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On a weighted Trudinger-Moser inequality in $\mathbb{R}^N$ ☆

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Abstract

We establish the Trudinger-Moser inequality on weighted Sobolev spaces in the whole space, and for a class of quasilinear elliptic operators in radial form of the type $Lu := -r^{-\theta}(r^{\alpha}|u'(r)|^{\beta}u'(r))'$, where $\theta, \beta \geq 0$ and $\alpha > 0$, are constants satisfying some existence conditions. It is worth emphasizing that these operators generalize the p-Laplacian and k-Hessian operators in the radial case. Our results involve fractional dimensions, a new weighted Pólya-Szegö principle, and a boundness value for the optimal constant in a Gagliardo-Nirenberg type inequality.
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1. Introduction

It is well known that the classical Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for any $p \leq q \leq Np/(N-p)$, where $p < N$ and Ω is a domain in $\mathbb{R}^N$, see [2,11]. If $p = N$, then $W^{1,N}(\Omega) \not\subset L^\infty(\Omega)$, although the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $N \leq$

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$q < \infty$. Motivated by this fact, Adams [2] proved that for every $0 < \mu \leq 1$ the Sobolev space $W^{1,N}(\Omega)(\Omega$ unbounded) is embedded in the Orlicz space $L_{\Psi_{\mu,N}}(\Omega)$, where

$$\Psi_{\mu,N}(t) = e^{\mu t^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\mu^j}{j!} t^{\frac{N}{N-1}j}.$$

Hempel, Morris and Trudinger [12] showed that the best Orlicz space $L_\Psi(\Omega)$ for the embedding of $W_0^{1,N}(\Omega)$ (where Ω is a bounded domain in $\mathbb{R}^N$) occurs when $\Psi = e^{t^{\frac{N}{N-1}}} - 1$.

The case when Ω is a bounded domain was studied by J. Moser [16], which showed the following sharp result

$$\sup_{u \in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{1}{|\Omega|} \int_\Omega e^{\mu\left(\frac{|u|}{\|\nabla u\|_{L^P}}\right)^{\frac{N}{N-1}}} dx \begin{cases} \leq C(N,\mu), & \text{if } \mu \leq \mu_N \\ = +\infty, & \text{if } \mu > \mu_N, \end{cases} \tag{1}$$

where $\mu_N := N\omega_{N-1}^{\frac{1}{N-1}}$, $|\Omega|$ is the Lebesgue measure of Ω, ω_{N-1} is the $(N-1)$-dimensional Hausdorff measure of the unit sphere in $\mathbb{R}^N$, and $C(N,\mu)$ is a positive constant depending only on N and μ.

The case $\Omega = \mathbb{R}^N$, was studied by Ruf [19] for $N = 2$, and Li and Ruf [15] for $N \geq 3$. In all cases a sharp result was obtained. Namely, there exists $D(N,\mu)$ which depends only on N and μ satisfying

$$\int_{\mathbb{R}^N} \Psi_{\mu,N}(u)dx \leq D(N,\mu) \tag{2}$$

for all $u \in W^{1,N}(\mathbb{R}^N)$ with $\|u\|_{W^{1,N}(\mathbb{R}^N)} = 1$ and $\mu \leq \mu_N$. Here, the inequality (2) is not valid if $\mu > \mu_N$.

The attainability of the best constant

$$d_{N,\mu} := \sup_{u \in W^{1,N}(\mathbb{R}^N): \|u\|_{W^{1,N}(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \Psi_{\mu,N}(u)dx, \tag{3}$$

associated with (2), was studied by Ishiwata [13] [see also section 2, Theorem 2.5 and Theorem 2.6]. For similar studies, see [14,7].

To establish the attainability of $d_{N,\mu}$, Ishiwata proved that the compactness of a maximizing sequence to (3) happens, excluding the concentration behavior and the vanishing behavior of the maximizing sequence. He also showed that the functional $J(u) := \int_{\mathbb{R}^2} \Psi_{2,\mu}(u)dx$ does not have critical points on $M := \{u \in W^{1,2}(\mathbb{R}^2) : \|u\|_{W^{1,2}(\mathbb{R}^N)} = 1\}$ for μ sufficiently small, which implies non-existence results in this case.

Our approach for Trudinger-Moser inequality will be done for the class of quasilinear elliptic operators in radial form of the type

$$Lu := -r^{-\theta}(r^\alpha |u'(r)|^\beta u'(r))',$$

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where $\theta, \beta \geq 0$ and $\alpha > 0$. For some problems involving the operator L, see [9,10]. It is worth emphasizing that these operators generalize the p -Laplacian and k-Hessian operators in the radial case, more precisely,

(i) Laplacian	$\alpha = \theta = N - 1, \beta = 0$
(ii) p-Laplacian ($p \geq 2$)	$\alpha = \theta = N - 1, \beta = p - 2$
(iii) k-Hessian ($1 \leq k \leq N$)	$\alpha = N - k, \theta = N - 1, \beta = k - 1$

where these operators act on the weighted Sobolev spaces

$$W^{1,p}_{\alpha,\theta}(0, R) := W^{1,p}((0, R), d\lambda_\alpha, d\lambda_\theta) \ \text{ for } \ 0 < R \leq \infty$$

defined in section 2. The Proposition 2.1, in section 2 (see also Kufner-Opic [18]), gives us the following Sobolev-type embedding

$$W^{1,p}_{\alpha,\theta}(0, R) \hookrightarrow L^q_\theta(0, R), \ \text{ where } \ 1 \leq q \leq q^\star, \alpha - p + 1 > 0, 0 < R < +\infty,$$

and the number $q^\star := \frac{(1+\theta)p}{\alpha - p + 1}$ is the critical exponent associated with the weighted Sobolev space $W^{1,p}_{\alpha,\theta}(0, R)$. We would like to emphasize that the embedding still holds for $\alpha - p + 1 = 0$, and $1 \leq q < \infty$.

As in the classical case, a function in $W^{1,p}_{\alpha,\theta}(0, R)$ (when $\alpha - (p - 1) = 0$) could have a local singularity, which proves that $W^{1,p}_{p-1,\theta}(0, R) \not\subset L^\infty_\theta(0, R)$. Motivated by this approach, Oliveira and Do Ó [17] studied this embedding, and they proved some results on validity and attainability of the Trudinger-Moser inequality (for bounded domains see section 2, Theorem 2.2, Theorem 2.3 and Theorem 2.4).

Our goal here is twofold: we prove a Trudinger-Moser type inequality for weighted Sobolev spaces involving fractional dimensions in the unbounded case $(0, \infty)$; and, we discuss the existence of extremal functions in such inequalities.

We will replace the constant $c_{\alpha,\theta}$ (which depends on α, θ and R) in Theorem 2.2 by a uniform constant $d(\alpha, \theta, \mu)$ (which depends on α, θ and μ), by replacing the Dirichlet norm with weight $\|u'\|_{L^p_\alpha}$ by the Sobolev norm with weights $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}$, in the same spirit of the results stated before [15,19]. Furthermore, we investigate the compactness on maximizing sequences for such inequalities in the same sense of the results established by Ishiwata [13].

Let

$$A_{p,\mu}(t) = e^{\mu t^{\frac{p}{p-1}}} - \sum_{j=0}^{\lfloor p \rfloor - 1} \frac{\mu^j}{j!} t^{\frac{p}{p-1} j}, \ \text{ with } \lfloor p \rfloor \text{ the largest integer less than p.}$$

Our main results are the following ones:

Theorem 1.1. *Let $p \geq 2$, $\theta, \alpha \geq 0$ and $\mu > 0$ be real numbers such that $\alpha - (p - 1) = 0$ and $\mu \leq \mu_{\alpha,\theta} := (1 + \theta)\omega_\alpha^{\frac{1}{\alpha}}$. Then there exists a constant $D(\theta, \alpha, \mu)$ which depends only on θ, α and μ such that*

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$$\int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x) \le D(\theta,\alpha,\mu) \tag{4}$$

for all $u \in W^{1,p}_{\alpha,\theta}(0,\infty)$ *with* $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$. *Furthermore, the inequality* (4) *fails if* $\mu > \mu_{\alpha,\theta}$, *that is, for any* $\mu > \mu_{\alpha,\theta}$ *there exists a sequence* $(u_j) \subset W^{1,p}_{\alpha,\theta}(0,\infty)$ *such that*

$$\int_0^\infty A_{p,\mu}\left(\frac{|u_j(x)|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right)d\lambda_\theta(x) \to \infty \ \ as\ \ j \to \infty.$$

To state our next results, we need to define the best constant associated with the inequality (4), namely

$$d(\theta,\alpha,\mu) := \sup_{0\neq u\in W^{1,p}_{\alpha,\theta}(0,\infty)} \int_0^\infty A_{p,\mu}\left(\frac{|u(x)|}{\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right)d\lambda_\theta(x), \tag{5}$$

where $\alpha - (p-1) = 0$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists a positive nonincreasing function u in* $W^{1,p}_{\alpha,\theta}(0,\infty)$ *with* $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$ *such that*

$$d(\theta,\alpha,\mu) = \int_0^\infty A_{p,\mu}(|u(x)|)d\lambda_\theta(x),$$

in the following cases:

(i) $p > 2$ *and* $0 < \mu < \mu_{\alpha,\theta}$,
(ii) $p = 2$ *and* $\frac{2}{B(2,\theta)} < \mu < \mu_{\alpha,\theta}$,

where

$$B(2,\theta)^{-1} := \inf_{0\neq u\in W^{1,2}_{1,\theta}(0,\infty)} \frac{\|u'\|^2_{L^2_1(0,\infty)}\cdot\|u\|^2_{L^2_\theta(0,\infty)}}{\|u\|^4_{L^4_\theta(0,\infty)}}.$$

Theorem 1.3. *Let* $p = 2$, $\theta \ge 0$ *and* $\alpha = 1$. *Then there exists* μ_0 *such that* $d(\theta,\alpha,\mu)$ *is not achieved for all* $0 < \mu < \mu_0$.

