mathcal{F}$ with $\mathcal{F} \in \dot H^\alpha_{p,0}(\mathbb{R}^n_+)$ or $\mathcal{F} \in \dot B^\alpha_{pq,0}(\mathbb{R}^n_+)$.*

$$\|\mathbb{Q}f\|_{\dot H^\alpha_p(\mathbb{R}^n_+)} \le c\|\mathcal{F}\|_{\dot H^\alpha_p(\mathbb{R}^n_+)} \qquad \alpha \ge 0,$$
$$\|\mathbb{Q}f\|_{\dot B^\alpha_{pq}(\mathbb{R}^n_+)} \le c\|\mathcal{F}\|_{\dot B^\alpha_{pq}(\mathbb{R}^n_+)} \qquad \alpha > 0.$$

Proof. The proof of Lemma 3.6 is given in Appendix B. □

Lemma 3.7. *Let $1 < p_1 \le p < \infty$, $1 < q_1 \le q < \infty$. Let $0 < \beta \le \alpha < \beta + 1 < 2$ such that $0 = \alpha - \beta - 1 + n(\frac{1}{p_1} - \frac{1}{p}) + \frac{2}{q_1} - \frac{2}{q}$. Let $f = div\mathcal{F}$ with $\mathcal{F} \in L^{q_1}(0,\infty, \dot B^\beta_{p_1 q,0}(\mathbb{R}^n_+))$. Then,*

$$\|\Gamma * \mathbb{P}f\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))},\ \|\Gamma^* * \mathbb{P}f\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1 q}(\mathbb{R}^n_+))}.$$

Here $\Gamma * f := \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x-y, t-s) f(y,s)dyds$ and $\Gamma^* * f := \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x'-y', x_n + y_n, t - s) f(y,s)dyds$.

Proof. The proof of Lemma 3.7 is given in Appendix C. □

4. Proof of Theorem 1.4

First, we decompose the Stokes equation (1.2) as the following two equations:

$$\begin{aligned} v_t - \Delta v + \nabla \pi = 0, \qquad \operatorname{div} v = 0 \qquad \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ v|_{t=0} = h \quad \text{and} \quad v|_{x_n=0} = 0, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} V_t - \Delta V + \nabla \Pi = \operatorname{div} \mathcal{F}, \qquad \operatorname{div} V = 0 \text{ in } \mathbb{R}^n_+ \times (0,\infty), \\ V|_{t=0} = 0, \qquad V|_{x_n=0} = 0. \end{aligned} \tag{4.2}$$

Let $u = V + v$ and $p = \pi + \Pi$. Then, (u, p) is a solution of (1.2).

4.1. *Estimate of* (v, π)

Define (v, π) by

$$v_i(x,t) = \int_{\mathbb{R}^n_+} G_{ij}(x,y,t) h_j(y) dy, \tag{4.3}$$

$$\pi(x,t) = \int_{\mathbb{R}^n_+} P(x,y,t) \cdot h(y) dy, \tag{4.4}$$

where G and P are defined by

$$\begin{aligned} G_{ij} &= \delta_{ij}(\Gamma(x-y,t) - \Gamma(x-y^*,t)) \\ &\quad + 4(1-\delta_{jn}) \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial N(x-z)}{\partial x_i} \Gamma(z-y^*,t) dz, \end{aligned} \tag{4.5}$$

$$\begin{aligned} P_j(x,y,t) &= 4(1-\delta_{jn}) \frac{\partial}{\partial x_j} \Big[\int_{\mathbb{R}^{n-1}} \frac{\partial N(x'-z',x_n)}{\partial x_n} \Gamma(z'-y', y_n, t) dz' \\ &\quad + \int_{\mathbb{R}^{n-1}} N(x'-z',x_n) \frac{\partial \Gamma(z'-y',y_n,t)}{\partial y_n} dz' \Big]. \end{aligned} \tag{4.6}$$

Then (v, π) satisfies (4.1) (see [24]).

From Section 4 in [11], we have the following estimate;

$$\|v\|_{L^q(0,\infty;\dot H^{\alpha_i}_p(\mathbb{R}^n_+))} \le c\|h\|_{\dot B^{\alpha_i - 2/q}_{pq,0}(\mathbb{R}^n_+)}, \qquad 0 \le \alpha_i \le 2, \quad i = 1, 2.$$

Using the property of real interpolation (see (2.1), (2.2) and (2.4)), we have

$$\|v\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c\|h\|_{\dot B^{\alpha-2/q}_{pq,0}(\mathbb{R}^n_+)}, \qquad 0\le\alpha\le 2. \tag{4.7}$$

For $\alpha > \frac{1}{p}$, the following estimates for π also hold.

Lemma 4.1. *Let $1<p,q<\infty$ and $\frac{1}{p}<\alpha<2$. Then π can be decomposed in the form $\pi = \sum_{j=1}^{n-1} D_{y_j}\pi_{0j} + D_t\pi_{00}$ with*

$$\|\pi_{0j}\|_{L^q(0,\infty;\dot H^\alpha_p(\mathbb{R}^n_+))} + \|\pi_{00}\|_{L^q(0,\infty;\dot H^{\alpha+1}_p(\mathbb{R}^n_+))} \le c\|h\|_{B^{\alpha-2/q}_{pq,0}(\mathbb{R}^n_+)}.$$

Proof. The proof of Lemma 4.1 is given in Appendix D. □

4.2. Estimate of (V,Π)

Let $f=\mathbb{P}f+\nabla\mathbb{Q}f$ be the decomposition of f, where $\mathbb{P}$ and $\mathbb{Q}$ are the operator defined in section 3.5. Note that $\operatorname{div}\mathbb{P}f=0$ and $(\mathbb{P}f)_n|_{x_n=0}=0$. We define (V,Π_0) by

$$V_i(x,t)=\int_0^t\int_{\mathbb{R}^n_+} G_{ij}(x,y,t-\tau)(\mathbb{P}f)_j(y,\tau)dyd\tau, \tag{4.8}$$

$$\Pi_0(x,t)=\int_0^t\int_{\mathbb{R}^n_+} P(x,y,t-\tau)\cdot(\mathbb{P}f)(y,\tau)dyd\tau, \tag{4.9}$$

where G and P are defined by (4.5) and (4.6). Then (V,Π_0) satisfies

$$V_t-\Delta V+\nabla\Pi_0=\mathbb{P}f, \qquad \operatorname{div} V=0, \text{ in } \mathbb{R}^n_+\times(0,\infty),$$
$$V|_{t=0}=0, \qquad V|_{x_n=0}=0.$$

(See [24].) Let $\Pi=\Pi_0+\mathbb{Q}f$. Then, (V,Π) is solution of (4.2).

