

On the Period Function of Liénard Systems

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We study the period function T of a center O of a Liénard system. A sufficient condition for the monotonicity of T , or for the isochronicity of O , is given. Such a condition is also necessary when f and g are analytic, and g is odd. In this case a characterization of isochronous centers of Liénard systems is given. Strict monotonicity and global monotonicity of T are also investigated. © 1999

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1. INTRODUCTION

Let us consider a second order differential equation of Liénard type:

$$x'' + f(x) x' + g(x) = 0. \quad (E_L)$$

Equation (E_L) has been widely studied for its interest for physical applications (see [C, SC] for a discussion of some applications). Since Van der Pol's paper [VdP], where a very special case was taken into account, hundreds of research papers have been published, considering problems like boundedness of solutions, oscillation of solutions, existence, uniqueness or multiplicity of periodic solutions. Equation (E_L) is usually studied by means of an equivalent plane differential system. The most used ones are:

$$\begin{cases} x' = y \\ y' = -g(x) - f(x)y, \end{cases} \quad (S_1)$$

and

$$\begin{cases} x' = y - F(x) \\ y' = -g(x), \end{cases} \quad (S_L)$$

where $F(x) = \int_0^x f(s) ds$. Other systems have been used in order to study the existence of positively bounded solutions and limit cycles [V].

Assume that $g(0) = 0$, so that $x(t) \equiv 0$ is a solution of (E_L) . In this case the origin O is a singular point of (S_1) and (S_L) . We say that O is a *center* of (S_1) —or of (S_L) —if it has a punctured neighborhood filled with cycles surrounding O . Let N_O , the *central region*, be the largest open connected set covered with cycles surrounding O . In this paper we do not assume $O \in N_O$. We can define a function $T: N_O \rightarrow \mathbb{R}$, by associating to every $(x, y) \in N_O$ the period of the cycle passing through (x, y) . T is called the *period function of O* . T is obviously constant on cycles, so that it is a first integral. We say that T is *increasing* if, for every couple of cycles γ_1 and γ_2 , with γ_1 contained in the interior of γ_2 , we have $T(\gamma_1) \leq T(\gamma_2)$. We say that O is an *isochronous center* if T is constant in a neighborhood of O , that is if every cycle of the center contained in a suitable neighborhood has the same period. If one of the systems equivalent to (E_L) has a center, then every other system equivalent to (E_L) has a center. If the period function is increasing (constant, decreasing) in one system, then it is increasing (constant, decreasing) in every other system. Thus, it is natural to say that (E_L) has a center if one of the systems equivalent to (E_L) has a center, and to extend similarly the definitions concerned with the period function.

Aside from its interest in physical applications, the study of the period function is essential in approaching some problems related to (E_L) . The monotonicity of T is strictly related to existence and uniqueness of solutions of some boundary value, bifurcation or perturbation problems. Moreover, isochronicity has a strong relationship to stability: a periodic solution of the central region is Liapunov stable if and only if the neighboring periodic solutions have constant period.

Classical results about the monotonicity of the period function of a center of (E_L) have been obtained almost only under the additional assumption $f(x) \equiv 0$. In this case the equation appears in the form

$$x'' + g(x) = 0, \quad (E_g)$$

and is equivalent to a plane hamiltonian system. The knowledge of the first integral $y^2 + 2 \int_0^x g(s) ds$ of (E_g) reduces significantly the difficulties connected to this problem. Two important results about (E_g) are due to Urabe [U1, U2] (see also [Op]) who showed

- (1) if g is an odd function and $xg(x) > 0$ for small $x \neq 0$, then O is an isochronous center if and only if $g(x) = \lambda^2 x$ for some $\lambda \neq 0 \in \mathbb{R}$;
- (2) let $T|_x$ be the restriction to the x -axis of a two-variables function; then for every assigned scalar function ϕ there exists an equation of the type (E_g) , having a center at O with period function T such that $T|_x \equiv \phi$.

Point (1) was extended in [CJ] replacing the condition “ g odd” with “ g polynomial”. Point (1) does not hold, in general, because there exist

examples of equations (E_g) , with g analytic and nonlinear, having isochronous centers [U1, CMV]. In this case, the first derivative $g^{(j)}$, $j > 1$ such that $g^{(j)}(0) \neq 0$, has to be of even order (see remark 3 in the next section).

The monotonicity of the period function of centers of (E_g) , or the isochronicity, was also studied in [Ch, LS, O, Op, S]. In [Ch] it was also proved that the period function of some classes of quadratic centers is monotonic. One of these systems is equivalent to the Liénard equation one obtains by setting $f(x) = \lambda x$, $\lambda \in \mathbb{R}$, $g(x) = x$.

Quite recently significant improvements in the study of the period function of (E_L) have been obtained. In [CD] a necessary condition for the existence of an isochronous center of (S_1) has been given. In [CGMM], under the assumptions $g(x) = x$ and f analytic, it was proved that T is increasing in a neighborhood of O . In [AFG], assuming f and g to be odd polynomials, a necessary and sufficient condition for (S_L) to have an isochronous center has been proved.

The aim of this paper is to study the monotonicity properties of the period function of (E_L) , and of its equivalent systems. The main tool used is a new plane differential system equivalent to (E_L) :

$$\begin{cases} x' = y - xB(x) \\ y' = -C(x) - yB(x), \end{cases} \quad (S_{BC})$$

where $x^2B(x) = \int_0^x sf(s) ds$, $C(x) = g(x) - xB^2(x)$. It can be obtained in two ways: either by imposing that it is equivalent to (E_L) , or by looking for a "triangular" change of variables $(x, y) \rightarrow (x, y - \zeta(x))$ that takes (S_L) into (S_{BC}) . System (S_{BC}) is convenient because its angular speed has the simple form $(-xC(x) - y^2)/(x^2 + y^2)$, so reducing the complexity of computations involved, with respect to those ones required by systems (S_1) and (S_L) .

