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Periodic Solutions of the Forced Pendulum: Exchange of Stability and Bifurcations

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We study the T-periodic solutions of the forced pendulum equation $u'' + cu' + a\sin(u) = \lambda f(t)$, where f satisfies $f(t + \frac{T}{2}) = -f(t)$. We prove that this equation always has at least two geometrically distinct T-periodic solutions u_0 and u_1. We then investigate the dependence of the set of T-periodic solutions on the forcing strength λ. We prove that under some restriction on the parameters a, c, the periodic solutions found before can be smoothly parameterized by λ, and that there are some λ-intervals for which $u_0(\lambda)$, $u_1(\lambda)$ are the only T-periodic solutions up to geometrical equivalence, but there are other λ-intervals in which additional T-periodic solutions bifurcate off the branches $u_0(\lambda)$, $u_1(\lambda)$. We characterize the global structure of the bifurcating branches. Related to this bifurcation phenomenon is the phenomenon of 'exchange of stability' – in some λ-intervals $u_0(\lambda)$ is dynamically stable and $u_1(\lambda)$ is unstable, while in other λ-intervals the reverse is true, a phenomenon which has important consequences for the dynamics of the forced pendulum, as we show by both theoretical analysis and numerical simulation.

Key Words: forced pendulum; periodic solutions; bifurcation; stability.

1. INTRODUCTION

In this work, we investigate the periodic solutions of the forced pendulum equation with damping ($c \geqslant 0$)

$$u'' + cu' + a\sin(u) = \lambda f(t), \tag{1}$$

where $f\colon \mathbf{R} \to \mathbf{R}$ is continuous and T-periodic. By periodic solutions of (1), we mean solutions satisfying

$$u(t + T) = u(t) \qquad \text{for all } t \in \mathbf{R}.$$

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Many studies have been devoted to the periodic solutions of (1) (see surveys [8–11]), with various methods contributing to our (still partial) understanding.

Here we study the periodic solutions of (1) when the forcing f is $\frac{T}{2}$-antiperiodic, that is

$$f\left(t+\frac{T}{2}\right) = -f(t) \qquad \text{for all} \qquad t \in \mathbf{R}. \tag{2}$$

For example, the important special case $f(t) = \sin(\frac{2\pi}{T}t)$ satisfies (2). We shall see that under assumption (2) the problem exhibits some interesting properties, including a striking bifurcation phenomenon, which are not present for the case of general f.

Our main aim is to understand how the set of periodic solutions varies as the parameter λ, measuring the strength of the forcing, varies in some interval $I \subset \mathbf{R}$, all other parameters remaining fixed. We use several methods of nonlinear analysis to piece together a description which we unfold in the following sections.

A summary of our main results will be presented after we introduce some notations that will serve us throughout this study.

We denote by Y the space of T-periodic functions $x : \mathbf{R} \to \mathbf{R}$ which are square-integrable on bounded intervals, with the norm

$$\|x\|_Y = \left(\frac{1}{T}\int_0^T (x(t))^2\,dt\right)^{\frac{1}{2}}$$

and by X the following subspace of Y:

$$X = \{x \in Y \mid x|_{[0,T]} \in H^2[0,T]\}.$$

If we assume that f is continuous, any solution $u \in X$ of (1) is, in fact, C^2 and T-periodic, so it is a classical periodic solution of (1).

We are mainly interested in the dependence of the set of periodic solutions on λ, so we denote

$$\Sigma = \Sigma(a) = \{(\lambda, u) \in \mathbf{R} \times X \mid u \text{ satisfies (1)}\}$$

(of course, Σ also depends on c and f, but we suppress this dependence, assuming that c and f are fixed). Many of our results may be considered as descriptions of the set Σ. Of course, we cannot plot Σ, which is a subset of an infinite-dimensional space, so when we wish to visualize our results, we shall plot the set $\tilde{\Sigma} \subset \mathbf{R}^2$ (the 'bifurcation diagram') defined by

$$\tilde{\Sigma} = \tilde{\Sigma}(a) = \{(\lambda, \|u\|_Y) \mid (\lambda, u) \in \Sigma\}. \tag{3}$$

We note that if u is a periodic solution of (1) then so is $u + 2\pi k$ for any $k \in \mathbf{Z}$. We shall say that two periodic solutions of (1) are *geometrically equivalent* if they differ by $2\pi k$ and *geometrically distinct* otherwise. It is often convenient to identify geometrically equivalent periodic solutions, since they correspond to the same physical motion.

We define

$$\omega = \frac{2\pi}{T}.$$

We now describe the picture that will emerge from the investigation to be carried out in the following sections.

In Section 2, we show that when f satisfies (2) there always exist at least two geometrically distinct solutions of (1) (this contrasts with the case of general f with zero mean value, as will be discussed). In particular, it seems that our result provides the first proof that the equation

$$u'' + cu' + a\sin(u) = \lambda \sin\left(\frac{2\pi}{T}t\right)$$

has at least two T-periodic solutions for *any* value of the parameters c, a, λ, and T.

We also prove that if a satisfies

$$0 < a < \omega\sqrt{\omega^2 + c^2}, \tag{4}$$

then there are two smooth branches of solutions $u_0(\lambda), u_1(\lambda)$ such that, for each $\lambda \in \mathbf{R}$, $u_0(\lambda)$ and $u_1(\lambda)$ are geometrically distinct periodic solutions of (1) for the corresponding λ. Of course together with $u_0(\lambda)$ and $u_1(\lambda)$, we have the periodic solutions which are geometrically equivalent to them, given by

$$u_{2j}(\lambda) = u_0(\lambda) + 2j\pi, \qquad u_{2j+1}(\lambda) = u_1(\lambda) + 2j\pi, \qquad j \in \mathbf{Z}.$$

The curves $\mathscr{S}_k \subset \mathbf{R} \times X$ defined by $\mathscr{S}_k = \{(\lambda, u_k(\lambda)) \mid \lambda \in \mathbf{R}\}$ $(k \in \mathbf{Z})$ are thus subsets of Σ, and will be called the curves of symmetric solutions (for a reason which will be explained). We will see that $u_k(0) \equiv k\pi$, so that the symmetric solutions $u_k(\lambda)$ may be considered as the continuations of the equilibrium solutions in the unforced case $(\lambda = 0)$. Moreover, since condition (4) implies that in the unforced case the only T-periodic solutions are the equilibria (in the case $c > 0$ this is trivial; in the case $c = 0$, (4) is equivalent to $0 < a < \omega^2$, while the condition for nonexistence of nonconstant T-periodic solutions is $|a| \leqslant \omega^2$), a naive intuition might suggest that when (4) holds the solutions $u_k(\lambda)$ will be the only periodic solutions of (1). This, however, is not the case. While it is true, as we show in Section 2, that there are λ-intervals for which $u_k(\lambda)$ are the only periodic solutions, we show in

Section 5 that in other λ-intervals more periodic solutions bifurcate off the curves $\mathscr{S}_k$ of symmetric solutions. This bifurcation phenomenon is related to another phenomenon which we call the 'exchange of stability', which is the subject of Section 4. Studying the dynamical stability of the symmetric periodic solutions $u_k(\lambda)$ (in the case $c>0$), we discover that, while for λ sufficiently close to 0, $u_k(\lambda)$ is stable if k is even and unstable if k is odd (which is not surprising if we recall that $u_k(\lambda)$ is the continuation of the equilibrium $u_k(0) \equiv k\pi$), for larger values of λ the solution curves which were stable become unstable and vice versa. This 'exchange of stability' occurs repeatedly as λ increases. This phenomenon has striking dynamic consequences, as we show by a numerical simulation of the forced pendulum in Section 4.

In Sections 6 and 7, we make a finer analysis of the phenomena described above, this time under the assumption that $a>0$ is *sufficiently small* (or in physical terms, that the pendulum is *sufficiently long*). Under this assumption, we can obtain a complete description of the set of all periodic solutions. To state these further results more precisely, we shall need to fix some $\tilde{\lambda}$ (which may be arbitrarily large) and assume that $\lambda \in I = [0, \tilde{\lambda}]$. Then, as we shall show, there exists $a_0 > 0$ (depending on $\tilde{\lambda}$) such that when $0<a<a_0$, we can give a complete description of the set $\Sigma \cap (I \times X)$ as follows (see Fig. 7):

(1) There exists a finite set of disjoint subintervals $I_i = I_i(a)$ $(1 \leqslant i \leqslant n)$ of I such that whenever $\lambda \in I$ and $\lambda \notin \bigcup_{i=1}^n I_i$, the symmetric solutions $u_k(\lambda)$ $(k \in \mathbf{Z})$ are the *only* periodic solutions of (1) (so that there are precisely two geometrically distinct solutions). When a tends to 0, the lengths of the intervals $I_i(a)$ go to 0, and they concentrate near zeroes of an explicitly given function.

(2) For each $1 \leqslant i \leqslant n$, we have a curve $\mathscr{C}_i \subset I_i \times X$, homeomorphic to the real line, with $\mathscr{C}_i \subset \Sigma$, such that $\mathscr{C}_i$ intersects each of the curves $\mathscr{S}_k$ exactly once. The point of intersection of $\mathscr{C}_i$ and $\mathscr{S}_k$ is a point of bifurcation from the curve $\mathscr{S}_k$ and it is the unique such point for $\lambda \in I_i$. All points of $\Sigma \cap (I_i \times X)$ are either on $\mathscr{C}_i$ or on one of the $\mathscr{S}_k$'s.

(3) An 'exchange of stability' occurs at every point of bifurcation from $\mathscr{S}_k$. Thus, if we look at the intervals forming the complement of $\bigcup_{i=1}^n I_i$ in I, then in each of them $u_0(\lambda), u_1(\lambda)$ are the only geometrically distinct solutions, and their stability properties alternate from one interval to another.

The phenomena of exchange of stability and bifurcation were observed by Schmitt and Sari [20] in their numerical study of the periodic solutions of the forced pendulum in the case $c = 0$ and $f(t) = \cos(\omega t)$. Our rigorous results provide an explanation of these phenomena, give a formula for the location of the bifurcation points in the limit of a 'long' pendulum, and show that these phenomena occur also for $c > 0$ and whenever the forcing term f satisfies (2) (we note that pure harmonic forcings, as considered in [20], have additional symmetry besides (2), which was essential for the numerical method used there, as was the assumption that $c = 0$).

2. EXISTENCE OF AT LEAST TWO PERIODIC SOLUTIONS

To introduce our first result, we recall the result of Mawhin and Willem [12], who proved that when $c = 0$, and when f satisfies

$$\int_0^T f(t)\,dt = 0, \tag{5}$$

Eq. (1) has at least two geometrically distinct periodic solutions. Note that our assumption (2) implies (5). For some years it was an open question whether the Mawhin–Willem result is true when $c > 0$, but Ortega provided a counterexample ([14], and see also [2, 17]). We now show that if we replace (5) by the stronger condition (2), the Mawhin–Willem result remains true for $c > 0$ (of course, the method of proof is quite different—topological rather than variational). The existence of a periodic solution when f satisfies (2) was already proved by Mawhin [9, p. 127], so our contribution in Theorem 1 is only the fact that a second solution exists. We repeat the entire (simple) argument, however, since it is also used in Theorem 2, which is fundamental for the whole development that follows.

THEOREM 1. *If f satisfies* (2) *then* (1) *has at least two geometrically distinct periodic solutions. In fact, it has a periodic solution u_0, which is $\frac{T}{2}$-antiperiodic, and also a periodic solution u_1 such that $u_1 - \pi$* is *$\frac{T}{2}$-antiperiodic.*

To prove Theorem 1, we first set up an appropriate functional–analytic framework.

We define $L : X \to Y$ (see the previous section for the definitions of X, Y) to be the linear operator

$$L(x) = x'' + cx'.$$

Assuming that f satisfies (5), we denote by x_0 the unique periodic solution of the linear equation

$$x_0'' + cx_0' = f(t)$$

satisfying

$$\int_0^T x_0(t)\,dt = 0.$$

For example, in the case $f(t) = \sin(\omega t)$, we obtain

$$x_0(t) = \frac{1}{\omega\sqrt{c^2 + \omega^2}} \sin(\omega t - \beta), \tag{6}$$

where β is defined by

$$\cos(\beta) = -\frac{\omega}{\sqrt{c^2 + \omega^2}}, \qquad \sin(\beta) = \frac{c}{\sqrt{c^2 + \omega^2}}.$$

We define $N\colon \mathbf{R} \times Y \to Y$ to be the nonlinear operator

$$N(\lambda, x)(t) = \sin(\lambda x_0(t) + x).$$

With this notation, we have that $u \in X$ is a periodic solution of (1) if and only if $u = \lambda x_0 + x$, where x satisfies the equation

$$L(x) + aN(\lambda, x) = 0, \qquad x \in X. \tag{7}$$

The study of periodic solutions of (1) is thus equivalent to the study of solutions of Eq. (7).

