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Linear Stability of the Elliptic Lagrangian Triangle Solutions in the Three-Body Problem

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This paper concerns the linear stability of the well-known periodic orbits of Lagrange in the three-body problem. Given any three masses, there exists a family of periodic solutions for which each body is at the vertex of an equilateral triangle and travels along an elliptic Kepler orbit. Reductions are performed to derive equations which determine the linear stability of the periodic solutions. These equations depend on two parameters – the eccentricity e of the orbit and the mass parameter $\beta = 27(m_1m_2 + m_1m_3 + m_2m_3)/(m_1 + m_2 + m_3)^2$. A combination of numerical and analytic methods is used to find the regions of stability in the βe-plane. In particular, using perturbation techniques it is rigorously proven that there are mass values where the truly elliptic orbits are linearly stable even though the circular orbits are not.

Key Words: n-body problem; linear stability; relative equilibria; Lagrange's equilateral triangle solutions.

1. INTRODUCTION

In 1772, Lagrange discovered one of the most simple and elegant solutions to the n-body problem [5]. It consists of three masses located at the vertices of an equilateral triangle, each traveling along a specific Kepler orbit. The triangular configuration of the bodies is maintained over the course of the motion. This family of solutions was one of the first explicit solutions given in the three-body problem. Contained in the family are two types of periodic orbits: rigid circular motion (choosing a circular Kepler orbit) and homographic motion (choosing an elliptic Kepler orbit). Although Lagrange thought his equilateral triangle solutions were of no great practical significance, it was later realized that the sun, Jupiter and the Trojan asteroids formed such a configuration in our galaxy. Thus, it became fruitful to study the behavior of nearby solutions.

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A crucial first step in analyzing the local behavior near a periodic solution is to compute the characteristic multipliers of the linearized equations. For the circular case, this was first accomplished by Gascheau in 1843 in his thesis [3]. He proved that linear stability was achieved only when the masses satisfied

$$\frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} < \frac{1}{27}.$$

This rather well-known inequality is also often attributed to Routh [10]. The circular problem is made easier by the fact that in a rotating coordinate frame, Lagrange's circular solution becomes a fixed point. One can obtain analytic expressions for the multipliers as functions of the mass parameter

$$\beta = 27(m_1 m_2 + m_1 m_3 + m_2 m_3)/(m_1 + m_2 + m_3)^2.$$

Then, using perturbation or continuation methods, the existence of nearby periodic solutions can be proven (see for example [6] or [12]).

Calculating the characteristic multipliers for the elliptic Lagrange orbits requires finding the fundamental matrix solution to the associated time-dependent linear system. Unfortunately, this is difficult in general and usually requires the use of numerical methods. In [1], Danby studies the elliptic restricted three-body problem and uses numerical integration to determine the linear stability of the elliptic Lagrange orbits. Using the traditional mass value μ and the eccentricity e as parameters, he obtains a stability diagram in the μe-plane and notes that there are cases where the elliptic orbits appear to be linearly stable even though the circular ones are not. We will show that the same phenomenon occurs in the unrestricted problem.

First, we carefully reduce the dimensions of the problem from 12 to 4. This is accomplished by eliminating the standard integrals and then making a clever change of coordinates which decouples the associated linear system. One of the resulting systems yields two $+1$ multipliers, expected due to the nature of the problem. The other system is four dimensional and governs the linear stability of the periodic solution. This system is surprisingly simple and only depends on the mass coefficient β and the eccentricity e.

We then analyze the behavior of the characteristic multipliers and how they vary with e and β. In general, as the eccentricity is increased, stability is lost through a period-doubling bifurcation. A simple numerical method is used to generate the period-doubling curve in the βe-plane. Two crucial values of β are located at $\beta = 3/4$ and 1. For $\beta = 3/4$, stability is immediately lost for $e > 0$ while for $\beta = 1$, stability is maintained locally for

$e > 0$. We prove this analytically by using perturbation techniques and perturbing away from the known case $e = 0$. This result is crucial to the linear stability analysis as it divides the stability diagram into two pieces and rigorously proves that there are mass values where the elliptic orbits are linearly stable even though their circular counterparts are not.

2. LAGRANGE'S EQUILATERAL TRIANGLE SOLUTIONS

We begin with the equations of motion for the planar n-body problem. Letting m_i and $\mathbf{q}_i \in \mathbb{R}^2$ denote the mass and position, respectively, of the ith body, the second-order equation for the ith body is

$$m_i \ddot{\mathbf{q}}_i = \sum_{i \neq j} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3} = \frac{\partial U}{\partial \mathbf{q}_i}, \tag{1}$$

where $U(\mathbf{q})$ is the Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}.$$

Following Meyer's approach in [6], we look for solutions of the form $\mathbf{q}_i(t) = \psi(t)\mathbf{z}_i$, where $\psi(t)$ is a scalar function and $\mathbf{z}_i$ is a constant vector. For the moment, identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ so that $\mathbf{q}_i(t)$, $\psi(t)$ and $\mathbf{z}_i$ are complex numbers. Substituting this guess into Eq. (1) yields

$$|\psi|^3 \psi^{-1} \ddot{\psi} m_i \mathbf{z}_i = \sum_{i \neq j} \frac{m_i m_j (\mathbf{z}_j - \mathbf{z}_i)}{\|\mathbf{z}_j - \mathbf{z}_i\|^3}.$$

This can be split into an equation for the scalar function $\psi(t)$

$$\ddot{\psi} = -\frac{\mu \psi}{|\psi|^3} \tag{2}$$

and an equation for the initial vectors $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$

$$\sum_{i \neq j} \frac{m_i m_j (\mathbf{z}_j - \mathbf{z}_i)}{\|\mathbf{z}_j - \mathbf{z}_i\|^3} + \mu m_i \mathbf{z}_i = 0. \tag{3}$$

The motion of our special solution is determined by Eq. (2), which is simply the planar Kepler problem. Among the solutions to this problem are periodic orbits on circles and ellipses. The initial shape of the solution in position space $\mathbb{R}^{2n}$ is determined by Eq. (3). This is a complicated set of nonlinear algebraic equations for which very few explicit solutions are known. Solutions to (3) are called *central configurations* and a great deal of effort has gone into understanding their properties (see for example [7, 11, 13]).

The above analysis shows that given a planar central configuration $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$, there exists a solution to the n-body problem where each body travels along an ellipse with one focus at the origin. Since multiplication by a complex number is geometrically the composition of a rotation and a scaling, the shape (but not necessarily the size) of the configuration is maintained for all time. In other words, the motion of the solution is homographic. The larger the eccentricity of the Kepler orbit, the more the size of the configuration is varied. A periodic solution of this form with a circular Kepler orbit is often called a *relative equilibrium*, for in a rotating coordinate frame this solution becomes a fixed point. We will refer to any periodic solution of the form $\mathbf{q}_i(t) = \psi(t)\mathbf{z}_i$ as a *relative periodic solution*.

Perhaps, the most famous central configuration is Lagrange's equilateral triangle solution. This solution is a central configuration for any choice of the three masses, a fact that can be understood geometrically from the symmetry of the configuration. Indeed, the gravitational force on any body in a central configuration is proportional to its position. This is the case for the equilateral triangle. If the initial velocity is zero, the configuration shrinks toward its center of mass (homothetic motion) resulting in collision.

The coordinates $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ for the equilateral triangle are functions of the three masses m_1, m_2 and m_3. Note that summing Eq. (3) over all i yields

$$\sum_{i=1}^{3} m_i \mathbf{z}_i = 0$$

so that the center of mass is at the origin. This will be beneficial later when we change coordinates. For simplicity, we choose $\mathbf{z}_1 = (\alpha, 0)$, with $\alpha > 0$. The other two coordinates $\mathbf{z}_2$ and $\mathbf{z}_3$ can be expressed in terms of α by requiring the center of mass to be at the origin and making the triangle equilateral. We then scale the triangle so that the parameter μ in Eq. (3) is set to one. This fixes a unique value of α as a function of the three masses.

Fortunately, we do not need explicit expressions for $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ to perform the linear stability analysis of the relative periodic solution. The only important feature is that the bodies lie at the vertices of an equilateral triangle with center of mass at the origin. Let τ denote the length of the side of the equilateral triangle and let $M = m_1 + m_2 + m_3$ represent the total mass. To find the scaling of the triangle which yields $\mu = 1$, we examine Eq. (3) for $i = 1$.