In order to state the main result of this paper, let us set $g_n(x) = g(x) - g'(0)x$, and let us define the following function:

$$\sigma(x) := 2x^2f(x) \int_0^x sf(s) ds - 4 \left(\int_0^x sf(s) ds \right)^2 + x^3g_n(x) - x^4g'_n(x).$$

We prove the following theorem (see Theorem 2, in next section):

THEOREM A. *Let $f, g \in C^1(a, b)$, for some $a < 0 < b$, and the origin O be a center of (E_L) . If $xC(x) > 0$ in a punctured neighborhood of 0, then we have:*

(1) *if $\sigma(x) \leq 0$ in a neighborhood of 0, then T is decreasing in a neighborhood of O ;*

(2) if $\sigma(x) \equiv 0$ in a neighborhood of 0, then T is constant in a neighborhood of O ;

(3) if $\sigma(x) \geq 0$ in a neighborhood of 0, then T is increasing in a neighborhood of O .

Points (1) and (3) can be modified in order to obtain strict monotonicity properties of T .

In case (2), our system takes the form:

$$\begin{cases} x' = y - xB(x) \\ y' = -x - yB(x), \end{cases}$$

and has constant angular speed. This system is equivalent to the Liénard equation:

$$x'' + f(x)x' + g'(0)x + \frac{1}{x^3} \left(\int_0^x sf(s) ds \right)^2 = 0. \quad (E_{\text{Iso}})$$

One of the main consequences of the above theorem is the possibility to give a necessary and sufficient condition for the isochronicity of centers, or for the monotonicity of the period function, when f and g are analytic, and g is odd. In this case we prove the following result (see Corollary 7 in next section). Let us set:

$$\tau(x) := \left(\int_0^x sf(s) ds \right)^2 - x^3 g_n(x).$$

THEOREM B. *Let f, g be analytic, g odd, $f(0) = g(0) = 0$, $g'(0) > 0$. Then O is a center if and only if f is odd and*

(1) *T is strictly decreasing in a neighborhood of O if and only if $\tau(x)$ has a maximum at 0;*

(2) *O is an isochronous center if and only if $\tau(x) \equiv 0$;*

(3) *T is strictly increasing in a neighborhood of O if and only if $\tau(x)$ has a minimum at 0.*

Theorem B, in case (2), gives an extension to analytic systems of the result obtained in [AFG] for polynomial systems. The same extension has been independently proved in [CDL], applying a different technique. All these results lead to the form (E_{Iso}) for Liénard equations with isochronous centers. An advantage of the approach presented here is the possibility to show that (E_{Iso}) has an isochronous center even when f and g are of class C^1 .

Theorems A and B allow also to give some simple conditions for the monotonicity of the period function of (E_g) , and a new proof of Urabe's

theorem about the isochronous centers of (E_g) . Moreover, a different proof of the quoted result in [CGMM] can be given, with an additional result about the estimate of the region where T is increasing.

For sake of simplicity, in this introduction theorems A and B have been stated as local results, but it is possible to state them as non-local ones. For instance, if $xC(x) > 0$ and $\sigma(x) > 0$ for all $x \neq 0$, then the period function is strictly increasing on all of the central region. At the end of next section some examples of Liénard equations with globally increasing, or globally decreasing period function are given.

2. DEFINITIONS AND RESULTS

In this paper we consider plane differential systems:

$$\begin{cases} x' = P(x, y) \\ y' = Q(x, y) \end{cases} \quad (S)$$

with P, Q locally lipschitzian functions defined in a neighborhood U of the origin O . We denote by $(x(t, x_0, y_0), y(t, x_0, y_0))$ the solution of (S) such that $(x(0, x_0, y_0), y(0, x_0, y_0)) = (x_0, y_0)$. If (S) has a center at O , we call N_O the largest open connected region covered with cycles surrounding O . We do not assume that $O \in N_O$. We define a function $T: N_O \rightarrow \mathbb{R}$, by associating to every $(x, y) \in N_O$ the period of the cycle passing through (x, y) . T is called the *period function of O* . T is constant on cycles. Let M be an invariant connected subset of N_O . We say that T is *increasing* in M if, for every couple of cycles $\gamma_1, \gamma_2 \subset M$, with γ_1 contained in the interior of γ_2 , we have $T(\gamma_1) \leq T(\gamma_2)$. We say that T is *strictly increasing* in M if, for every couple of cycles $\gamma_1, \gamma_2 \subset M$, with γ_1 contained in the interior of γ_2 , we have $T(\gamma_1) < T(\gamma_2)$. We say that T is (strictly) increasing at O if it is (strictly) increasing in a neighborhood of O . We say that O is an *isochronous center* if T is constant in a neighborhood of O , that is if every cycle of the center contained in a suitable neighborhood has the same period.

For every connected $A \subset \mathbb{R}^2$ we call \mathcal{S}_A the family of cycles contained in $A \cap N_O$, and we set $N_A = \bigcup \{ \gamma : \gamma \in \mathcal{S}_A \} \subset N_O$. In this paper, J will always denote an interval containing the origin 0 (possibly, $J \equiv \mathbb{R}$). We set $J_0 := J \setminus \{0\}$. We call W_J the vertical strip $\{(x, y) : x \in J\}$, \mathcal{S}_J the family of cycles contained in $W_J \cap N_O$ and we set $N_J = \bigcup \{ \gamma : \gamma \in \mathcal{S}_J \} \subset N_O$.

Together with (S) we consider the corresponding system in polar coordinates (r, θ) :

$$\begin{cases} r' = \rho(r, \theta) \\ \theta' = \omega(r, \theta). \end{cases} \quad (S_p)$$

We denote by $(r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ the solution of (S_p) such that $(r(0, r_0, \theta_0), \theta(0, r_0, \theta_0)) = (r_0, \theta_0)$. If Δ is a subset of the (x, y) -plane, we shall denote by Δ_p the corresponding subset in the (r, θ) -coordinates. For $\theta \in [0, 2\pi)$, we set $r_\Delta(\theta) = \sup\{r: (r, \theta) \in \Delta_p\}$.