To prove the statement (1), Moser [16] used the well known Schwarz Symmetrization arguments, which provide a radially symmetric function $u^\#$ defined on the ball $B_R(0)$, where $\mathcal{L}^N(\Omega) = \mathcal{L}^N(B_R(0))$ and all the balls $\{x \in B_R(0); u^\#(x) > t\}$ have the same $\mathcal{L}^N$ measure of the sets $\{x \in \Omega; u(x) > t\}$. Furthermore, $u^\#$ satisfies the Pólya-Szegö inequality

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$$\int_{B_R(0)} |\nabla u^{\#}|^N dx \leq \int_{\Omega} |\nabla u|^N dx \tag{6}$$

which was crucial in the proof. In our case, Pólya-Szegö inequality for $W^{1,p}_{\alpha,\theta}$ is not available, which was an additional difficulty in this type of problem. See, for instance, [17].

In this paper we present the half weighted Schwarz symmetrization with the purpose of obtaining a Pólya-Szegö inequality for $W^{1,p}_{\alpha,\theta}$ which will permit again to reduce the proof of the Trudinger-Moser inequality to symmetric non-increasing functions.

The paper is organized as follows. In section 2, we define some elements and present some previous results about Trudinger-Moser inequality on $W^{1,p}_{p-1,\theta}(0,R)$, where $R<\infty$. In section 3, we prove a new Pólya-Szegö Principle on $W^{1,p}_{\alpha,\theta}$ using a new class of isoperimetric inequalities on $\mathbb{R}$ with respect to weights $|x|^k$. In section 4, we establish the Trudinger-Moser inequality on $W^{1,p}_{\alpha,\theta}(0,\infty)$, under the assumptions of Theorem 1.1. In the section 5, we obtain the Theorem 1.2 studying the compactness of a maximizing sequence (u_n) for (5). In the section 6, we show the Theorem 1.3 proving that the functional $F(u)=\int_0^\infty A_{2,\mu}(|u(x)|)d\lambda_\theta(x)$ does not have critical points on $\{u\in W^{1,2}_{1,\theta}(0,\infty) : \|u\|_{W^{1,2}_{1,\theta}(0,\infty)}=1\}$. Finally, in the section 7 we present a brief discourse about Gagliardo-Nirenberg-Sobolev type inequality and we show that $2/B(2,\theta)<2\pi(1+\theta)=\mu_{1,\theta}$. Thus, the case (ii) of the Theorem 1.2 makes sense.

2. Basic definitions and previous results

Let $0<R\leq+\infty$, $1\leq p<+\infty$ and $\theta\geq 0$. Let us denote by $L^p_\theta(0,R)$ the weighed Lebesgue space defined as the set of all measurable functions u on $(0,R)$ for which

$$\|u\|_{L^p_\theta(0,R)} := \left[\int_0^R |u(x)|^p d\lambda_\theta(x)\right]^{1/p} < \infty,$$

where

$$d\lambda_\theta(x)=\omega_\theta x^\theta dx,\ \ \omega_\theta=\frac{2\pi^{\frac{1+\theta}{2}}}{\Gamma\left(\frac{1+\theta}{2}\right)},\ \text{for all } \theta\geq 0,$$

with $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ the Gamma Function. Finally, let us denote by $W^{1,p}_{\alpha,\theta}(0,R)$ the space of all locally absolutely continuous functions such that $u(x)\to 0$, as $x\to R^-$ and

$$\|u\|_{W^{1,p}_{\alpha,\theta}(0,R)} := \left(\|u'\|^p_{L^p_\alpha(0,R)}+\|u\|^p_{L^p_\theta(0,R)}\right)^{\frac{1}{p}} < +\infty.$$

In the following proposition, see [18] for more details, we collect some embedding results for the weighted spaces $W^{1,p}_{\alpha,\theta}$, which will be used in this paper.

Proposition 2.1. *Let $u:(0,R]\to\mathbb{R}$ be an absolutely continuous function. If $R<\infty$, $u(R)=0$ and*

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(1) *for* $1 \le p \le q < \infty$ *one has*
  (a) $\alpha > p-1$, $\theta \ge \alpha\frac{q}{p} - q\frac{p-1}{p} - 1$, *or*
  (b) $\alpha \le p-1$, $\theta > -1$.
(2) *for* $1 \le q < p < \infty$ *one has*
  (c) $\alpha > p-1$, $\theta > \alpha\frac{q}{p} - q\frac{p-1}{p} - 1$, *or*
  (d) $\alpha \le p-1$, $\theta > -1$

then

$$\left(\int_0^R |u|^q x^\theta dx\right)^{\frac{1}{q}} \le C\left(\int_0^R |u'|^p x^\alpha dx\right)^{\frac{1}{p}},$$

where C *is a constant which does not depend on* u.

Next, we present results due to Oliveira and Do Ó [17].

Theorem 2.2. *Let* $\alpha, \theta \ge 0$ *and* $p \ge 2$ *be real numbers such that* $\alpha - (p-1) = 0$. *Then there exists a constant* $c_{\alpha,\theta}$ *depending only on* α, θ *and* R *such that*

$$\sup_{u \in W^{1,p}_{\alpha,\theta}(0,R)} \int_0^R e^{\mu(|u|)^{\frac{p}{p-1}}} d\lambda_\theta(r) \begin{cases} \le c_{\alpha,\theta}, & \text{if } \mu \le \mu_{\alpha,\theta} := (1+\theta)\omega_\alpha^{\frac{1}{\alpha}} \\ = \infty, & \text{if } \mu > \mu_{\alpha,\theta}, \end{cases} \tag{7}$$

where $\|u'\|_{L^p_\alpha} = 1$.

They also showed the existence of extremal functions for inequality (7), as follows

Theorem 2.3. *Under the assumptions of Theorem 2.2, there are extremal functions for* $C_{\alpha,\theta,R}(\mu)$ *when* $\mu \le \mu_{\alpha,\theta}$; *that is, there exists* $u \in W^{1,p}_{\alpha,\theta}(0,R)$ *such that*

$$C_{\alpha,\theta,R}(\mu) = \int_0^R e^{\mu|u|^{\frac{p}{p-1}}} d\lambda_\theta(r),$$

where

$$C_{\alpha,\theta,R}(\mu) := \sup_{u \in W^{1,p}_{\alpha,\theta}(0,\infty):\|u'\|_{L^p_\alpha}=1} \int_0^R e^{\mu(|u|)^{\frac{p}{p-1}}} d\lambda_\theta(r).$$

In the same spirit of Adachi and Tanaka (see [1]), Oliveira and Do Ó showed the following result

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Theorem 2.4. *Let $\theta, \alpha \geq 0$ and $p \geq 2$ be real numbers such that $\alpha - (p-1) = 0$. Then for any $\mu \in (0, \mu_{\alpha,\theta})$ there is a constant $C_{\mu,p,\theta}$ depending only on μ, p and θ such that*

$$\int_0^\infty A_{p,\mu}\left(\frac{|u(r)|}{\|u'\|_{L^p_\alpha(0,\infty)}}\right) d\lambda_\theta(r) \leq C_{\mu,p,\theta}\left(\frac{\|u\|_{L^p_\theta(0,\infty)}}{\|u'\|_{L^p_\alpha(0,\infty)}}\right)^p \tag{8}$$

for all $u \in W^{1,p}_{\alpha,\theta}(0,R)\backslash\{0\}$. Besides that, for any $\mu \geq \mu_{\alpha,\theta}$ there is a sequence $(u_j) \subset W^{1,p}_{\alpha,\theta}(0,\infty)$ such that $\|u'_j\|_{L^p_\alpha(0,\infty)} = 1$ and

$$\frac{1}{\|u'_j\|_{L^p_\alpha(0,\infty)}}\int_0^\infty A_{p,\mu}\left(|u_j(r)|\right) d\lambda_\theta(r) \to \infty \text{ as } j \to \infty,$$

where

$$A_{p,\mu}(t) = e^{\mu t^{\frac{p}{p-1}}} - \sum_{j=0}^{\lfloor p\rfloor-1}\frac{\mu^j}{j!}t^{\frac{p}{p-1}j}, \text{ with } \lfloor p\rfloor \text{ is the largest integer less than } p.$$

As mentioned in the Introduction, Ishiwata [13] studied the attainability of $d_{N,\mu}$ (3) in the classical case. He emphasized the importance of evaluating vanishing behavior on maximizing sequence in unbounded case. Next, the main results in [13] are presented.

Theorem 2.5. *Let $N \geq 2$ and*

$$B_2 := \sup_{0\neq\psi\in W^{1,2}(\mathbb{R}^2)}\frac{\|\psi\|^4_{L^4}}{\|\nabla\psi\|^2_{L^2}\,\|\psi\|^2_{L^2}}.$$

Then $d_{N,\mu}$ is attained for $0 < \mu < \mu_N$ if $N \geq 3$ and for $2/B_2 < \mu \leq \mu_2 = 4\pi$ if $N = 2$.

Theorem 2.6. *Let $N = 2$. If $\mu \ll 1$, then $d_{2,\mu}$ is not attained.*

3. Pólya-Szegö principle on $W^{1,p}_{\alpha,\theta}$

Now we are going to define a half weighted Schwarz symmetrization to prove a Pólya-Szegö Principle, see the inequality (6).

We define the measure μ_l by $d\mu_l(x) = |x|^l dx$. Besides, if $M \subset \mathbb{R}$ is a measurable set with finite μ_l-measure, then let M^* denote the interval $(0,R)$ such that

$$\mu_l((0,R)) = \mu_l(M).$$

Further, if $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that

$$\mu_l\left(\{y \in \mathbb{R}; |u(y)| > t\}\right) < \infty \text{ for all } t > 0,$$

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then let u^* denote the half weighted Schwarz symmetrization of u, or in short, the half μ_l-symmetrization of u, given by

$$u^*(x) = \sup\{t \geq 0; \mu_l(\{y \in \mathbb{R}; |u(y)| > t\}) > \mu_l(0,x)\},$$

for every $x > 0$.

Remark 3.1. The word "half" appears here because our symmetrization is slightly different in three aspects:

(i) it is defined on $(0, \infty)$;
(ii) we are comparing the distribution $\rho(t) := \mu_l(\{y \in \mathbb{R}; |u(y)| > t\})$ with the measure of $(0, x)$, instead of $B_{|x|}(0)$;
(iii) the set M^* is a semi ball with the same measure of M, instead of a ball.