We denote by Y_* the subspace of Y consisting of $\frac{T}{2}$-antiperiodic functions:

$$Y_* = \left\{ x \in Y \,|\, x\left(t + \frac{T}{2}\right) = -x(t) \qquad \text{for all } t \in \mathbf{R} \right\} \tag{8}$$

and $X_* = Y_* \cap X$.

We note that $L(X_*) \subset Y_*$, so the restriction $L|_{X_*}$ is a linear operator from X_* to Y_*. Moreover, $L|_{X_*}$ is a Fredholm operator of index 0, and the kernel of $L|_{X_*}$ is trivial, hence $L|_{X_*}$ is an isomorphism from X_* to Y_*, so $[L|_{X_*}]^{-1}\colon Y_* \to X_*$ is well defined. Since $X_* \subset Y_*$, we will also view $[L|_{X_*}]^{-1}$ as an operator from Y_* to itself, and we note that as an operator from Y_* to itself $[L|_{X_*}]^{-1}$ is *compact*.

We now wish to consider Eq. (7), restricted to the subspace X_*. If this restriction is made, we may rewrite the equation as

$$x = -a[L|_{X_*}]^{-1}(N(\lambda, x)), \qquad x \in X_*. \tag{9}$$

We note now that by the oddness of the sine function, and by the fact that

$$(2) \quad \text{implies that } x_0 \in X_*, \tag{10}$$

we have that $x \in Y_*$ implies $N(\lambda, x) \in Y_*$. Therefore, we may define a nonlinear operator from Y_* to itself by

$$H_\lambda(x) = -a[L|_{X_*}]^{-1}(N(\lambda, x)),$$

so that (9) is equivalent to

$$H_\lambda(x) = x, \qquad x \in Y_* \tag{11}$$

(note that any solution $x \in Y_*$ of (11) is automatically in X_*, so that (9) and (11) are indeed equivalent). Moreover, by the compactness of $[L|_{X_*}]^{-1} : Y_* \to Y_*$, H_λ is compact. We also note that since

$$\|N(\lambda, x)\|_Y \leqslant 1 \qquad \text{for all} \qquad \lambda \in \mathbf{R}, \qquad x \in Y \tag{12}$$

we have the bound

$$\|H_\lambda(x)\|_{Y_*} \leqslant a\|[L|_{X_*}]^{-1}\|_{Y_*, Y_*} \qquad \text{for all} \qquad x \in Y_*. \tag{13}$$

We may, therefore, apply the Schauder fixed point theorem in a closed ball of radius $r = a\|[L|_{X_*}]^{-1}\|_{Y_*, Y_*}$ around the origin in Y_* to conclude that (11) has a solution $x \in Y_*$ (which is in fact in X_*), so (7) has a solution $x \in X_*$. Defining $u_0 = \lambda x_0 + x$, we have that u_0 is a $\frac{T}{2}$-antiperiodic solution of (1).

In order to prove the existence of a second periodic solution, we consider the equation

$$x = -H_\lambda(x), \qquad x \in Y_*. \tag{14}$$

Applying the Schauder fixed point theorem as before, we obtain the existence of a solution $x \in X_*$. We now define $\bar{x} = x + \pi$. We claim that $\bar{x}$ is a solution of (7). Indeed,

$$\begin{aligned} L(\bar{x}) + aN(\lambda, \bar{x}) &= L(x) - aN(\lambda, x) = L(x - a[L|_{X_*}]^{-1} \circ N(\lambda, x)) \\ &= L(x + H_\lambda(x)) = 0. \end{aligned}$$

Therefore, $u_1 = \lambda x_0 + \bar{x}$ is a periodic solution of (1), and $u_1 - \pi = \lambda x_0 + x$ is $\frac{T}{2}$-antiperiodic.

We note that u_0 and u_1 are indeed geometrically distinct, since if we had $u_0 - u_1 = 2\pi k$ for some integer k then it would be impossible for both u_0 and $u_1 - \pi$ to be $\frac{T}{2}$-antiperiodic. We have thus proven Theorem 1.

The Schauder fixed point theorem does not guarantee uniqueness of the fixed point, but if we assume that a is sufficiently small then we may show that H_λ is a contraction (for any λ), so we may apply the Banach fixed point theorem to (11) and (14) in order to conclude that each of them has a *unique* solution. Moreover, in this case we can apply the analytic implicit function theorem to conclude that these solutions depend analytically on the parameters a and λ. To obtain the conditions under which H_λ is a contraction, we shall need an upper estimate for $\|[L|_{X_*}]^{-1}\|_{Y_*,Y_*}$, which will follow from the following inequality, which can be found in [5].

LEMMA 1. *If $x \in X$ satisfies $\int_0^T x(t)\,dt = 0$ and $y = L(x)$, then*

$$\|x\|_Y \leqslant \frac{1}{\omega\sqrt{\omega^2 + c^2}}\|y\|_Y. \tag{15}$$

From Lemma 1, we conclude that

$$\|[L|_{X_*}]^{-1}\|_{Y_*,Y_*} \leqslant \sigma = \sigma(T, c) = \frac{1}{\omega\sqrt{\omega^2 + c^2}}. \tag{16}$$

For all $x, y \in Y$, we have

$$\begin{aligned}\|N(\lambda, x) - N(\lambda, y)\|_Y &= \left(\frac{1}{T}\int_0^T |\sin(\lambda x_0(t) + x(t)) - \sin(\lambda x_0(t) + y(t))|^2\,dt\right)^{\frac{1}{2}} \\ &\leqslant \left(\frac{1}{T}\int_0^T |x(t) - y(t)|^2\right)^{\frac{1}{2}} = \|x - y\|_Y. \end{aligned} \tag{17}$$

Using (16) and (17) we conclude that, for all $x, y \in Y_*$,

$$\|H_\lambda(x) - H_\lambda(y)\|_{Y_*} \leqslant a\sigma\|x - y\|_{Y_*}.$$

Therefore, the condition $a\sigma < 1$ implies that H_λ is a contraction, so we obtain

THEOREM 2. *Assume f satisfies* (2). *If a satisfies*

$$0 \leqslant a < \frac{1}{\sigma} = \omega\sqrt{\omega^2 + c^2}, \tag{18}$$

then, for any $\lambda \in \mathbf{R}$, there exists a unique periodic solution $u_0 = u_0(a, \lambda)$ of (1) *which is also $\frac{T}{2}$-antiperiodic, and a unique periodic solution $u_1 = u_1(a, \lambda)$ of* (1) *such that $u_1 - \pi$ is $\frac{T}{2}$-antiperiodic. The mappings $u_k{:}[0, \frac{1}{\sigma}) \times \mathbf{R} \to X$ $(k = 0, 1)$ are real analytic.*

In the above theorem, we interpreted restriction (18) as a smallness condition on a, and since a is inversely proportional to the length of the pendulum, we may say that the theorem holds for a sufficiently long pendulum. It would be equally possible to regard (18) as a condition on c, ensuring that the damping is sufficiently large, or as a condition on T, ensuring that the frequency of the forcing be sufficiently large. Later in this work, we shall meet conditions similar to (18), and we shall continue to regard them as smallness conditions on a.

We note that using (9) and the fact that $[L|_{X_*}]^{-1}$ is bounded as an operator from Y_* to the space C^0 of continuous T-periodic functions with the maximum norm, we have

$$||u_0(a,\lambda) - \lambda x_0||_{C^0} = ||a[L|_{X_*}]^{-1} \circ N(\lambda, u_0(a,\lambda) - \lambda x_0)||_{C^0} \leqslant a||[L|_{X_*}]^{-1}||_{Y_*,C^0},$$

$$\begin{aligned} ||u_1(a,\lambda) - \lambda x_0 - \pi||_{C^0} &= ||a[L|_{X_*}]^{-1} \circ N(\lambda, u_1(a,\lambda) - \lambda x_0 - \pi)||_{C^0} \\ &\leqslant a||[L|_{X_*}]^{-1}||_{Y_*,C^0}, \end{aligned}$$

so that

$$||u_0(a,\lambda) - \lambda x_0||_{C^0} = O(a) \qquad \text{as } a \to 0, \tag{19}$$

$$||u_1(a,\lambda) - \lambda x_0 - \pi||_{C^0} = O(a) \qquad \text{as } a \to 0 \tag{20}$$

uniformly in $\lambda \in \mathbf{R}$, and in particular $u_0(0,\lambda) = \lambda x_0$, $u_1(0,\lambda) = \lambda x_0 + \pi$. We note also that since when $\lambda = 0$ and a satisfies (18) the only T-periodic solutions of (1) are $u \equiv 0$ and $u \equiv \pi$, we have $u_0(a,0) \equiv 0$, $u_1(a,0) \equiv \pi$ for all a satisfying (18).

We introduce the following notation for the periodic solutions that are geometrically equivalent to $u_0(a,\lambda)$ and $u_1(a,\lambda)$:

$$u_k(a,\lambda) = \begin{cases} u_0(a,\lambda) + \pi k & k \in \mathbf{Z} \text{ even}, \\ u_1(a,\lambda) + \pi(k-1) & k \in \mathbf{Z} \text{ odd}. \end{cases}$$

Before proceeding, we would like to remark on a certain symmetry of Eq. (1) under assumption (2). We define the linear mapping $\mathscr{T}: Y \to Y$ by

$$\mathscr{T}(x)(t) = -x\left(t + \frac{T}{2}\right). \tag{21}$$

It is easy to check that L and $N(\lambda,\cdot)$ both commute with $\mathscr{T}$ (for $N(\lambda,\cdot)$ this follows from the fact that $x_0 \in X_*$, which follows by (10)). This implies that if u is a periodic solution of (1) then so is $\mathscr{T}(u)$.

Let us call a periodic solution u of (1) *symmetric* if $\mathscr{T}(u)$ and u are geometrically equivalent. We note that $\mathscr{T}(u_k(a,\lambda)) = u_k(a,\lambda)$ while $\mathscr{T}(u_1(a,\lambda)) = u_1(a,\lambda) - 2\pi$, so all the $u_k(a,\lambda)$ $(k \in \mathbf{Z})$ are symmetric in the above sense. We would like to show that, when (18) holds, $u_0(a,\lambda)$ and $u_1(a,\lambda)$ are the *only* symmetric periodic solutions up to geometric equivalence. Indeed, assume u is a symmetric periodic solution so that

$$\mathscr{T}(u) - u = 2\pi k$$

for some integer k. If k is even then $\bar{u} = u + \pi k$ is a periodic solution satisfying

$$\mathscr{T}(\bar{u}) = \mathscr{T}(u) - \pi k = u + 2\pi k - \pi k = \bar{u},$$

that is, $\bar{u}$ is $\frac{T}{2}$-antiperiodic, hence, by Theorem 2, $\bar{u} = u_0(a, \lambda)$, so u is geometrically equivalent to $u_0(a, \lambda)$. Similarly, if k is odd then $\bar{u} = u + (k+1)\pi$ is a periodic solution satisfying $\mathscr{T}(\bar{u} - \pi) = \bar{u} - \pi$, so $\bar{u} - \pi$ is $\frac{T}{2}$-antiperiodic, hence, by Theorem 2, $\bar{u} = u_1(a, \lambda)$, so $u = u_1(a, \lambda) - (k+1)\pi$ is geometrically equivalent to u_1. We have therefore proved

THEOREM 3. *Assume f satisfies* (2) *and a satisfies* (18). *Then the only symmetric periodic solutions of* (1) *are $u_k(a, \lambda)$* ($k \in \mathbf{Z}$).

Since in our further investigations we shall assume that (18) holds, we can refer to $u_k(a, \lambda)$ ($k \in \mathbf{Z}$) as *the symmetric periodic solutions*. We note that $u_0(a, \lambda), u_1(a, \lambda)$ are also symmetric in the geometrical sense that $u_0(a, \lambda)$ is symmetric about the downward position of the pendulum, and $u_1(\lambda)$ is symmetric about the vertical position.