We say that a scalar function ϕ is increasing if $x_1 \leq x_2 \Rightarrow \phi(x_1) \leq \phi(x_2)$. When dealing with strict monotonicity properties, this will be explicitly stated. For sake of brevity, we say that ϕ is increasing at a point x if it is increasing in a neighborhood of x . If ϕ is defined only in a punctured right neighborhood of x , $(x, x + \varepsilon)$, $\varepsilon > 0$, we say that ϕ is increasing at x if there exists $\delta \in (0, \varepsilon]$, with ϕ increasing in $(x, x + \delta)$.

The results about Liénard equation presented in this paper will be consequences of next theorem. Its main hypotheses are a star-shapedness condition on the orbits (S) and a monotonicity condition on $\omega(r, \theta)$.

THEOREM 1. *Let (S) have a center at O . Assume that there exists a star-shaped set $\Delta \subset R^2$ such that $\omega(r, \theta)$ does not vanish for all $(r, \theta) \in \Delta_p$. Then one has:*

(1) *if there exist a zero-measure set $Z \subset [0, 2\pi)$, such that, for all $\theta \in [0, 2\pi) \setminus Z$, the function $r \mapsto |\omega(r, \theta)|$ is increasing in $(0, r_\Delta(\theta))$, then T is decreasing in N_Δ ;*

(2) *if point (1) holds, and for every orbit γ in a neighborhood V of O there exists a point $(r_\gamma, \theta_\gamma) \in \gamma$ such that $r \mapsto \omega(r, \theta_\gamma)$ is strictly increasing at r_γ , then T is strictly decreasing in N_Δ ;*

(3) *if there exist a zero-measure set $Z \subset [0, 2\pi)$, such that, for all $\theta \in [0, 2\pi) \setminus Z$, the function $r \mapsto |\omega(r, \theta)|$ is constant in $(0, r_\Delta(\theta))$, then T is constant in N_Δ ;*

(4) *as point (1), exchanging the words “increasing” and “decreasing”;*

(5) *as point (2), exchanging the words “increasing” and “decreasing”.*

Proof. We prove points (1) and (2). The other cases can be easily proved similarly.

(1) Without loss of generality, we can assume that $\omega(r, \theta) > 0$. As a consequence, for every $r_0 > 0$ such that $\forall t: (r(t, r_0, 0), \theta(t, r_0, 0)) \in \Delta_p$, the function $t \mapsto \theta(t, r_0, 0)$ is strictly increasing on the interval $[0, 2\pi)$. Let us denote by $\theta \mapsto t(\theta, r_0, 0)$ its inverse function, defined on the interval $[0, T(r_0))$, where $T(r_0)$ is the period of $(r(t, r_0, 0), \theta(t, r_0, 0))$. Also $t(\theta, r_0, 0)$ is strictly increasing, with positive derivative.

Let us consider $r_1 < r_2$, with r_2 such that $\forall t: (r(t, r_2, 0), \theta(t, r_2, 0)) \in \Delta_p$. Then, by uniqueness of solutions, also the solution starting at $(r_1, 0)$

remains in \mathcal{A}_p : $\forall t: (r(t, r_1, 0), \theta(t, r_1, 0)) \in \mathcal{A}_p$. If $\theta \in [0, 2\pi) \setminus Z$ then $r(t(\theta, r_1, 0), r_1, 0) < r(t(\theta, r_2, 0), r_2, 0)$, so that $\omega(r(t(\theta, r_1, 0), r_1, 0), \theta) \leq \omega(r(t(\theta, r_2, 0), r_2, 0), \theta)$. Hence we have:

$$\begin{aligned} T(r_1) &= \int_{[0, 2\pi)} \frac{dt(\theta, r_1, 0)}{d\theta} d\theta = \int_{[0, 2\pi)} \frac{d\theta}{\omega(r(t(\theta, r_1, 0), r_1, 0), \theta)} \\ &= \int_{[0, 2\pi) \setminus Z} \frac{d\theta}{\omega(r(t(\theta, r_1, 0), r_1, 0), \theta)} \\ &\geq \int_{[0, 2\pi) \setminus Z} \frac{d\theta}{\omega(r(t(\theta, r_2, 0), r_2, 0), \theta)} \\ &= \int_{[0, 2\pi)} \frac{d\theta}{\omega(r(t(\theta, r_2, 0), r_2, 0), \theta)} \\ &= T(r_2). \end{aligned}$$

(2) Let γ_1, γ_2 be the orbits corresponding to the solutions $(r(t, r_1, 0), \theta(t, r_1, 0)), (r(t, r_2, 0), \theta(t, r_2, 0))$. By hypothesis, there exists $(r_{\gamma_1}, \theta_{\gamma_1}) \in \gamma_1$ such that the function $r \mapsto \omega(r, \theta_{\gamma_1})$ is strictly increasing at r_{γ_1} . Hence $\omega(r(t(\theta_{\gamma_1}, r_1, 0), r_1, 0), \theta) < \omega(r(t(\theta_{\gamma_1}, r_2, 0), r_2, 0), \theta)$. By the continuity of all the functions involved, this inequality holds in a neighborhood of θ_{γ_1} . As a consequence, we can write:

$$\begin{aligned} T(r_1) &= \int_{[0, 2\pi) \setminus Z} \frac{d\theta}{\omega(r(t(\theta, r_1, 0), r_1, 0), \theta)} \\ &> \int_{[0, 2\pi) \setminus Z} \frac{d\theta}{\omega(r(t(\theta, r_2, 0), r_2, 0), \theta)} = T(r_2). \quad \blacksquare \end{aligned}$$

Remark 1. Theorem 1 allows us to study also the case of a globally monotone period function. If $\mathcal{A} = N_O$, then the properties of theorem 1 hold globally for T . Some examples will be given at the end of this section.