We will carry out the proof of the next result based on Isoperimetric Inequality on $\mathbb{R}$ with weight $|x|^k$ [see [4], Theorem 6.1]. Besides, It is worth noting that the Theorem 8.1 in [4] does not cover the case $k < l + 1$ when $N = 1$. For negative values of k, the proof is a consequence of the well-known Hardy-Littlewood inequality. See also Cabré and Ros-Oton [5] for monomial weights, and Talenti [20] for some cases when $N \geq 2$.

Theorem 3.2. *Let k, l be real numbers satisfying $0 < k \leq l + 1$. Besides, let $1 \leq p < \infty$ and $m := pk + (1 - p)l$. Then*

$$\int_0^\infty |u'|^p |x|^{pk+(1-p)l} \, dx \geq \int_0^\infty |(u^*)'|^p |x|^{pk+(1-p)l} \, dx, \tag{9}$$

for every $u \in W^{1,p}_{m,l}(0, \infty)$, where u^* denotes the half μ_l-symmetrization of u.

Proof. Observe that it is sufficient to consider u a non-negative function. Let

$$I := \int_0^\infty |u'|^p |x|^{pk+(1-p)l} \, dx \text{ and}$$

$$I^* := \int_0^\infty |(u^*)'|^p |x|^{pk+(1-p)l} \, dx.$$

The Coarea Formula holds

$$I := \int_0^\infty \int_{u=t} |u'|^{p-1} |x|^{pk+(1-p)l} \, d\mathcal{H}^0(x) dt \text{ and}$$

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$$I^* := \int_0^\infty \int_{u^*=t} \left|(u^*)'\right|^{p-1} |x|^{pk+(1-p)l} \, d\mathcal{H}^0(x) dt.$$

If $p = 1$, we get

$$I := \int_0^\infty \int_{u=t} |x|^k \, d\mathcal{H}^0(x) dt \text{ and}$$

$$I^* := \int_0^\infty \int_{u^*=t} |x|^k \, d\mathcal{H}^0(x) dt,$$

hence, we obtain from Isoperimetric Inequality on $\mathbb{R}$ with weight $|x|^k$ [see [4], Theorem 6.1] and definition of $u^\star$ that

$$\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \geq \int_{u^*=t} |x|^k \, d\mathcal{H}^0(x).$$

Therefore, $I \geq I^*$ when $p = 1$.

Now, assume that $1 < p < \infty$. By Holder's Inequality we have

$$\int_{u=t} |x|^k \, dH^0(x) \leq \left(\int_{u=t} |x|^{kp+(1-p)l} \left|u'\right|^{p-1} d\mathcal{H}^0(x) \right)^{\frac{1}{p}} \left(\int_{u=t} \frac{|x|^l}{|u'|} d\mathcal{H}^0(x) \right)^{\frac{p-1}{p}}$$

for $a.e \; t \in [0, \infty)$, thus we get

$$I \geq \int_0^\infty \left(\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \right)^p \left(\int_{u=t} \frac{|x|^l}{|u'|} d\mathcal{H}^0(x) \right)^{1-p} dt. \tag{10}$$

Since that $|(u^*)'|$ and $|x|$ are constants along with $\{u^* = t\}$, hence, for u^* we obtain the equality, i.e.,

$$I^* = \int_0^\infty \left(\int_{u^*=t} |x|^k \, d\mathcal{H}^0(x) \right)^p \left(\int_{u^*=t} \frac{|x|^l}{|(u^*)'|} d\mathcal{H}^0(x) \right)^{1-p} dt. \tag{11}$$

In addition, by definition of u^*, we have

$$\int_{u>t} |x|^l \, dx = \int_{u^*>t} |x|^l \, dx,$$

and as a consequence of Coarea Formula we get

$$\int\limits_{u=t} \frac{|x|^l}{|u'|} d\mathcal{H}^0(x) = \int\limits_{u^*=t} \frac{|x|^l}{|(u^*)'|} d\mathcal{H}^0(x), \tag{12}$$

for $a.e\ t \in [0, \infty)$ which is sometimes called Fleming-Rishel's Formula.

Again, by Isoperimetric Inequality on $\mathbb{R}$ with weight $|x|^k$ [see [4], Theorem 6.1] and the definition of u^* we obtain

$$\int\limits_{u=t} |x|^k \, d\mathcal{H}^0(x) \geq \int\limits_{u^*=t} |x|^k \, d\mathcal{H}^0(x). \tag{13}$$

Therefore, from (10), (11), (12), and (13) we have

$$I \geq I^*,$$

thus, (9) follows. ■

4. Trudinger-Moser inequality on $W^{1,p}_{\alpha,\theta}(0,\infty)$

In this section, we establish a Trudinger-Moser type inequality on $W^{1,p}_{\alpha,\theta}(0,\infty)$ (Theorem 1.1) via the Pólya-Szegö Principle presented in section 3.

Lemma 4.1.

(i) *Let u be a function in $W^{1,p}_{\alpha,\theta}(0,\infty)$. Then,*

$$|u(x)|^p \leq p\omega_\theta^{-\frac{p-1}{p}} \omega_\alpha^{-\frac{1}{p}} x^{-\frac{(p-1)\theta+\alpha}{p}} \|u\|^{p-1}_{L^p_\theta(0,\infty)} \left\|u'\right\|_{L^p_\alpha(0,\infty)} \quad \textit{for all} \;\; x > 0. \tag{14}$$

Consequently, the embedding $W^{1,p}_{\alpha,\theta}(0,\infty) \hookrightarrow L^q_\theta(0,\infty)$ is compact for all q satisfying

$$\frac{p^2(1+\theta)}{(p-1)\theta+\alpha} \leq q < \frac{p(1+\theta)}{\alpha-(p-1)} := p^\star,$$

where $\alpha \geq (p-1)$ and $\alpha \leq p+\theta$.

(ii) *Let $u \in L^p_\theta(0,R)$ be a nonincreasing function, then*

$$|u(x)| \leq \left(\frac{1+\theta}{\omega_\theta x^{1+\theta}}\right)^{1/p} \left[\int\limits_0^R |u(s)|^p d\lambda_\theta(s)\right]^{1/p}, \;\; \textit{for all} \;\; 0 < x < R. \tag{15}$$

Hence, if $(u_n) \subset W^{1,p}_{\alpha,\theta}(0,\infty)$ is a nonincreasing sequence converging weakly to u in $W^{1,p}_{\alpha,\theta}(0,\infty)$, then $u_n \to u$ strongly in $L^q_\theta(0,\infty)$, for each $p < q < p^\star$ $(\alpha \geq p-1)$.

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Proof. It is easy to check out (15) for a nonincreasing function. Then, we will do only (14).

For every $0 < x < y$ we have

$$|u(x)|^p \leq |u(y)|^p + p \int_x^y |u(t)|^{p-1} \left|u'(t)\right| dt.$$

By Holder Inequality and $\lim_{y\to\infty} u(y) = 0$, we get

$$\begin{aligned} |u(x)|^p &\leq p \int_x^\infty |u(t)|^{p-1} \left|u'(t)\right| dt \\ &\leq p\omega_\theta^{-\frac{p-1}{p}} \omega_\alpha^{-\frac{1}{p}} x^{-\frac{(p-1)\theta+\alpha}{p}} \left(\int_0^\infty |u(t)|^p \, d\lambda_\theta(t)\right)^{\frac{p-1}{p}} \cdot \left(\int_0^\infty \left|u'(t)\right|^p d\lambda_\alpha(t)\right)^{\frac{1}{p}}, \end{aligned}$$

which proves (14). ■

The next remark will be used in the proof of Theorem 1.1.

Remark 4.2. By inequality (14), we have $|u(x)| \leq 1$, for all

$$x \geq \left(\frac{p}{\omega_\theta^{\frac{p}{p-1}} \omega_\alpha^{\frac{1}{p}}}\right)^{\frac{p}{(p-1)(1+\theta)}} := a_0$$

whenever $u \in W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} \leq 1$ and $\alpha - (p-1) = 0$. It is worth noting that a_0 depends only on p, and θ.

Proof of Theorem 1.1. We can assume by Theorem 3.2 that u is a nonincreasing positive function on $(0,\infty)$ and we also recall that $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} \leq 1$.

Let $a \geq a_0$ (see Remark 4.2) to be chosen later. Next, we divide the integral at (4) into two parts, that is,

$$\int_0^\infty A_{p,\mu}(|u(x)|) d\lambda_\theta(x) = \int_0^a A_{p,\mu}(|u|) d\lambda_\theta(x) + \int_a^\infty A_{p,\mu}(|u|) d\lambda_\theta(x). \tag{16}$$

It follows from Lemma 4.1 and Remark 4.2 that

$$\int_a^\infty A_{p,\mu}(|u|) d\lambda_\theta(x) = \sum_{j=\lfloor p \rfloor}^\infty \frac{\mu^j}{j!} \int_a^\infty |u|^{\frac{p}{p-1} j} r^\theta \omega_\theta dr$$

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$$
\begin{aligned}
&\le \omega_\theta \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor!} \int_0^\infty |u|^p r^\theta dr \\
&+ \omega_\theta \sum_{j=\lfloor p \rfloor+1}^{\infty} \frac{\mu^j (1+\theta)^{\frac{j}{p-1}}}{j! \omega_\theta^{\frac{j}{p-1}}} \left[\omega_\theta \int_0^\infty |u|^p r^\theta dr \right]^{\frac{j}{p-1}} \cdot \int_a^\infty r^{\theta - \frac{(1+\theta)j}{p-1}} dr \\
=& \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor!} + \sum_{j=\lfloor p \rfloor+1}^{\infty} \frac{\mu^j (1+\theta)^{\frac{j}{p-1}} (p-1)\omega_\theta}{j! \omega_\theta^{j/p-1} (1+\theta)(j-(p-1)) a^{\frac{(1+\theta)(j-(p+1))}{p-1}}}.
\end{aligned} \tag{17}
$$