Let $1<p<\infty$ and $0\le\alpha\le 2$. In Section 3 in [11], the authors showed that V, Π_0 defined by (4.8) and (4.9) have the following estimates (using real interpolations); if $0<\alpha<2$, then

$$\|V\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c\big(\|\Gamma*\mathbb{P}f\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} + \|\Gamma^* * \mathbb{P}f\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}\big).$$

Using Lemma 3.6 and Lemma 3.7, the following theorem holds: Let $1<p_1\le p<\infty$, $1<q_1\le q<\infty$, $0\le\beta\le\alpha$ and $0=\alpha-\beta-1+n(\frac{1}{p_1}-\frac{1}{p})+\frac{2}{q_1}-\frac{2}{q}$. Then, for $\mathcal{F}\in L^{q_1}(0,\infty,\dot B^\beta_{p_1q,0}(\mathbb{R}^n_+))$, we have

$$\|V\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1q}(\mathbb{R}^n_+))}, \qquad 0<\alpha<2. \tag{4.10}$$

On the other hand, by Lemma 3.6 the following estimate hold for $\mathbb{Q}f$.

$$\|\mathbb{Q}f\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))} \le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))}. \tag{4.11}$$

For $\alpha > \frac{1}{p}$, the following estimates for Π_0 also hold.

Lemma 4.2. *Let $1 < p, q < \infty$ and $\frac{1}{p} < \alpha < 2$. Let p_1, q_1 and β satisfy the same conditions in Lemma 3.7. Let $f = \operatorname{div}\mathcal{F}$ for $\mathcal{F} \in L^{q_1}(0,\infty; \dot B^{\beta}_{p_1q,0}(\mathbb{R}^n_+))$. Then $\Pi_0 = \sum_{j=1}^{n-1} D_{y_j}\Pi_{0j} + D_t\Pi_{00}$ with*

$$\|\Pi_{0j}\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} + \|\Pi_{00}\|_{L^q(0,\infty;\dot B^{\alpha+1}_{pq}(\mathbb{R}^n_+))} \le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))}.$$

Proof. The proof of Lemma 4.2 is given in Appendix E. □

4.3. Estimate of (u, p)

Note that (u, p) defined by $u = V + v$ and $p = \pi + \Pi_0 + \mathbb{Q}\operatorname{div}\mathcal{F}$ are solution of (1.2). From (4.7) and (4.10), we obtain the estimate (1.3) for u.

Let $p_1 = \pi_{00} + \Pi_{00}$, $P_j = \pi_{0j} + \Pi_{0j}$ and $P_0 = \mathbb{Q}\operatorname{div}\mathcal{F}$, where π_{00} and π_{0j} are defined in Lemma 4.1, and Π_{00} and Π_{0j} are defined in Lemma 4.2. Then, the corresponding pressure p is decomposed by $p = D_t p_1 + \sum_{j=1}^{j=n-1} P_j + P_0$. From (4.11), Lemma 4.1 and Lemma 4.2, we get the estimate (1.4) for p, p_0 and p_j. This completes the proof Theorem 1.4.

5. Nonlinear problem

In this section, we would like to give proofs of Theorem 1.1 and Theorem 1.2. For the purpose of them, we construct approximate velocities and then derive uniform convergence in $L^q(0,\infty; \dot B^{\alpha}_{pq}(\mathbb{R}^n_+))$.

5.1. p_0, q_0, p_1, p_2, q_1 and β

Lemma 5.1. *Let (α, p, q) satisfy $1 < p, q < \infty$, $0 < \alpha < 2$, $\frac{2}{q} < 1 + \frac{n}{p}$ and $\alpha + 1 = \frac{n}{p} + \frac{2}{q}$. Then, there are $1 < p_0, p_1, p_2, q_0, q_1 < \infty$ and $0 < \beta$ satisfying that*

$$\begin{aligned} &1 < p_1 < p < \min(p_0, p_2),\ 1 < q_1 < q < q_0,\ \frac{n}{p_0} + \frac{2}{q_0} = 1,\\ &\beta = \alpha - 1 + n(\frac{1}{p_1} - \frac{1}{p}) + \frac{2}{q_1} - \frac{2}{q}, \qquad 0 < \beta \le \alpha \le \beta + 1 < 2,\\ &\beta - \frac{n}{p_2} = \alpha - \frac{n}{p}, \qquad \frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_0}, \qquad \frac{1}{q_1} = \frac{1}{q} + \frac{1}{q_0}. \end{aligned} \tag{5.1}$$

Proof. We take $\epsilon_1, \epsilon_2 \in (0,1)$ satisfying

$$\begin{aligned} \max(0, 1 - \frac{n}{p}) <&\epsilon_1 < \min(1, 2 - \frac{2}{q}, \frac{2}{q}, 2 - \alpha),\\ \max(0, 1 - \alpha) <&\epsilon_1 + \epsilon_2 < \min(1, 2 - \alpha). \end{aligned} \tag{5.2}$$

Let $\frac{2}{q_1} = \frac{2}{q} + \epsilon_1$, $\frac{n}{p_1} = \frac{n}{p} + \epsilon_2$, $\frac{n}{p_0} = 1 - \epsilon_1$, $\frac{2}{q_0} = \epsilon_1$, $\frac{n}{p_2} = -1 + \epsilon_1 + \epsilon_2 + \frac{n}{p}$ and $\beta = \alpha - 1 + \epsilon_1 + \epsilon_2$. Then $1 < p_0, p_1, p_2, q_0, q_1 < \infty$ and $0 < \beta$ satisfy the conditions in Lemma 5.1. □

Remark 5.2. If the condition $\frac{2}{q} < 1 + \frac{n}{p}$ is dropped in Lemma 5.1, then we choose $\epsilon_1, \epsilon_2 \in (0, 1)$ satisfying

$$0 < \epsilon_1 < \min(1, 2 - \frac{2}{q}, 2 - \alpha), \qquad \max(0, 1 - \alpha) < \epsilon_1 + \epsilon_2 < \min(1, 2 - \alpha).$$

Then, the numbers p_0, p_1, p_2, q_0, q_1 and $0 < \beta$ defined in the proof of Lemma 5.1 satisfy (5.1) without $p < p_0$ and $q < q_0$.

5.2. *Approximating solutions*

Let (u^1, p^1) be the solution of the Stokes equations

$$\begin{aligned} u^1_t - \Delta u^1 + \nabla p^1 = 0, \qquad \text{div}\, u^1 = 0, \ \text{in}\ \mathbb{R}^n_+ \times (0, \infty),\\ u^1|_{t=0} = h, \qquad u^1|_{x_n=0} = 0. \end{aligned} \tag{5.3}$$

Let $m \geq 1$. After obtaining $(u^1, p^1), \cdots, (u^m, p^m)$ construct (u^{m+1}, p^{m+1}) which satisfies the following equations

$$\begin{aligned} u^{m+1}_t - \Delta u^{m+1} + \nabla p^{m+1} = f^m, \qquad \text{div}\, u^{m+1} = 0, \ \text{in}\ \mathbb{R}^n_+ \times (0, \infty),\\ u^{m+1}|_{t=0} = h, \qquad u^{m+1}|_{x_n=0} = 0, \end{aligned} \tag{5.4}$$

where $f^m = -\text{div}(u^m \otimes u^m)$.