In case (3) of theorem 1, O is a *uniformly isochronous center*, that is a center for which the angular speed of solutions is constant on rays $\{(r, \theta): r > 0, \theta = \bar{\theta}\}$. In this case the system in polar coordinates appears as follows:

$$\begin{cases} r' = \rho(r, \theta) \\ \theta' = \omega(\theta). \end{cases}$$

Examples of systems with uniformly isochronous centers can be found in [CGG1, CGG2, CGG3, Cl, Co, MRT, MS].

In order to study the period function of Liénard centers, we shall look for a suitable differential system, equivalent to Liénard equation, and such that the computations required by theorem 1 can be actually performed. Following [V], we consider systems of the form:

$$\begin{cases} x' = y - A(x) \\ y' = -C(x) - yB(x). \end{cases} \quad (S_{ABC})$$

If A , B , C are all defined on an interval J , the natural choice for U is the strip W_J . The angular speed of the solutions of (S_{ABC}) is given by $(-xC(x) - yB(x) - y^2 + yA(x))/(x^2 + y^2)$. If we choose $A(x) = xB(x)$, then the angular speed has the simpler form $(-xC(x) - y^2)/(x^2 + y^2)$, that is, it coincides with that of the system:

$$\begin{cases} x' = y \\ y' = -C(x). \end{cases} \quad (S_C)$$

This makes much easier to study the period function of a center. Before proving the main result, we need some preliminary lemmata. Let us set

$$I(x) := \int_0^x sf(s) ds.$$

LEMMA 1. *Let $f, g \in C^0(J, R)$. Let us define the functions B and C in J as follows:*

$$B(x) := \begin{cases} \frac{I(x)}{x^2}, & x \neq 0 \\ \frac{f(0)}{2}, & x = 0 \end{cases}$$

$$C(x) := \begin{cases} g(x) - xB^2(x) = g(x) - \frac{1}{x^3} I^2(x) & x \neq 0 \\ 0, & x = 0. \end{cases}$$

-

Then we have:

- (1) B and C are continuous;
- (2) if $f, g \in C^1(J, R)$, $f(0) = 0$, then $B, C \in C^1(J, R)$, $B'(0) = f'(0)/3$, $C'(0) = g'(0)$.

Proof. In both cases, it is sufficient to prove the regularity of B and C at 0.

(1) Applying de l'Hopital theorem to $I(x)/x^2$ at 0, we get $\lim_{x \rightarrow 0} B(x) = f(0)/2$.

(2) Applying again de l'Hopital theorem it is easy to prove that if $f(0) = 0$, then B is differentiable at 0, $B'(0) = f'(0)/3$ and $B'(x)$ is continuous at 0. $C'(0) = g'(0)$ is immediate. ■

For the study of centers, the condition $f(0) = 0$ is not a restriction, because if $f(0) \neq 0$ the origin is not a center of S_I , hence it is not a center of any system equivalent to (E_L) .

LEMMA 2. Let $f, g \in C^1(J, \mathbb{R})$, $f(0) = 0$. Then the differential system:

$$\begin{cases} x' = y - xB(x) \\ y' = -C(x) - yB(x), \end{cases} \quad (S_{BC})$$

is of class C^1 in a neighborhood of O and equivalent to (E_L) .

Proof. Lemma 1 gives the regularity of (S_{BC}) .

Let us prove that, if $(x(t), y(t))$ is a solution to (S_{BC}) , then $x(t)$ is a solution to (E_L) . First let us observe that

$$xB'(x) = f(x) - 2B(x).$$

In fact, such equality is trivially true for $x = 0$, while for $x \neq 0$, we have:

$$xB'(x) = x \frac{x^3 f(x) - 2xI(x)}{x^4} = f(x) - 2B(x).$$

Then, since $y = x' + xB(x)$, one has:

$$\begin{aligned} x'' &= y' - x'B(x) - xB'(x) x' \\ &= -C(x) - yB(x) - x'B(x) - x'B'(x) \\ &= -g(x) + xB^2(x) - x'B(x) - xB^2(x) - x'B(x) - x'f(x) + 2x'B(x) \\ &= -g(x) - f(x) x'. \quad \blacksquare \end{aligned}$$

Remark 2. There is an alternative way to obtain system (S_{BC}) . Let us consider the following change of variables:

$$\begin{cases} u = x \\ v = y - F(x) + xB(x) \end{cases} \quad \begin{cases} x = u \\ y = v + F(u) - uB(u). \end{cases}$$

Recalling that $x B'(x) = f(x) - 2B(x)$ (Lemma 2), system (S_L) is transformed into the system

$$\begin{cases} u' = x' = y - F(x) = v + F(u) - uB(u) - F(u) = v - uB(u) \\ v' = y' + x'(-f(x) + B(x) + xB'(x)) \\ \quad = -g(x) + (y - F(x))(-f(x) + B(x) + f(x) - 2B(x)) \\ \quad = -g(u) - (v + F(u) - uB(u) - F(u)) B(u) \\ \quad = -g(u) + uB^2(u) - vB(u), \end{cases}$$

that is just (S_{BC}) in the new coordinates (u, v) .

Similarly, (S_{BC}) can be obtained from (S_1) by means of another transformation of the type $(x, y) \mapsto (x, y - \zeta(x))$.

Let us write $g(x) = g'(0)x + g_n(x)$ and $C(x) = C'(0)x + C_n(x)$. Since $C'(0) = g'(0)$, we have $C_n(x) = g_n(x) - xB^2(x)$.