To estimate the other part at (16), let

$$
v(r) = \begin{cases} u(r) - u(a), & 0 < r \le a \\ 0, & r \ge a. \end{cases}
$$

Note that if $1 < q \le 2$ and $b \ge 0$, we have $(x+b)^q \le |x|^q + q b^{q-1} x + b^q$ for all $x \ge -b$. Then, by Lemma 4.1 we obtain

$$
\begin{aligned}
u(r)^{\frac{p}{p-1}} &\le v(r)^{\frac{p}{p-1}} + \frac{p}{p-1} v(r)^{\frac{1}{p-1}} u(a) + u(a)^{\frac{p}{p-1}} \\
&\le v(r)^{\frac{p}{p-1}} + v(r)^{\frac{p}{p-1}} u(a)^p + u(a)^{\frac{p}{p-1}} + \frac{1}{(p-1)^{1/p-1}} \\
&\le v(r)^{\frac{p}{p-1}} \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr \right) \right] + \left(\frac{1+\theta}{a^{1+\theta}\omega_\theta} \right)^{1/p-1} \\
&+ \frac{1}{(p-1)^{1/p-1}} \\
&:= v(r)^{\frac{p}{p-1}} \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr \right) \right] + d(a).
\end{aligned} \tag{18}
$$

Setting

$$
w(r) := v(r) \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr \right) \right]^{\frac{p-1}{p}},
$$

it follows that

$$
\omega_\alpha \int_0^a |w'|^p r^\alpha dr = \omega_\alpha \int_0^a |u'|^p \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta} \left(\omega_\theta \int_0^\infty |u|^p r^\theta dr \right) \right]^{p-1} r^\alpha dr
$$

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$$
\begin{aligned}
&= \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta}\left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right]^{p-1} \omega_\alpha \int_0^a |u'|^p r^\alpha dr \\
&\le \left[1 + \frac{1+\theta}{a^{1+\theta}\omega_\theta}\left(\omega_\theta \int_0^\infty |u|^p r^\theta dr\right)\right]^{p-1} \left[1 - \omega_\theta \int_0^\infty |u|^p r^\theta dr\right] \\
&\le 1,
\end{aligned} \tag{19}
$$

where the last inequality comes from the non-positivity of the function $f : [0, 1] \to \mathbb{R}$ defined by $f(t) = (1 + \gamma t)^{p-1}(1 - t) - 1$ for any γ fixed in the interval $(0, 1/(p - 1))$, whence the inequality (19) is valid with

$$
\left(\frac{(p-1)(1+\theta)}{\omega_\theta}\right)^{1/(1+\theta)} \le a < \infty.
$$

Next, it follows from (18) that

$$
u(r)^{\frac{p}{p-1}} \le w(r)^{\frac{p}{p-1}} + d(a),
$$

and consequently, we obtain

$$
\begin{aligned}
\int_0^a A_{p,\mu}(|u(x)|) d\lambda_\theta(x) &\le \omega_\theta \int_0^a e^{\mu|u|^{\frac{p}{p-1}}} r^\theta dr \\
&\le \omega_\theta e^{\mu d(a)} \int_0^a e^{\mu|w|^{\frac{p}{p-1}}} r^\theta dr.
\end{aligned} \tag{20}
$$

We combine (17), (19), (20) and Theorem 2.2 to conclude the first part of the proof of the theorem.

For the second part, we will make a change of variable as in [17]. We define $w(t) = \omega_\alpha^{\frac{1}{\alpha+1}}(1 + \theta)^{\frac{\alpha}{1+\alpha}} u(Re^{-\frac{t}{1+\theta}})$ for all $u \in W^{1,p}_{\alpha,\theta}(0, R)$, where $\alpha - (p - 1) = 0$. Then, we get

$$
\int_0^R |u'(r)|^p d\lambda_\alpha(r) = \int_0^\infty |w'(t)|^p dt, \tag{21}
$$

$$
\int_0^R |u(r)|^p d\lambda_\theta(r) = \frac{R^{1+\theta}\omega_\theta}{(1+\theta)^p \omega_\alpha} \int_0^\infty |w(t)|^p e^{-t} dt \tag{22}
$$

and

$$\int_0^R e^{\mu|u|^{\frac{p}{p-1}}} d\lambda_\theta(r) = \frac{\omega_\theta R^{1+\theta}}{1+\theta} \int_0^\infty e^{\frac{\mu}{\mu_{\alpha,\theta}}|w|^{\frac{p}{p-1}}-t} dt. \tag{23}$$

We consider Moser's functions

$$w_j(t) = \begin{cases} \frac{t}{j^{\frac{1}{p}}} & 0 \le t \le j \\ j^{\frac{p-1}{p}} & t \ge j. \end{cases}$$

Hence, we obtain from (21), (22) and (23) that

$$\begin{aligned} \int_0^R e^{\mu\left(\frac{|u_j|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,R)}}\right)^{\frac{p}{p-1}}} d\lambda_\theta(r) &= \frac{\omega_\theta R^{1+\theta}}{1+\theta} \int_0^\infty e^{\frac{\mu|w_j|^{\frac{p}{p-1}}}{\mu_{\alpha,\theta}\left(1+\rho(\alpha,\theta,R)a_j\right)^{\frac{1}{p-1}}}-t} dt \\ &\ge e^{\left(\frac{\mu}{\mu_{\alpha,\theta}(1+\rho(\alpha,\theta,R)a_j)^{\frac{1}{p-1}}}-1\right)j}, \end{aligned}$$

where $\rho(\alpha,\theta,R) = \frac{R^{1+\theta}\omega_\theta}{(1+\theta)^p\omega_\alpha}$, $a_j = \frac{1}{j}\int_0^j e^{-t}t^p dt + j^{p-1}e^{-j}$ and $w_j(t) = \omega_\alpha^{\frac{1}{1+\alpha}}(1+\theta)^{\frac{\alpha}{\alpha+1}} u_j(Re^{-\frac{t}{(1+\theta)}})$. Thus, if $\mu > \mu_{\alpha,\theta}$

$$\begin{aligned} \lim_{j\to\infty} \int_0^R e^{\mu\left(\frac{|u_j|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,R)}}\right)^{\frac{p}{p-1}}} d\lambda_\theta(r) &\ge \lim_{j\to\infty} e^{\left(\frac{\mu}{\mu_{\alpha,\theta}\left(1+\rho(\alpha,\theta,R)a_j\right)^{\frac{1}{p-1}}}-1\right)j} \\ &= +\infty, \end{aligned}$$

which concludes the theorem. ■

5. Proof of the Theorem 1.2

In this section, we are going to show the Theorem 1.2. To show the attainability, we study the behavior of maximizing sequences to (5). Throughout this section we assume that (u_n) is a bounded sequence in $W^{1,p}_{\alpha,\theta}(0,\infty)$ satisfying

$$u_n \rightharpoonup u \text{ in } W^{1,p}_{\alpha,\theta}(0,\infty), \quad \text{where } \alpha-(p-1)=0.$$

We begin with

Lemma 5.1. *Let $0<\mu<\mu_{\alpha,\theta}$. Assume that (u_n) is a positive maximizing sequence to (5). Then, we have*

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$$\int_0^\infty A_{p,\mu}(|u_n|) - \frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}|u_n|^{\frac{p\lfloor p\rfloor}{p-1}}\,d\lambda_\theta - \int_0^\infty A_{p,\mu}(|u|) - \frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}|u|^{\frac{p\lfloor p\rfloor}{p-1}}\,d\lambda_\theta \to 0$$

$$as\ n\to\infty. \tag{24}$$

Proof. We can rewrite (24) as follows

$$\int_0^\infty B_{\lfloor p\rfloor+1,\mu}(|u_n|)\;d\lambda_\theta - \int_0^\infty B_{\lfloor p\rfloor+1,\mu}(|u|)\,d\lambda_\theta \to 0$$

as $n\to\infty$, where

$$B_{k,\mu}(t) := \sum_{j=k}^{\infty}\frac{\mu^j}{j!}t^{\frac{p}{p-1}j},\ \text{ where } k\in\mathbb{N} \text{ and } t\in[0,\infty).$$

It follows from Mean Value Theorem and convexity of $B_{\lfloor p\rfloor,\mu}$ that

$$\begin{aligned}
&\left|B_{\lfloor p\rfloor+1,\mu}(u_n(x)) - B_{\lfloor p\rfloor+1,\mu}(u(x))\right|\\
&\le \left(B_{\lfloor p\rfloor+1,\mu}\right)'(\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x))\cdot|u_n(x)-u(x)|\\
&= \mu\frac{p}{p-1}|\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x)|^{\frac{1}{p-1}}\\
&\cdot B_{\lfloor p\rfloor,\mu}(\gamma_n(x)u_n(x)+(1-\gamma_n(x))u(x))\cdot|u_n(x)-u(x)|\\
&\le \mu\frac{p}{p-1}|\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x)|^{\frac{1}{p-1}}\\
&\cdot\left[\gamma_n(x)B_{\lfloor p\rfloor,\mu}(u_n(x))+(1-\gamma_n(x))B_{\lfloor p\rfloor,\mu}(u(x))\right]\cdot|u_n(x)-u(x)|\\
&\le \mu\frac{p}{p-1}|\gamma_n(x)u_n(x)+(1-\gamma_n(x)u(x)|^{\frac{1}{p-1}}\cdot\left[A_{p,\mu}(u_n(x))+A_{p,\mu}(u(x))\right]\\
&\cdot|u_n(x)-u(x)|
\end{aligned} \tag{25}$$