This gives

$$\begin{aligned}\mathbf{z}_1 &= \frac{m_2(\mathbf{z}_1 - \mathbf{z}_2) + m_3(\mathbf{z}_1 - \mathbf{z}_3)}{\tau^3}\\ &= \frac{-m_2\mathbf{z}_2 - m_3\mathbf{z}_3 + (m_2 + m_3)\mathbf{z}_1}{\tau^3}\\ &= \frac{m_1\mathbf{z}_1 + (m_2 + m_3)\mathbf{z}_1}{\tau^3}\\ &= \frac{M}{\tau^3}\mathbf{z}_1.\end{aligned}$$

Since $\mathbf{z}_1 \neq 0$, we have

$$\tau = M^{1/3}. \tag{4}$$

Kepler's equation (2) is solvable up to quadrature [6]. In polar coordinates (r, θ), the solution with $\mu = 1$ is given by

$$r(t) = \frac{\omega^2}{1 + e\cos\theta(t)}, \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0,$$

where e, the eccentricity of the ellipse, and ω, the angular momentum, are two parameters. We have chosen the argument of the perihelion and $\theta(0)$ both to be zero. This means the true anomaly begins at zero and is measured from the positive horizontal axis. While these choices clearly do not affect the stability of the periodic orbit, the parameters e and ω could. As e gets larger, the orbit becomes more eccentric and the triangle dilates and expands to a greater amount. This suggests a loss of stability for larger values of e, although we show in Section 4 that this is not always the case. As ω is varied, the angular speed of the orbit and hence the period is altered. By reducing the dimensions of the problem, we show that this has no effect on the linear stability of Lagrange's triangle solutions.

If we write our central configuration in polar coordinates, $\mathbf{z}_i = \overline{r_i}(\cos\overline{\theta_i}, \sin\overline{\theta_i})$, then the position component of the periodic orbit is written as

$$\mathbf{q}_i(t) = \overline{r_i}\, r(t)\begin{pmatrix}\cos(\theta(t) + \overline{\theta_i})\\ \sin(\theta(t) + \overline{\theta_i})\end{pmatrix}. \tag{5}$$

The period of the orbit T, which is equivalent to the period of the Kepler orbit, is given by

$$T = \omega^3\int_0^{2\pi}\frac{1}{(1 + e\cos\theta)^2}\,d\theta = \frac{2\pi\omega^3}{(1 - e^2)^{3/2}},$$

where the last equality is a consequence of Kepler's third law.

3. REDUCING FROM 12 TO 4 DIMENSIONS

We are interested in studying the linear stability of Lagrange's equilateral triangle solutions and how this stability depends on the masses and eccentricity of the orbit. To do this, we must compute a fundamental matrix solution $X(t)$ to the equations of motion linearized about the periodic orbit. The monodromy matrix is the matrix C satisfying $X(t+T) = X(t)C$ (see [4, Ch. 37]). In particular, if we choose initial conditions $X(0) = I$, then $C = X(T)$. The monodromy matrix is the linear part of the time T flow about the periodic orbit. Stability is governed by the eigenvalues of the monodromy matrix, called the characteristic multipliers. Since we are dealing with a Hamiltonian system, C is symplectic and the multipliers are symmetric about the unit circle. In order to have linear stability, it is necessary that all the multipliers have modulus one.

As is well known, the n-body problem is a Hamiltonian system with several integrals. These integrals show up in the linear stability analysis as characteristic multipliers with a value of $+1$. For example, if we vary the center of mass for the relative periodic solution but maintain the total linear momentum equal to zero, then the elliptic orbits for each mass will revolve around a different point. In the full phase space $\mathbb{R}^{2n}$, this corresponds to a two-dimensional surface of periodic orbits. Any vector tangent to this surface corresponds to a left eigenvector of the monodromy matrix with eigenvalue $+1$ (see [6, p. 134]). If the total linear momentum is not zero, then nearby orbits will have different periods and drift apart in the full phase space, leading to instability in the classical sense. It follows that the monodromy matrix has two 2×2 Jordan blocks of the form

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

corresponding to the center of mass and total linear momentum integrals.

It is natural then to define the linear stability of the relative periodic solution on a reduced space by fixing all the integrals and obtaining the non-trivial multipliers [8, 9]. For the planar n-body problem there will always be 8 trivial multipliers for any relative periodic solution. There are 2 from the center of mass, 2 from the total linear momentum, 2 from the SO(2) symmetry and angular momentum, 1 from the Hamiltonian and 1 from the periodic orbit itself.

DEFINITION. A relative periodic solution of the planar n-body problem has 8 trivial characteristic multipliers of value $+1$. The solution is *spectrally stable* if the remaining multipliers lie on the unit circle and *linearly stable*, if in addition, the monodromy matrix restricted to the reduced space is diagonalizable.

3.1. Eliminating the Standard Integrals

We now reduce the dimensions of our problem from 12 to 6. Let the momenta of each body be $\mathbf{p}_i = m_i\dot{\mathbf{q}}_i$. The Hamiltonian for the three-body problem is

$$H_1 = \frac{||\mathbf{p}_1||^2}{2m_1} + \frac{||\mathbf{p}_2||^2}{2m_2} + \frac{||\mathbf{p}_3||^2}{2m_3} - \frac{m_1m_2}{||\mathbf{q}_2 - \mathbf{q}_1||} - \frac{m_1m_3}{||\mathbf{q}_3 - \mathbf{q}_1||} - \frac{m_2m_3}{||\mathbf{q}_3 - \mathbf{q}_2||}.$$

We first eliminate the center of mass and total linear momentum by using Jacobi coordinates. Specifically, set

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{q}_2 - \mathbf{q}_1, & \mathbf{v}_1 &= -\frac{m_2}{m_1 + m_2}\mathbf{p}_1,\\
\mathbf{u}_2 &= \mathbf{q}_3 - \frac{1}{m_1 + m_2}(m_1\mathbf{q}_1 + m_2\mathbf{q}_2), & \mathbf{v}_2 &= -\frac{m_3}{M}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) + \mathbf{p}_3,\\
\mathbf{u}_3 &= \frac{1}{M}(m_1\mathbf{q}_1 + m_2\mathbf{q}_2 + m_3\mathbf{q}_3), & \mathbf{v}_3 &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3.
\end{aligned}$$

This is a symplectic change of variables. By setting $\mathbf{u}_3 = 0$ and $\mathbf{v}_3 = 0$, the new Hamiltonian becomes

$$H_2 = \frac{||\mathbf{v}_1||^2}{2M_1} + \frac{||\mathbf{v}_2||^2}{2M_2} - \frac{m_1m_2}{||\mathbf{u}_1||} - \frac{m_1m_3}{||\mathbf{u}_2 + M_3\mathbf{u}_1||} - \frac{m_2m_3}{||\mathbf{u}_2 + M_4\mathbf{u}_1||},$$

where

$$\begin{aligned}
M_1 &= \frac{m_1m_2}{m_1 + m_2}, & M_2 &= \frac{m_3(m_1 + m_2)}{M},\\
M_3 &= \frac{m_2}{m_1 + m_2}, & M_4 &= -\frac{m_1}{m_1 + m_2}.
\end{aligned}$$

Note that we are justified in choosing $\mathbf{u}_3 = 0$ and $\mathbf{v}_3 = 0$ because on our periodic orbit we have

$$\sum_{i=1}^{3} m_i\mathbf{q}_i = \psi \sum_{i=1}^{3} m_i\mathbf{z}_i = 0$$

and similarly,

$$\sum_{i=1}^{3} \mathbf{p}_i = \dot{\psi} \sum_{i=1}^{3} m_i\mathbf{z}_i = 0.$$

This reduction reduces the dimensions from 12 to 8.