Let us assume that the hypotheses of lemma 2 hold. In order to state next results, let us write

$$\sigma(x) := 2x^2 f(x) I(x) - 4I^2(x) + x^3 g_n(x) - x^4 g'_n(x).$$

THEOREM 2. *Let $f, g \in C^1(J, R)$, $f(0) = g(0) = 0$. Let the origin O be a center of (E_L) . If $xC(x) > 0$ for $x \in J_0$, and*

- (1) $\sigma(x) \leq 0$ for $x \in J$, then T is decreasing in N_J ;
- (2) (1) holds, and there exists a sequence $x_n \in J$, $x_n \rightarrow 0$ with $\sigma(x_n) < 0$, then T is strictly decreasing in N_J ;
- (3) $\sigma(x) \equiv 0$ in J , then T is constant in N_J ;
- (4) $\sigma(x) \geq 0$ for $x \in J$, then T is increasing in N_J ;
- (5) (4) holds, and there exists a sequence $x_n \in J$, $x_n \rightarrow 0$ with $\sigma(x_n) > 0$, then T is strictly increasing in N_J ;

Proof. Let us prove points (1) and (2). The other ones follow similarly. We apply theorem 1, taking $A = W_J$.

- (1) In polar coordinates (r, θ) , system (S_{BC}) has the following form:

$$\begin{cases} r' = r \cos \theta \sin \theta - rB(r \cos \theta) - \sin \theta C(r \cos \theta) = \rho(r, \theta) \\ \theta' = -g'(0) \cos^2 \theta - \sin^2 \theta - \frac{\cos \theta C_n(r \cos \theta)}{r} = \omega(r, \theta), \end{cases}$$

Let us observe that, for (r, θ) such that $(r \cos \theta, r \sin \theta) \in W_J$, and $\theta \neq \pi/2, 3\pi/2$, we have:

$$r^2 \omega(r, \theta) = r^2 \theta' = -xC - y^2 < 0.$$

By theorem 1, it is sufficient to prove that $\sigma \leq 0$ implies that $\partial |\omega(r, \theta)| / \partial r = -\partial \omega(r, \theta) / \partial r \geq 0$, for almost all values of $\theta \in [0, 2\pi)$.

We have

$$\begin{aligned} -\frac{\partial \omega(r, \theta)}{\partial r} &= \frac{\partial}{\partial r} \frac{\cos \theta C_n(r \cos \theta)}{r} \\ &= \frac{r \cos^2 \theta C'_n(r \cos \theta) - \cos \theta C_n(r \cos \theta)}{r^2} \\ &= \frac{r^2 \cos^2 \theta C'_n(r \cos \theta) - r \cos \theta C_n(r \cos \theta)}{r^3} \\ &= \frac{x^2 C'_n(x) - x C_n(x)}{(x^2 + y^2)^{3/2}}. \end{aligned}$$

In order to study the sign of the above fraction it is sufficient to study the sign of its numerator,

$$\begin{aligned} x^2 C'_n(x) - x C_n(x) &= x^2 (g_n(x) - x B^2(x))' - x (g_n(x) - x B^2(x)) \\ &= x^2 g'_n(x) - x^2 B^2(x) - 2x^3 B(x) B'(x) - x g_n(x) + x^2 B^2(x) \\ &= x^2 g'_n(x) - x g_n(x) - 2x^2 f(x) B(x) + 4x^2 B^2(x). \end{aligned}$$

The sign is the same as that of the function

$$\begin{aligned} x^4 g'_n(x) - x^3 g_n(x) - 2x^4 f(x) B(x) + 4x^4 B^2(x) \\ = x^4 g'_n(x) - x^3 g_n(x) - 2x^2 f(x) I(x) + 4I^2(x) = -\sigma(x) \geq 0. \end{aligned}$$

Hence, for $\theta \neq \pi/2, 3\pi/2$, we have $\partial |\omega(r, \theta)| / \partial r \geq 0$. Then, Theorem 1 gives the statement.

(2) Let γ be a cycle of (S_{BC}) contained in W_J . γ meets the line $x = x_n$ at some point (x_n, y_n) , corresponding to (r_n, θ_n) in polar coordinates. Since $\sigma(x_n) < 0$, we have $\partial |\omega(r_n, \theta_n)| / \partial r > 0$. Then, by theorem 1, point (2), we have the thesis. ■

Let N_{Ox} be the projection of N_O on the x -axis.

COROLLARY 1. *Let $f, g \in C^1(J, R)$, $f(0) = g(0) = 0$. Let the origin O be a center of (E_L) . If $xC(x) > 0$ for $x \in N_{Ox} \setminus \{0\}$, then the statements of Theorem 2 hold on all of N_O .*

Proof. It is an immediate consequence of Theorems 1, 2 and Remark 1. ■

The condition $xC(x) > 0$ ensures that the orbits of (S_{BC}) are strictly star-shaped in a neighborhood of O . This condition is equivalent to:

$$x^3g(x) > I^2(x).$$

It is always satisfied in a suitable neighborhood of the origin if $g'(0) > 0$, as shown in next corollary.

COROLLARY 2. *Let $f, g \in C^1(J, R)$, $f(0) = g(0) = 0$, $g'(0) > 0$. Let the origin O be a center of (E_L) . Then the statements of theorem 2 hold, in a suitable subinterval of J .*

Proof. We have $xg(x) = g'(0)x^2 + xg_n(x)$, with $g_n(x) = o(x)$. Since $\lim_{x \rightarrow 0} B(x) = 0$, there exists $\varepsilon > 0$ such that $xC(x) = g'(0)x^2 + xg_n(x) - x^2B^2(x) > 0$ for $0 < |x| < \varepsilon$. ■

The isocronicity condition $\sigma(x) \equiv 0$ can be expressed in a simpler form, as in next corollary. Let us write

$$\tau(x) := I^2(x) - x^3g_n(x).$$

For $x \neq 0$, we have

$$xC(x) = g'(0)x^2 - \frac{\tau(x)}{x^2}. \quad (C\tau)$$

$$\sigma(x) = x^5 \left(\frac{\tau(x)}{x^4} \right)'. \quad (\sigma\tau)$$

COROLLARY 3. *Let $f, g \in C^1(J, R)$, $f(0) = g(0) = 0$, $g'(0) > 0$. Let the origin O be a center of (E_L) . If $\sigma(x) \equiv 0$ in a neighborhood of 0, then $\tau(x) \equiv 0$ and (E_L) has the form:*

$$x'' + f(x)x' + g'(0)x + \frac{1}{x^3} \left(\int_0^x sf(s) ds \right)^2 = 0. \quad (E_{Iso})$$