Now, by Hölder's and Minkowski's Inequalities and (25) we get

$$\begin{aligned}
&\left|\int_0^\infty B_{\lfloor p\rfloor+1,\mu}(|u_n|)\,d\lambda_\theta - \int_0^\infty B_{\lfloor p\rfloor+1,\mu}(|u|)\,d\lambda_\theta\right|\\
&\le \mu\frac{p}{p-1}\left(\int_0^\infty |\gamma_n(x)u_n(x)+(1-\gamma_n(x))u(x)|^{\frac{r}{p-1}}\,d\lambda_\theta(x)\right)^{\frac{1}{r}}\\
&\cdot\left(\int_0^\infty \left[A_{p,\mu}(u_n(x))+A_{p,\mu}(u(x))\right]^q\,d\lambda_\theta(x)\right)^{\frac{1}{q}}\left(\int_0^\infty |u_n(x)-u(x)|^t\,d\lambda_\theta(x)\right)^{\frac{1}{t}}
\end{aligned}$$

$$\begin{aligned}
&\le \mu \frac{p}{p-1}\left(\|u_n\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)} + \|u\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)}\right)\left(\int_0^\infty \left(A_{p,\mu}(u_n(x))\right)^q d\lambda_\theta(x)\right)^{\frac{1}{q}} \\
&\cdot \left(\int_0^\infty \left(A_{p,\mu}(u(x))\right)^q d\lambda_\theta(x)\right)^{\frac{1}{q}} \cdot \|u_n - u\|_{L_\theta^t(0,\infty)} \\
&\le \mu \frac{p}{p-1}\left(\|u_n\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)} + \|u\|^{\frac{1}{p-1}}_{L_\theta^{\frac{r}{p-1}}(0,\infty)}\right)\left(\int_0^\infty A_{p,q\mu}(u_n(x)) d\lambda_\theta(x)\right)^{\frac{1}{q}} \\
&\cdot \left(\int_0^\infty A_{p,q\mu}(u(x)) d\lambda_\theta(x)\right)^{\frac{1}{q}} \|u_n - u\|_{L_\theta^t(0,\infty)}, \qquad (26)
\end{aligned}$$

where $q, r, t > 1$ are real numbers satisfying $\frac{1}{r} + \frac{1}{q} + \frac{1}{t} = 1$, $q\mu < \mu_{\alpha,\theta}$, $\frac{r}{p-1} \ge p$ and $t > \frac{p^2}{p-1}$. Besides, in the last inequality at (26), we used the following inequality

$$\left(e^{\mu t^{\frac{p}{p-1}}} - \sum_{j=0}^{\lfloor p\rfloor -1} \frac{\mu^j}{j!} t^{\frac{p}{p-1}j}\right)^q \le e^{q\mu t^{\frac{p}{p-1}}} - \sum_{j=0}^{\lfloor p\rfloor -1} \frac{(q\mu)^j}{j!} t^{\frac{p}{p-1}j}.$$

Therefore, from (26), Lemma 4.1, and compactness embedding, we conclude the proof of the Lemma. ■

To study the compactness of a maximizing sequence to (5) based on the concentration-compactness type argument, we analyze the possibility of a lack of compactness which is called vanishing. For this, we will introduce some components as follows

$$\mu_0 = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_0^R |u_n(x)|^p d\lambda_\theta(x) + \int_0^R |(u_n)'(x)|^p d\lambda_\alpha(x)\right)$$

$$\mu_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_R^\infty |u_n(x)|^p d\lambda_\theta(x) + \int_R^\infty |(u_n)'(x)|^p d\lambda_\alpha(x)\right),$$

$$\nu_0 = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_0^R A_{p,\mu}(|u_n|)\, d\lambda_\theta(x)\right)$$

$$\nu_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\left(\int_R^\infty A_{p,\mu}(|u_n(x)|)\, d\lambda_\theta(x)\right),$$

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$$\eta_0 = \lim_{R\to\infty}\lim_{n\to\infty}\int_0^R |u_n(x)|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta(x), \text{ and}$$

$$\eta_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\int_R^\infty |u_n(x)|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta(x)$$

taking an appropriate subsequence if necessary. It is easy to see that, if $(u_n)_{n\in\mathbb{N}}$ is a maximizing sequence to (5) ($\|u_n\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$), then

$$\begin{aligned} &\nu_i \geq \frac{\mu^{\lfloor p\rfloor}}{\lfloor p\rfloor!}\eta_i,\ 1 = \mu_0 + \mu_\infty,\ d(p,\theta,\mu) = \nu_0 + \nu_\infty \text{ and} \\ &1 \geq \eta_0 + \eta_\infty \,(\text{if } p \text{ is an integer}), \end{aligned} \tag{27}$$

where $i = 0$ or $i = \infty$.

Definition 5.2. (u_n) is a normalized vanishing sequence, (NVS) in short, if (u_n) satisfies $\|u_n\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} = 1$ (with $\alpha - (p-1) = 0$), $u = 0$ and $\nu_0 = 0$.

Example 5.3. Let ϕ be a smooth nonincreasing function with compact support on $[0, +\infty)$ satisfying $\|\phi\|_{L^p_\theta(0,\infty)} = 1$. We set

$$\phi_n(x) := \frac{\lambda_n \phi(\lambda_n^\gamma x)}{(1+\lambda_n^p \lambda_0)^{\frac{1}{p}}},$$

where $\gamma = \frac{p}{1+\theta}$, $\lambda_0 := \|\phi'\|^p_{L^p_\alpha(0,\infty)}$, and (λ_n) is a positive sequence such that $\lambda_n \to 0$ as $n \to \infty$. Thus, $(\phi_n)_{n\in\mathbb{N}}$ is a normalized vanishing sequence.

The main aim here is to show that $d(\alpha,\theta,\mu)$ is greater than the vanishing level. More precisely,

$$d(\alpha,\theta,\mu) > \sup_{\left\{(u_n)\subset W^{1,p}_{\alpha,\theta}(0,\infty):(u_n)\text{ is a NVS}\right\}} \int_0^\infty A_{p,\mu}(|u_n(x)|)\, d\lambda_\theta(x).$$

Thus, we define the *normalized vanishing limit* as follows

Definition 5.4. The number

$$d_{nvl}(\alpha,\theta,\mu) = \sup_{\left\{(u_n)\subset W^{1,p}_{\alpha,\theta}(0,\infty):(u_n)\text{ is a NVS}\right\}} \int_0^\infty A_{p,\mu}(|u_n(x)|)\, d\lambda_\theta(x), \tag{28}$$

is called a *normalized vanishing limit*.

The *normalized vanishing limit* will depend only on α and μ.

Next, we rewrite the elements defined above. For this purpose, given a real number $R > 0$, we take a function $\phi_R \in C^\infty(\mathbb{R})$, such that

$$\begin{cases} \phi_R(x) = 1, & \text{if } 0 \le x < R, \\ 0 \le \phi_R(x) \le 1, & \text{if } R \le x \le R+1, \\ \phi_R(x) = 0, & \text{if } R+1 \le x, \\ |\phi_R'(x)| \le 2, & \text{if } x \in \mathbb{R}. \end{cases}$$

After that, we define the functions ϕ_R^0 and ϕ_R^∞ by

$$\phi_R^0(x) := \phi_R(x), \;\; \phi_R^\infty(x) := 1 - \phi_R^0(x).$$

Lemma 5.5. *Let $R > 0$, $n \in \mathbb{N}$, and $u_{n,R}^i = \phi_R^i u_n$ $(i = 0, \infty)$. We have*

$$\mu_i = \lim_{R\to\infty} \lim_{n\to\infty} \left(\int_0^\infty |u_{n,R}^i(x)|^p d\lambda_\theta(x) + \int_0^\infty |(u_{n,R}^i(x))'|^p d\lambda_\alpha(x) \right), \tag{29}$$

$$\nu_i = \lim_{R\to\infty} \lim_{n\to\infty} \int_0^\infty A_{p,\mu}\left(|u_{n,R}^i(x)|\right) d\lambda_\theta(x), \;\; \textit{and} \tag{30}$$

$$\eta_i = \lim_{R\to\infty} \lim_{n\to\infty} \int_0^\infty |u_{n,R}^i|^{\frac{p}{p-1}\lfloor p \rfloor} d\lambda_\theta(x). \tag{31}$$

Proof. We will prove only (29) with $i = 0$. On the one hand,

$$\int_0^R |u_n|^p d\lambda_\theta \le \int_0^\infty |\phi_R^0 u_n|^p d\lambda_\theta \le \int_0^{R+1} |u_n|^p d\lambda_\theta. \tag{32}$$

On the other hand, from the Mean Value Theorem we obtain

$$\begin{aligned} \int_0^\infty |(u_{n,R}^0)'|^p d\lambda_\alpha &= \int_0^\infty |\phi_R^0 u_n' + (\phi_R^0)' u_n|^p d\lambda_\alpha \\ &= \int_0^\infty |\phi_R^0 u_n'|^p d\lambda_\alpha + \rho_{n,R}, \end{aligned} \tag{33}$$

where

$$\rho_{n,R} = p \int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^{p-2} \phi_R^0 u_n' (\phi_R^0)' u_n \, d\lambda_\alpha(x)$$

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$$+ p \int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^{p-2} t_n(x)(\phi_R^0)' u_n \cdot (\phi_R^0)' u_n d\lambda_\alpha(x)$$

and $0 \le t_n(x) \le 1$.

We get

$$\begin{aligned}|\rho_{n,R}| &\le p \left[\int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^p d\lambda_\alpha \right]^{\frac{p-1}{p}} \left[\int_0^\infty |(\phi_R^0)' u_n|^p d\lambda_\alpha \right]^{\frac{1}{p}} \\ &\le 2p[\|u_n'\|_{L^p_\alpha} + 2\|u_n\|_{L^p_\alpha(R,R+1)}]^{p-1} \|u_n\|_{L^p_\alpha(R,R+1)} \\ &\le 2p\left[1 + 2\|u_n\|_{L^p_\alpha(R,R+1)}\right]^{p-1} \|u_n\|_{L^p_\alpha(R,R+1)}.\end{aligned}$$

By compactness embedding, $W^{1,p}((R,R+1)) \hookrightarrow L^p((R,R+1))$ with $R > 0$, we have $\lim_{n\to\infty} \|u_n\|_{L^p_\alpha(R,R+1)} = \|u\|_{L^p_\alpha(R,R+1)}$. Thus,

$$\lim_{R\to\infty} \lim_{n\to\infty} \rho_{n,R} = 0. \tag{34}$$

We conclude (29) (with $i = 0$) from (32), (33) and (34). The other cases follow from similar arguments. ■

Next, our goal is determining the normalized vanishing limit defined at (28).