5.3. *Uniform boundedness in $L^{q_0}(0, \infty; L^{p_0}(\mathbb{R}^n_+))$*

We denote $L^q(L^p) := L^q(0, \infty; L^p(\mathbb{R}^n_+))$. Let $1 < p_0, q_0 < \infty$ with

$$\frac{n}{p_0} + \frac{2}{q_0} = 1. \tag{5.5}$$

(Observe that $n < p_0 < \infty$, $2 < q_0 < \infty$ and $-1 + \frac{n}{p_0} = -\frac{2}{q_0}$.) From Theorem 1.4, we have

$$\|u^1\|_{L^{q_0}(L^{p_0})} \leq c_0 \|h\|_{\dot{B}^{-1+n/p_0}_{p_0 q_0, 0}(\mathbb{R}^n_+)} := N_0. \tag{5.6}$$

From Theorem 1.4, taking $p_1 = \frac{p_0}{2}$, $q_1 = \frac{q_0}{2}$ and $\alpha = \beta = 0$, we have

$$\begin{aligned} \|u^{m+1}\|_{L^{q_0}(L^{p_0})} &\leq c\big(\|h\|_{\dot{B}^{-2/q_0}_{p_0 q_0, 0}(\mathbb{R}^n_+)} + \|u^m \otimes u^m\|_{L^{q_0/2}(L^{p_0/2})}\big)\\ &\leq c_1\big(\|h\|_{\dot{B}^{-2/q_0}_{p_0 q_0, 0}(\mathbb{R}^n_+)} + \|u^m\|^2_{L^{q_0}(L^{p_0})}\big). \end{aligned} \tag{5.7}$$

Under the hypothesis $\|u^m\|_{L^{q_0}(L^{p_0})} \le M_0$, (5.7) leads to the estimate

$$\|u^{m+1}\|_{L^{q_0}(L^{p_0})} \le c_1\big(N_0 + M_0^2\big).$$

Choose M_0 and N_0 so small that

$$M_0 \le \frac{1}{2c_1} \text{ and } N_0 < \frac{M_0}{2c_1}. \tag{5.8}$$

By the mathematical induction argument, we conclude

$$\|u^m\|_{L^{q_0}(L^{p_0})} \le M_0 \ \text{ for all } \ m = 1, 2 \cdots. \tag{5.9}$$

5.4. Uniform boundedness in $L^q(0,\infty; \dot B^\alpha_{pq}(\mathbb{R}^n_+))$

Let

$$\frac{n}{p} + \frac{2}{q} = 1 + \alpha. \tag{5.10}$$

(Observe that $\frac{n}{\alpha+1} < p < \infty$, $\frac{2}{\alpha+1} < q < \infty$ and $-1 + \frac{n}{p} = \alpha - \frac{2}{q}$.) From Theorem 1.4, we have

$$\|u^1\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c_2\|h\|_{\dot B^{-1+n/p}_{pq,0}(\mathbb{R}^n_+)} := N.$$

Let p_0, q_0, p_1, p_2, q_1 and β be the constants defined in the proof of Lemma 5.1. We apply Theorem 1.4 to obtain

$$\|u^{m+1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c\big(\|h\|_{\dot B^{-1+n/p}_{pq,0}(\mathbb{R}^n_+)} + \|u^m \otimes u^m\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1q}(\mathbb{R}^n_+))}\big). \tag{5.11}$$

By Lemma 3.5 and Besov imbedding (since $\beta - \frac{n}{p_2} = \alpha - \frac{n}{p}$ and $\beta \le \alpha$), we have

$$\begin{aligned}\|(u^m \otimes u^m)\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1q}(\mathbb{R}^n_+))} &\le c\|u^m\|_{L^q(0,\infty;\dot B^\beta_{p_2q}(\mathbb{R}^n_+))}\|u^m\|_{L^{q_0}(0,\infty;L^{p_0}(\mathbb{R}^n_+))}\\ &\le c\|u^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}\|u^m\|_{L^{q_0}(0,\infty;L^{p_0}(\mathbb{R}^n_+))}.\end{aligned} \tag{5.12}$$

From (5.11)-(5.12), we have

$$\|u^{m+1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c_1\big(N + \|u^m\|_{L^{q_0}(0,\infty;L^{p_0}(\mathbb{R}^n_+))}\|u^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}\big). \tag{5.13}$$

Under the hypothesis $\|u^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le M$, (5.13) leads to the estimate

$$\|u^{m+1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c_1\big(N + M_0M\big).$$

Choose that M_0 is small and M is large so that

$$M_0 \le \frac{1}{2c_1}, \qquad 2c_1N \le M. \tag{5.14}$$

By the mathematical induction argument, we conclude

$$\|u^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le M \text{ for all } m = 1, 2\cdots. \tag{5.15}$$

5.5. Uniform convergence

Let $U^m = u^{m+1} - u^m$ and $P^m = p^{m+1} - p^m$. Then, (U^m, P^m) satisfy the equations

$$U^m_t - \Delta U^m + \nabla P^m = -\mathrm{div}(u^m \otimes U^{m-1} + U^{m-1} \otimes u^{m-1}), \qquad \mathrm{div}\, U^m = 0, \text{ in } \mathbb{R}^n_+ \times (0,\infty),$$
$$U^m|_{t=0} = 0, \qquad U^m|_{x_n=0} = 0.$$

Recall the uniform estimates (5.9) and (5.15) for the approximate solutions. From Theorem 1.4 and Lemma 3.5 we have

$$\begin{aligned}\|U^m\|_{L^{q_0}(L^{p_0})} &\le c\big(\|u^{m-1}\|_{L^{q_0}(L^{p_0})} + \|u^m\|_{L^{q_0}(L^{p_0})}\big)\|U^{m-1}\|_{L^{q_0}(L^{p_0})}\\ &\le c_5 M_0 \|U^{m-1}\|_{L^{q_0}(L^{p_0})},\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}\|U^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} &\le c\|u^m \otimes U^{m-1} + U^{m-1}\otimes u^{m-1}\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1q_1}(\mathbb{R}^n_+))}\\ &\le c\big(\|u^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} + \|u^{m-1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}\big)\|U^{m-1}\|_{L^{q_0}(L^{p_0})}\\ &\quad + c\big(\|u^m\|_{L^{q_0}(L^{p_0})} + \|u^{m-1}\|_{L^{q_0}(L^{p_0})}\big)\|U^{m-1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}\big)\\ &\le c_6 M\|U^{m-1}\|_{L^{q_0}(L^{p_0})} + c_6 M_0 \|U^{m-1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}.\end{aligned} \tag{5.17}$$

Here, $\alpha, \beta, p, p_0, p_1, q, q_0, q_1, N_0, M_0$ are the same constants defined in the previous subsection, and we take the constant c_6 greater than c_5, that is,

$$c_6 > c_5. \tag{5.18}$$

From (5.16), if $c_5 M_0 < 1$, then $\sum_{m=1}^\infty \|U^m\|_{L^{q_0}(L^{p_0})}$ converges, that is,

$$\sum_{m=1}^\infty U^m \text{ converges in } L^{q_0}(L^{p_0}).$$

Take $A > 0$ satisfying $A(c_6 - c_5)M_0 \ge c_6 M$. Then from (5.16) and (5.17) it holds that

$$\begin{aligned}&\|U^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} + A\|U^m\|_{L^{q_0}(L^{p_0})}\\ &\le c_6 M_0(\|U^{m-1}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} + A\|U^{m-1}\|_{L^{q_0}(L^{p_0})})\end{aligned}$$

Again if $c_6 M_0 < 1$, then $\sum_{m=1}^\infty (\|U^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} + A\|U^m\|_{L^{q_0}(L^{p_0})})$ converges. This implies that $\sum_{m=1}^\infty \|U^m\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}$ converges, that is,

$$\sum_{m=1}^{\infty} U^m \text{ converges in } L^q(0,\infty; \dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+)).$$

Therefore, if M_0 satisfies the condition (5.14) with the additional conditions

$$M_0 < \frac{1}{c_6}, \tag{5.19}$$

then $u^m = u^1 + \sum_{k=1}^m U^k$ converges to $u^1 + \sum_{k=1}^\infty U^k$ in $L^q(0,\infty; \dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+)) \cap L^{q_0}(L^{p_0})$. Set $u := u^1 + \sum_{k=1}^\infty U^k$.