Proof. If $\sigma \equiv 0$, then $(\tau(x)/x^4)' \equiv 0$, hence $(\tau(x)/x^4) \equiv c \in R$. Let us set

$$f(x) = c_m x^m + o(|x|^m) \quad g_n(x) = d_l x^l + o(|x|^l).$$

with $m \geq 1$, $l \geq 2$. Then

$$\tau(x) = \kappa_m x^{2m+4} + o(|x|^{2m+4}) - d_l x^{l+3} - o(|x|^{l+3})$$

and

$$\lim_{x \rightarrow 0} \frac{\tau(x)}{x^4} = \lim_{x \rightarrow 0} \frac{\kappa_m x^{2m+4} + o(|x|^{2m+4}) - d_l x^{l+3} - o(|x|^{l+3})}{x^4} = 0.$$

Hence $\tau(x) \equiv c = 0$.

Hence, if $\sigma \equiv 0$, we have, from $\tau(x) \equiv 0$:

$$g_n(x) = \frac{1}{x^3} \left(\int_0^x sf(s) ds \right)^2,$$

from which we derive the form of (E_{Iso}) . ■

In [AFG] the authors prove that system (S_L) , with f and g odd polynomials, has an isochronous center at O if and only if it is equivalent to equation (E_{Iso}) . In theorem 3 we extend their result to the case of f, g odd and analytic.

By Corollary 3, we can state the isochronicity condition in a simpler form than the one considered in Theorem 2. In the following we shall refer to such a condition as (Iso) :

$$\tau(x) = \left(\int_0^x sf(s) ds \right)^2 - x^3 g_n(x) \equiv 0. \quad (Iso)$$

In Corollary 3 we cannot delete the hypothesis $g'(0) > 0$. In fact, if $g'(0) < 0$, an eigenvalue analysis shows that the origin is not a rotation point of (S_L) , hence it is not a center. If $g'(0) = 0$, then $g(x) = g_n(x)$, $C(x) \equiv 0$, and (S_{BC}) takes the form:

$$\begin{cases} x' = y - xB(x) \\ y' = -yB(x). \end{cases}$$

The x -axis is an invariant line for such system, that does not have a center at O .

In the case considered in Corollary 3, (S_{BC}) assumes a very simple form. This allows us to prove that O is its unique singular point. For polynomial systems, a similar result was proved in [CD]. Next corollary holds regardless of O being a center or not.

COROLLARY 4. *Let $f, g \in C^1(J, \mathbb{R})$, $f(0) = g(0) = 0$, $g'(0) > 0$. If (Iso) holds, then O is the unique singular point of (S_{BC}) .*

Proof. We can assume, without loss of generality, that $g'(0) = 1$. When (Iso) holds, system (S_{BC}) takes the following form:

$$\begin{cases} x' = y - xB(x) \\ y' = -x - yB(x). \end{cases}$$

In this case the angular speed is -1 at every point of W_J , but the origin. As a consequence we have that O is the unique singular point of the system. ■

A relevant case is that of conservative second order equations:

$$x'' + g(x) = 0. \quad (E_g)$$

In this case $f(x) \equiv 0$, that greatly simplifies computations. Let us set

$$\sigma_g := xg_n(x) - x^2g'_n(x).$$

COROLLARY 5. *Let $g \in C^1(J, R)$, $g(0) = 0$, $xg(x) > 0$ for $x \neq 0$. Then the origin O is a center of (E_g) , and the statements of Theorem 2 hold, replacing σ with σ_g .*

Proof. The origin is a center because the function $y^2 + 2 \int_0^x g(s) ds$ is a first integral with an isolated minimum at O. Since in this case we have:

$$\sigma(x) = x^2\sigma_g(x),$$

the other statements follow immediately from Theorem 2. ■

If g is a nonlinear function of class C^k , then we can give a simple condition for the monotonicity of the period function.

COROLLARY 6. *Let $f(x) \equiv 0$, $g \in C^k(J, R)$, $g(0) = 0$, $xg(x) > 0$ for $x \neq 0$. Assume that there exists an integer $l \leq k$ such that $g^{(l)}(0) \neq 0$. Let j be the minimum integer > 1 such that $g^{(j)}(0) \neq 0$. If j is odd, then:*

- (1) T is strictly decreasing at O if and only if $g^{(j)}(0) > 0$;
- (2) T is strictly increasing at O if and only if $g^{(j)}(0) < 0$.

Proof. By Corollary 5, the origin is a center.

(1) and (2). We have

$$\begin{aligned} \sigma_g(x) &= xg_n(x) - x^2g'_n(x) \\ &= x \left(\frac{g^{(j)}(0) x^j}{j!} + o(|x|^j) \right) - x^2 \left(\frac{g^{(j)}(0) x^{j-1}}{(j-1)!} + o(|x|^{j-1}) \right) \\ &= \frac{1-j}{j!} g^{(j)}(0) x^{j+1} + o(|x|^{j+1}). \end{aligned}$$

If j is odd, then σ_g is sign-definite in a neighborhood of 0, and T can only be strictly increasing or strictly decreasing. For small values of x , the sign of $\sigma_g(x)$ is opposite to that of $g^{(j)}(0)$. Then we can apply Corollary 5. ■

Remark 3. As a consequence of Corollary 6, (E_g) can have an isochronous center at O only if the first non-zero derivative of $g_n(x)$ is of even order.