Next, we change to symplectic polar coordinates to eliminate the integrals due to the angular momentum and rotational symmetry. Set

$$\mathbf{u}_i = \begin{pmatrix} r_i \cos\theta_i \\ r_i \sin\theta_i \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_i = \begin{pmatrix} R_i \cos\theta_i - \dfrac{\Theta_i}{r_i} \sin\theta_i \\ R_i \sin\theta_i + \dfrac{\Theta_i}{r_i} \cos\theta_i \end{pmatrix}$$

for $i = 1, 2$. The new Hamiltonian then becomes

$$H_3 = \frac{1}{2M_1}\left(R_1^2 + \frac{\Theta_1^2}{r_1^2}\right) + \frac{1}{2M_2}\left(R_2^2 + \frac{\Theta_2^2}{r_2^2}\right) - \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2},$$

where

$$\delta_1 = \sqrt{r_2^2 + M_3^2 r_1^2 + 2M_3 r_1 r_2 \cos(\theta_2 - \theta_1)},$$

$$\delta_2 = \sqrt{r_2^2 + M_4^2 r_1^2 + 2M_4 r_1 r_2 \cos(\theta_2 - \theta_1)}.$$

The angles only enter the Hamiltonian H_3 in the form $\theta_2 - \theta_1$. This suggests making a final symplectic change of coordinates by leaving the radial variables alone and setting

$$\begin{aligned} \upsilon &= \theta_1, & \Upsilon &= \Theta_1 + \Theta_2, \\ \phi &= \theta_2 - \theta_1, & \Phi &= \Theta_2. \end{aligned}$$

The new Hamiltonian will be independent of υ which means that Υ (angular momentum) is an integral, and υ is an ignorable variable. Setting $\Theta_1 + \Theta_2 = \Upsilon = c$ and plugging into the Hamiltonian H_3 yields

$$H = \frac{1}{2M_1}\left(R_1^2 + \frac{(c-\Phi)^2}{r_1^2}\right) + \frac{1}{2M_2}\left(R_2^2 + \frac{\Phi^2}{r_2^2}\right) - \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2},$$

where we now have

$$\delta_1 = \sqrt{r_2^2 + M_3^2 r_1^2 + 2M_3 r_1 r_2 \cos\phi},$$

$$\delta_2 = \sqrt{r_2^2 + M_4^2 r_1^2 + 2M_4 r_1 r_2 \cos\phi}.$$

This reduces the system to six dimensions, with the variables $(r_1, r_2, \phi, R_1, R_2, \Phi)$.

The equations of motion in these new variables are

$$\dot{r}_1 = \frac{R_1}{M_1}, \qquad \dot{r}_2 = \frac{R_2}{M_2}, \qquad \dot{\phi} = \frac{\Phi - c}{M_1 r_1^2} + \frac{\Phi}{M_2 r_2^2},$$

$$\dot{R}_1 = \frac{(\Phi - c)^2}{M_1 r_1^3} - \frac{m_1 m_2}{r_1^2} - \frac{m_1 m_3 M_3 (r_1 M_3 + r_2 \cos\phi)}{\delta_1^3} - \frac{m_2 m_3 M_4 (r_1 M_4 + r_2 \cos\phi)}{\delta_2^3},$$

$$\dot{R}_2 = \frac{\Phi^2}{M_2 r_2^3} - \frac{m_1 m_3 (r_2 + r_1 M_3 \cos\phi)}{\delta_1^3} - \frac{m_2 m_3 (r_2 + r_1 M_4 \cos\phi)}{\delta_2^3},$$

$$\dot{\Phi} = m_3 r_1 r_2 \sin\phi \left(\frac{m_1 M_3}{\delta_1^3} + \frac{m_2 M_4}{\delta_2^3} \right).$$

Using (4) and (5), a short calculation reveals that the relative periodic solution, denoted in general as $\gamma(t)$, is written as

$$r_1(t) = C_1 r(t), \qquad R_1(t) = M_1 C_1 R(t),$$

$$r_2(t) = C_2 r(t), \qquad R_2(t) = M_2 C_2 R(t),$$

$$\phi = \overline{\theta_3} + \eta, \qquad \Phi = \omega M_2 C_2^2, \tag{6}$$

where $C_1 = \tau = M^{1/3}$, $C_2 = \bar{r}_3 M/(m_1 + m_2)$, and η is the angle between the vector $\mathbf{z}_2 - \mathbf{z}_1$ and the positive horizontal axis. Recall that

$$r(t) = \frac{\omega^2}{1 + e\cos\theta(t)}, \qquad \dot{r} = R, \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0$$

is the periodic solution to Kepler's problem mentioned earlier. In addition to the three masses, the two parameters in this solution are the eccentricity e and the angular momentum ω of the elliptic orbits. The angular momentum integral for the full problem has the value $\Upsilon = c = \omega(M_1 C_1^2 + M_2 C_2^2)$.

Note that along the periodic orbit $\gamma(t)$, we have

$$r_1(t) = \delta_1(t) = \delta_2(t)$$

since each of the above terms gives the length of one side of the triangle formed by the three bodies. Simplifying $\delta_1(t) = \delta_2(t)$ gives

$$2\bar{r}_3 \cos\phi = M^{-2/3}(m_1 - m_2) \tag{7}$$

and then solving $r_1(t) = \delta_1(t)$ gives

$$\bar{r}_3^2 = M^{-4/3}(m_1^2 + m_1 m_2 + m_2^2). \tag{8}$$

These expressions will be useful in simplifying the associated linear system.

Linearizing the six-dimensional system about the periodic solution $\gamma(t)$ gives the time-dependent periodic linear Hamiltonian system

$$\dot{\xi} = J_3 D^2 H(\gamma(t))\xi,$$

where J_3 is the canonical matrix

$$J_3 = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}.$$

After a good deal of calculation and simplification, using expressions (6)–(8), we have

$$J_3 D^2 H(\gamma(t)) = \begin{bmatrix} 0 & 0 & 0 & \dfrac{1}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{M_2} & 0 \\ \dfrac{2\omega}{C_1 r^3} & \dfrac{-2\omega}{C_2 r^3} & 0 & 0 & 0 & \dfrac{c}{\omega M_1 M_2 C_1^2 C_2^2 r^2} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & \dfrac{-2\omega}{C_1 r^3} \\ a_{12} & a_{22} & -\dfrac{C_1}{C_2} a_{13} & 0 & 0 & \dfrac{2\omega}{C_2 r^3} \\ a_{13} & -\dfrac{C_1}{C_2} a_{13} & a_{33} & 0 & 0 & 0 \end{bmatrix},$$

where

$$a_{12} = \frac{9 m_1 m_2 m_3}{4M\sqrt{m_1^2 + m_1 m_2 + m_2^2}} \frac{1}{r^3}, \qquad a_{13} = \frac{3\sqrt{3}\, m_1 m_2 m_3 (m_1 - m_2)}{4M^{2/3}(m_1 + m_2)^2} \frac{1}{r^2}$$

and

$$a_{33} = \frac{9m_1m_2m_3}{4M^{1/3}(m_1+m_2)}\frac{1}{r}.$$

The coefficients $a_{ij},\ i,j \in \{1,2\}$ satisfy the important relation

$$\frac{1}{C_1M_1}(C_1a_{11}+C_2a_{12}) = \frac{1}{C_2M_2}(C_1a_{12}+C_2a_{22}) = -\frac{3\omega^2}{r^4}+\frac{2}{r^3}. \tag{9}$$

3.2. *Decoupling the linear system*

A linear, time-dependent periodic Hamiltonian system is one of the form

$$\dot{\xi} = JD^2H(t)\xi, \tag{10}$$

where J is the canonical matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and $D^2H(t+T) = D^2H(t)$.
When such a system results from linearizing about a periodic solution, there are at least two $+1$ characteristic multipliers. One of these is attributable to the periodic orbit and another arises from the existence of an integral, which in this case is the Hamiltonian H. This fact is easily proven via differentiation (see [6] for example). Indeed, given a periodic solution $\gamma(t)$ to a Hamiltonian system $\dot{x} = J\nabla H(x)$, plugging in $\gamma(t)$ and differentiating with respect to t yields

$$\ddot{\gamma} = JD^2H(\gamma(t))\dot{\gamma}. \tag{11}$$

Thus, $\dot{\gamma}$ is a solution of the associated linear system. Since $\gamma(t)$ is periodic, so is its derivative. If we choose coordinates so that $\dot{\gamma}(0) = (1,0,\ldots,0)$, the first column of the monodromy matrix is $(1,0,\ldots,0)$ and $+1$ is an eigenvalue. But relation (11) is important for another reason: It suggests a useful change of coordinates. Choosing variables so that the periodic orbit is easily represented helps decouple the system. This follows from a standard result in the theory of Hamiltonian systems.