5.6. Existence

Let u be the same one constructed in the previous Section. Since $u_m \to u$ in $L^{q_0}(0,\infty; L^{p_0}(\mathbb{R}^n_+))$ and $L^q(0,\infty; \dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+))$, by (5.9) and (5.15), we have

$$\|u\|_{L^{q_0}(L^{p_0})} \le M_0, \qquad \|u\|_{L^q(0,\infty;\dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+))} \le M.$$

In this section, we will show that u satisfies weak formulation of Navier-Stokes equations (2.6), that is, u is a weak solution of Navier-Stokes equations (1.1) with appropriate distribution p. Let $\Phi \in C^\infty_0(\mathbb{R}^n_+ \times [0,\infty))$ with div $\Phi = 0$. Observe that

$$-\int_0^\infty \int_{\mathbb{R}^n_+} u^{m+1} \cdot \Delta\Phi dxdt = \int_0^\infty \int_{\mathbb{R}^n_+} u^{m+1} \cdot \Phi_t + (u^m \otimes u^m) : \nabla\Phi dxdt + < h, \Phi(\cdot,0) > .$$

Now, send m to the infinity, then, $u^m \to u$ in $L^{q_0}(L^{p_0})$. Since $n < p_0$ and $2 < q_0$, $u^m \otimes u^m \to u \otimes u$ in $L^1_{loc}(\mathbb{R}^n_+ \times [0,\infty))$. Hence, we have the identity

$$-\int_0^\infty \int_{\mathbb{R}^n_+} u \cdot \Delta\Phi dxdt = \int_0^\infty \int_{\mathbb{R}^n_+} u \cdot \Phi_t + (u \otimes u) : \nabla\Phi dxdt + < h, \Phi(\cdot,0) > .$$

Therefore u is a weak solution of Navier-Stokes equations (1.1). This completes the proof of the existence part of Theorem 1.1.

5.7. Uniqueness in space $L^{q_0}(L^{p_0})$

Let $u_1 \in L^{q_0}(L^{p_0})$ be another weak solution of Navier-Stokes equations (1.1) with pressure p_1. Then, $(u - u_1, p - p_1)$ satisfies the equations

$$\begin{aligned}
(u-u_1)_t - \Delta(u-u_1) + \nabla(p-p_1) &= -\mathrm{div}(u \otimes (u-u_1) + (u-u_1)\otimes u_1) \text{ in } \mathbb{R}^n_+ \times (0,\infty),\\
\mathrm{div}\,(u-u_1) &= 0, \text{ in } \mathbb{R}^n_+ \times (0,\infty),\\
(u-u_1)|_{t=0} = 0, &\quad (u-u_1)|_{x_n=0} = 0.
\end{aligned}$$

Applying the estimate of Theorem 1.4 in [9] to the above Stokes equations, we have

$$\|u-u_1\|_{L^{q_0}(0,\tau;L^{p_0}\mathbb{R}^n_+))} \le c\|u\otimes(u-u_1)+(u-u_1)\otimes u_1\|_{L^{q_0/2}(0,\tau;L^{p_0/2}\mathbb{R}^n_+))}$$
$$\le c_5(\|u\|_{L^{q_0}(0,\tau;L^{p_0}(\mathbb{R}^n_+))}+\|u_1\|_{L^{q_0}(0,\tau;L^{p_0}(\mathbb{R}^n_+))})\|u-u_1\|_{L^{q_0}(0,\tau;L^{p_0}(\mathbb{R}^n_+))},\ \ \tau<\infty.$$

Since $u, u_1 \in L^{q_0}(0,\infty; L^{p_0}(\mathbb{R}^n_+))$, by absolutely continuity, there is $0<\delta$ such that if $\tau_2-\tau_1\le \delta$ for $\tau_1<\tau_2$, then

$$\|u\|_{L^{q_0}(\tau_1,\tau_2;L^{p_0}(\mathbb{R}^n_+))}+\|u_1\|_{L^{q_0}(\tau_1,\tau_2;L^{p_0}(\mathbb{R}^n_+))}<\frac{1}{c_5}.$$

Hence, we have

$$\|u-u_1\|_{L^{q_0}(0,\delta;L^{p_0}(\mathbb{R}^n_+))}<\|u-u_1\|_{L^{q_0}(0,\delta;L^{p_0}(\mathbb{R}^n_+))}.$$

This implies that $\|u-u_1\|_{L^{q_0}(0,\delta;L^{p_0}(\mathbb{R}^n_+))}=0$, that is, $u\equiv u_1$ in $\mathbb{R}^n_+\times(0,\delta]$. Observe that $u-u_1$ satisfies the Stokes equations

$$\begin{aligned}(u-u_1)_t-\Delta(u-u_1)+\nabla(p-p_1)&=-\mathrm{div}(u\otimes(u-u_1)+(u-u_1)\otimes u_1)\text{ in }\mathbb{R}^n_+\times(\delta,\infty),\\ \mathrm{div}\,(u-u_1)&=0\text{ in }\mathbb{R}^n_+\times(\delta,\infty),\\ (u-u_1)|_{t=\delta}=0,&\quad (u-u_1)|_{x_n=0}=0.\end{aligned}$$

Again, applying the estimate of Theorem 1.4 in [9] to the above Stokes equations, we have

$$\begin{aligned}\|u-u_1\|_{L^{q_0}(\delta,2\delta;L^{p_0}(\mathbb{R}^n_+))}&\le c_5(\|u\|_{L^{q_2}(\delta,2\delta;L^{p_0}(\mathbb{R}^n_+))}\\ &\quad+\|u_1\|_{L^{q_0}(\delta,2\delta;L^{p_0}(\mathbb{R}^n_+))})\|u-u_1\|_{L^{q_0}(\delta,2\delta;L^{p_0}(\mathbb{R}^n_+))}\\ &<\|u-u_1\|_{L^{q_0}(\delta,2\delta;L^{p_0}(\mathbb{R}^n_+))}.\end{aligned}$$

This implies that $\|u-u_1\|_{L^{q_0}(\delta,2\delta;L^{p_0}(\mathbb{R}^n_+))}=0$, that is, $u\equiv u_1$ in $\mathbb{R}^n_+\times[\delta,2\delta]$. After iterating this procedure infinitely, we obtain the conclusion that $u=u_1$ in $\mathbb{R}^n_+\times(0,\infty)$. This completes the proof of the uniqueness part of Theorem 1.1.