Condition (18) appearing in Theorems 2 and 3 is not merely an artifact of our method of proof: indeed when $c = 0$, $f \equiv 0$, and $a > \omega^2$, there is a nonconstant periodic solution of (1) which is $\frac{T}{2}$-antiperiodic, so the conclusions of these theorems fail to hold. However, some information may be obtained for *arbitrary a* and for *sufficiently large* values of λ. Indeed, by using the method which was developed in [7] one may prove the following: assuming the set of critical points of x_0 is of measure 0 and denoting by C^1_{ap} the space of functions in Y_* which are also C^1 with the norm $\|x\|_{C^1_{ap}} = \max_{t \in \mathbf{R}} |x'(t)|$ (note that C^1_{ap} is invariant under H_λ), we have that for any $r > 0$,

$$\lim_{\lambda \to \infty} \sup_{\|x\|_{C^1_{ap}} \leqslant r} \|DH_\lambda(x)|_{C^1_{ap}}\|_{C^1_{ap}, C^1_{ap}} = 0. \tag{22}$$

Since the range of $H_\lambda|_{C^1_{ap}}$ is bounded in C^1_{ap}, we may choose $r_0 > 0$ such that the ball of radius r_0 around the origin in C^1_{ap} is invariant under H_λ (for all λ), and all solutions of (11) and (14) are in this ball. Equation (22) implies that for sufficiently large λ, $H_\lambda|_{C^1_{ap}}$ is a contraction in this ball, so that we may conclude that the result of Theorem 2 holds. We thus have

THEOREM 4. *Assume f satisfies* (2) *and the set of critical points of* x_0 *has measure* 0 (*which happens, for example, if f is real analytic and not identically* 0). *Then for any* $a > 0$ *there exists a* $\lambda_c(a) \geqslant 0$ *such that when* $\lambda \geqslant \lambda_c(a)$ *there exists a unique periodic solution* $u_0 = u_0(a, \lambda)$ *of* (1) *which is also* $\frac{T}{2}$*-antiperiodic, and a unique periodic solution* $u_1 = u_1(a, \lambda)$ *of* (1) *such that* $u_1 - \pi$ *is* $\frac{T}{2}$*-antiperiodic.*

Theorem 4 will not be used in our further investigations, and we will henceforth assume that (18) does hold so that we have precisely two geometrically distinct symmetric solutions for *all* values of λ.

3. λ-INTERVALS FOR WHICH THERE EXIST PRECISELY TWO GEOMETRICALLY DISTINCT PERIODIC SOLUTIONS

It is important to note that Theorem 3 does not imply that $u_k(a, \lambda)$ $(k \in \mathbf{Z})$ are the only periodic solutions of (1), but merely that they are the only *symmetric* ones. Indeed, we shall later see that as λ varies, more periodic solutions of (1) bifurcate off the curves of symmetric solutions that we have found. However, as we shall now show, if we fix a compact interval $I \subset \mathbf{R}$, then as a approaches 0 the set of values of $\lambda \in I$ for which there exist other periodic solutions of (1) besides the two we have found becomes smaller and smaller, and concentrates near a discrete set of values of λ. This discrete set is the set of zeroes of the function h: $\mathbf{R} \to \mathbf{R}$, defined by

$$h(\lambda) = \frac{1}{T} \int_0^T \cos(\lambda x_0(t))\, dt.$$

We denote the set of zeroes of h by $\mathscr{Z}$. Since h is real analytic and not identically zero ($h(0) = 1$), the set $\mathscr{Z}$ is discrete. We note also that it may be shown that for generic forcing f $\mathscr{Z}$ is infinite (see the arguments in [4, p. 182]).

We note that in the case $f(t) = \sin(\omega t)$ we get, using (6),

$$h(\lambda) = \int_0^T \cos\left(\frac{\lambda}{\omega\sqrt{c^2 + \omega^2}} \sin(\omega t - \beta)\right) dt = J_0\left(\frac{\lambda}{\omega\sqrt{c^2 + \omega^2}}\right), \quad (23)$$

where J_0 is the 0th-order Bessel function of the first kind.

Theorem 5 below shows the significance of the function h for our problem.

THEOREM 5. *Assume f satisfies* (2), $a > 0$ *satisfies* (18) *and* λ *satisfies*

$$|h(\lambda)| > a\sigma\left[1 + \left(\frac{2 - a\sigma}{1 - a\sigma}\right)^2\right]^{\frac{1}{2}}. \tag{24}$$

Then (1) *has precisely two geometrically distinct periodic solutions, the solutions* $u_0(a,\lambda), u_1(a,\lambda)$ *given by Theorem* 2. *These solutions are nondegenerate.*

The nondegeneracy statement above means that for all $k \in \mathbf{Z}$, the linearized equation

$$v'' + cv' + a\cos(u_k(a,\lambda))v = 0$$

has no nontrivial periodic solutions.

To understand the implications of Theorem 5 let us define, for each $a > 0$, the 'good' set

$$\mathscr{G}_a = \left\{\lambda \in \mathbf{R} \mid |h(\lambda)| > a\sigma\left[1 + \left(\frac{2 - a\sigma}{1 - a\sigma}\right)^2\right]^{\frac{1}{2}}\right\}.$$

Theorem 5 tells us that if $a > 0$ satisfies (18), then whenever $\lambda \in \mathscr{G}_a$, the only periodic solutions of (1) are the symmetric ones $u_k(a,\lambda)$ ($k \in \mathbf{Z}$). We note that if we fix a compact interval $I \subset \mathbf{R}$ satisfying $I \cap \mathscr{Z} = \emptyset$, then for $a > 0$ sufficiently small we will have $I \subset \mathscr{G}_a$ so that (1) has precisely two geometrically distinct solutions for all $\lambda \in I$. More generally, if we fix *any* compact interval $I \subset \mathbf{R}$, then $\mathscr{G}_a \cap I$ is the union of a finite number of intervals, the sum of whose lengths tends to the length of I as a tends to 0, that is, we have

$$\lim_{a\to 0} \text{meas}(\mathscr{G}_a \cap I) = \text{meas}(I)$$

and note also that

$$a' < a \quad \text{implies} \quad \mathscr{G}_a \subset \mathscr{G}_{a'}, \tag{25}$$

which follows from the fact that the right-hand side of (24) is monotonically increasing with respect to a. We also have $\bigcup_{a>0} \mathscr{G}_a = \mathbf{R} - \mathscr{Z}$. It is to be noted, however, that since we have $\lim_{|\lambda|\to\infty} h(\lambda) = 0$, the set $\mathscr{G}_a$ is always bounded for each $a > 0$. Thus, if we restrict λ to a *finite* interval, we may say that, when $a > 0$ is sufficiently small, then, for most values of λ, (1) has precisely two geometrically distinct periodic solutions.

Theorem 5 will follow from the following more general theorem which is valid for any forcing f, not necessarily satisfying (2).

THEOREM 6. *Assume f satisfies* (5) *and $a > 0$ satisfies* (18). *Define*

$$\bar{h}(\lambda) = \frac{1}{T}\left[\left(\int_0^T \cos(\lambda x_0(t))\,dt\right)^2 + \left(\int_0^T \sin(\lambda x_0(t))\,dt\right)^2\right]^{\frac{1}{2}}$$

and assume that λ satisfies

$$\bar{h}(\lambda) > a\sigma\left[1 + \left(\frac{2 - a\sigma}{1 - a\sigma}\right)^2\right]^{\frac{1}{2}}. \tag{26}$$

Then (1) *has precisely two geometrically distinct periodic solutions. These solutions are nondegenerate.*

To see that Theorem 5 follows from Theorem 6, note that if f satisfies (2) then by (10) $x_0 \in X_*$, from which it follows that the second integral in the definition of $\bar{h}$ vanishes, hence we conclude that $\bar{h} = |h|$.

The quantity $\bar{h}(\lambda)$ already appears in the work of Fournier and Mawhin [5]. Theorem 3 of that paper implies that, for f satisfying (5), the condition

$$\bar{h}(\lambda) \geqslant a\sigma \tag{27}$$

is sufficient to ensure that there exist *at least* two geometrically distinct periodic solutions of (1). Here we see that by somewhat strengthening (27), we obtain the existence of *precisely* two periodic solutions. It would be of interest to know whether some strengthening of (27) is really necessary in order to obtain the existence of precisely two periodic solutions, or whether, on the contrary, one may replace (26) by (27) in the statement of Theorem 6.

Conditions ensuring that the number of geometrically distinct periodic solutions of (1) is precisely two have already been obtained by Tarantello [21] (see also [3]), but the conditions obtained here are different in that here we do not impose any restriction on the norm of f.

To prove Theorem 6, we use a Lyapunov–Schmidt reduction. The kernel of L is the set of constant functions, which we identify with $\mathbf{R}$. We denote by $\tilde{Y}$ the subspace of Y consisting of functions $x \in Y$ that satisfy

$$\int_0^T x(t)\,dt = 0$$

and we denote $\tilde{X} = X \cap \tilde{Y}$. We have $X = \mathbf{R} \oplus \tilde{X}$, $Y = \mathbf{R} \oplus \tilde{Y}$. The range of L is equal to $\tilde{Y}$. We denote by $Q{:}Y \to Y$ the orthogonal projection onto

$\tilde{Y}$ given by

$$Q(x) = x - \frac{1}{T}\int_0^T x(t)\,dt.$$

Setting $x = s + \tilde{x}$ with $s \in \mathbf{R}$, $\tilde{x} \in \tilde{X}$, we see that, for $a > 0$, (7) is equivalent to the pair of equations

$$L(\tilde{x}) + aQ(N(\lambda, s + \tilde{x})) = 0, \tag{28}$$

$$(I - Q)(N(\lambda, s + \tilde{x})) = 0. \tag{29}$$

Since $L|_{\tilde{X}}$ is invertible, (28) may be rewritten as

$$\tilde{x} = -a[L|_{\tilde{X}}]^{-1} \circ Q(N(\lambda, s + \tilde{x})). \tag{30}$$

Fixing $s \in \mathbf{R}$, the right-hand side of (30) defines a nonlinear mapping from $\tilde{Y}$ to itself, which we want to show is a contraction.

We shall need

Lemma 2.

$$\|[L|_{\tilde{X}}]^{-1} \circ Q\|_{Y,Y} \leqslant \sigma(T, c). \tag{31}$$

Proof. Assume $y \in Y$, $\tilde{x} = [L|_{\tilde{X}}]^{-1} \circ Q(y)$. Then $Q(y) = L(\tilde{x})$, so by (15)

$$\|\tilde{x}\|_Y \leqslant \sigma \|Q(y)\|_Y.$$

Clearly $\|Q(y)\|_Y \leqslant \|y\|_Y$, so

$$\|[L|_{\tilde{X}}]^{-1} \circ Q(y)\|_Y = \|\tilde{x}\|_Y \leqslant \sigma \|Q(y)\|_Y \leqslant \sigma \|y\|_Y.$$

This gives the desired result. ∎

Using (31) and (17), we have

$$\|a[L|_{\tilde{X}}]^{-1} \circ Q(N(\lambda, s + \tilde{x})) - a[L|_{\tilde{X}}]^{-1} \circ Q(N(\lambda, s + \tilde{y}))\|_{\tilde{Y}} \leqslant a\sigma \|\tilde{x} - \tilde{y}\|_{\tilde{Y}}$$

for all $\tilde{x}, \tilde{y} \in \tilde{Y}$. Therefore, if $|a| < \frac{1}{\sigma}$, the mapping defined by the right-hand side of (30) is a contraction in $\tilde{Y}$ (for any $(\lambda, s) \in \mathbf{R} \times \mathbf{R}$). This implies that (30), considered as an equation for $\tilde{x}$, has a unique solution $\tilde{x} = S(\lambda, a, s) \in \tilde{X}$, and S is real analytic in all three variables. S: $\mathbf{R} \times (-\frac{1}{\sigma}, \frac{1}{\sigma}) \times \mathbf{R} \to \tilde{X}$ is a mapping satisfying

$$S(\lambda, a, s) = -a[L|_{\tilde{X}}]^{-1} \circ Q[N(\lambda, s + S(\lambda, a, s))]. \tag{32}$$

Moreover, using (12) and (31), we have

$$\begin{aligned}\|S(\lambda, a, s)\|_Y &= \| - a[L|_{\tilde{X}}]^{-1} \circ Q[N(\lambda, s + S(\lambda, a, s))]\|_Y \\ &\leq |a| \, \|[L|_{\tilde{X}}]^{-1} \circ Q\|_{\tilde{Y},\tilde{Y}} \leqslant |a|\sigma \qquad (33)\end{aligned}$$

for all $(\lambda, a, s) \in \mathbf{R} \times (-\frac{1}{\sigma}, \frac{1}{\sigma}) \times \mathbf{R}$.