THEOREM 3. *Let $f, g \in C^\omega((-\eta, \eta), \mathbb{R})$ for some $\eta > 0$, f, g odd, $f(0) = g(0) = 0$, $xC(x) > 0$ for $x \neq 0$. If O is a center, then the following hold:*

(1) *T is strictly decreasing at 0 if and only if $\tau(x)$ has a proper maximum at 0;*

(2) *O is an isochronous center if and only if $\tau(x) \equiv 0$ in a neighborhood of 0;*

(3) *T is strictly increasing at 0 if and only if $\tau(x)$ has a proper minimum at 0.*

Proof. First, let us prove that T is strictly increasing at 0 (strictly decreasing at 0, constant) if and only if $\sigma(x)$ has a proper minimum at 0 (decreasing at 0, is constant).

Let us set

$$\mu(x) := \begin{cases} B^2(x) - \frac{g_n(x)}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The function μ is even and analytic. In fact, working as in the proof of Corollary 3, we can show that if

$$f(x) = c_m x^m + o(|x|^m),$$

with $m \geq 1$, then

$$B(x) = \frac{1}{x^2} I(x) = \frac{1}{x^2} \left(\frac{c_m}{m+2} x^{m+2} + o(|x|^{m+2}) \right) = \frac{c_m}{m+2} x^m + o(|x|^m).$$

Also, $g_n(x) = d_l x^l + o(|x|^l)$, with $l \geq 2$, hence μ is analytic in J .

We have

$$\sigma(x) = x^5 \mu'(x), \quad \tau(x) = x^4 \mu(x),$$

that implies

$$\sigma(x) > 0 \Leftrightarrow x\mu'(x) > 0, \quad \tau(x) > 0 \Leftrightarrow \mu(x) > 0.$$

Both f and g are odd and analytic, hence σ is even and analytic. Analytic functions either have isolated zeroes, or are identically constant. Since σ is even and $\sigma(0)=0$, only three possibilities can occur in a neighborhood of 0: either 0 is a proper minimum of σ , or it is a proper maximum, or σ is identically zero. By Theorem 2, T is strictly increasing, strictly decreasing, constant, respectively.

Vice-versa, if T is strictly increasing, then σ cannot be constant, otherwise, by Theorem 2 the center would be isochronous. Similarly, if T is strictly increasing, then σ cannot have a proper maximum at 0, that would imply T to be strictly decreasing, by Theorem 2. Hence, if T is strictly increasing, then σ has a proper minimum.

The other cases can be treated similarly.

Now let us show that σ has a proper maximum at 0 (has a proper minimum at 0, is constant) if and only if τ has a proper maximum at 0 (has a proper minimum at 0, is constant). We shall use the fact that also τ is even and analytic, and $\tau(0)=\mu(0)=0$. Also for τ only three possibilities can occur: a proper maximum at 0, a proper minimum at 0, identically zero.

If 0 is a proper minimum of σ , then $\sigma(x)=x\mu'(x)>0$ for small $x\neq 0$, hence μ has a proper minimum at 0: $\mu(x)>0$ for small $x\neq 0$. Hence $\tau(x)>0$ for small $x\neq 0$, that is, also τ has a proper minimum at 0. One can prove similarly that if 0 is a proper maximum of σ , then 0 is a proper maximum of τ . If $\sigma\equiv 0$, then $\mu'(x)\equiv 0$, hence $\mu'(x)\equiv 0$ and $\tau\equiv 0$.

Vice-versa, if τ has a proper minimum at 0, then by what above, σ cannot have a proper maximum, nor be constant, hence it has a proper minimum at 0. The other cases follow similarly. ■

The couple (f, g) satisfies condition (Iso) if and only if the couple $(-f, g)$ satisfies (Iso). This is a consequence of the fact that, if $x(t)$ is a solution to (E_L) , then $x(-t)$ is a solution to

$$x'' - f(x) x' + g(x) = 0.$$

For every given f , there exists a g such that the couple (f, g) satisfies (Iso). It is sufficient to choose g such that $g_n(x) = 1/x^3(\int_0^x sf(s) ds)^2$.

Vice versa, for every given g , with $xg_n(x)>0$ for $x\neq 0$, there exists an f such that the couple (f, g) satisfies (Iso):

$$f(x) = \frac{3xg_n(x) + x^2g_n'(x)}{2\sqrt{x^3g_n(x)}}.$$

COROLLARY 7. *Let $f, g \in C^\omega((-\eta, \eta), \mathbb{R})$ for some $\eta>0$, g odd, $f(0)=g(0)=0$, $g'(0)>0$. Then O is a center if and only if f is odd. Moreover, (1), (2), (3) of Theorem 3 hold.*

Proof. If O is a center, then by theorem 2 in [V], f is odd.

Vice-versa, if f is odd, then the solutions of (S_{BC}) are symmetric with respect to the y -axis. Since $g'(0) > 0$, the origin is a rotation point, hence the origin is a center. Then the statement comes from Theorem 3. ■

As a special case of Theorem 3, we get a classical result about second order equations without a dissipative term ([Op,U1]).

COROLLARY 8. *Under the hypotheses of Corollary 7, assume further that $f(x) \equiv 0$. Then O is an isochronous center if and only if $g(x) = \lambda^2 x$, for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$.*

Proof. Assume that O is an isochronous center. By Theorem 3, $\tau(x) \equiv 0$, hence, $x^3 g_n(x) \equiv 0$. This implies that g is linear. ■

Let us consider the analytic Liénard equation with linear restoring term:

$$x'' + f(x) x' + x = 0. \quad (L_x)$$

This equation has been considered in [CGMM], where its monotonicity at the origin is proved by computing the period constants. In next corollary we provide an estimate of the region of monotonicity of T .