5.8. *Pressure estimate*

Let $\alpha,\beta,p,p_1,p_0,q,q_1,q_0$ be the same as in subsection 5.4. By Hölder type inequality (3.5) we have $u\otimes u\in L^{q_1}(0,\infty;\dot B^\beta_{p_1q_1}(\mathbb{R}^n_+))$ with

$$\|(u\otimes u)\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1q}(\mathbb{R}^n_+))}\le c\|u\|_{L^{q_0}(L^{p_0})}\|u\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}.$$

According to Theorem 1.4, if $\alpha>\frac{1}{p}$, then there is P_0, P_j, p_1 so that $p=D_tp_0+\sum_{j=1}^{n-1}D_{x_j}P_j+p_1$ with

$$\|p_1\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q_1}(\mathbb{R}^n_+))} + \sum_{j=1}^{n-1}\|P_j\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} + \|P_0\|_{L^q(0,\infty;\dot B^{\alpha+1}_{pq}(\mathbb{R}^n_+))}$$
$$\le c\big(\|h\|_{\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)} + \|(u\otimes u)\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))}\big).$$

This completes the proof of Theorem 1.2.

Appendix A. Proof of Lemma 3.1

Applying the proof of Theorem 6.6.1 in [4], we can obtain the estimate (1) in Lemma 3.1.

Now, we derive the estimate (2) in Lemma 3.1. Let $\phi \in B^{\frac{1}{p}}_{p'p'}(\mathbb{R}^{n-1})$ be a real-valued function and $\Phi = D_{x_n}N\phi$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $N\phi$ is defined in (3.1). From Lemma 3.2, we have $\Phi \in \dot H^1_{p'}(\mathbb{R}^n_+)$ with $\|\Phi\|_{\dot H^1_{p'}(\mathbb{R}^n_+)} \le c\|\phi\|_{\dot B^{\frac{1}{p}}_{p'p'}(\mathbb{R}^{n-1})}$. Since div $f=0$, we have

$$< f_n|_{x_n=0},\phi >= \int_{\mathbb{R}^n_+} f(x)\cdot\nabla\Phi(x)dx \le \|f\|_{L^p(\mathbb{R}^n_+)}\|\Phi\|_{\dot H^1_{p'}(\mathbb{R}^n_+)} \le c\|f\|_{L^p(\mathbb{R}^n_+)}\|\phi\|_{\dot B^{1/p}_{p'p'}(\mathbb{R}^{n-1})}.$$

Since $\dot B^{-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1})$ is the dual space of $\dot B^{\frac{1}{p}}_{p'p'}(\mathbb{R}^{n-1})$, by duality argument, we have $f_n|_{x_n=0} \in \dot B^{-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1})$ with $\|f_n|_{x_n=0}\|_{\dot B^{-\frac{1}{p}}_{pp}(\mathbb{R}^{n-1})} \le c\|f\|_{L^p(\mathbb{R}^n_+)}$. Hence, we complete the proof of Lemma 3.1.

Appendix B. Proof of Lemma 3.6

Lemma B.1. *Let $f = \mathrm{div}\mathcal{F}$ (here $\mathcal{F} = (F_{ki})_{k,i=1,\cdots,n}$, $f_i = D_{x_k}F_{ki}$) with $\mathcal{F}|_{x_n=0} = 0$. Let $F_k = (F_{k1},\cdots,F_{kn})$. Then,*

$$\mathbb{Q}_1 f(x) = \sum_{k\neq n} D_{x_k}\mathbb{Q}_1 F_k(x) + D_{x_n}A(x),$$
$$\mathbb{Q}_2 f(x) = \sum_{k\neq n} D_{x_k}\mathbb{Q}_2 F_k(x) - \sum_{k\neq n} D^2_{x_k}B(x),$$

where

$$A(x) = -\int_{\mathbb{R}^n_+} \nabla_y\big(N(x-y)+N(x-y^*)\big)\cdot F_n(y)dy,$$
$$B(x) = \int_{\mathbb{R}^{n-1}} N(x'-y',x_n)A(y',0)dy'.$$

Proof. From (3.8), we have

$$\begin{aligned}\mathbb{Q}_1 f(x) &= -\sum_{k\neq n} D_{x_k}\int_{\mathbb{R}^n_+} \nabla_y\big(N(x-y)-N(x-y^*)\big)\cdot F_k(y)dy\\ &\quad - D_{x_n}\int_{\mathbb{R}^n_+} \nabla_y\big(N(x-y)+N(x-y^*)\big)\cdot F_n(y)dy\\ &:= \sum_{k\neq n} D_{x_k}\mathbb{Q}_1 F_k(x) + D_{x_n}A(x). \qquad \text{(B.1)}\end{aligned}$$

Since $\Delta A = \operatorname{div} F_n = f_n$, from (B.1), we have $D_{y_n}\mathbb{Q}_1 f(y) = D_{y_n}\sum_{k\neq n} D_{y_k}\mathbb{Q}_1 F_k(y) + f_n(y) - \Delta' A(y)$. Hence, we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} N(x'-y',x_n)D_{y_n}\mathbb{Q}_1 f(y',0)dy' &= \sum_{k\neq n} D_{x_k}\int_{\mathbb{R}^{n-1}} N(x'-y',x_n)D_{y_n}\mathbb{Q}_1 F_k(y',0)dy'\\ &\quad + \int_{\mathbb{R}^{n-1}} N(x'-y',x_n)f_n(y',0)dy'\\ &\quad + \sum_{k\neq n} D_{x_k}\int_{\mathbb{R}^{n-1}} N(x'-y',x_n)D_{y_k}A(y',0)dy'. \qquad \text{(B.2)}\end{aligned}$$

Hence, from (3.9) and (B.2), we have

$$\begin{aligned}\mathbb{Q}_2 f(x) &= -\sum_{k\neq n} D_{x_k}\int_{\mathbb{R}^{n-1}} N(x'-y',x_n)D_{y_n}\mathbb{Q}_1 F_k(y',0)dy'\\ &\quad -\sum_{k\neq n} D_{x_k}\int_{\mathbb{R}^{n-1}} N(x'-y',x_n)D_{y_k}A(y',0)dy'. \quad \square \qquad \text{(B.3)}\end{aligned}$$

Now, we will show that for nonnegative integers k,

$$\|\mathbb{Q} f\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|\mathcal{F}\|_{\dot H^k_p(\mathbb{R}^n_+)}. \qquad \text{(B.4)}$$

Using the property of complex interpolation, (B.4) implies Lemma 3.6.

From Lemma B.1, we have

$$\begin{aligned}\|\mathbb{Q} f\|_{\dot H^k_p(\mathbb{R}^n_+)} &\le c\Big(\sum_{k\neq n}\|D_{x_k}\mathbb{Q}_1 F_k\|_{\dot H^k_p(\mathbb{R}^n_+)} + \sum_{k\neq n}\|D_{x_k}\mathbb{Q}_2 F_k\|_{\dot H^k_p(\mathbb{R}^n_+)}\\ &\quad + \|D_{x_n}A\|_{\dot H^k_p(\mathbb{R}^n_+)} + \sum_{k\neq n}\|D^2_{x_k}B\|_{\dot H^k_p(\mathbb{R}^n_+)}\Big),\end{aligned}$$

where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are defined in (3.8) and (3.9), and A and B are defined in Lemma B.1, respectively. From Lemma 3.3, we have

$$\|D_{x_k}\mathbb{Q}_1 F_k\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|F\|_{\dot H^k_p(\mathbb{R}^n_+)}, \qquad \|D_{x_n} A\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|F\|_{\dot H^k_p(\mathbb{R}^n_+)}.$$

From (3.2), Lemma 3.1 and Lemma 3.3 continuously, we get

$$\|D^2_{x_k} B\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|A|_{x_n=0}\|_{\dot B^{k+1-1/p}_{pp}(\mathbb{R}^{n-1})} \le c\|D_x A\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|F_n\|_{\dot H^k_p(\mathbb{R}^n_+)}.$$