Substituting $\tilde{x} = S(\lambda, a, s)$ into (29), we obtain that, when $0 < |a| < \frac{1}{\sigma}$, (7) is equivalent to

$$B(\lambda, a, s) = (I - Q)[N(\lambda, s + S(\lambda, a, s))] = 0, \qquad s \in \mathbf{R} \qquad (34)$$

in the sense that there is a one-to-one correspondence between solutions s of (34) and solutions x of (7), given by $s \to s + S(\lambda, a, s)$. It should be noted that S is 2π-periodic with respect to s. Indeed, using (32) and the fact that $N(\lambda, x + 2\pi) = N(\lambda, x)$, we have

$$\begin{aligned}S(\lambda, a, s + 2\pi) &= - a[L|_{\tilde{X}}]^{-1} \circ Q[N(\lambda, s + 2\pi + S(\lambda, a, s + 2\pi))] \\ &= - a[L|_{\tilde{X}}]^{-1} \circ Q[N(\lambda, s + S(\lambda, a, s + 2\pi))],\end{aligned}$$

but since $\tilde{x} = S(\lambda, a, s)$ is the *unique* solution of (30), we must have

$$S(\lambda, a, s + 2\pi) = S(\lambda, a, s). \qquad (35)$$

This implies that B is also 2π-periodic with respect to s. We note that solutions of (34), which differ by a multiple of 2π, correspond to geometrically equivalent periodic solutions of (1), hence the number of geometrically distinct periodic solutions of (1) is equal to the number of solutions of (34) in $[-\pi, \pi)$, and we want to show that, when (26) holds, this number is two. Moreover, it is an easy matter to show that nondegeneracy of a solution s of (34) implies nondegeneracy of the corresponding solution $u = \lambda x_0 + s + S(\lambda, a, s)$ of (1), hence what we need to show is

LEMMA 3. *If $|a| < \omega\sqrt{\omega^2 + c^2}$ and λ satisfies*

$$\bar{h}(\lambda) > |a|\sigma \left[1 + \left(\frac{2 - |a|\sigma}{1 - |a|\sigma}\right)^2\right]^{\frac{1}{2}}, \qquad (36)$$

then (34) *has precisely two solutions $s \in [-\pi, \pi)$, and they are nondegenerate in the sense that if s is a solution of* (34) *then $D_s B(\lambda, a, s) \neq 0$.*

Once we have proven Lemma 3, we will have completed the proof of Theorem 5. To prove Lemma 3, we begin by writing B more explicitly as

$$B(\lambda, a, s) = \frac{1}{T}\int_0^T \sin(s + \lambda x_0(t) + S(\lambda, a, s)(t))\, dt.$$

In particular, since, by (32),

$$S(\lambda, 0, s) = 0, \tag{37}$$

we have

$$\begin{aligned} B(\lambda, 0, s) &= \frac{1}{T}\int_0^T \sin(s + \lambda x_0(t))\, dt \\ &= \sin(s)\frac{1}{T}\int_0^T \cos(\lambda x_0(t))\, dt + \cos(s)\frac{1}{T}\int_0^T \sin(\lambda x_0(t))\, dt, \end{aligned} \tag{38}$$

or, defining α by

$$\cos(\alpha) = \frac{1}{\bar{h}(\lambda)}\frac{1}{T}\int_0^T \cos(\lambda x_0(t))\, dt, \qquad \sin(\alpha) = \frac{1}{\bar{h}(\lambda)}\frac{1}{T}\int_0^T \sin(\lambda x_0(t))\, dt,$$

we have

$$B(\lambda, 0, s) = \bar{h}(\lambda)\sin(s + \alpha). \tag{39}$$

We note for future use that when (2) holds then, as was noted before, the second integral on the right-hand side of (38) vanishes, so (38) reduces to

$$B(\lambda, 0, s) = h(\lambda)\sin(s). \tag{40}$$

We will use the following lemma, which provides a sufficient condition for two functions to have the same number of zeroes in an interval, and was already used in [6].

LEMMA 4. *Suppose $g_1, g_2 : \mathbf{R} \to \mathbf{R}$ are C^1 and 2π-periodic, and*

$$|g_2(s) - g_1(s)| \leqslant \beta_1, \tag{41}$$

$$|g_2'(s) - g_1'(s)| \leqslant \beta_2 \tag{42}$$

for all $s \in \mathbf{R}$. Suppose also that g_1 satisfies

$$|g_1(s)| \leqslant \beta_1 \qquad \textit{implies} \quad |g_1'(s)| > \beta_2. \tag{43}$$

Then g_1 *and* g_2 *have the same number of zeroes in the interval* $[-\pi, \pi)$, *and the zeroes of* g_2 *are nondegenerate* (*that is*: $g_2(s) = 0$ *implies* $g_2'(s) \neq 0$).

Returning to the proof of Lemma 3, We will use Lemma 4 to prove that when (26) holds, (34) has the same number of solutions in $[-\pi, \pi)$ as

$$B(\lambda, 0, s) = 0, \qquad s \in \mathbf{R} \tag{44}$$

and since, by (39), (44) has exactly two solutions in $[-\pi, \pi)$, we then conclude that (34) has exactly two solutions in $[-\pi, \pi)$, which will complete the proof of Lemma 3.

To apply Lemma 4, we set $g_1(s) = B(\lambda, 0, s)$, $g_2(s) = B(\lambda, a, s)$. Using (33), (17), and the fact that $\|I - Q\|_{Y,Y} = 1$, we have

$$\begin{aligned} |g_2(s) - g_1(s)| &= |(I - Q)[N(\lambda, s + S(\lambda, a, s)) - N(\lambda, s)]| \\ &\leq \|S(\lambda, a, s)\|_Y \leqslant |a|\sigma, \end{aligned} \tag{45}$$

hence (41) holds with

$$\beta_1 = |a|\sigma. \tag{46}$$

Differentiating $B(\lambda, a, s)$ with respect to s, we have

$$D_s B(\lambda, a, s) = (I - Q) \circ D_x N(\lambda, s + S(\lambda, a, s))(1 + D_s S(\lambda, a, s)) \tag{47}$$

(note that in the above 1 denotes the constant function with value 1). To compute $D_s S(\lambda, a, s)$, we differentiate relation (32) with respect to s, obtaining

$$D_s S(\lambda, a, s) = -a[L|_{\tilde{X}}]^{-1} \circ Q \circ D_x N(\lambda, s + S(\lambda, a, s))(1 + D_s S(\lambda, a, s)), \tag{48}$$

which may be rewritten as

$$[I - A] \circ D_s S(\lambda, a, s) = A(1), \tag{49}$$

where $A{:}Y \to Y$ is the linear operator defined by

$$A = -a[L|_{\tilde{X}}]^{-1} \circ Q \circ D_x N(\lambda, s + S(\lambda, a, s)). \tag{50}$$

We have

$$D_x N(\lambda, x)(v) = \cos(\lambda x_0(t) + x(t))v(t) \qquad \text{for all } x, v \in Y, \tag{51}$$

hence $\|D_x N(\lambda, x)\|_{Y,Y} \leqslant 1$ for all $x \in Y$, and in particular

$$\|D_x N(\lambda, s + S(\lambda, a, s))\|_{Y,Y} \leqslant 1. \tag{52}$$

From (31), (52) and the assumption that $|a| < \omega\sqrt{\omega^2 + c^2}$, we obtain

$$\|A\|_{Y,Y} \leqslant |a| \, \|[L|_{\tilde{X}}]^{-1} \circ Q\|_{Y,Y} \leqslant |a|\sigma < 1, \tag{53}$$

which implies that $I - A$ is invertible and

$$\|[I - A]^{-1}\|_{Y,Y} \leqslant \frac{1}{1 - \|A\|_{Y,Y}} \leqslant \frac{1}{1 - |a|\sigma}. \tag{54}$$

Since $I - A$ is invertible, we may rewrite (49) as

$$D_s S(\lambda, a, s) = [I - A]^{-1} \circ A(1). \tag{55}$$

Using (53)–(55), we obtain

$$\|D_s S(\lambda, a, s)\|_Y \leqslant \|[I - A]^{-1}\|_{Y,Y} \|A\|_{Y,Y} \leqslant \frac{|a|\sigma}{1 - |a|\sigma}. \tag{56}$$

We note, also, that by (51),

$$\|D_x N(\lambda, x) - D_x N(\lambda, y)\|_{Y,Y} \leqslant \|x - y\|_Y \qquad \text{for all } x, y \in Y. \tag{57}$$

Therefore, using (33), (47), (52), (56) and (57),

$$\begin{aligned}
|g_2'(s) - g_1'(s)| &= |D_s B(\lambda, a, s) - D_s B(\lambda, 0, s)| \\
&= |(I - Q) \circ [D_x N(\lambda, s + S(\lambda, a, s))(1 + D_s S(\lambda, a, s)) \\
&\quad - D_x N(\lambda, s)(1)]| \\
&\leqslant \|D_x N(\lambda, s + S(\lambda, a, s))(1 + D_s S(\lambda, a, s)) - D_x N(\lambda, s)(1)\|_Y \\
&\leqslant \|[D_x N(\lambda, s + S(\lambda, a, s)) - D_x N(\lambda, s)](1)\|_Y \\
&\quad + \|D_x N(\lambda, s + S(\lambda, a, s))(D_s S(\lambda, a, s))\|_Y \\
&\leqslant \|S(\lambda, a, s)\|_Y + \|D_s S(\lambda, a, s)\|_Y \leqslant |a|\sigma \\
&\quad + \frac{|a|\sigma}{1 - |a|\sigma} = |a|\sigma \frac{2 - |a|\sigma}{1 - |a|\sigma}.
\end{aligned} \tag{58}$$

Therefore, (42) holds with

$$\beta_2 = |a|\sigma \frac{2 - |a|\sigma}{1 - |a|\sigma}. \tag{59}$$

To see that condition (43) of Lemma 4 holds, we assume by way of contradiction that it does not, so there exists some $s \in [-\pi, \pi)$ for which $|g_1(s)| \leqslant \beta_1$ and $|g_1'(s)| \leqslant \beta_2$, that is, using (39), (46) and (59)

$$\bar{h}(\lambda)\,|\sin(s+\alpha)| \leqslant a\sigma, \qquad \bar{h}(\lambda)\,|\cos(s+\alpha)| \leqslant a\sigma\frac{2-a\sigma}{1-a\sigma}.$$

Squaring these two inequalities and adding them, we obtain

$$\bar{h}(\lambda) \leqslant a\sigma\left[1+\left(\frac{2-a\sigma}{1-a\sigma}\right)^2\right]^{\frac{1}{2}},$$

which contradicts assumption (24). Hence the conditions of Lemma 4 hold, and it yields the desired result.

At this point in our investigation one may wonder whether the 'good' set $\mathscr{G}_a$ is not merely an artifact of our method of proof, that is, whether it may not, in fact, be true that fixing a compact interval $I \subset \mathbf{R}$, for $a > 0$ sufficiently small there are precisely two geometrically distinct periodic solutions of (1) for *all* $\lambda \in I$. We shall later show that this is not the case: if $\mathscr{X} \cap I \neq \emptyset$ then for *any* $a > 0$ sufficiently small there are certain subintervals of I such that for λ in those intervals (1) has *four* geometrically distinct periodic solutions.

4. THE EXCHANGE OF STABILITY

In this section, we investigate the dynamical stability of the symmetric periodic solutions. We first recall the definition of asymptotic stability.

We define the Poincaré mapping $P_{a,\lambda}$ corresponding to (1) as follows. For $\theta, \theta' \in \mathbf{R}$, let u be the solution of Eq. (1) with initial conditions

$$u(0) = \theta, \qquad u'(0) = \theta' \tag{60}$$

and set

$$P_{a,\lambda}(\theta, \theta') = (u(T), u'(T)).$$

$P_{a,\lambda}$: $\mathbf{R}^2 \to \mathbf{R}^2$ is a diffeomorphism, and its fixed points are in one-to-one correspondence with the periodic solutions of (1). A periodic solution $\bar{u}$ of (1) is *asymptotically stable* iff

(i) For any neighborhood $U \subset \mathbf{R}^2$ of $(\bar{u}(0), \bar{u}'(0))$ there exists a neighborhood $V \subset U$ of $(\bar{u}(0), \bar{u}'(0))$ such that $P^n_{a,\lambda}(V) \subset U$ for all $n \geqslant 1$, where $P^n_{a,\lambda}$ denotes the n-fold composition of $P_{a,\lambda}$ with itself.

(ii) There exists a neighborhood W of $(\bar{u}(0), \bar{u}'(0))$ such that for any $(\theta, \theta') \in W$, we have

$$\lim_{n\to\infty} P^n_{a,\lambda}(\theta, \theta') = (\bar{u}(0), \bar{u}'(0)).$$

A periodic solution which is not asymptotically stable will be called unstable.