COROLLARY 9. *Let f be analytic, $f(x) \not\equiv 0$, O be a center of (L_x) . Let us set $\alpha = \inf\{x: B^2(x) < 1\}$, $\beta = \sup\{x: B^2(x) < 1\}$, $\bar{J} = (\alpha, \beta)$. Then T is strictly increasing in $N_{\bar{J}}$.*

Proof. The star-shapedness condition $xC(x) > 0$ becomes $x^2 - x^2 B^2(x) > 0$, that is, equivalent to $B^2(x) < 1$. It is satisfied in the strip $W_{\bar{J}}$. Since $g_n(x) \equiv 0$, we have $\tau(x) = I^2(x) > 0$ in a punctured neighborhood of 0. Then the conclusion comes from Theorem 3. ■

The same holds for another class of analytic second order equations with linear restoring term, known as Rayleigh equations [C]:

$$x'' + h(x') + x = 0. \quad (R_x)$$

In order to state next corollary, let us set

$$B_h(x) = -\frac{1}{x^2} \int_0^x sh'(s) ds.$$

COROLLARY 10. *Let h be analytic, $h(x) \not\equiv 0$, O be a center of (R_x) . Let us set $\alpha_h = \inf\{x: B_h^2(x) < 1\}$, $\beta_h = \sup\{x: B_h^2(x) < 1\}$, $J_h = (\alpha_h, \beta_h)$. Then T is strictly increasing in N_{J_h} .*

Proof. Let us consider the system

$$\begin{cases} x' = y \\ y' = -x - h(y), \end{cases} \quad (S_R)$$

equivalent to (R_x) . (S_R) has a unique singular point at $(-h(0), 0)$. Without loss of generality, we can assume that $h(0) = 0$. Exchanging variables and multiplying the vector field by -1 the system becomes:

$$\begin{cases} x' = y + h(x) \\ y' = -x, \end{cases}$$

that is a system of Liénard type, with $g(x) = x$, and $f(x) = -h'(x)$. Then we can apply Corollary 9. ■

We conclude this paper by considering some examples exhibiting the behaviors studied in Theorems 2 and 3. We choose $f(x) = \lambda x^{2d+1}$, where d is a non-negative integer, $\lambda > 0$, and $g(x) = x + x^{4d+3}$:

$$x'' + \lambda x^{2d+1} x' + x + x^{4d+3} = 0.$$

O is a center of the above equation for every value of λ . We have

$$xC(x) = xg(x) - x^2 B^2(x) = x^2 + \frac{(2d+3)^2 - \lambda^2}{(2d+3)^2} x^{4d+4}$$

$$\tau(x) = I^2(x) - x^3 g_n(x) = \frac{\lambda^2 - (2d+3)^2}{(2d+3)^2} x^{4d+6}$$

If $\lambda < 2d+3$, then $xC(x) > 0$ and $\tau(x) < 0$ for all $x \neq 0$, so that T is strictly decreasing on all of N_O . If $\lambda = 2d+3$, then O is an isochronous center. If $\lambda > 2d+3$, then $xC(x) > 0$ for

$$-\left(\frac{(2d+3)^2}{\lambda^2 - (2d+3)^2}\right)^{1/(4d+2)} < x < \left(\frac{(2d+3)^2}{\lambda^2 - (2d+3)^2}\right)^{1/(4d+2)},$$

while $\tau(x) > 0$ for all $x \neq 0$. In this case T is increasing on the set of cycles contained in the vertical strip defined by the above inequalities.

We can also give an example of analytic Liénard equation with globally increasing period function. Let us choose $f(x) = (3x - 2x^3)e^{-x^2}$, $g(x) = x$

$$x'' + (3x - 2x^3) e^{-x^2} x' + x = 0.$$

Then both $xC(x) = x^2 - x^2 B^2(x) = x^2 - x^4 e^{-2x^2}$ and $\tau(x) = I^2(x) = x^6 e^{-2x^2}$ are strictly positive for all $x \neq 0$. This implies that T is strictly increasing on all of N_O .

We also give an example of a non-analytic equation having a globally increasing period function. Let us choose $g(x)=x$, and the following function f :

$$f(x) = \begin{cases} \operatorname{sgn}(x) e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The origin is a center because f and g are odd. Let us prove that $\sigma(x) = 2x^2f(x)I(x) - 4I^2(x) > 0$ for $x \neq 0$. Since f is odd, then also I is odd and σ is even, so that it is sufficient to prove that $\sigma(x) > 0$ for $x > 0$. We have $(x^2f(x))' = 2xf(x) + x^2f'(x) > 2xf(x) = (2I(x))'$. Since $f(0)=0$, $I(0)=0$, this implies that, for $x > 0$, $x^2f(x) > 2I(x) > 0$. Then we have $\sigma(x) = 2x^2f(x)I(x) - 4I^2(x) > 0$ for $x > 0$.

As for the star-shapedness condition, we have

$$xC(x) = x^2 - \frac{(\int_0^x \operatorname{sgn}(s) s e^{-1/s^2} ds)^2}{x^2} > x^2 - \frac{x^2}{4} > 0$$

for $x \neq 0$. As a consequence, the period function is increasing on all of N_O .

APPENDIX

In [AFG] some systems of the type (S_L) are studied. The authors give a necessary and sufficient condition for O to be an isochronous center, when F is an even polynomial and g is an odd polynomial. They show that in this case the system has the following form:

$$\begin{cases} x' = -y - F(x) \\ y' = x + \frac{1}{x^3} \left(\int_0^x F(s) ds - xF(x) \right)^2. \end{cases} \quad (S_{\text{AFG}})$$

System (S_{AFG}) is equivalent to the equation

$$x'' + f(x)x' + x + \frac{1}{x^3} \left(\int_0^x F(s) ds - xF(x) \right)^2 = 0,$$

where $F'(x) = f(x)$. This equation coincides with the one given in Corollary 3. In fact, integrating by parts, we can write:

$$\left(\int_0^x sf(s) ds \right)^2 = \left(\int_0^x sF'(s) ds \right)^2 = \left(xF(x) - \int_0^x F(s) ds \right)^2.$$

After this paper was written, we received a preliminary version of [CDL], concerned with analytic Liénard systems with isochronous centers. The authors prove a necessary and sufficient condition for (S_I) to have an isochronous center. When g is odd, they find the isochronicity condition stated in our paper as point (2) of Corollary 7.

The techniques used in [AFG], [CDL] and in the present paper are all different from each other.

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