From (3.2), we have

$$\|D_{x_k}\mathbb{Q}_2 F_k\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|\big(F_k - \nabla\mathbb{Q}_1 F_k\big)_n|_{x_n=0}\|_{\dot B^{k-1/p}_{pp}(\mathbb{R}^{n-1})}.$$

If $k > 0$, then by (1) of Lemma 3.1, we have

$$\|\big(F_k - \nabla\mathbb{Q}_1 F_k\big)_n|_{x_n=0}\|_{\dot B^{k-1/p}_{pp}(\mathbb{R}^{n-1})} \le \|F_k - \nabla\mathbb{Q}_1 F_k\|_{\dot H^k_p(\mathbb{R}^n_+)} \le c\|F\|_{\dot H^k_p(\mathbb{R}^n_+)}.$$

If $k = 0$, then since $F_k - \nabla\mathbb{Q}_1 F_k$ is divergence free in $\mathbb{R}^n_+$, by (2) of Lemma 3.1, we have

$$\|\big(F_k - \nabla\mathbb{Q}_1 F_k\big)_n|_{x_n=0}\|_{\dot B^{-1/p}_{pp}(\mathbb{R}^{n-1})} \le \|F_k - \nabla\mathbb{Q}_1 F_k\|_{L^p(\mathbb{R}^n_+)} \le c\|F\|_{L^p(\mathbb{R}^n_+)}.$$

Therefore we obtain the estimate (B.4).

Using the properties of real interpolation and complex interpolation (see (2.1)), we complete the proof of Lemma 3.6.

Appendix C. Proof of Lemma 3.7

Since the proofs will be done by the same way, we prove only the case of $\Gamma^* * \mathbb{P} f$.

A crucial step in the proof of Lemma 3.7 is the following lemma, which is probably known to experts and we couldn't, however, find it in the literature and thus, we provide its proof.

Lemma C.1. *Let X_i and Y_i, $i = 1, 2$ be Banach spaces and $0 < t$ be fixed. Let $T : L^1(0, t; X_i) \to Y_i$, $i = 1, 2$ be linear operator such that*

$$\|Tf\|_{Y_i} \le M_i \int_0^t (t-s)^{-\beta_i} \|f(s)\|_{X_i} ds, \quad i = 1, 2 \quad \forall f \in L^1(0, t; X_i).$$

Then, for $0 < \theta < 1$ and $1 \le q \le \infty$,

$$\|Tf\|_{(Y_1,Y_2)_{\theta,q}} \le M_1^\theta M_2^{1-\theta} \int_0^t (t-s)^{-\beta} \|f(s)\|_{(X_1,X_2)_{\theta,q}} ds,$$

where $\beta = \beta_1\theta + \beta_2(1-\theta)$.

Proof. Note that

$$
\begin{aligned}
K(r,Tf,Y_1,Y_2) &= \inf_{Tf(t)=u_1+u_2, u_i\in Y_i, i=1,2} (\|u_1\|_{Y_1} + r\|u_2\|_{Y_2})\\
&\le \inf_{f=f_1+f_2, f_i\in L_{\beta_i}(0,t;X_i), i=1,2} (\|Tf_1\|_{Y_1} + r\|Tf_2\|_{Y_2})\\
&\le \inf_{f=f_1+f_2, f_i\in L_{\beta_i}(0,t;X_i), i=1,2} (M_1\int_0^t (t-s)^{-\beta_1}\|f_1(s)\|_{X_1}ds\\
&\quad + M_2 r\int_0^t (t-s)^{-\beta_2}\|f_2(s)\|_{X_2}ds)\\
&= \int_0^t \inf_{f(s)=f_1(s)+f_2(s), f_i(s)\in X_i, i=1,2} (M_1(t-s)^{-\beta_1}\|f_1(s)\|_{X_1}\\
&\quad + M_2 r(t-s)^{-\beta_2}\|f_2(s)\|_{X_2})ds\\
&= M_1\int_0^t (t-s)^{-\beta_1} K(M_1^{-1}M_2 r(t-s)^{-\beta_2+\beta_1}, f(s), X_1, X_2)ds.
\end{aligned}
$$

For the fourth equality, see the proofs of Theorem 1.18.4 and Theorem 1.18.5 in [27]. Hence, we have

$$
\begin{aligned}
\|Tf\|_{(Y_1,Y_2)_{\theta,q}} &= \Big(\int_0^\infty (r^{-\theta}K(r,Tf,Y_1,Y_2))^q \frac{dr}{r}\Big)^{\frac{1}{q}}\\
&\le M_1\int_0^t \Big(\int_0^\infty r^{-\theta q}(t-s)^{-\beta_1 q} K(M_1^{-1}M_2 r(t-s)^{-\beta_2+\beta_1}, f(s), X_1, X_2)^q \frac{dr}{r}\Big)^{\frac{1}{q}} ds\\
&\le M_1^{\theta}M_2^{1-\theta}\int_0^t \Big(\int_0^\infty r^{-\theta q}(t-s)^{-q(\beta_1\theta+\beta_2(1-\theta))} K(r, f(s), X_1, X_2)^q \frac{dr}{r}\Big)^{\frac{1}{q}} ds\\
&= M_1^{\theta}M_2^{1-\theta}\int_0^t (t-s)^{-\beta}\|f(s)\|_{(X_1,X_2)_{\theta,q}}ds.
\end{aligned}
$$

Therefore, we complete the proof of Lemma C.1. □

Proof of Lemma 3.7 Recall that $(\mathbb{P} f)_j = f_j - D_{x_j}\mathbb{Q} f = \mathrm{div} F_j - D_{x_j}\mathbb{Q} f$. For $1 \le j \le n-1$ we have

$$\Gamma^* * (\mathbb{P} f)_j(x,t) = -\int_0^t \int_{\mathbb{R}^n_+} \nabla_y \Gamma(x-y^*, t-\tau) \cdot F_j(y,\tau) dy d\tau$$
$$+ \int_0^t \int_{\mathbb{R}^n_+} D_{y_j} \Gamma(x-y^*, t-\tau) \mathbb{Q} f(y,\tau) dy d\tau. \tag{C.1}$$

From the relation $D^2_{x_n} A = -\Delta' A + \sum_{k=1}^n D_{x_k} F_{nk}$ and from Lemma B.1, we have the identity $(\mathbb{P}\ f)_n = \Delta' A + D_{x_n} \Delta' B - \sum_{k\neq n} D_{x_n} D_{x_k} Q F_k$. Hence we have

$$\Gamma^* * (\mathbb{P}\ f)_n(x,t) = \sum_{1\le k\le n-1} \int_0^t \int_{\mathbb{R}^n_+} D_{y_k} \Gamma(x-y^*, t-\tau) D_{y_n} \mathbb{Q} F_k(y,\tau) dy d\tau$$
$$+ \int_0^t \int_{\mathbb{R}^n_+} \nabla_{x'} \Gamma(x-y^*, t-\tau) \cdot \nabla_{y'} A(y,\tau) dy d\tau$$
$$+ \int_0^t \int_{\mathbb{R}^n_+} \nabla'_{x'} \Gamma(x-y^*, t-\tau) \cdot \nabla_{y'} D_{y_n} B(y,\tau) dy d\tau. \tag{C.2}$$