We now discuss the stability of the u_k's in the case $c > 0$, revealing the following phenomenon, which we call the 'exchange of stability': when $a > 0$ is sufficiently small, there are λ-intervals in which $u_0(a, \lambda)$ is asymptotically stable while $u_1(a, \lambda)$ is unstable, and λ-intervals for which the reverse is true.

Theorem 7. *Assume f satisfies* (2), $c > 0$, $a > 0$ *satisfy* (18) *and*

$$a \leqslant \frac{1}{4}(\omega^2 + c^2). \tag{61}$$

Then whenever $\lambda \in \mathcal{G}_a$, the only periodic solutions of (1) *are the symmetric solutions $u_k(a, \lambda)$ $(k \in \mathbf{Z})$, and*

(i) *If $h(\lambda) > 0$, $u_k(a, \lambda)$ is asymptotically stable for k even, and unstable for k odd.*

(ii) *If $h(\lambda) < 0$, $u_k(a, \lambda)$ is unstable for k even and asymptotically stable for k odd.*

Fig. 1 represents the information one obtains from Theorem 7. We plot three of the branches of symmetric periodic solutions and indicate their stability. The question marks indicate that when $\lambda \notin \mathcal{G}_a$ we do not, at this point in our investigation, know anything about the stability of $u_k(a, \lambda)$.

Since it is only the stable periodic solutions which are observable in an experiment (or computer simulation), this 'exchange of stability' will result in an interesting dynamical phenomenon. Assume that we fix $a > 0$, and start with $\lambda = 0$, increasing λ very slowly. For λ small the solution u_0 will be stable (since $h(0) = 1$), so that after an initial transient, the behavior of the pendulum will be described by u_0 (or one of the periodic solutions geometrically equivalent to it, depending on the initial conditions). As λ increases, it will eventually leave the set $\mathcal{G}_a$, and when it reenters $\mathcal{G}_a$ we shall have $h(\lambda) < 0$, so that u_0 will become unstable, while u_1 will become stable, and thus the behavior of the pendulum will be described by u_1 (or one of the periodic solutions geometrically equivalent to it). For λ still larger, u_0 will become stable again, so that once again the motion of the pendulum will be described by u_0, and so on. We note that, since when a is small the intervals of the 'bad' set $\mathbf{R} - \mathcal{G}_a$ become very narrow, we shall observe, when a is small, very rapid transitions from u_0 to u_1 and vice versa take place, which

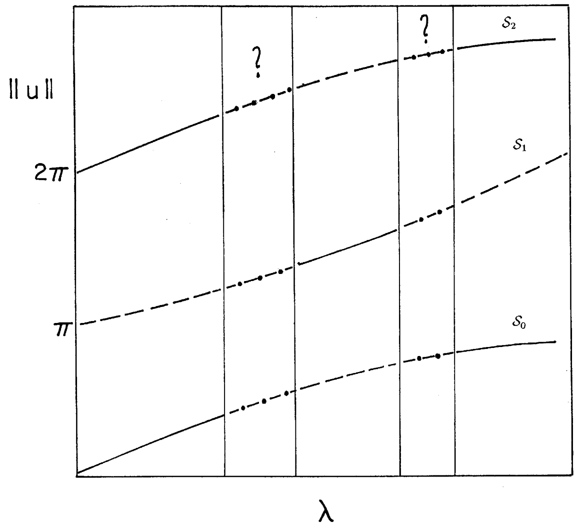

FIG. 1. The exchange of stability, as given by Theorem 7.

look like jumps. The precise nature of these transitions (that is the behavior when λ passes through the 'bad' set) will be illuminated in the following sections. Results of numerical simulation exhibiting this phenomenon will be discussed below.

In the above heuristic description, we have glossed over the possibility that after u_0 loses stability we might observe subharmonic or chaotic behavior, rather than asymptotic approach to u_1. In fact, if we restrict a further, we can use the results of [13] to exclude subharmonic or chaotic behavior, so that the behavior of the pendulum for $\lambda \in \mathscr{G}_a$ is indeed described by u_0 or u_1, depending on the sign of $h(\lambda)$. Indeed the results of [13], after a suitable rescaling, tell us that if

$$a < \frac{c^2}{4}, \tag{62}$$

then there is a curve in $\mathbf{R}^2$ which is invariant under the Poincaré mapping $P_{a,\lambda}$, and which attracts all orbits. Moreover, from Theorems 2.5 and 2.6 of [13] it follows that in our case (in which we know that fixed points of $P_{a,\lambda}$ exist), almost all orbits are attracted to the set of fixed points of $P_{a,\lambda}$. Therefore, combining the results of [13] and Theorem 7, we obtain

THEOREM 8. *Assume f satisfies* (2), $c > 0$, *and* $a > 0$ *satisfy* (18) *and* (62). *Let* $\lambda \in \mathscr{G}_a$, *and assume* $h(\lambda) > 0$ $(h(\lambda) < 0)$. *Then for almost all initial* *conditions* $(\theta, \theta') \in \mathbf{R}^2$, *we have, defining u to be the solution of* (1) *satisfying* (60), *that there exists* $k \in \mathbf{Z}$ *even* (*odd*) *such that*

$$\lim_{t\to\infty} (|u(t) - u_k(a,\lambda)(t)| + |u'(t) - u_k'(a,\lambda)(t)|) = 0.$$

We thus have, under the stated assumptions, a practically complete description of the dynamic behavior of the forced pendulum.

Fig. 2 presents the results of a numerical simulation of the forced pendulum which demonstrates the results of Theorem 8 (it is interesting to note that these computations were performed only after the phenomenon was discovered by the theoretical analysis to be presented in this section). We numerically solved (using MAPLE) Eq. (1) with initial conditions

$$u(0) = u'(0) = 0, \tag{63}$$

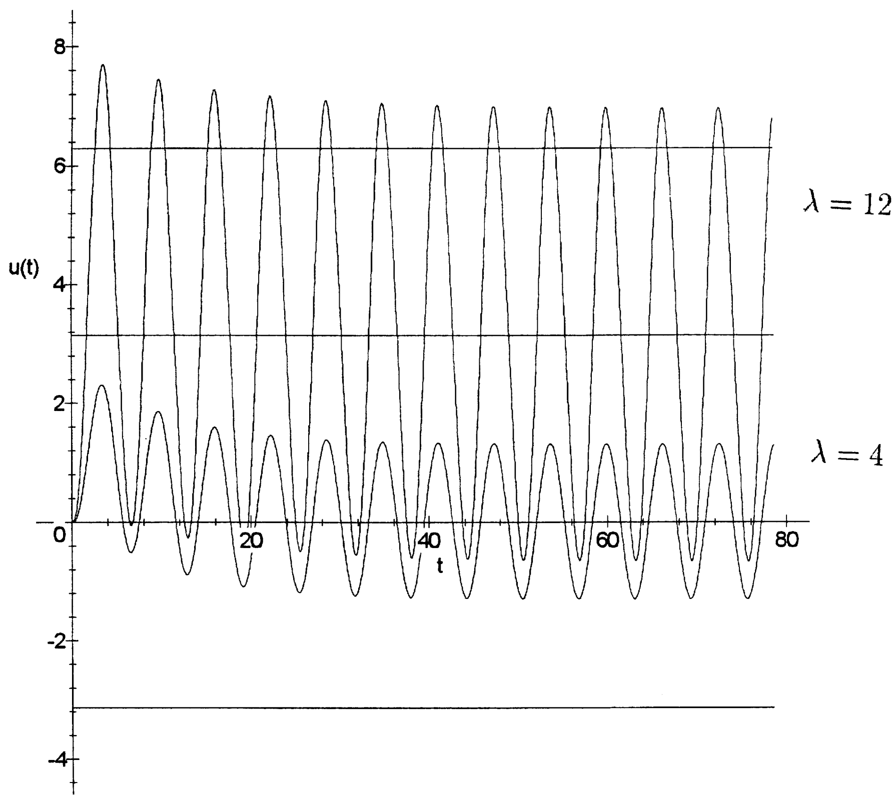

FIG. 2. Numerical solution of (1) with initial conditions (63), taking $a = \frac{1}{2}$, $c = 3$, and $\lambda = 4, 12$.

taking $f(t) = \sin(t)$, $a = \frac{1}{2}$, $c = 3$. The figure displays the solutions for two values of λ:$\lambda = 4$ and 12. We note that the parameters were chosen so that all the assumptions of Theorem 8 are satisfied. In this case, we have (see (23))

$$h(\lambda) = J_0\left(\frac{\lambda}{\sqrt{10}}\right),$$

so that we have $h(4) > 0$ and $h(12) < 0$. Theorem 8 thus tells us that when $\lambda = 4$ almost all solutions will be attracted to one of the periodic solutions u_k with k even, and indeed we see that our solution is attracted to $u_0(\frac{1}{2}, 4)$ (whose mean value is 0; the horizontal lines in the figure indicate multiples of π), and similarly the theorem tells us that when $\lambda = 12$ almost all solutions will be attracted to some u_k with k odd, and we see that our solution is attracted to $u_1(\frac{1}{2}, 12)$ (whose mean value is π). Further computations for intermediate values of λ show that when $\lambda \in [4, 7.5]$, the solution is attracted to $u_0(\frac{1}{2}, \lambda)$, while for $\lambda \in [7.8, 12]$ the solution is attracted to $u_1(\frac{1}{2}, \lambda)$, so that the exchange of stability occurs in the interval (7.5,7.8). We note that our theorems show that in the limit of $a > 0$ small the exchange of stability will occur at a λ which is a root of h, and the only root of h in the interval [4, 12] is $\bar{\lambda} = 7.6047$, which is in good agreement with what we found, even though a here is not so small.

We now present the results of another type of numerical simulation which demonstrates the exchange of stability in a striking way. Here, instead of solving (1) for various values of λ, we increase λ very slowly during the simulation. In Fig. 3, we plot the numerically computed solution of the equation

$$u'' + u' + \sin(u) = \frac{t}{100}\sin(t) \tag{64}$$

(with initial conditions (63)). Note that the strength of the forcing increases very slowly in comparison with the period 2π of the oscillations. As t increases, starting at $t = 0$, the amplitude of the oscillations of the pendulum increases but their mean value remains 0, until the first ‘jump’ point at which the mean value of the oscillations becomes π. This jump is caused by the fact that u_0 loses stability while u_1 becomes stable. As t increases, we have a series of jumps, where at each one we have a transition to behavior described by a different u_k, so that the mean value of the oscillations is always a multiple of π.

In Fig. 4, we present the result of computing the same solution of the same equation, but with a small change in the integration step, and here we see that we obtain a different picture: the transitions from one u_k to another

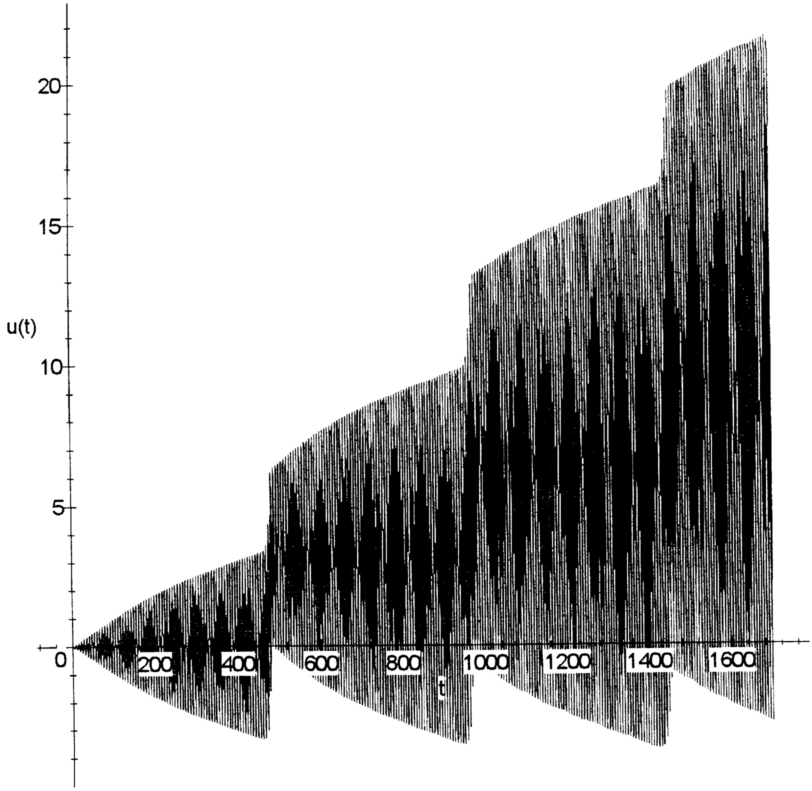

FIG. 3. Numerical solution of (64), with integration step = 0.21.

occur at precisely the same values of t, but whereas in the first computation the sequence of transitions was $u_0 \to u_1 \to u_2 \to u_3$, in the second computation it was $u_0 \to u_1 \to u_0 \to u_{-1}$. Slight changes in the integration steps will lead to various other sequences of transition, but the transition is always from u_k to u_{k+1} or to u_{k-1}. The explanation of this phenomenon is that when u_k loses stability, the system may jump to either u_{k+1} or u_{k-1}, and which way the jump will occur depends on 'noise', which in the numerical simulation is provided by the integration error (in an ideal situation – impossible both physically and computationally – with no noise, transition to another solution would not occur). Thus the precise sequence of transitions is unpredictable, although their time of occurrence is!