Define the skew-inner product of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^{4n}$ as

$$\Omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T J\mathbf{w}.$$

Note that $J^T = -J = J^{-1}$ so that J is orthogonal and skew-symmetric. A key trait of linear Hamiltonian systems is that the skew-orthogonal complement of an invariant subspace is also invariant.

LEMMA 3.1. *Suppose W is an invariant subspace of the matrix $JD^2H(t)$, then the skew-orthogonal complement of W, defined as $W^{\perp} = \{\mathbf{v} \in \mathbb{C}^{4n} : \Omega(\mathbf{v}, \mathbf{w}) = 0 \ \forall \mathbf{w} \in W\}$, is also an invariant subspace of $JD^2H(t)$.*

Proof. Suppose $\mathbf{v} \in W^{\perp}$. Then, for any $\mathbf{w} \in W$ we have

$$\begin{aligned}\Omega(JD^2H(t)\mathbf{v}, \mathbf{w}) &= \mathbf{v}^T D^2H(t) J^T J\mathbf{w} \\ &= \mathbf{v}^T D^2H(t)\mathbf{w} \\ &= -\mathbf{v}^T J\hat{\mathbf{w}} \\ &= 0,\end{aligned}$$

where $\hat{\mathbf{w}} = JD^2H(t)\mathbf{w} \in W$. Thus, $JD^2H(t)\mathbf{v} \in W^{\perp}$. ∎

Given an invariant subspace, Lemma 3.1 shows that a simple linear change of variables will decouple the system. Specifically, suppose that W is an invariant subspace for $JD^2H(t)$ with basis B_1 and let B_2 be a basis for $W^{\perp}$. Denote $S = [B_1 \ B_2]$ as the matrix whose columns are the elements of the two bases. Making the linear change of variables $\xi = S\zeta$ in (10) yields a decoupled linear system of the form

$$\dot{\zeta} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \zeta,$$

where S_1 and S_2 are the restrictions of $JD^2H(t)$ to W and $W^{\perp}$, respectively. The characteristic multipliers remain the same since the transformation is linear.

To apply these ideas to our problem, we need to find an invariant subspace for $J_3 D^2H(\gamma(t))$. As mentioned before, the periodic orbit itself provides an excellent suggestion. We make use of the fact that the Kepler solution $r(t)$ satisfies

$$\ddot{r} = \frac{\omega^2}{r^3} - \frac{1}{r^2}. \tag{12}$$

Differentiating this with respect to t yields

$$\dddot{r} = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}. \tag{13}$$

From this we have

$$\dot{\gamma} = \begin{pmatrix} C_1\dot{r} \\ C_2\dot{r} \\ 0 \\ M_1C_1\ddot{r} \\ M_2C_2\ddot{r} \\ 0 \end{pmatrix} \quad \text{and} \quad \ddot{\gamma} = \begin{pmatrix} C_1\ddot{r} \\ C_2\ddot{r} \\ 0 \\ M_1C_1\left(-\dfrac{3\omega^2}{r^4} + \dfrac{2}{r^3}\right)\dot{r} \\ M_2C_2\left(-\dfrac{3\omega^2}{r^4} + \dfrac{2}{r^3}\right)\dot{r} \\ 0 \end{pmatrix}$$

as expressions for the first and second derivatives of the periodic orbit. Using relation (9), it is clear that

$$\begin{aligned} J_3D^2H(\gamma(t))\dot{\gamma} &= \ddot{\gamma}, \\ J_3D^2H(\gamma(t))\ddot{\gamma} &= \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{\gamma}. \end{aligned}$$

Then, the vectors $\mathbf{w}_1 = (C_1, C_2, 0, 0, 0, 0)$ and $\mathbf{w}_2 = (0, 0, 0, M_1C_1, M_2C_2, 0)$ will span an invariant subspace W for $J_3D^2H(\gamma(t))$.

Consider the change of variables determined by

$$\begin{pmatrix} r_1 \\ r_2 \\ \phi \\ R_1 \\ R_2 \\ \Phi \end{pmatrix} = \begin{bmatrix} C_1 & 0 & \omega C_2M_2/c & 0 & 0 & 0 \\ C_2 & 0 & -\omega C_1M_1/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & M_1C_1 & 0 & \omega C_2/c & 0 & 0 \\ 0 & M_2C_2 & 0 & -\omega C_1/c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ X \\ y \\ Y \\ z \\ Z \end{pmatrix} \tag{14}$$

(recall that $c = \omega(M_1C_1^2 + M_2C_2^2)$). The last four columns of the above matrix are chosen to form a basis for the skew-orthogonal complement of W. Consequently, by the remark after Lemma 3.1, this change of variables will decouple our linear system into a 2×2 and a 4×4 system. The new coordinates are

$$\begin{aligned} x &= \frac{\omega}{c}(M_1C_1r_1 + M_2C_2r_2), & X &= \frac{\omega}{c}(C_1R_1 + C_2R_2), \\ y &= C_2r_1 - C_1r_2, & Y &= M_2C_2R_1 - M_1C_1R_2, \\ z &= \phi, & Z &= \Phi. \end{aligned}$$

Note that along the periodic orbit $\gamma(t)$, $x = r$, $X = R$ while $y = Y = 0$. Thus, we expect the 2×2 system in the x and X variables to identify the two remaining $+1$ multipliers, leaving the remaining four variables to decide the linear stability of the relative periodic solution.

The equations for the x and X variables give a simple 2×2 periodic, linear Hamiltonian system:

$$\begin{aligned}\dot{x} &= X,\\ \dot{X} &= \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)x.\end{aligned}$$

For the initial condition $x(0) = 0,\ X(0) = 1$, making use of (13), we have as a solution $x = k\dot{r},\ X = k\ddot{r}$, where $k = \omega^4/(e(1+e)^2)$ is chosen so that $X(0) = 1$. Since this is a periodic solution with the same period as the system itself, the second column of the monodromy matrix for this system will be $(0, 1)$. Since we have a Hamiltonian system, the monodromy matrix is symplectic, with a determinant one, and must have the form

$$\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}.$$

As expected, this yields the remaining two $+1$ characteristic multipliers.

The equations for the remaining four variables come from computing the restriction of $J_3\, D^2H(\gamma(t))$ to the space spanned by the last four columns of the matrix in (14). Using relation (9), this gives

$$\begin{pmatrix} \dot{y} \\ \dot{Y} \\ \dot{z} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} 0 & \dfrac{1}{M_1M_2} & 0 & 0 \\ M_1M_2\left(-\dfrac{3\omega^2}{r^4} + \dfrac{2}{r^3}\right) - \dfrac{a_{12}c}{\omega C_1C_2} & 0 & \dfrac{a_{13}c}{\omega C_2} & -\dfrac{2c}{C_1C_2r^3} \\ \dfrac{2\omega}{C_1C_2r^3} & 0 & 0 & d \\ \dfrac{a_{13}}{C_2} & 0 & a_{33} & 0 \end{bmatrix} \begin{pmatrix} y \\ Y \\ z \\ Z \end{pmatrix}.$$

We now make a time-dependent scaling of the variables using the transformation $\hat{y} = y/r,\ \hat{Y} = rY,\ \hat{z} = C_1C_2\, z,\ \hat{Z} = Z/(C_1C_2)$. Since this is a linear transformation, it will not change the characteristic multipliers. Next, we change the independent variable from t to θ. In other words, use

$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = y'\,\frac{\omega}{r^2}$$

and similar expressions for $\dot{Y}$, $\dot{z}$, $\dot{Z}$ and $\dot{r}$. Dropping the hats off the variables and letting $'$ represent the derivative with respect to θ, the new system is

$$\begin{aligned}
y' &= -\frac{r'}{r}\,y + \frac{1}{\omega M_1 M_2}\,Y,\\
Y' &= \frac{r^4}{\omega}\left(M_1 M_2\left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) - \frac{a_{12}c}{\omega C_1 C_2}\right) y + \frac{a_{13}cr^3}{\omega^2 C_1 C_2^2}\,z + \frac{r'}{r}\,Y - \frac{2c}{\omega}\,Z,\\
z' &= 2y + \frac{c}{\omega^2 M_1 M_2}\,Z,\\
Z' &= \frac{a_{13}r^3}{\omega C_1 C_2^2}\,y + \frac{a_{33}r^2}{\omega C_1^2 C_2^2}\,z.
\end{aligned} \tag{15}$$