Proposition 5.6. *It holds that*

$$d_{nvl}(p,\theta,\mu) = \begin{cases} \dfrac{\mu^{p-1}}{(p-1)!} & \text{if } p \text{ is integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can suppose that u_n is non-increasing (see Theorem 3.2), then by Lemma 4.1

$$|u_n(x)| \le \left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{1}{p}} \cdot \frac{1}{x^{\frac{1+\theta}{p}}} \left(\int_0^\infty |u_n(y)|^p d\lambda_\theta(y) \right)^{\frac{1}{p}}.$$

Assuming that $1 \le R < \infty$, then

$$\begin{aligned}\sum_{j=\lfloor p\rfloor+1}^\infty \frac{\mu^j}{j!} \int_R^\infty |u_n|^{\frac{p}{p-1}j} d\lambda_\theta &\le \sum_{j=\lfloor p\rfloor+1}^\infty \frac{\mu^j}{j!} \left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{j}{p-1}} \omega_\theta \int_R^\infty x^{\theta - \frac{(1+\theta)}{p-1} j} dx \\ &\le \sum_{j=\lfloor p\rfloor+1}^\infty \frac{\mu^j}{j!} \left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{j}{p-1}} \frac{\omega_\theta (p-1)}{R^{(1+\theta)\left(\frac{j}{p-1}-1\right)}}\end{aligned}$$

$$\leq \frac{\omega_\theta(p-1)}{R^{(1+\theta)\left(\frac{\lfloor p\rfloor+1}{p-1}-1\right)}} \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{j}{p-1}}.$$

Thus

$$\lim_{R\to\infty}\lim_{n\to\infty} \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\int_R^\infty |u_n|^{\frac{p}{p-1}j}\, d\lambda_\theta = 0. \tag{35}$$

If p is not integer, we get

$$\int_R^\infty |u_n|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta \leq \left(\frac{1+\theta}{\omega_\theta}\right)^{\frac{\lfloor p\rfloor}{p-1}} \frac{\omega_\theta(p-1)}{(\lfloor p\rfloor-(p-1))\, R^{(1+\theta)\left(\frac{\lfloor p\rfloor}{p-1}-1\right)}}. \tag{36}$$

Hence, using (35) and (36), we obtain $\nu_\infty = 0$, if p is not integer.

Now, if p is integer, then $\lfloor p\rfloor = p-1$ and passing to subsequence if necessary, we have

$$\begin{aligned}
\nu_\infty &= \lim_{R\to\infty}\lim_{n\to\infty} \sum_{j=\lfloor p\rfloor+1}^{\infty} \frac{\mu^j}{j!}\int_R^\infty |u_n|^{\frac{p}{p-1}j}\, d\lambda_\theta \\
&\quad + \lim_{R\to\infty}\lim_{n\to\infty} \frac{\mu^{p-1}}{(p-1)!}\|u_n\|^p_{L^p_\theta(R,\infty)} \\
&= \frac{\mu^{p-1}}{(p-1)!}\lim_{R\to\infty}\lim_{n\to\infty}\|u_n\|^p_{L^p_\theta(R,\infty)} \\
&\leq \frac{\mu^{p-1}}{(p-1)!}.
\end{aligned} \tag{37}$$

Taking $u_n := \phi_n$ as in the Example 5.3 we obtain (35) as well. Besides, we get

$$\lim_{R\to\infty}\lim_{n\to\infty}\|u_n\|^p_{L^p_\theta(R,+\infty)} = \lim_{R\to\infty}\lim_{n\to\infty} \frac{\|\phi\|^p_{L^p_\theta(\lambda_n^\sigma R,\infty)}}{(1+\lambda_n^{p\gamma}\lambda_0)} = 1. \tag{38}$$

From (35), (36), (37) and (38) the proposition follows. ■

Proposition 5.7. *Let $p \geq 2$ be an integer number. Then*

$$d(p,\theta,\mu) > \begin{cases} \dfrac{\mu^{p-1}}{(p-1)!}, & \text{if } p>2 \text{ and } \mu \in (0,\mu_{\alpha,\theta}], \\[2ex] \dfrac{\mu^{p-1}}{(p-1)!}, & \text{if } p=2 \text{ and } \mu \in \left(\frac{2}{B(2,\theta)},\mu_{\alpha,\theta}\right], \end{cases}$$

where $B(2,\theta)$ is defined as in Theorem 1.2.

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Proof. Let $\sigma = \frac{p}{1+\theta}$, and $v \in W^{1,p}_{\alpha,\theta}(0,\infty)$. We set

$$v_t(x) = tv(t^{\sigma}x), \quad \text{for all } t, x \in (0,\infty).$$

We get

$$\begin{aligned}
&\int_0^\infty A_{p,\mu}\left(\frac{|v_t|}{\|v_t\|_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\right) d\lambda_\theta \\
&\geq \frac{\mu^{p-1}}{(p-1)!}\left[\frac{\|v\|^p_{L^p_\theta}}{\|v\|^p_{L^p_\theta} + t^p\|v'\|^p_{L^p_\alpha}} + \frac{\mu}{p}\frac{t^{\frac{p}{p-1}}\|v\|^{\frac{p}{p-1}p}_{L^{\frac{p}{p-1}p}_\theta}}{\left(\|v\|^p_{L^p_\theta} + t^p\|v'\|^p_{L^p_\alpha}\right)^{\frac{p}{p-1}}}\right] \\
&:= \frac{\mu^{p-1}}{(p-1)!}h_{p,\theta,\mu}(t).
\end{aligned}$$

Note that $\lim_{t\to 0} h_{p,\theta,\mu}(t) = 1$. Thus, it is sufficient to show that $h'_{p,\theta,\mu}(t) > 0$ for $0 < t \ll 1$. Through straightforward calculation we obtain

$$\begin{aligned}
h'_{p,\theta,\mu}(t) = &\frac{pt^{\frac{p}{p-1}-1}}{\left(\|v\|^p_{L^p_\theta} + t^p\|v'\|^p_{L^p_\alpha}\right)^2} \\
&\cdot\left[\frac{\mu}{p(p-1)}\|v\|^{\frac{p}{p-1}p}_{L^{\frac{p}{p-1}p}_\theta}\left(\|v\|^p_{L^p_\theta} + t^p\left\|v'\right\|^p_{L^p_\alpha}\right)^{\frac{p-2}{p-1}}\right. \\
&- \frac{\mu}{p-1}t^p\|v\|^{\frac{p}{p-1}p}_{L^{\frac{p}{p-1}p}_\theta}\left\|v'\right\|^p_{L^p_\alpha}\left(\|v\|^p_{L^p_\theta} + t^p\left\|v'\right\|^p_{L^p_\alpha}\right)^{-\frac{1}{p-1}} \\
&\left.- \|v\|^p_{L^p_\theta}\left\|v'\right\|^p_{L^p_\alpha}t^{p-\frac{p}{p-1}}\right].
\end{aligned}$$

Thus we get $h'_{p,\theta,\mu}(t) > 0$ for $0 < t \ll 1$ if $p > 2$. Now, for $p = 2$ it is slightly different, because

$$h'_{2,\theta,\mu}(t) = \frac{2t}{\left(\|v\|^2_{L^2_\theta} + t^2\|v'\|^2_{L^2_1}\right)^2}\cdot\left[\frac{\mu}{2}\|v\|^4_{L^4_\theta} - \frac{\mu t^2\|v\|^4_{L^4_\theta}\left\|v'\right\|^2_{L^2_1}}{\left(\|v\|^2_{L^2_\theta} + t^2\|v'\|^2_{L^2_1}\right)} - \|v\|^2_{L^2_\theta}\left\|v'\right\|^2_{L^2_1}\right].$$

Taking $v \in W^{1,p}_{\alpha,\theta}(0,\infty)$, such that $B(2,\theta)^{-1} = B(v)^{-1}$, we obtain $h'_{2,\theta,\mu}(t) > 0$ for $0 < t \ll 1$, if $\frac{2}{B(2,\theta)} < \mu \leq 2\pi(1+\theta)$, [see Proposition 7.1]. ■

Lemma 5.8. *Let $\mu_i < 1$ $(i = 0, \infty)$ and let $p \geq 2$ be an integer. Then, we obtain*

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$$
\begin{aligned}
&d(p,\theta,\mu)\left\|u^i_{n,R}\right\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)}\\
&\geq \int_0^\infty A_{p,\mu}\left(\left|u^i_{n,R}\right|\right)d\lambda_\theta+\left[\left(\frac{1}{\left\|u^i_{n,R}\right\|^{\frac{p}{p-1}}_{W^{1,p}_{\alpha,\theta}(0,\infty)}}-1\right)\right.\\
&\left.\cdot\int_0^\infty A_{p,\mu}\left(\left|u^i_{n,R}\right|\right)-\frac{\mu^{p-1}}{(p-1)!}\left|u^i_{n,R}\right|^p d\lambda_\theta\right],
\end{aligned}
$$

whenever n and R are sufficiently large.