We now turn to the proof of Theorem 7. We will use some results of Ortega, together with the following perturbational result (we note that the

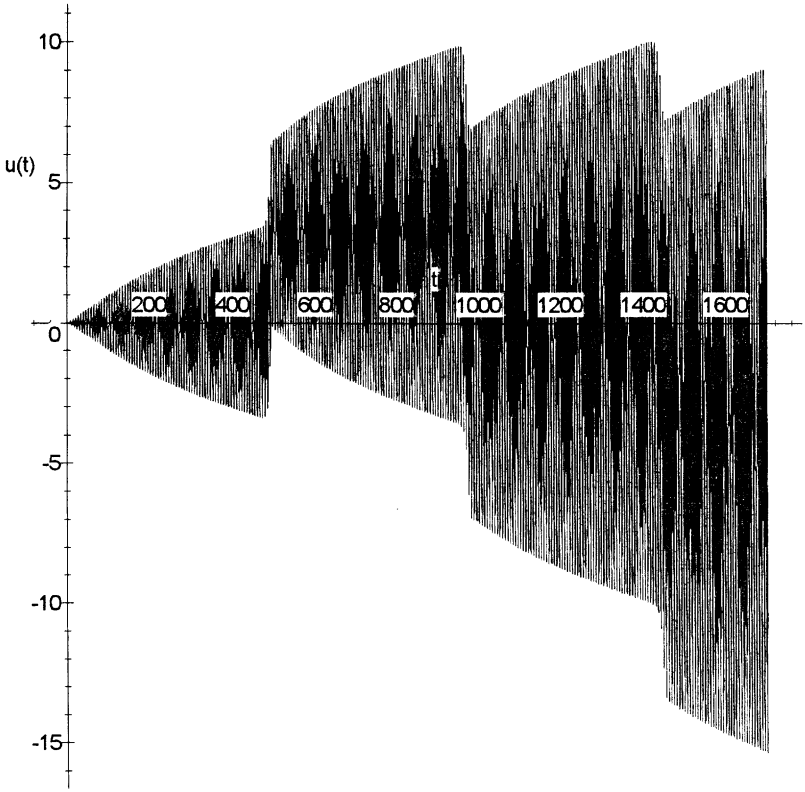

FIG. 4. Numerical solution of (64), with integration step = 0.20.

following lemma is in fact a corollary of Theorem 7, but here we use it to prove Theorem 7).

LEMMA 5. *Assume f satisfies* (2). *Assume* $c > 0$. *Fix* $\lambda \in \mathbf{R}$. *If* $h(\lambda) > 0$ *then for* $a > 0$ *sufficiently small* $u_k(a, \lambda)$ *is asymptotically stable for k even and unstable for k odd. If* $h(\lambda) < 0$ *then for* $a > 0$ *sufficiently small* $u_k(a, \lambda)$ *is asymptotically stable for k odd and unstable for k even.*

To prove Lemma 5, we recall that in order to investigate the asymptotic stability of u_k we must determine whether the characteristic multipliers of the linearized equation

$$v'' + cv' + a\cos(u_k(a, \lambda)(t))v = 0 \tag{65}$$

lie inside the unit circle of the complex plane when a is small. Since, by (19) and (20), we have

$$a\cos(u_k(a,\lambda)(t)) = a(-1)^k \cos(\lambda x_0(t)) + O(a^2) \qquad \text{as } a \to 0 \tag{66}$$

uniformly with respect to $t \in \mathbf{R}$, it will suffice to use the following lemma:

LEMMA 6. *Assume $q : [0,\infty) \times \mathbf{R} \to \mathbf{R}$ is continuous and T-periodic with respect to the second variable, and satisfies*

$$q(a,t) = aq_0(t) + O(a^2) \quad \text{as } a \to 0$$

uniformly with respect to $t \in \mathbf{R}$. Assume $c > 0$. If $\int_0^T q_0(t)\,dt > 0$, then the characteristic multipliers of the linear equation

$$v'' + cv' + q(a,t)v = 0 \tag{67}$$

are inside the unit circle for $a > 0$ sufficiently small. If $\int_0^T q_0(t)\,dt < 0$, then one of the characteristic multipliers of (67) is outside the unit circle for $a > 0$ sufficiently small.

We postpone the proof of Lemma 6 to the end of this section. Returning to the proof of Lemma 5 we now see, by (66) and Lemma 6, that $u_k(a,\lambda)$ is asymptotically stable for small $a > 0$ if

$$\int_0^T (-1)^k \cos(\lambda x_0(t))\,dt > 0$$

and unstable if the reverse inequality holds. But the last integral is equal to $(-1)^k h(\lambda)$, from which Lemma 5 follows.

We now return to the proof of Theorem 7. We recall that for each isolated periodic solution u of (1) there is associated an integer index $\gamma_T(u)$ which is defined by

$$\gamma_T(u) = \operatorname{ind}[P_{a,\lambda}, (u(0), u'(0))],$$

where $P_{a,\lambda}$ is the Poincaré mapping associated with (1), as defined above, and ind is the standard fixed-point index. When $(u(0), u'(0))$ is a nondegenerate fixed point of $P_{a,\lambda}$, we have

$$\gamma_T(u) = \operatorname{sign}[\det[I - DP_{a,\lambda}(u(0), u'(0))]]$$

(equivalently, $\gamma_T(u)$ can be defined as the degree of a mapping between Banach spaces associated with (1), see [16]). This index has the standard homotopy-invariance property. We shall use the following results of Ortega (for $c > 0$):

(1) If (61) holds then a nondegenerate periodic solution u of (1) is asymptotically stable if and only if $\gamma_T(u) = 1$ ([15, Theorem 1.1]).

(2) If the number of geometrically distinct periodic solutions of (1) is finite, then the sum of the $\gamma_T(u)$ over all the geometrically distinct solutions is 0 ([16, Chap. 3, Proposition 2]).

Theorem 7 follows from result (1) of Ortega together with the following 'exchange of index' theorem, which holds without assumption (61), and also for $c = 0$.

THEOREM 9. *Assume f satisfies* (2) *and* $a > 0$ *satisfies* (18). *Then, whenever* $\lambda \in \mathcal{G}_a$,

(i) *If* $h(\lambda) > 0$ *then* $\gamma_T(u_k(a, \lambda)) = (-1)^k$ *for all* $k \in \mathbf{Z}$.

(ii) *If* $h(\lambda) < 0$ *then* $\gamma_T(u_k(a, \lambda)) = (-1)^{k+1}$ *for all* $k \in \mathbf{Z}$.

We note that in the numerical work of Schmitt and Sari [20] which was mentioned in the introduction, in which the periodic solutions of (1) with $c = 0$ and $f(t) = \cos(\omega t)$ were computed and the phenomenon of exchange of stability was observed, the condition $\gamma_T(u) = 1$ was taken as the *definition* of stability, rather than the stricter notion of asymptotic stability which we use here (which of course can never occur for the conservative case $c = 0$). Hence it is Theorem 9 which provides an explanation for the observations in [20].

Proof of Theorem 9. We will assume $c > 0$. The case $c = 0$ follows by going to the limit $c \to 0$. We fix λ_0 and a_0 satisfying the conditions of the theorem. Assume $h(\lambda_0) > 0$ (the proof in case $h(\lambda_0) < 0$ is analogous). Since a_0 satisfies (18), we have that (18) holds for any $a \in (0, a_0)$. The conditions $\lambda_0 \in \mathcal{G}_{a_0}$ and (25) imply that $\lambda_0 \in \mathcal{G}_a$ for all $a \in (0, a_0)$. Therefore by Theorem 5, we know that (1) with $\lambda = \lambda_0$ has precisely two geometrically distinct periodic solutions, which are nondegenerate, for any $a \in (0, a_0)$, namely $u_0(a, \lambda_0), u_1(a, \lambda_0)$. Also, from Lemma 5, we know that when a is very small $u_0(a, \lambda_0)$ is stable and $u_1(a, \lambda_0)$ is unstable. Moreover, if a is very small then (61) holds, so from result (1) of Ortega quoted above we conclude that for $a > 0$ very small

$$\gamma_T(u_0(a, \lambda_0)) = 1. \tag{68}$$

By the homotopy invariance of the index and the fact that $u_0(a,\lambda_0)$ is nondegenerate for $a \in (0, a_0]$, we conclude that (68) remains true for $a \in (0, a_0]$, and in particular, we have

$$\gamma_T(u_0(a_0,\lambda_0)) = 1. \tag{69}$$

By (69), result (2) of Ortega and the fact that $u_0(a_0,\lambda_0), u_1(a_0,\lambda_0)$ are the only periodic solutions up to geometric equivalence, we conclude that

$$\gamma_T(u_1(a_0,\lambda_0)) = -1. \tag{70}$$

Since clearly $\gamma_T(u_k(a,\lambda))$ is equal to $\gamma_T(u_0(a,\lambda))$ when k is even and to $\gamma_T(u_1(a,\lambda))$ when k is odd, this concludes the proof of Theorem 9. ∎

We now give the

Proof of Lemma 6. We denote by $v_1(a,t)$, $v_2(a,t)$ the solutions of (67) satisfying the initial conditions

$$v_1(a,0) = 1, \qquad \frac{dv_1}{dt}(a,0) = 0,$$

$$v_2(a,0) = 0, \qquad \frac{dv_2}{dt}(a,0) = 1.$$

A standard computation shows that

$$v_1(a,t) = 1 + \frac{a}{c}\int_0^t (e^{c(s-t)} - 1) q_0(s)\, ds + O(a^2),$$

$$\frac{dv_1}{dt}(a,t) = -a\int_0^t e^{c(s-t)} q_0(s)\, ds + O(a^2),$$

$$v_2(a,t) = \frac{1}{c}(1 - e^{-ct}) + \frac{a}{c^2}\left(e^{-ct}\int_0^t (e^{cs} - 1) q_0(s)\, ds + \int_0^t (e^{-cs} - 1) q_0(s)\, ds\right) + O(a^2),$$

$$\frac{dv_2}{dt}(a,t) = e^{-ct} - \frac{a}{c}e^{-ct}\int_0^t (e^{cs} - 1) q_0(s)\, ds + O(a^2).$$

The characteristic multipliers of (67) are the eigenvalues of the monodromy matrix:

$$M(a) = \begin{pmatrix} v_1(a,T) & v_2(a,T) \\ \dfrac{dv_1}{dt}(a,T) & \dfrac{dv_2}{dt}(a,T) \end{pmatrix}.$$

Computing the characteristic polynomial of $M(a)$, we obtain

$$p(a,v) = v^2 - \left(1 + e^{-cT} + \frac{a}{c}(e^{-cT} - 1)\int_0^T q_0(s)\,ds\right)v + e^{-cT} + O(a^2).$$

When $a = 0$ the roots of $p(0,v)$ are $v_1 = 1$ and $v_2 = e^{-cT} < 1$. For $a > 0$ small, v_2 will remain smaller than 1, so we are interested in the direction in which v_1 moves, that is we are interested in the curve $v(a)$ satisfying $v(0) = 1$ and

$$p(a, v(a)) = 0 \tag{71}$$

for a sufficiently small. Differentiating (71) and setting $a = 0$, we obtain

$$v'(0) = -\frac{D_a p(0,1)}{D_v p(0,1)} = -\frac{1}{c}\int_0^T q_0(t)\,dt.$$

Therefore, if $\int_0^T q_0(t)\,dt > 0$ then for $a > 0$ small $|v(a)| < 1$, and if $\int_0^T q_0(t)\,dt < 0$ then for $a > 0$ small $v(a) > 1$, as we wanted to show. ∎

5. BIFURCATION FROM THE SYMMETRIC SOLUTIONS

As we have seen in Theorem 7, if we fix $c > 0$, $a > 0$ satisfying (18) and (61), and vary λ in an interval $[\lambda_*, \lambda^*]$, where $\lambda_*, \lambda^* \in \mathcal{G}_a$ and

$$h(\lambda_*)h(\lambda^*) < 0, \tag{72}$$

the periodic solutions u_0 and u_1 'exchange' their stability. Our aim now is to understand how this exchange of stability takes place. A well-known heuristic principle says that loss of stability is related to bifurcation. Note also that by (72), the interval $[\lambda_*, \lambda^*]$ intersects the 'bad' set $\mathbf{R} - \mathcal{G}_a$, concerning which Theorem 5 tells us nothing, so at least, in principle, it is possible that when λ passes through $\mathbf{R} - \mathcal{G}_a$, we will have periodic solutions other than the u_k's. This suspicion can be justified by using the Index Jump Principle (see [22, Theorem 15.A]), from which we obtain Theorem 10 below (in fact, we only need the exchange of index, given by Theorem 9, and not the exchange of stability, so conditions (61) and $c > 0$ are not needed).