It helps to convert the above system into two second-order equations. Differentiating Eq. (15) with respect to θ will yield terms involving $(r')^2$ and r''. The relation

$$\frac{r}{\omega^2} - 1 = \frac{2(r')^2 - rr''}{r^2}$$

can be derived from Eq. (12). Using this, we obtain

$$\begin{aligned}
y'' + 2z' &= \left(\frac{3r}{\omega^2} - \frac{a_{12}cr^4}{\omega^3 M_1 M_2 C_1 C_2}\right) y + \frac{a_{13}cr^3}{\omega^3 M_1 M_2 C_1 C_2^2}\,z,\\
z'' - 2y' &= \frac{a_{13}cr^3}{\omega^3 M_1 M_2 C_1 C_2^2}\,y + \frac{a_{33}cr^2}{\omega^3 M_1 M_2 C_1^2 C_2^2}\,z,
\end{aligned}$$

which reduces to

$$\begin{aligned}
y'' + 2z' &= \frac{r}{\omega^2}\left((3 - v_1)\,y + v_2 z\right),\\
z'' - 2y' &= \frac{r}{\omega^2}\left(v_2\,y + v_1 z\right),
\end{aligned}$$

where

$$v_1 = \frac{9K(m_1 + m_2)}{4M(m_1^2 + m_1 m_2 + m_2^2)}, \qquad v_2 = \frac{3\sqrt{3}\,K(m_1 - m_2)}{4M(m_1^2 + m_1 m_2 + m_2^2)}$$

$$\text{and} \quad K = m_1 m_2 + m_1 m_3 + m_2 m_3.$$

Our final change of variables comes from the symmetry of the previous second-order equations. Specifically, let B be the symmetric matrix

$$\begin{bmatrix} 3 - v_1 & v_2 \\ v_2 & v_1 \end{bmatrix}$$

and let Q be its orthogonal matrix of eigenvectors so that $Q^T B Q = \Lambda$ is diagonal. The change of variables

$$\begin{pmatrix} y \\ z \end{pmatrix} = Q \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix}$$

gives

$$\begin{pmatrix} \hat{y}'' \\ \hat{z}'' \end{pmatrix} + Q^T \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} Q \begin{pmatrix} \hat{y}' \\ \hat{z}' \end{pmatrix} = \frac{r}{\omega^2} \Lambda \begin{pmatrix} \hat{y} \\ \hat{z} \end{pmatrix}.$$

Since Q is orthogonal, it commutes with

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

The final second-order equations are then (dropping the hats)

$$\begin{aligned} y'' + 2z' &= \frac{r}{\omega^2} \lambda_1 y, \\ z'' - 2y' &= \frac{r}{\omega^2} \lambda_2 z, \end{aligned}$$

where $\lambda_{1,2}$ are the eigenvalues of B or roots of the quadratic

$$\lambda^2 - 3\lambda + \frac{27K}{4M^2} = \lambda^2 - 3\lambda + \frac{\beta}{4}.$$

Recall that $\beta = 27(m_1 m_2 + m_1 m_3 + m_2 m_3)/(m_1 + m_2 + m_3)^2$. Our final four-dimensional system for the linearization about the relative periodic orbit $\gamma(t)$ is

$$\begin{pmatrix} y' \\ z' \\ Y' \\ Z' \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\lambda_1}{1+e\cos\theta} & 0 & 0 & -2 \\ 0 & \dfrac{\lambda_2}{1+e\cos\theta} & 2 & 0 \end{bmatrix} \begin{pmatrix} y \\ z \\ Y \\ Z \end{pmatrix}, \tag{16}$$

where the derivative is with respect to θ. One crucial fact about this system is that the masses only enter through $\lambda_{1,2}$ which are functions of the mass coefficient β. Remarkably, the stability depends on one mass parameter β rather than the three mass parameters m_1, m_2, m_3. Moreover, since ω is not present, the angular momentum of the elliptic Kepler orbit does not affect the linear stability.

4. LINEAR STABILITY ANALYSIS

In this section, we analyze the linear stability of the Lagrange equilateral triangle solutions in terms of the parameters e and β. This entails computing the fundamental matrix solution $X(\theta)$ to system (16) with initial conditions $X(0) = I_4$. The monodromy matrix, subsequently denoted by C, is then $X(2\pi)$ and the eigenvalues of this matrix are the characteristic multipliers. The map $x \mapsto Cx$ can be interpreted as the linearization of the Poincare map on our reduced space. Since system (16) has been derived from a Hamiltonian system with coordinate changes which do not alter the multipliers, the characteristic polynomial of C will be reciprocal [6]. In other words, λ is an eigenvalue of C if and only if $1/\lambda$ is also an eigenvalue. Thus, to have linear stability, we require that the eigenvalues reside on the unit circle.

The characteristic polynomial of C has the form

$$\lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1, \tag{17}$$

where

$$a = -\mathrm{tr}(C) \quad \text{and} \quad b = \tfrac{1}{2}((\mathrm{tr}(C))^2 - \mathrm{tr}(C^2)). \tag{18}$$

Given that the multipliers are on the unit circle, there are three ways in which stability can be lost:

- period-doubling bifurcation (two -1 eigenvalues), occurring when $b = 2a - 2$,

- two $+1$ eigenvalues, occurring when $b = -2a - 2$,
- Krein collision (repeated eigenvalues on the unit circle), occurring when $b = a^2/4 + 2$.

In the first two cases, a pair of eigenvalues meets and then breaks off onto the real line yielding an eigenvalue with modulus greater than one and an eigenvalue with modulus less than one. Eigenvalues experiencing a Krein collision lose stability when two pairs meet on the unit circle and then split off into a complex quartet $(\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda})$. In terms of a and b, a straightforward calculation shows that the stability region for quartic (17) is given by the four conditions $b \geqslant 2a - 2$, $b \geqslant -2a - 2$, $b \leqslant a^2/4 + 2$ and $-4 \leqslant a \leqslant 4$ (see Fig. 1).

We begin by analyzing the behavior of the multipliers for the circular case $e = 0$. In this case, the matrix in system (16) is constant and therefore, the multipliers can be explicitly computed. The characteristic polynomial

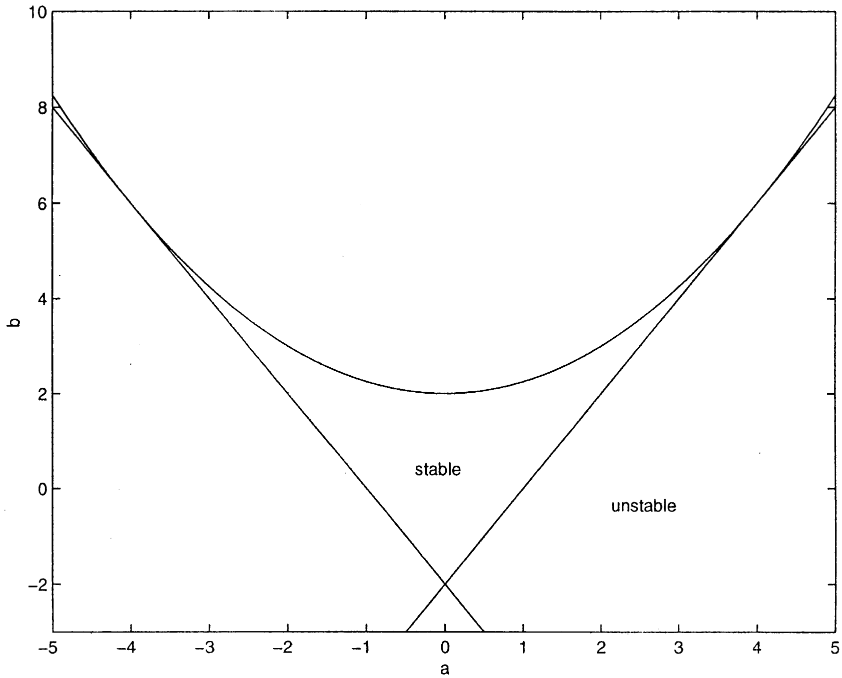

FIG. 1. Stability region for $\lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1$ in the ab-plane. The period-doubling boundary is $b = 2a - 2$ and the Krein collision curve is the parabola $b = a^2/4 + 2$.

of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & 0 & -2 \\ 0 & \lambda_2 & 2 & 0 \end{bmatrix}$$

is given by

$$\rho^4 + \rho^2 + \beta/4. \tag{19}$$

If ρ is a root of this polynomial, then $e^{2\pi\rho}$ is a characteristic multiplier. We follow the behavior of these multipliers as β varies. In order to have stability, the roots of (19) must be purely imaginary. It is easy to see that this occurs only when $0 \leqslant \beta \leqslant 1$.