Proof. By definition, we have

$$
\begin{aligned}
d(\alpha,\theta,\mu)&\geq\sum_{j=p-1}^\infty\frac{\mu^j}{j!}\frac{\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}}{\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\\
&\geq\frac{1}{\|u^i_{n,R}\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\sum_{j=p-1}^\infty\frac{\mu^j}{j!}\|u^i_{n,R}\|^{\frac{jp}{p-1}}_{L_\theta^{\frac{jp}{p-1}}}\\
&+\frac{1}{\|u^i_{n,R}\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)}}\sum_{j=p}^\infty\left(\frac{1}{\|u^i_{n,R}\|^{\frac{p}{p-1}(j-(p-1))}_{W^{1,p}_{\alpha,\theta}(0,\infty)}}-1\right)\frac{\mu^j}{j!}\|u^i_{n,R}\|^{\frac{jp}{p-1}}_{L_\theta^{\frac{jp}{p-1}}}
\end{aligned}
\tag{39}
$$

From $\mu_i<1$ and (39) we obtain

$$
\begin{aligned}
&d(\alpha,\theta,\mu)\|u^i_{n,R}\|^p_{W^{1,p}_{\alpha,\theta}(0,\infty)}\\
&\geq\sum_{j=p-1}^\infty\frac{\mu^j}{j!}\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}+\sum_{j=p}^\infty\left(\frac{1}{\|u^i_{n,R}\|^{\frac{p}{p-1}}_{W^{1,p}_{\alpha,\theta}(0,\infty)}}-1\right)\frac{\mu^j}{j!}\|u^i_{n,R}\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}\\
&=\int_0^\infty A_{p,\mu}(|u^i_{n,R}|)d\lambda_\theta\\
&+\left(\frac{1}{\|u^i_{n,R}\|^{\frac{p}{p-1}}_{W^{1,p}_{\alpha,\theta}(0,\infty)}}-1\right)\int_0^\infty A_{p,\mu}(|u^i_{n,R}|)-\frac{\mu^{p-1}}{(p-1)!}|u^i_{n,R}|^p d\lambda_\theta
\end{aligned}
$$

for large R and large n. ■

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Proposition 5.9. *Let $p \geq 2$ be an integer. Assume that (u_n) is a positive maximizing sequence to (5). Then*

$$(\mu_0, \nu_0) = (1, d(p, \theta, \mu)) \text{ and } (\mu_\infty, \nu_\infty) = (0, 0).$$

Proof. By contradiction, suppose that $0 < \mu_0 < 1$. Then $0 < \mu_\infty < 1$, by relation (27). From Lemma 5.5 and Lemma 5.8 we have

$$d(\alpha, \theta, \mu)\mu_i \geq \nu_i + \left[\frac{1}{\mu_i^{\frac{1}{p-1}}} - 1\right]\left[\nu_i - \frac{\mu^{p-1}}{(p-1)!}\eta_i\right]. \tag{40}$$

By relation (27) and together with (40) we get

$$d(\alpha, \theta, \mu)\mu_i \geq \nu_i, \quad \text{for } i = 0, \infty.$$

Thus,

$$d(\alpha, \theta, \mu) = d(\alpha, \theta, \mu)(\mu_0 + \mu_\infty) \geq \nu_0 + \nu_\infty = d(\alpha, \theta, \mu)$$

and consequently

$$d(\alpha, \theta, \mu)\mu_i = \nu_i.$$

From the last relation and (40) we obtain

$$\nu_i \leq \frac{\mu^{p-1}}{(p-1)!}\eta_i,$$

whence,

$$d(\alpha, \theta, \mu) = \nu_0 + \nu_\infty \leq \frac{\mu^{p-1}}{(p-1)!}(\eta_0 + \eta_\infty) \leq \frac{\mu^{p-1}}{(p-1)!},$$

which contradicts the Proposition 5.7.

Now, again, by contradiction, suppose that $\mu_0 = 0$. Thus, by Lemma 5.8

$$\begin{aligned} & d(p, \theta, \mu) \left\| u_{n,R}^0 \right\|_{W_{\alpha,\theta}^{1,p}(0,\infty)}^p \\ & \geq \int_0^\infty A_{p,\mu}\left(\left|u_{n,R}^0\right|\right) d\lambda_\theta \\ & + \frac{1}{2}\int_0^\infty \left(A_{p,\mu}\left(\left|u_{n,R}^0\right|\right) - \frac{\mu^{p-1}}{(p-1)!}\left|u_{n,R}^0\right|^p\right) d\lambda_\theta, \end{aligned} \tag{41}$$

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for large R and large n.

Taking the double limit in (41), $\lim_{R\to\infty}\lim_{n\to\infty}$, we obtain

$$d(\alpha,\theta,\mu)\mu_0 \geq \nu_0 + \frac{1}{2}\left(\nu_0 - \frac{\mu^{p-1}}{(p-1)!}\eta_0\right) \geq \nu_0,$$

hence, $\nu_0 = 0$ from relation (27), and $\mu_0 = 0$, getting a contradiction from Proposition 5.6, relation (27), and Proposition 5.7. Finally, using the same arguments we can get $\nu_\infty = 0$ whenever $\mu_\infty = 0$. Therefore, the proposition follows. ■

Proof of Theorem 1.2. First of all, assume that (u_n) is a non-increasing positive maximizing sequence to (5). We will show that

$$\lim_{n\to\infty}\int_0^\infty |u_n|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta = \int_0^\infty |u|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta. \tag{42}$$

Indeed, given $R > 0$, note that

$$\begin{aligned}\left|\int_0^\infty \left(|u_n|^{\frac{p}{p-1}\lfloor p\rfloor} - |u|^{\frac{p}{p-1}\lfloor p\rfloor}\right) d\lambda_\theta\right| &\leq \left|\int_0^R \left(|u_n|^{\frac{p}{p-1}\lfloor p\rfloor} - |u|^{\frac{p}{p-1}\lfloor p\rfloor}\right) d\lambda_\theta\right| \\ &\quad + \int_R^\infty |u_n|^{\frac{p}{p-1}\lfloor p\rfloor} d\lambda_\theta + \int_R^\infty |u|^{\frac{p}{p-1}\lfloor p\rfloor} dx \\ &=: I(n,R) + II(n,R) + III(R).\end{aligned}$$

By compact embedding we have $\lim_{R\to\infty}\lim_{n\to\infty} I(n,R) = 0$. From Dominated Convergence Theorem, we obtain $\lim_{R\to\infty}\lim_{n\to\infty} III(R) = 0$. If p is an integer, we get $\lim_{R\to\infty}\lim_{n\to\infty} II(n,R) = 0$ from $\mu_\infty = 0$ (Proposition 5.9). If $p \notin \mathbb{N}$, we obtain $\lim_{R\to\infty}\lim_{n\to\infty} II(n,R) = 0$ from inequality (36). Hence, (42) follows. Now, assume that either $p > 2$ and $\mu \in (0, \mu_{\alpha,\theta})$ or $p = 2$ and $\mu \in (2/B(2,\theta), \mu_{1,\theta})$. Writing

$$\begin{aligned}d(p,\theta,\mu) - \int_0^\infty A_{p,\mu}(|u|)\, d\lambda_\theta &= \int_0^\infty A_{p,\mu}(|u_n|)\, d\lambda_\theta - \int_0^\infty A_{p,\mu}(|u|)\, d\lambda_\theta \\ &\quad + \left(d(p,\theta,\mu) - \int_0^\infty A_{p,\mu}(|u_n|)\, d\lambda_\theta\right) \\ &=: IV(n) + V(n),\end{aligned}$$

where

$$V(n) := d(p,\theta,\mu) - \int_0^\infty A_{p,\mu}(|u_n|)\, d\lambda_\theta$$

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and

$$IV(n) := \int_0^\infty A_{p,\mu}(|u_n|)\, d\lambda_\theta - \int_0^\infty A_{p,\mu}(|u|)\, d\lambda_\theta .$$

We get, by definition of $d(\alpha, \theta, \mu)$, that

$$\lim_{R\to\infty} \lim_{n\to\infty} V(n) = 0.$$

Since

$$\begin{aligned} IV(n) &= \int_0^\infty \left(A_{p,\mu}(|u_n|) - \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor !} |u_n|^{\frac{p}{p-1}\lfloor p \rfloor} \right) d\lambda_\theta \\ &\quad - \int_0^\infty \left(A_{p,\mu}(|u|) - \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor !} |u|^{\frac{p}{p-1}\lfloor p \rfloor} \right) d\lambda_\theta \\ &\quad + \frac{\mu^{\lfloor p \rfloor}}{\lfloor p \rfloor !} \int_0^\infty \left(|u_n|^{\frac{p}{p-1}\lfloor p \rfloor} - |u|^{\frac{p}{p-1}\lfloor p \rfloor} \right) d\lambda_\theta . \end{aligned}$$

From Lemma 5.1 and relation (42) we obtain

$$\lim_{n\to\infty} IV(n) = 0.$$

Thus, $u \neq 0$ and

$$d(p, \theta, \mu) = \int_0^\infty A_{p,\mu}(|u|)\, d\lambda_\theta .$$

Now, we assert that $\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1$. Indeed, on the one hand,

$$\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} \leq \liminf_{n\to\infty} \|u_n\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1.$$

On the other hand,

$$\begin{aligned} d(p,\theta,\mu) &\geq \int_0^\infty A_{p,\mu}\left(\left(\frac{|u|}{\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}} \right) \right) d\lambda_\theta \\ &= \sum_{j=\lfloor p \rfloor}^{\infty} \frac{\mu^j}{j!} \frac{\|u\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}}{\|u\|^{\frac{p}{p-1}j}_{W^{1,p}_{p-1,\theta}(0,\infty)}} \end{aligned}$$

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$$\geq \frac{1}{\|u\|^{\frac{p}{p-1}\lfloor p\rfloor}_{W^{1,p}_{p-1,\theta}(0,\infty)}} \sum_{j=\lfloor p\rfloor}^{\infty} \frac{\mu^j}{j!} \|u\|^{\frac{p}{p-1}j}_{L_\theta^{\frac{p}{p-1}j}}$$

$$\geq \frac{1}{\|u\|^{\frac{p}{p-1}\lfloor p\rfloor}_{W^{1,p}_{p-1,\theta}(0,\infty)}} \cdot d(p,\theta,\mu).$$

Therefore, $\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1$ and

$$d(p,\theta,\mu) = \int_0^\infty A_{p,\mu}(|u|)\, d\lambda_\theta. \quad \blacksquare$$

6. Proof of the Theorem 1.3

Throughout this section, we assume that $p = 2$ and $\mu \leq \pi(1+\theta)/3$.
By Theorem 2.4 (inequality (8)), we get

$$\frac{\|u\|^{2j}_{L^{2j}_\theta}}{\|u'\|^2_{L^2_1} \cdot \|u\|^2_{L^2_\theta}} \leq C_{\gamma,2,\theta} \frac{j!}{\gamma^j} \|u'\|^{2(j-2)}_{L^2_1} \tag{43}$$

for all $u \in W^{1,2}_{1,\theta}(0,\infty)$, $j \in \mathrm{N}$, and $0 < \gamma < (1+\theta)\omega_1$. We are going to use the inequality (43) to prove the Theorem 1.3.