We first precisely define the notion of bifurcation from the symmetric solutions. We denote by $\mathscr{S}_k(a) \subset \Sigma$ ($k \in \mathbf{Z}$) the curves of symmetric periodic solutions:

$$\mathscr{S}_k = \mathscr{S}_k(a) = \{(\lambda, u_k(a, \lambda)) \mid \lambda \in \mathbf{R}\}.$$

We set

$$\mathscr{S} = \mathscr{S}(a) = \bigcup_{k \in \mathbf{Z}} \mathscr{S}_k(a).$$

For given a, we shall say that a bifurcation from $\mathscr{S}_k$ occurs at $\lambda = \lambda_k$ if in every neighborhood of $(\lambda_k, u_k(a, \lambda_k))$ in $\mathbf{R} \times X$ there exists a point $(\lambda', u) \notin \mathscr{S}$ such that $(\lambda', u) \in \Sigma$.

From the Index Jump Principle we obtain:

Theorem 10. *Assume f satisfies* (2) *and* $a > 0$ *satisfies* (18). *Assume that* $\lambda_*, \lambda^* \in \mathscr{G}_a$ *with* $\lambda_* < \lambda^*$ *and that* (72) *holds. Then, for each* $k \in \mathbf{Z}$, *there exists* $\lambda_k \in (\lambda_*, \lambda^*)$ *such that a bifurcation from* $\mathscr{S}_k$ *occurs at* $\lambda = \lambda_k$.

We note that the formulation of the Index Jump Principle in [22, Theorem 15.A], is for bifurcation from a trivial branch of solutions, while here we use it for bifurcation from a smooth nontrivial branch $\mathscr{S}_k$, but a more general formulation of the Index Jump Principle valid for bifurcation from smooth branches is an easy consequence of the Index Jump Principle for bifurcation from trivial branches.

Using arguments of the type appearing in the proof of the Rabinowitz global bifurcation theorem [19, 22, Theorem 15.C], one can obtain some information about the global structure of the bifurcating branches (the Rabinowitz theorem itself cannot be used here, since we are dealing with a bifurcation from a curve of solutions for which we do not have an explicit representation). However, we shall use a different approach to study the bifurcation that we have just shown to occur. In fact, we shall rederive Theorem 10 by a different approach, and also obtain additional information which cannot be derived from general bifurcation theory. We shall show, for example, that for each $k \in \mathbf{Z}$ the branch of solutions bifurcating from $\mathscr{S}_k$ and the branch bifurcating from $\mathscr{S}_{k+1}$ meet, so in fact there is a single branch of solutions lying in $(\lambda_*, \lambda^*) \times X$ and connecting $\mathscr{S}_k$ and $\mathscr{S}_{k+1}$.

We refer to Fig. 5 for a representation of the contents of the following theorem, obtained by plotting the set $\tilde{\Sigma}$, defined by (3).

Theorem 11. *Assume f satisfies* (2) *and* $a > 0$ *satisfies* (18). *Assume that* $\lambda_*, \lambda^* \in \mathscr{G}_a$ *with* $\lambda_* < \lambda^*$ *and that* (72) *holds. Then, for each* $k \in \mathbf{Z}$, *there*

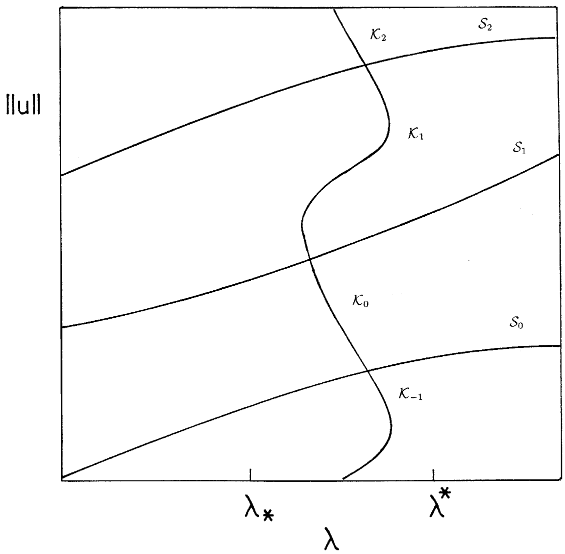

FIG. 5. Bifurcation from the symmetric solutions, as described by Theorem 11.

exists a connected set $\mathcal{K}_k = \mathcal{K}_k(a) \subset \Sigma(a) \cap ((\lambda_*, \lambda^*) \times X)$ *such that, for each* $k \in \mathbf{Z}$,

(i) $\bar{\mathcal{K}}_k \cap \mathcal{S}_k = \bar{\mathcal{K}}_{k-1} \cap \mathcal{S}_k \neq \emptyset$.

(ii) *If* $(\lambda, u) \in \mathcal{K}_k$ *then* $\frac{1}{T}\int_0^T u(t)\,dt \in (k\pi, (k+1)\pi)$ (*this implies that* $\mathcal{K}_j \cap \mathcal{K}_k = \emptyset$ *for* $j \neq k$).

(iii) $\mathcal{K}_{k+2} = \{(\lambda, u + 2\pi) \mid (\lambda, u) \in \mathcal{K}_k\}$.

Thus if we define $\mathcal{C} = \bigcup_{k \in \mathbf{Z}} \mathcal{K}_k$ then $\mathcal{C}$ is a connected subset of Σ, and it intersects all the $\mathcal{S}_k$'s. We may call it the 'bifurcating continuum' of periodic solutions.

To prove Theorem 11, we will exploit the fact that (18) implies the existence of a Lyapunov–Schmidt reduction, as we showed in the proof of Theorem 6, which implies a one-to-one correspondence between periodic solutions of (1) and the solutions of (34). In other words, if we define, for $|a| < \omega\sqrt{\omega^2 + c^2}$, F_a: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$F_a(\lambda, s) = B(\lambda, a, s),$$

then for $a > 0$ satisfying (18) there is a one-to-one correspondence between the zeroes of F_a in the (λ, s)-plane and the set $\Sigma(a)$: defining $\Phi_a : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times X$ by

$$\Phi_a(\lambda, s) = (\lambda, s + S(\lambda, a, s) + \lambda x_0).$$

We have

$$\Sigma(a) = \Phi_a(F_a^{-1}(0)), \tag{73}$$

so our problem reduces to that of studying the set $F_a^{-1}(0) \subset \mathbf{R} \times \mathbf{R}$.
We first derive some properties of the function F_a.

Lemma 7. *Assume f satisfies (2) and $|a| < \omega\sqrt{\omega^2 + c^2}$. Then*

(i) *$F_a(\lambda, s)$ is 2π-periodic with respect to s.*
(ii) *$F_a(\lambda, s)$ is odd with respect to s.*
(iii) *$F_a(\lambda, \pi k) = 0$ for all $\lambda \in \mathbf{R}$, $k \in \mathbf{Z}$.*
(iv) *If λ satisfies*

$$|h(\lambda)| > |a|\sigma \left[1 + \left(\frac{2 - |a|\sigma}{1 - |a|\sigma}\right)^2\right]^{1/2} \tag{74}$$

and $k \in \mathbf{Z}$, then $(-1)^k h(\lambda) F_a(\lambda, s) > 0$ for all $s \in (k\pi, (k+1)\pi)$.

Proof. Part (i) was already proved in Section 3 (we showed that B is 2π-periodic with respect to s). Part (ii) follows from the fact that $\mathscr{T}$, as defined by (21), commutes with L, $N(\lambda, \cdot)$, and Q, so that, using (32), we have

$$\mathscr{T}(S(\lambda, a, s)) = -a[L|_{\tilde{X}}]^{-1} \circ Q[N(\lambda, -s + \mathscr{T}(S(\lambda, a, s)))]$$

and since $\tilde{x} = S(\lambda, a, -s)$ is the unique solution of the equation

$$\tilde{x} = -a[L|_{\tilde{X}}]^{-1} \circ Q(N(\lambda, -s + \tilde{x})),$$

we must have $S(\lambda, a, -s) = \mathscr{T}(S(\lambda, a, s))$. Therefore,

$$\begin{aligned} F_a(\lambda, -s) &= (I - Q)[N(\lambda, -s + S(\lambda, a, -s))] = (I - Q)[N(\lambda, -s + \mathscr{T}(S(\lambda, a, s)))] \\ &= \mathscr{T}((I - Q)[N(\lambda, s + S(\lambda, a, s))]) = -F_a(\lambda, s). \end{aligned}$$

Part (iii) follows from (i) and (ii).
We now prove part (iv). We note that it suffices to prove it for $k = 0$, since the general statement follows from the case $k = 0$ by parts (i) and (ii). We assume $h(\lambda) > 0$ and show that $F_a(\lambda, s) > 0$ for $s \in (0, \pi)$ (the proof in the case

$h(\lambda) < 0$ is analogous). By Lemma 3 (which may be invoked since by (74) λ satisfies (36)), $F_a(\lambda, \cdot)$ vanishes exactly twice in the interval $[-\pi, \pi)$, and since, by part (iii) of our lemma, it vanishes at 0 and at π, it does not vanish in $(0, \pi)$. This is also true when a is replaced by any $a' \in (0, a)$, so the sign of $F_a(\lambda, s)$ for $s \in (0, \pi)$ is equal to the sign of $F_0(\lambda, s)$ for $s \in (0, \pi)$, and since, by (40),

$$F_0(\lambda, s) = B(\lambda, 0, s) = h(\lambda) \sin(s), \tag{75}$$

we conclude that $F_a(\lambda, s)$ is positive for $s \in (0, \pi)$. ∎

We remark that part (iii) of Lemma 7 and (73) imply that, for all $\lambda \in \mathbf{R}$ and $k \in \mathbf{Z}$, $\Phi_a(\lambda, \pi k) \in \Sigma$. However, we have not thereby found any new periodic solutions; in fact we claim that

Lemma 8. *Assume f satisfies* (2) *and $a > 0$ satisfies* (18). *Then*

$$\Phi_a(\lambda, \pi k) = (\lambda, u_k(a, \lambda)) \quad \textit{for all } k \in \mathbf{Z}, \;\; \lambda \in \mathbf{R} \tag{76}$$

so that

$$\Phi_a(\mathbf{R} \times \{k\pi\}) = \mathscr{S}_k.$$

Proof. We shall show this for k even (the proof for k odd is similar). Since $u_0(a, \lambda)$ is a solution of (1), so that, defining $x = u_0(a, \lambda) - \lambda x_0$, x is a solution of (7), we have, by the properties of the Lyapunov–Schmidt reduction, defining $s = (I - Q)(x) = \frac{1}{T} \int_0^T x(t)\, dt$,

$$x = s + S(\lambda, a, s),$$

but since $x \in X_*$ we have $s = 0$, hence $u_0(a, \lambda) - \lambda x_0 = x = S(\lambda, a, 0)$, so $u_0(a, \lambda) = S(\lambda, a, 0) + \lambda x_0$. The result for k even follows from this since when k is even (using (35))

$$\Phi_a(\lambda, \pi k) = (\lambda, \pi k + S(\lambda, a, \pi k) + \lambda x_0) = (\lambda, \pi k + S(\lambda, a, 0) + \lambda x_0)$$

$$= (\lambda, \pi k + u_0(a, \lambda)) = (\lambda, u_k(a, \lambda)). \quad ∎$$

Returning to the proof of Theorem 11, let D be the rectangle in the (λ, s)-plane defined by

$$D = \{(\lambda, s) \mid \lambda \in (\lambda_*, \lambda^*), s \in (0, \pi)\}.$$

We have $\partial D = I_1 \cup I_2 \cup I_3 \cup I_4$, where

$$I_1 = \{(\lambda_*, s) \,|\, s \in (0, \pi)\}, \qquad I_2 = \{(\lambda^*, s) \,|\, s \in (0, \pi)\},$$

$$I_3 = \{(\lambda, 0) \,|\, \lambda \in [\lambda_*, \lambda^*]\}, \qquad I_4 = \{(\lambda, \pi) \,|\, \lambda \in [\lambda_*, \lambda^*]\}.$$

We shall now assume, for definiteness, that $h(\lambda_*) > 0$ and $h(\lambda^*) < 0$. From part (iv) of Lemma 7 we have that F_a is positive on I_1 and negative on I_2. We now apply the following topological result.