Let $q(\kappa) = \kappa^2 + \kappa + \beta/4$ and denote the two roots of $q(\kappa)$ as κ_1 and κ_2 where $\kappa_2 \leqslant \kappa_1 \leqslant 0$. As β increases from 0 to 1, κ_1 decreases monotonically from 0 to 1/2 and κ_2 increases monotonically from -1 to 1/2. In the interval $\beta \in [0, 1]$, the four multipliers occur in pairs $e^{\pm i\theta_1}$ and $e^{\pm i\theta_2}$, where $\theta_1 = 2\pi\sqrt{-\kappa_1}$ and $\theta_2 = 2\pi\sqrt{-\kappa_2}$. There are three important values of β:

- $\beta = 0 \Rightarrow \theta_1 = 0$ and $\theta_2 = 2\pi$ (four $+1$ multipliers),
- $\beta = 3/4 \Rightarrow \theta_1 = \pi$ and $\theta_2 = \sqrt{3}\pi$ (period doubling),
- $\beta = 1 \Rightarrow \theta_1 = \theta_2 = \sqrt{2}\pi$ (Krein collision).

The value $\beta = 3/4$ is especially significant as it represents a possible location where we may lose stability for $e \neq 0$. Indeed, we will show that although the circular orbit is stable, the orbits with eccentricity nonzero are linearly unstable. In contrast, as β passes through the Krein collision value 1, stability is lost in the circular case, but is actually preserved in the elliptic case. In other words, at $\beta = 1$, the multipliers pass through each other and remain on the unit circle as the eccentricity increases from $e = 0$. This yields values of masses where the truly elliptic orbits are linearly stable even though the circular orbits are not.

The above β values are the only place where bifurcations can occur. The angle θ_1 increases from 0 to π as β increases from 0 to 3/4, while the angle θ_2 decreases from 2π to $\sqrt{3}\pi$ for the same β values. There are no Krein collisions for $\beta \in (0, 3/4)$ because θ_1 increases away from 0 faster than θ_2 decreases away from 2π. (Alternatively, the equation $2\pi = 2\pi(\sqrt{-\kappa_1} + \sqrt{-\kappa_2})$ is only satisfied when $\beta = 0$.) After passing through the period-doubling point at $\beta = 3/4$, θ_1 increases from π to $\sqrt{2}\pi$ while θ_2 decreases from $\sqrt{3}\pi$ to $\sqrt{2}\pi$, ending in the Krein collision at $\beta = 1$. The orbit is linearly stable for $0 < \beta < 1$ but only spectrally stable at $\beta = 0, 1$. For values of $\beta > 1$, Eq. (19) has four complex roots and thus there will be a complex quartet of multipliers leading to instability. This agrees with the classical result of Gascheau mentioned in the introduction [3].

For $e = 0$, we visualize the characteristic multipliers in Fig. 2 by sketching the eigenvalue curve (dashed) in the ab-plane for different values of β, where a and b are the coefficients of the quartic (17). For $\beta \in [0, 1]$, the values of a and b are given by

$$a = -2(\cos\theta_1 + \cos\theta_2) \quad \text{and} \quad b = 2 + 4\cos\theta_1\cos\theta_2.$$

Note that the eigenvalue curve is tangent to the period-doubling boundary at $\beta = 3/4$ and crosses the Krein collision curve at $\beta = 1$.

We now investigate the linear stability of the periodic orbits which are truly elliptic ($e \neq 0$). We accomplish this by numerically integrating system (16) and calculating the eigenvalues of the monodromy matrix C.

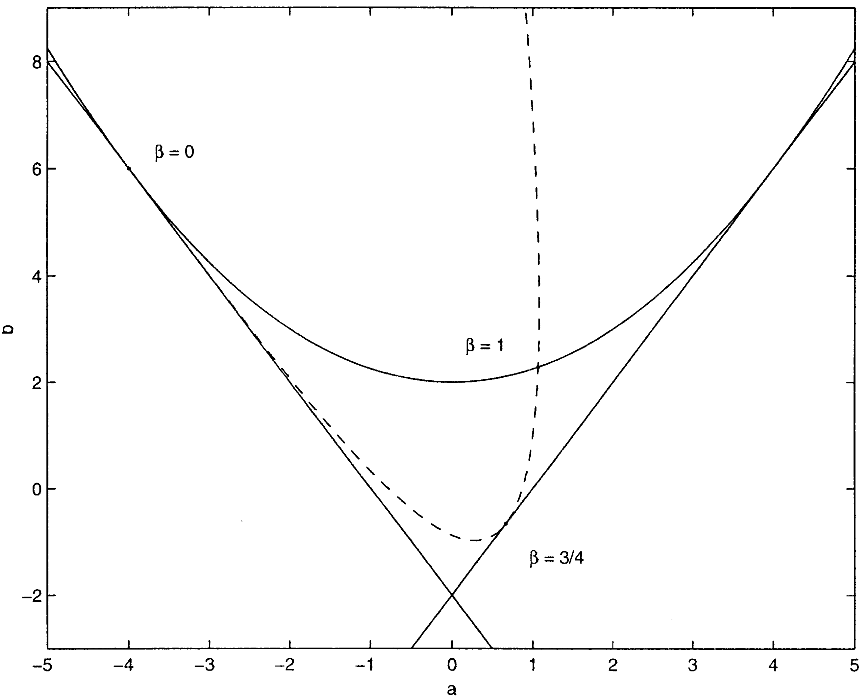

FIG. 2. Stability diagram for the case $e = 0$. The dashed curve describes the characteristic multipliers. Two key β values are $\beta = 3/4$, where two multipliers are -1 and $\beta = 1$, where the two pairs of multipliers meet on the unit circle. Stability is lost for $\beta > 1$.

For a fixed β value, we start with $e = 0$ and increase e, watching the multipliers move around the unit circle. Stability is lost through a period-doubling bifurcation, when two eigenvalues meet at -1 and then break off onto the real axis. At this precise point, the orbit is spectrally stable but not linearly stable. This was determined numerically by following the eigenvectors as the period-doubling point was approached. To numerically locate the bifurcation value, a simple bisection method was used in the equation for period doubling, $b = 2a - 2$. Specifically, for a fixed β, we think of a and b as functions of e and consider the quantity $b - 2a + 2$. If this quantity is positive, we are below the period-doubling curve and thus choose to average the current value of e with a larger one for which $b - 2a + 2$ is negative. This gives us a new e value. If $b - 2a + 2$ is negative, then we are above the curve and average with the previous value found below the curve. Since $b - 2a + 2$ is positive for $e = 0$ (except for the key value $\beta = 3/4$) and is negative for $e = 1$, we can always apply this numerical method. One advantage of this method is that it is easy to obtain a bound on the error of the final approximation. Using MATLAB, a simple program was written to obtain the period-doubling bifurcation curve in the βe-plane. This curve is shown in Fig. 3 and is given with a relative error less than 5×10^{-7}.

Note that for $\beta = 0$, when two of the masses necessarily vanish, we obtain four eigenvalues of $+1$ for any value of the eccentricity e. This follows because the problem decouples into two Kepler problems, one for each of the infinitesimal masses. The multipliers for linearizing the two-dimensional Kepler problem about a periodic orbit are all $+1$ (two coming from the angular momentum integral and SO(2) action, and two coming from the period orbit and the Hamiltonian). Consequently, for $\beta = 0$, $a(e) = -4$ and $b(e) = 6$ are constant functions of the eccentricity.