Using (C.1), (C.2), Young's inequality and the proof of Lemma 3.6 ($\alpha = 0$), for $p_0 \le p$, we have

$$\|\Gamma^* * (\mathbb{P}\ f)(t)\|_{L^p(\mathbb{R}^n_+)} \le c \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p})} (\sum_{j=1}^{n-1} \|F_j(s)\|_{L^{p_0}(\mathbb{R}^n_+)} + \|\mathbb{Q} f(s)\|_{L^{p_0}(\mathbb{R}^n_+)}$$
$$+ \sum_{j=1}^{n-1} \|D_{y_n} \mathbb{Q} F_j(s)(s)\|_{L^{p_0}(\mathbb{R}^n_+)} + \|\nabla A(s)\|_{L^{p_0}(\mathbb{R}^n_+)} + \|\nabla^2 B(s)\|_{L^{p_0}(\mathbb{R}^n_+)}) ds$$
$$\le c \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p})} \|\mathcal{F}(s)\|_{L^{p_0}(\mathbb{R}^n_+)} ds. \tag{C.3}$$

Similarly, we get

$$\|\nabla \Gamma^* * (\mathbb{P}\ f)(t)\|_{L^p(\mathbb{R}^n_+)}$$
$$\le c \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p})} (\sum_{j=1}^{n-1} \|\nabla F_j(s)\|_{L^{p_0}(\mathbb{R}^n_+)} + \|\nabla' \mathbb{Q} f(s)\|_{L^{p_0}(\mathbb{R}^n_+)}$$

$$+\sum_{j=1}^{n-1}\|\nabla' D_{y_n}\mathbb{Q}F_j(s)(s)\|_{L^{p_0}(\mathbb{R}^n_+)}+\|\nabla'\nabla A(s)\|_{L^{p_0}(\mathbb{R}^n_+)}+\|\nabla'\nabla^2 B(s)\|_{L^{p_0}(\mathbb{R}^n_+)})ds$$

$$\le c\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p})}\|\mathcal{F}(s)\|_{\dot H^1_{p_0}(\mathbb{R}^n_+)}ds. \tag{C.4}$$

Then, from (C.3), (C.4), Lemma C.1 and (2.1), we get

$$\|\Gamma^* * (\mathbb{P} f)(t)\|_{\dot B^\beta_{pq}(\mathbb{R}^n_+)}\le c\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p})}\|\mathcal{F}(s)\|_{\dot B^\beta_{p_0q}(\mathbb{R}^n_+)}ds,\quad 0<\beta<1.$$

From Hardy-Littlewood-Sobolev inequality (see [27]), we have

$$\|\Gamma^* * (\mathbb{P} f)\|_{L^q(0,\infty;\dot B^\beta_{pq}(\mathbb{R}^n_+))}\le c\|\mathcal{F}\|_{L^{q_0}(0,\infty;\dot B^\beta_{p_0q}(\mathbb{R}^n_+))}, \tag{C.5}$$

when

$$\frac{1}{q}+1=\frac{1}{q_0}+\frac{1}{2}+\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p}),\quad p_0\le p,\quad q_0\le q,\quad 0<\beta<1. \tag{C.6}$$

By parabolic type's Calderón-Zygmund Theorem and Lemma 3.6, for $1<p,q<\infty$, we have

$$\begin{aligned}\|\nabla^2\Gamma^* * (\mathbb{P} f)\|_{L^q(L^p)}&\le c\|\mathbb{P} f\|_{L^q(L^p)}\\ \le c\big(\|f\|_{L^q(L^p)}+\|\nabla\mathbb{Q} f\|_{L^q(L^p)}\big)&\le c\|\mathcal{F}\|_{L^q(0,\infty;\dot H^1_p(\mathbb{R}^n_+))}\end{aligned} \tag{C.7}$$

and

$$\|\nabla\Gamma^* * (\mathbb{P} f)\|_{L^q(L^p)}\le c\|\mathcal{F}\|_{L^q(L^p)}. \tag{C.8}$$

By the property of real interpolation to (C.7) and (C.8) we have

$$\|\Gamma^* * (\mathbb{P} f)\|_{L^q(0,\infty;\dot B^{\beta+1}_{pq}(\mathbb{R}^n_+)}\le c\|\mathcal{F}\|_{L^q(0,\infty;\dot B^\beta_{pq}(\mathbb{R}^n_+))},\ 0<\beta<1. \tag{C.9}$$

Let $0<\theta<1$, $\frac{1}{p_1}=\frac{\theta}{p_0}+\frac{1-\theta}{p}$ and $\frac{1}{q_1}=\frac{\theta}{q_0}+\frac{1-\theta}{q}$, where q,q_0,p,p_0 satisfy the condition (C.6):

$$\frac{1}{q}+1=\frac{1}{q_0}+\frac{1}{2}+\frac{n}{2}(\frac{1}{p_0}-\frac{1}{p}),\quad p_0\le p,\quad q_0\le q,\quad 0<\beta<1.$$

By the complex interpolation (see (2.3) and (2.2)) with exponent θ to (C.5) and (C.9), we have

$$\|\Gamma^* * (\mathbb{P} f)\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))}\le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot B^\beta_{p_1q}(\mathbb{R}^n_+))}.$$

Now let $\alpha:=1+\beta-\theta$. Then $\alpha,\beta,p_0,p_1,q_0,q_1$ satisfy the conditions in Lemma 3.7. This completes the proof of Lemma 3.7.

Appendix D. Proof of Lemma 4.1

Recalling the formula (4.4) with (4.6), we split π in two terms, i.e. $\pi(x,t) = \pi_1(x,t) + \pi_2(x,t)$, where

$$\pi_1(x,t) = 4 \sum_{1\le j\le n-1} \int_{\mathbb{R}^{n-1}} \frac{\partial^2 N(x'-z',x_n)}{\partial x_j \partial x_n} \int_{\mathbb{R}^n_+} \Gamma(z'-y',y_n,t) h_j(y,\tau) dy dz',$$

$$\pi_2(x,t) = 4 \sum_{1\le j\le n-1} \int_{\mathbb{R}^{n-1}} \frac{\partial N(x'-z',x_n)}{\partial x_j} \int_{\mathbb{R}^n_+} \frac{\partial \Gamma(z'-y',y_n,t)}{\partial y_n} h_j(y,\tau) dy dz'.$$

Note that π_1 is represented by

$$\pi_1(x,t) = 4 \sum_{1\le j\le n-1} D_{x_j} D_{x_n} N(\Gamma *_t h_j|_{x_n=0})(x) = 4 \sum_{1\le j\le n-1} D_{x_j} \pi_{1j}(x,t),$$

where $\pi_{1j}(x,t) = 4D_{x_n} N(\Gamma *_t h_j|_{x_n=0})$. From (3.2), Lemma 3.1 and Lemma 3.4, for $\alpha > \frac{1}{p}$ we have

$$\begin{aligned}\|\pi_{1j}\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} &\le c\|\Gamma_t * h_j|_{x_n=0}\|_{L^q(0,\infty;\dot B^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}))} \\ &\le c\|\Gamma_t * h_j\|_{L^q(0,\infty;\dot B^\alpha_{pq}(\mathbb{R}^n_+))} \le c\|h_j\|_{\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}.\end{aligned} \tag{D.1}$$