Proof of Theorem 1.3. Let $S := \{v \in W^{1,2}_{1,\theta}(0,\infty) : \|v\|_{W^{1,2}_{1,\theta}(0,\infty)} = 1\}$. For each $u \in S$, we define a family of functions by

$$u_t(x) := t^{\frac{1}{2}} u(t^{\frac{1}{1+\theta}} x),$$

where $t > 0$ is a parameter. Besides, let $v_t := u_t/\|u_t\|_{W^{1,2}_{1,\theta}(0,\infty)}$. Thus, v_t is a curve in S passing through u when $t = 1$. Then, it is sufficient to show that

$$\frac{d}{dt} F(v_t)\,|_{t=1} < 0,$$

where $F(w) := \int_0^\infty A_{2,\mu}(w(x))\, d\lambda_\theta(x)$.

Through a direct calculation we have that $\|u_t\|^{2j}_{L^{2j}_\theta} = t^{j-1}\|u\|^{2j}_{L^{2j}_\theta}$, $\|(u_t)'\|^2_{L^2_1} = t\|u'\|^2_{L^2_1}$ and

$$F(v_t) = \sum_{j=1}^{\infty} \frac{\mu^j}{j!} \frac{t^{j-1}\|u\|^{2j}_{L^{2j}_\theta}}{\left(\|u\|^2_{L^2_\theta} + t\|u'\|^2_{L^2_1}\right)^j}.$$

Thus, we obtain

$$\begin{aligned}\frac{d}{dt}F(v_t)\,|_{t=1} &= \sum_{j=1}^{\infty}\frac{\mu^j}{j!}\|u\|^{2j}_{L^{2j}_\theta}\left[(j-1)\|u\|^2_{L^2_\theta} - j\|u'\|^2_{L^2_1}\right]\\ &= -\mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1} + \sum_{j=2}^{\infty}\frac{\mu^j}{j!}\|u\|^{2j}_{L^{2j}_\theta}\left[(j-1)\|u\|^2_{L^2_\theta} - j\|u'\|^2_{L^2_1}\right]\\ &\le \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\left(-1+\sum_{j=2}^{\infty}\frac{\mu^{j-1}}{(j-1)!}\frac{\|u\|^{2j}_{L^{2j}_\theta}}{\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}}\right). \end{aligned} \tag{44}$$

From inequality (43) (with $\gamma := 2\pi(1+\theta)/3$) and (44) we get

$$\begin{aligned}&\frac{d}{dt}F(v_t)\,|_{t=1}\\ &\le \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\left(-1+C_{\frac{2}{3}\pi(1+\theta),2,\theta}\sum_{j=2}^{\infty}\frac{\mu^{j-1}}{(j-1)!}j!\left(\frac{3}{2\pi(1+\theta)}\right)^j\right)\\ &= \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\\ &\cdot\left(-1+C_{\frac{2}{3}\pi(1+\theta),2,\theta}\left(\frac{3}{2\pi(1+\theta)}\right)^2\mu\sum_{j=2}^{\infty}\mu^{j-2}j\left(\frac{3}{2\pi(1+\theta)}\right)^{j-2}\right)\\ &\le \mu\|u\|^2_{L^2_\theta}\|u'\|^2_{L^2_1}\left(-1+C_{\frac{2}{3}\pi(1+\theta),2,\theta}\left(\frac{3}{2\pi(1+\theta)}\right)^2\mu\sum_{j=2}^{\infty}j\left(\frac{1}{2}\right)^{j-2}\right).\end{aligned}$$

Thus, taking $\mu_0 := \frac{1}{a\cdot C_{\frac{2}{3}\pi(1+\theta),2,\theta}}\left(\frac{2\pi(1+\theta)}{3}\right)^2$, where $a := \sum_{j=2}^{\infty} j\left(\frac{1}{2}\right)^{j-2}$, the proof of the theorem follows. ■

7. Gagliardo-Nirenberg inequalities

In this section, we discuss about the best constant of the Gagliardo-Niremberg inequality, and we will explore some ideas contained in [3,6,8].

It is known the interpolation inequality with weights

$$\|u\|_{L^q_\theta(0,\infty)} \le K(p,q,\alpha,\theta)\,\|u'\|^{\gamma}_{L^p_\alpha(0,\infty)}\,\|u\|^{1-\gamma}_{L^p_\theta(0,\infty)}, \tag{45}$$

where $1 < p \le q < p^\star = \frac{p(1+\theta)}{\alpha-(p-1)}$, $\alpha \ge p-1$, $\theta \ge 0$ and $1-\gamma = \frac{p}{q}\cdot\frac{(p^\star-q)}{(p^\star-p)}$. It is worth noting that when $\alpha = p-1$ we have $1-\gamma = \frac{p}{q}$.

Throughout this section we will assume that $\alpha \le p+\theta$. So, we can compute the optimal $k = K(p,q,\alpha,\theta)$ in (45) if we determine the explicit solution of the minimization problem

$$\inf\left\{E(u):=\frac{1}{p}\int_0^\infty |u'|^p\,d\lambda_\alpha+\frac{1}{p}\int_0^\infty |u|^p\,d\lambda_\theta : \|u\|_{L^q_\theta((0,\infty))}=1\right\}. \tag{46}$$

It follows from Lemma 4.1 and Theorem 3.2 (with $\alpha=m$, and $l=\theta$) the existence of a minimizer for (46). Now if u_∞ is a minimizer of the variational problem (46), then

$$E(u_\infty)\le E(u)=\frac{1}{p}\,\|u'\|^p_{L^p_\alpha((0,\infty))}+\frac{1}{p}\,\|u\|^p_{L^p_\theta((0,\infty))}$$

for all $u\in W^{1,p}_{\alpha,\theta}((0,\infty))$ satisfying $\|u\|_{L^q_\theta((0,\infty))}=1$. Thus,

$$E(u_\infty)\le\frac{1}{p}\frac{\|u'\|^p_{L^p_\alpha((0,\infty))}}{\|u\|^p_{L^q_\theta((0,\infty))}}+\frac{1}{p}\frac{\|u\|^p_{L^p_\theta((0,\infty))}}{\|u\|^p_{L^q_\theta((0,\infty))}}$$

for every $0\neq u\in W^{1,p}_{\alpha,\theta}(0,\infty)$. Scaling u as $u_t(x)=u(tx)$, we get

$$E(u_\infty)\le t^{p-(\alpha+1)+\frac{p}{q}(1+\theta)}\frac{\|u'\|^p_{L^p_\alpha((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}}+t^{(1+\theta)\left(\frac{p}{q}-1\right)}\frac{\|u\|^p_{L^p_\theta((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}}.$$

A direct computation proves that the minimum over t is achieved at

$$t=\left[\frac{(1+\theta)(q-p)}{pq+p(1+\theta)-q(1+\alpha)}\,\frac{B}{A}\right]^{\frac{1}{p+\theta-\alpha}},$$

where

$$A=\frac{\|u'\|^p_{L^p_\alpha((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}}\quad\text{and}\quad B=\frac{\|u\|^p_{L^p_\alpha((0,\infty))}}{p\,\|u\|^p_{L^q_\theta((0,\infty))}}.$$

Therefore,

$$E(u_\infty)\le\left[\left(\tfrac{(1+\theta)(q-p)}{qp+p(1+\theta)-q(1+\alpha)}\right)^{1-\gamma}+\left(\tfrac{(1+\theta)(q-p)}{qp+p(1+\theta)-q(1+\alpha)}\right)^{\gamma}\right]\frac{\|u'\|^{p\gamma}_{L^p_\alpha}\|u\|^{p(1-\gamma)}_{L^p_\theta}}{p\|u\|^p_{L^q_\theta}}$$

and the equality holds when $u=u_\infty$.

The next result will be important in the study of attainability in the Trudinger-Moser inequality with weight when $p=2$.

Proposition 7.1. *If $p=2$, $\alpha=p-1$, and $\theta\ge 0$, then the infimum*

$$B(2,\theta)^{-1}:=\inf_{0\neq u\in W^{1,2}_{1,\theta}(0,\infty)}\frac{\|u'\|^2_{L^2_1((0,\infty))}\cdot\|u\|^2_{L^2_\theta((0,\infty))}}{\|u\|^4_{L^4_\theta((0,\infty))}},$$

is attained by a positive nonincreasing function in $W^{1,2}_{1,\theta}((0,\infty))$. *Moreover,*

$$B(2,\theta)^{-1} < \pi(1+\theta).$$

Proof. The first part has been discussed at the beginning of this section. Then, we focus on the second part. Set

$$B(u)^{-1} := \frac{\|u'\|^2_{L^2_1((0,\infty))} \cdot \|u\|^2_{L^2_\theta((0,\infty))}}{\|u\|^4_{L^4_\theta((0,\infty))}}.$$

Note that it is sufficient to exhibit a function $u \in W^{1,2}_{1,\theta}((0,\infty))$ such that $B(u)^{-1} = \pi(1+\theta)$ and it is not a solution of

$$-(u'x)'\omega_1 + ux^\theta\omega_\theta - \lambda u^3 x^\theta \omega_\theta = 0, \tag{47}$$

for all $\lambda > 0$.

On the one hand, through a direct calculation we can see that for every positive function v in $W^{1,2}_{1,\theta}((0,\infty))$ of the form

$$v(x) = a_1(1 + a_2 x^{a_3})^{a_4},$$

where a_1, a_2, a_3, a_4 are real numbers, it is not a solution for (47). On the other hand, choosing

$$u(x) = \frac{1}{1 + x^{1+\theta}},$$

then

$$\|u\|^4_{L^4_\theta((0,\infty))} = \tfrac{\omega_\theta}{3(1+\theta)}$$
$$\|u\|^2_{L^2_\theta((0,\infty))} = \tfrac{\omega_\theta}{(1+\theta)}$$
$$\|u'\|^2_{L^2_1((0,\infty))} = \tfrac{\omega_1(1+\theta)}{6}$$

Therefore, $B(u)^{-1} = \pi(1+\theta)$, and then the proposition follows. ∎

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