Surprisingly, for $\beta \geqslant 1$ there exist regions of stability for the elliptic orbits, even though their circular counterparts are linearly unstable. For these cases, the eigenvalues are initially off the unit circle for $e = 0$ but then return in the form of a Krein collision at a value $e = e_{kc}$. Stability is then lost through a period-doubling bifurcation at $e = e_{pd}$. The Krein collision curve was calculated using the same numerical method as for the period-doubling curve, but with the quantity $a^2/4 + 2 - b$ instead (see Fig. 3). As for the case $\beta < 1$, the periodic orbit is only spectrally stable at parameter values along the Krein collision curve and period-doubling curve. This was checked numerically by examining the corresponding eigenvectors along these curves. It should be pointed out that the stability diagram for the elliptic restricted problem in [1] does resemble the one found here for the full problem.

Note that the period-doubling curve reaches $e = 0$ at $\beta = 3/4$, in agreement with our calculation of the eigenvalues in the circular case. This

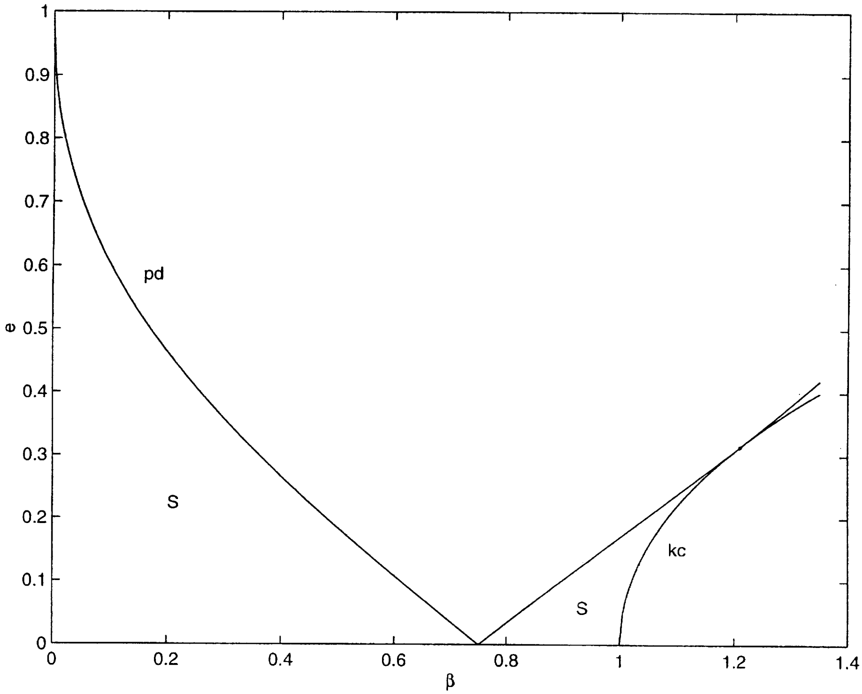

FIG. 3. The linear stability diagram in terms of the parameters β and e. There are two regions of linear stability labeled S. The period-doubling curve (pd) reaches the β-axis at $\beta = 3/4$. The Krein collision curve (kc) begins at $\beta = 1$ and is tangent to the period-doubling curve at the point $(1.209133, 0.314508)$, where all four multipliers are -1.

point is significant as it divides the stability region into two components, one for $\beta < 3/4$ and one for $\beta > 3/4$. We can interpret Fig. 3 for fixed values of e, moving horizontally across the graph rather than vertically. In the circular case, we have linear stability for the interval $0 < \beta < 1$. However, because of the two stability components, we have two intervals of stability whenever $0 < e < 0.314508$ and one interval of stability only when $e \geqslant 0.314508$. For example, when $e = 0.1$, we have linear stability for $\beta \in (0, 0.61)$ and $\beta \in (0.895, 1.02)$.

We now prove analytically that the picture is accurate near the key values $\beta = 3/4$ and 1. We show that when $\beta = 3/4$ perturbing in the positive e direction leads to instability, while for $\beta = 1$ perturbing in the positive e direction leads to stability. This means that the linear system at $\beta = 3/4$, $e = 0$ is stable under a constant Hamiltonian perturbation (as β increases the multipliers pass through -1 without moving off onto the real axis) but unstable under a time-dependent Hamiltonian perturbation (increasing e pushes the multipliers onto the real axis). In complete contrast, the linear system at $\beta = 1$, $e = 0$ is unstable under a constant Hamiltonian perturbation (increasing β pushes the multipliers off the unit circle) yet stable under a time-dependent Hamiltonian perturbation (as e increases, the multipliers pass through each other on the unit circle).

THEOREM 4.1. *For $\beta = 3/4$, two of the characteristic multipliers move off the unit circle as the eccentricity e is increased away from* 0, *while for $\beta = 1$, all of the multipliers remain on the unit circle as e increases away from* 0. *Consequently, there exist mass values $\beta > 1$ where the elliptic periodic orbits are linearly stable although the circular orbits are not.*

Proof. The last statement follows directly from the first. We will show that the multipliers are on the unit circle for $\beta = 1$ and e slightly positive. Since they are not at a Krein collision or period doubling, they cannot move off the unit circle under any small perturbation in the βe-plane. Thus, linear stability is guaranteed for β slightly larger than 1 and e slightly larger than 0.

To prove the theorem, we use a perturbation method to expand the coefficients of the monodromy matrix in powers of the eccentricity e. Since we know the criterion for stability in terms of a and b, we do not need to calculate the eigenvalues explicitly. Moreover, since a and b can be written in terms of the trace of C, we can obtain expansions for a and b by expanding the solution $X(\theta)$ to system (16) in powers of e.

Writing system (16) in powers of e yields

$$\dot{x} = (A_0 + eA_1(\theta) + e^2 A_2(\theta) + \cdots)\,x, \tag{20}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & 0 & -2 \\ 0 & \lambda_2 & 2 & 0 \end{bmatrix}, \qquad A_1(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda_1 \cos\theta & 0 & 0 & 0 \\ 0 & -\lambda_2 \cos\theta & 0 & 0 \end{bmatrix}$$

and

$$A_2(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_1 \cos^2\theta & 0 & 0 & 0 \\ 0 & \lambda_2 \cos^2\theta & 0 & 0 \end{bmatrix}.$$

If we expand a solution to (16) in powers of e, $x(\theta) = x_0(\theta) + ex_1(\theta) + e^2x_2(\theta) + \cdots$, and substitute this into (20), we obtain

$$\begin{aligned} \dot{x_0} &= A_0x_0, \\ \dot{x_1} &= A_0x_1 + A_1(\theta)x_0, \\ \dot{x_2} &= A_0x_2 + A_1(\theta)x_1 + A_2(\theta)x_0 \end{aligned} \tag{21}$$

as equations for each term. Writing the fundamental matrix solution as $X(\theta) = X_0(\theta) + eX_1(\theta) + e^2X_2(\theta) + \cdots$, where $X_0(0) = I_4$, $X_1(0) = 0$, $X_2(0) = 0, \ldots$, and solving the three differential equations above in succession yields the following expansion:

$$X_0(\theta) = e^{A_0\theta}, \qquad X_1(\theta) = e^{A_0\theta}\int_0^\theta e^{-A_0s}A_1(s)e^{A_0s}\,ds$$

and

$$X_2(\theta) = e^{A_0\theta}\int_0^\theta e^{-A_0s}(A_1(s)X_1(s) + A_2(s)e^{A_0s})\,ds.$$

From the formulas for a and b in (18) and using the fact that $X(\theta + 2\pi) = X(\theta)X(2\pi)$, we have

$$a = -\operatorname{tr} X(2\pi) \quad \text{and} \quad b = \frac{1}{2}((\operatorname{tr} X(2\pi))^2 - \operatorname{tr} X(4\pi)).$$

Using these formulas we can expand a and b in powers of e. One important fact is that the trace of $X_1(2\pi)$ and the trace of $X_1(4\pi)$ both vanish. This eliminates the first-order terms in the expansion for a and b, forcing us to consider the X_2 term. To see that these traces vanish, we make use of the fact that $\operatorname{tr}(P_1P_2) = \operatorname{tr}(P_2P_1)$ for any two square matrices P_1, P_2. We have

$$\begin{aligned} \operatorname{tr} X_1(2\pi) &= \int_0^{2\pi} \operatorname{tr}(e^{A_0(2\pi - s)}A_1(s)e^{A_0s})\,ds \\ &= \int_0^{2\pi} \operatorname{tr}(e^{A_02\pi}A_1(s))\,ds \\ &= \operatorname{tr}\left(e^{A_02\pi}\int_0^{2\pi} A_1(s)\,ds\right) \\ &= 0. \end{aligned}$$

An identical argument works to show that the trace of $X_1(4\pi)$ also vanishes.