By integrating by parts, by the identity $D_{y_n} h_n = -\sum_{j=1}^{n-1} D_{y_j} h_j$ and by the identity $D^2_{y_n}\Gamma_t * h_n = D_t\Gamma_t * h_n - \Delta_{y'}\Gamma_t * h_n$, π_2 can be rewritten by

$$\begin{aligned}\pi_2 &= -4 \int_{\mathbb{R}^{n-1}} N(x'-z',x_n) \int_{\mathbb{R}^n_+} D_{y_n}\Gamma(z'-y',y_n,t) D_{y_n} h_n(y) dy dz' \\ &= D_t\Big(4 \int_{\mathbb{R}^{n-1}} N(x'-z',x_n) \int_{\mathbb{R}^n_+} \Gamma(z'-y',y_n,t) h_n(y) dy dz'\Big) \\ &\quad + 4\sum_{k=1}^{n-1} D^2_{x_k} \int_{\mathbb{R}^{n-1}} N(x'-z',x_n) \int_{\mathbb{R}^n_+} \Gamma(z'-y',y_n,t) h_n(y) dy dz' \\ &= D_t \pi_{00} + \sum_{k\ne n} D_{x_k}\pi_{2k}.\end{aligned}$$

From (3.2), Lemma 3.1 and Lemma 3.4, for $\alpha > \frac{1}{p}$ we have

$$\begin{aligned}\|\pi_{2k}\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} &\le c\|\Gamma_t * h_n|_{x_n=0}\|_{L^q(0,\infty;\dot B^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}))}\\ &\le c\|\Gamma_t * h_n\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} \le c\|h_n\|_{\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}\end{aligned} \tag{D.2}$$

and

$$\begin{aligned}\|\pi_{00}\|_{L^q(0,\infty;\dot B^{\alpha+1}_{pq}(\mathbb{R}^n_+))} &\le c\|\Gamma_t * h_n|_{x_n=0}\|_{L^q(0,\infty;\dot B^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}))}\\ &\le c\|\Gamma_t * h_n\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} \le c\|h_n\|_{\dot B^{\alpha-\frac{2}{q}}_{pq,0}(\mathbb{R}^n_+)}.\end{aligned} \tag{D.3}$$

This completes the proof of Lemma 4.1.

Appendix E. Proof of Lemma 4.2

Recalling the formulae (4.9) with (4.6), we split Π_0 in two terms, i.e. $\Pi_0(x,t) = \Pi_1(x,t) + \Pi_2(x,t)$, where

$$\Pi_1(x,t) = 4\sum_{1\le j\le n-1}\int_{\mathbb{R}^{n-1}} \frac{\partial^2 N(x'-z',x_n)}{\partial x_j \partial x_n}\int_0^t\int_{\mathbb{R}^n_+}\Gamma(z'-y',y_n,t-\tau)(\mathbb{P}f)_j(y,\tau)dyd\tau dz',$$

$$\Pi_2(x,t) = 4\sum_{1\le j\le n-1}\int_{\mathbb{R}^{n-1}} \frac{\partial N(x'-z',x_n)}{\partial x_j}\int_0^t\int_{\mathbb{R}^n_+}\frac{\partial\Gamma(z'-y',y_n,t-\tau)}{\partial y_n}(\mathbb{P}f)_j(y,\tau)dyd\tau dz'.$$

Note that Π_1 is represented by

$$\Pi_1(x,t) = 4\sum_{1\le j\le n-1} D_{x_j}D_{x_n}N(\Gamma*(\mathbb{P}f)_j|_{x_n=0})(x,t) = 4\sum_{1\le j\le n-1} D_{x_j}\Pi_{1j}(x,t),$$

where $\Pi_{1j}(x,t) = 4D_{x_n}N(\Gamma*(\mathbb{P}f)_j|_{x_n=0})$. From (3.2), Lemma 3.1 and Lemma 3.7, for $\alpha > \frac{1}{p}$ we have

$$\begin{aligned}\|\Pi_{1j}\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} &\le c\|\Gamma*(\mathbb{P}f)_j|_{x_n=0}\|_{L^q(0,\infty;\dot B^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}))}\\ &\le c\|\Gamma*(\mathbb{P}f)_j\|_{L^q(0,\infty;\dot B^{\alpha}_{pq}(\mathbb{R}^n_+))} \le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot B^{\beta}_{p_1q}(\mathbb{R}^n_+))}.\end{aligned} \tag{E.1}$$

By integrating by parts, by the identity $D_{y_n}(\mathbb{P}f)_n = -\sum_{j=1}^{n-1}D_{y_j}(\mathbb{P}f)_j$ and by the identity $D^2_{y_n}\Gamma_t*(\mathbb{P}f)_n = D_t\Gamma_t*(\mathbb{P}f)_n - \Delta_{y'}\Gamma_t*(\mathbb{P}f)_n$, Π_2 can be rewritten by

$$\Pi_2 = -4\int_{\mathbb{R}^{n-1}} N(x'-z',x_n)\int_0^t\int_{\mathbb{R}^n_+} D_{y_n}\Gamma(z'-y',y_n,t-s)D_{y_n}(\mathbb{P}f)_n(y,s)dydsdz'$$

$$= D_t\Big(4\int\limits_{\mathbb{R}^{n-1}} N(x'-z',x_n)\int\limits_{\mathbb{R}^n_+}\Gamma(z'-y',y_n,t)h_n(y)dydz'\Big)$$

$$+4\sum_{k=1}^{n-1}D^2_{x_k}\int\limits_{\mathbb{R}^{n-1}} N(x'-z',x_n)\int\limits_0^t\int\limits_{\mathbb{R}^n_+}\Gamma(z'-y',y_n,t-s)(\mathbb{P}f)_n(y,s)dydsdz'$$

$$= D_t\Pi_{00}+\sum_{k\neq n}D_{x_k}\Pi_{2k}.$$

From (3.2), Lemma 3.1 and Lemma 3.7, for $\alpha > \frac{1}{p}$ we have

$$\begin{aligned}\|\Pi_{2k}\|_{L^q(0,\infty;\dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+))} &\le c\|\Gamma*(\mathbb{P}f)_n|_{x_n=0}\|_{L^q(0,\infty;\dot{B}^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}))}\\ &\le c\|\Gamma*(\mathbb{P}f)_n\|_{L^q(0,\infty;\dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+))}\le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot{B}^{\beta}_{p_1q}(\mathbb{R}^n_+))}\end{aligned}\tag{E.2}$$

and

$$\begin{aligned}\|\Pi_{00}\|_{L^q(0,\infty;\dot{B}^{\alpha+1}_{pq}(\mathbb{R}^n_+))} &\le c\|\Gamma*(\mathbb{P}f)_n|_{x_n=0}\|_{L^q(0,\infty;\dot{B}^{\alpha-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}))}\\ &\le c\|\Gamma*(\mathbb{P}f)_n\|_{L^q(0,\infty;\dot{B}^{\alpha}_{pq}(\mathbb{R}^n_+))}\le c\|\mathcal{F}\|_{L^{q_1}(0,\infty;\dot{B}^{\beta}_{p_1q}(\mathbb{R}^n_+))}.\end{aligned}\tag{E.3}$$

This completes the proof of Lemma 4.2.

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