We therefore have $a = a_0 + e^2 a_2 + \cdots$ and $b = b_0 + e^2 b_2 + \cdots$ with

$$a_0 = -\mathrm{tr}\, X_0(2\pi), \qquad b_0 = \tfrac{1}{2}((\mathrm{tr}\, X_0(2\pi))^2 - \mathrm{tr}\, X_0(4\pi)),$$

$$a_2 = -\mathrm{tr}\, X_2(2\pi), \qquad b_2 = \mathrm{tr}\, X_0(2\pi)\mathrm{tr}\, X_2(2\pi) - \tfrac{1}{2}\,\mathrm{tr}\, X_2(4\pi).$$

When $\beta = 3/4$, we obtain

$$a_0 = -2(\cos \pi + \cos \sqrt{3}\pi) = 0.667738,$$
$$b_0 = 2 + 4 \cos \pi \cos \sqrt{3}\pi = -0.6645237.$$

As expected, since $b = 2a - 2$ at $\beta = 3/4$, $e = 0$, we have $b_0 = 2a_0 - 2$. Finding $X_2(\theta)$ involves double integration since one integral is first needed to calculate $X_1(\theta)$. Using Mathematica, we found that $a_2 = 19.0878$ and $b_2 = -29.6561$. These calculations were checked by numerically integrating the three equations in system (21) and computing the traces directly. Therefore, at $\beta = 3/4$, we obtain

$$b - 2a + 2 = -67.8317e^2 + O(e^3).$$

It follows that $b < 2a - 2$ for e sufficiently small and stability is lost when the eccentricity is made nonzero.

When $\beta = 1$, we have

$$a_0 = -4 \cos \sqrt{2}\pi = 1.065021,$$
$$b_0 = 2 + 4 \cos^2 \sqrt{2}\pi = 2.283568.$$

As expected, since $b = a^2/4 + 2$ at $\beta = 1$, $e = 0$, we have $b_0 = a_0^2/4 + 2$. Again using Mathematica, we found that $a_2 = 22.5806$ and $b_2 = -24.6553$. These calculations were also checked by numerically integrating the three equations in system (21) and computing the traces directly. Therefore, at $\beta = 1$, we have

$$b - a^2/4 - 2 = -152.1262\, e^2 + O(e^3).$$

Thus, $b < a^2/4 + 2$ for e sufficiently small and stability is preserved when the eccentricity is made nonzero. This completes the proof of the theorem. ∎

We visualize the result of the theorem in Fig. 4. Beginning on the dashed curve corresponding to $e = 0$, we fix $\beta = 3/4$ and calculate the values a and b as functions of e. The calculations in the theorem show that the curve initially moves to the right and down, hence out of the stability region. In contrast, when $\beta = 1$, we begin on the Krein collision curve. We then move

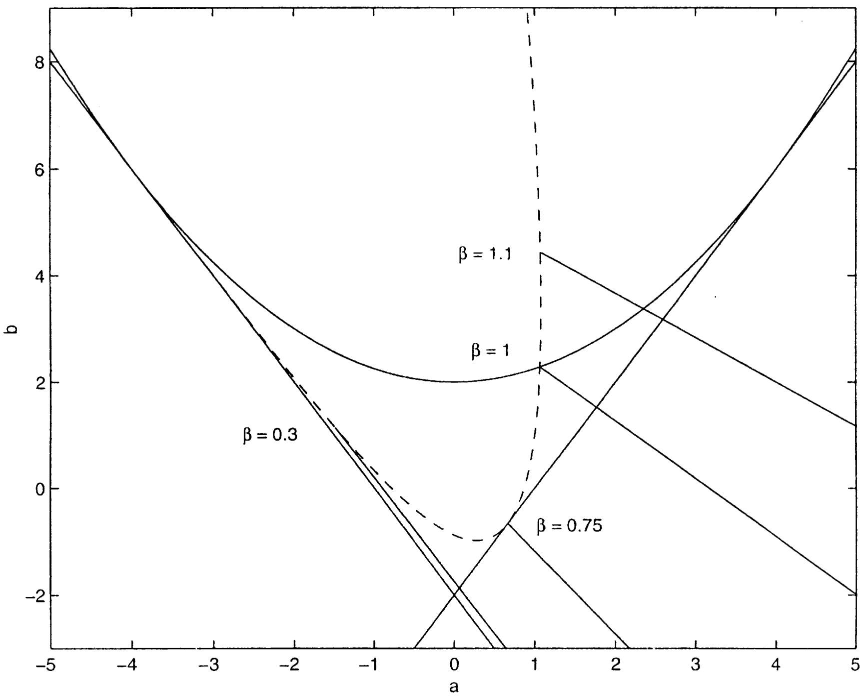

FIG. 4. Examining the multiplier curves for fixed β and increasing eccentricity e in the ab-plane. Remarkably, the curves are almost linear.

to the right and down, but this pushes us into the stability region. Eventually, the curve must cross the period-doubling curve and stability is lost. Note that a similar phenomenon happens for β values larger than 1. Here, however, we begin outside the stability region and as e increases, head towards it. Thus, stability begins with a Krein collision and ends with a period-doubling bifurcation (see Fig. 4).

Remark. (1) Surprisingly, if we fix β and treat e as a variable, then the relationship between $a(e)$ and $b(e)$ appears to be linear in Fig. 4. Denote $F(\beta, e) = (a(\beta, e), b(\beta, e))$ as the map which sends the parameters β and e to the coefficients of the quartic of the monodromy matrix. Then F would send vertical lines in the βe-plane to lines with negative slope in the ab-plane. A careful examination of the data, however, reveals that this is actually not the case although it appears so in the figure.

(2) In Fig. 3, the period-doubling curve does not look smooth at $\beta = 3/4,\ e = 0$. This is correct and can be rigorously justified by considering the

defining function $G(\beta, e) = b(\beta, e) - 2a(\beta, e) + 2$ in the entire βe-plane. (Even though negative eccentricity is physically meaningless, it is perfectly defined in system (16). In fact, as functions of eccentricity, a and b are both symmetric with respect to the β-axis.) The period-doubling curve is given by the level curve $G = 0$ which contains the point $\beta = 3/4,\ e = 0$. From the proof of Theorem 4.1, we know that $\partial G/\partial e$ vanishes along the β-axis. It is a straightforward calculation to show that $\partial G/\partial \beta$ also vanishes at $\beta = 3/4,\ e = 0$. In other words, $(3/4, 0)$ is a critical point of G. Computing the Hessian of G at $(3/4, 0)$ shows it to be a saddle point. It follows that the level curve $G = 0$ is locally an X at $\beta = 3/4,\ e = 0$.

(3) In an attempt to generalize his work on the elliptic restricted problem, Danby studied the linear stability of the elliptic triangle orbits in the full three-body problem in [2]. However, the analysis there is incomplete. Danby states in his abstract that, "this configuration is stable (against infinitesimal displacements) if $(m_1m_2 + m_1m_3 + m_2m_3)/(m_1 + m_2 + m_3)^2$ is less than some quantity that depends only on the eccentricity of the Keplerian orbit". This is incorrect since there are *two* regimes of stability for small values of the eccentricity. When the eccentricity e is zero, we have linear stability for $\beta < 1$. But for $e \in (0, 0.314508)$, we obtain two intervals of stability. This essential dichotomy between the circular case and the elliptic case is overlooked by Danby. This paper completes and corrects his work in the full three-body problem.

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