Lemma 9. *Let $D = (a_1, a_2) \times (b_1, b_2)$ be an open rectangle in the plane, $F{:}\bar{D} \to \mathbf{R}$ a continuous function which is positive on the left side $\{a_1\} \times (b_1, b_2)$ of $\bar{D}$ and negative on the right side $\{a_2\} \times (b_1, b_2)$. Then there exists a connected set $K \subset D$ such that $F|_K = 0$ and $\bar{K}$ intersects both the lower side $[a_1, a_2] \times \{b_1\}$ and the upper side $[a_1, a_2] \times \{b_2\}$* of $\bar{D}$.

In the case when F is, in fact, positive on $\{a_1\} \times [b_1, b_2]$ and negative on $\{a_2\} \times [b_1, b_2]$, this lemma is a very special case of Lemma A.5 in [18]. The general case where F can vanish at the vertices of $\bar{D}$ requires a simple additional limiting argument, of the type found in [1].

From Lemma 9 it follows that there exists a connected set

$$K_0 \subset D \cap F_a^{-1}(0) \tag{77}$$

with

$$\bar{K}_0 \cap I_3 \neq \emptyset, \qquad \bar{K}_0 \cap I_4 \neq \emptyset. \tag{78}$$

We define

$$K_{-1} = \{(\lambda, -s) \,|\, (\lambda, s) \in K_0\}. \tag{79}$$

By (77) and part (ii) of Lemma 7,

$$K_{-1} \subset F_a^{-1}(0). \tag{80}$$

We now define K_k for $k \in \mathbf{Z}$ by

$$K_k = \begin{cases} \{(\lambda, s) \,|\, (\lambda, s - k\pi) \in K_0\} & k \in \mathbf{Z} \text{ even}, \\ \{(\lambda, s) \,|\, (\lambda, s - (k+1)\pi) \in K_{-1}\} & k \in \mathbf{Z} \text{ odd}. \end{cases} \tag{81}$$

By (77), (80) and part (i) of Lemma 7, $K_k \subset F_a^{-1}(0)$ for all k. Therefore, defining $\mathscr{K}_k$ ($k \in \mathbf{Z}$) by

$$\mathscr{K}_k = \{\Phi_a(\lambda, s) \,|\, (\lambda, s) \in K_k\}$$

and using (73) we have $\mathcal{K}_k \subset \Sigma \cap ((\lambda_*, \lambda^*) \times X)$. The fact that $\mathcal{K}_k$ satisfies (ii) and (iii) of Theorem 11 follows immediately from the definition of $\mathcal{K}_k$. To show that (i) of Theorem 11 holds, it suffices, in view of Lemma 8, to show that

$$\bar{K}_k \cap (\mathbf{R} \times \{k\pi\}) = \bar{K}_{k-1} \cap (\mathbf{R} \times \{k\pi\}) \neq \emptyset. \tag{82}$$

For $k = 0$, the equality in (82) is immediate from (79), and the fact that the set is nonempty follows from (78). We show that (82) holds when k is even (the argument for k odd is similar). Indeed, using (81) and the fact that (82) holds for $k = 0$, we obtain

$$\begin{aligned}
\bar{K}_k \cap (\mathbf{R} \times \{k\pi\}) &= \{(\lambda, k\pi) \mid (\lambda, k\pi) \in \bar{K}_k\} \\
&= \{(\lambda, k\pi) \mid (\lambda, 0) \in \bar{K}_0\} = \{(\lambda, k\pi) \mid (\lambda, 0) \in \bar{K}_{-1}\} \\
&= \{(\lambda, ((k-1)+1)\pi) \mid (\lambda, 0) \in \bar{K}_{-1}\} \\
&= \{(\lambda, ((k-1)+1)\pi) \mid (\lambda, k\pi) \in \bar{K}_{k-1}\} \\
&= \{(\lambda, k\pi) \mid (\lambda, k\pi) \in \bar{K}_{k-1}\} = \bar{K}_{k-1} \cap (\mathbf{R} \times \{k\pi\}),
\end{aligned}$$

so we have the equality in (82). To see that the set is nonempty, note that by (78), the set $\bar{K}_k \cap (\mathbf{R} \times \{k\pi\}) = \{(\lambda, k\pi) \mid (\lambda, 0) \in \bar{K}_0\}$ is nonempty.

6. EXISTENCE OF BIFURCATING CURVES

Since we have used a general topological result, Lemma 9, in our proof of Theorem 11, we could not obtain much information on the structure of the bifurcating continuum $\mathcal{C} = \bigcup_{k \in \mathbf{Z}} \mathcal{K}_k$. In particular, we do not know that it is a curve rather than a more complicated set (for example 'secondary bifurcations' might occur). Moreover, there may be other periodic solutions for $\lambda \in [\lambda_*, \lambda^*]$ besides the symmetric ones and those lying on the set $\mathcal{C}$.

A more refined analysis, which we undertake now, will yield further information. We will show that in fact, when $a > 0$ is sufficiently small ('long pendulum'), the set $\mathcal{C}$ is a smooth curve and all nonsymmetric periodic solutions in $[\lambda_*, \lambda^*] \times X$ are on $\mathcal{C}$. In contrast to the theorems presented in the previous sections, in which we obtained explicit expressions for the range of parameters for which the results were valid, in this section the results are of a perturbational nature, in that we prove that certain statements are true for sufficiently small $a > 0$ without giving an explicit value of a below which the statements are true. It should be noted, however, that while the results to be proved are perturbational with respect to a, for those values of a for which they are valid they give *global* descriptions of the set of periodic solutions of (1) for all λ in a fixed bounded interval, so these results are of a rather different nature than those of local bifurcation theory.

We continue to investigate the set $F_a^{-1}(0)$. We will prove the following:

LEMMA 10. *Assume f satisfies* (2). *Assume* $\lambda_* < \lambda^*$ *satisfy* (72) *and*

$$|h'(\lambda)| > 0 \qquad \text{for all } \lambda \in [\lambda_*, \lambda^*]. \tag{83}$$

Then there exists $a_0 > 0$ *such that when* $0 < a < a_0$ *there exists a smooth curve* C *in the* (λ, s)*-plane*,

$$C = C(a) = \{(\lambda_a(s), s) \mid s \in \mathbf{R}\},$$

where $\lambda_a \colon \mathbf{R} \to \mathbf{R}$ *is a* 2π*-periodic, even, and real-analytic function, such that*

$$F_a^{-1}(0) \cap ([\lambda_*, \lambda^*] \times \mathbf{R}) = C(a) \cup \{(\lambda, \pi k) \mid \lambda \in [\lambda_*, \lambda^*], k \in \mathbf{Z}\}.$$

The proof of Lemma 10 will be presented later in this section. We then define $w_a(s) = \Phi_a(\lambda_a(s), s)$, and obtain the following:

THEOREM 12. *Assume f satisfies* (2), $\lambda_* < \lambda^*$ *satisfies* (72) *and* (83). *Then there exists an* $a_0 > 0$ *such that when* $0 < a < a_0$ *there exists a smooth curve* $\mathscr{C}(a) \subset \mathbf{R} \times X$:

$$\mathscr{C} = \mathscr{C}(a) = \{(\lambda_a(s), w_a(s)) \mid s \in \mathbf{R}\},$$

where $\lambda_a \colon \mathbf{R} \to \mathbf{R}$, $w_a \colon \mathbf{R} \to X$ *are real-analytic*, λ_a *is* 2π*-periodic and even, such that*

(i) $\Sigma \cap ([\lambda_*, \lambda^*] \times X) = \mathscr{C} \cup (\mathscr{S} \cap ([\lambda_*, \lambda^*] \times X))$.

(ii) *For all* $k \in \mathbf{Z}$, $\lambda = \lambda_a(k\pi)$ *is the unique point of bifurcation from the curve* $\mathscr{S}_k$ *in* $[\lambda_*, \lambda^*]$, *and we have* $w_a(k\pi) = u_k(a, \lambda_a(k\pi))$ (*since* $\lambda_a(s)$ *is* 2π*-periodic, if we define* $\lambda_k = \lambda_a(k\pi)$, *we have* $\lambda_k = \lambda_0$ *for k even and* $\lambda_k = \lambda_1$ *for k odd*).

(iii) *For all* $k \in \mathbf{Z}$,

$$\gamma_T(u_k(a, \lambda)) = \begin{cases} \operatorname{sign}(h(\lambda_*))(-1)^k & \text{for all } \lambda \in [\lambda_*, \lambda_k), \\ \operatorname{sign}(h(\lambda_*))(-1)^{k+1} & \text{for all } \lambda \in (\lambda_k, \lambda^*]. \end{cases}$$

(iv) *If, in addition, we have* $c > 0$ *and* (61) *holds, and* $h(\lambda_*) > 0$ ($h(\lambda_*) < 0$), *then for k even* $u_k(a, \lambda)$ *is stable* (*unstable*) *when* $\lambda \in [\lambda_*, \lambda_0)$ *and unstable* (*stable*) *when* $\lambda \in (\lambda_0, \lambda^*]$, *while for k odd* $u_k(a, \lambda)$ *is unstable* (*stable*) *when* $\lambda \in [\lambda_*, \lambda_1)$ *and stable* (*unstable*) *when* $\lambda \in (\lambda_1, \lambda^*]$.

(v) *Each point of bifurcation from $\mathcal{S}$ which lies in* $[\lambda_*, \lambda^*] \times X$ *is either subcritical or supercritical.*

In Fig. 6, we represent the information given by Theorem 12, in the case $h(\lambda_*) > 0$. Note that in our plot the bifurcation from $\mathcal{S}_0$ is supercritical and the bifurcation from $\mathcal{S}_1$ is subcritical, but the reverse may also be the case. Note also that we have not determined the stability of the solutions on the bifurcating branch $\mathcal{C}$. Also, we do not yet know that there are no other turning points on the curve $\mathcal{C}$ besides the points of bifurcation. These matters will be addressed in the next section.

It is important to note that Theorem 12, together with Theorem 5, provides a good description of the set of all periodic solutions of (1) when λ varies in a compact interval I, for $a > 0$ sufficiently small. Let us assume that the zeroes of h in I are nondegenerate (that is, $\lambda \in \mathcal{Z} \cap I$ implies $h'(\lambda) \neq 0$), and denote these by $\bar{\lambda}_i$ $(1 \leqslant i \leqslant n)$. This assumption is not too restrictive since it may be proven that for generic f satisfying (2) *all* zeroes of h are nondegenerate. We also note that this assumption holds in the case $f(t) = a\sin(\omega t)$, by (23) and known properties of J_0. We assume also that h does not vanish at the endpoints of I. Under these assumptions we can choose a $\delta > 0$ such that the intervals $[\bar{\lambda}_i - \delta, \bar{\lambda}_i + \delta]$ $(1 \leqslant i \leqslant n)$ are disjoint and

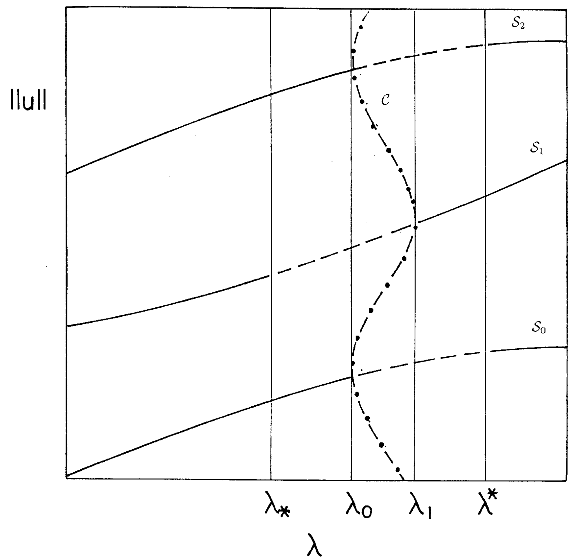

FIG. 6. Representation of the information on the bifurcation diagram which is given by Theorem 12, in the case $h(\lambda_*) > 0$.

contained in I such that $h'(\lambda) \neq 0$ for any λ in one of these intervals. Therefore for $a > 0$ sufficiently small the conclusions of Theorem 12 will hold in each of these intervals, and we shall also have

$$I - \bigcup_{i=1}^{n} [\bar{\lambda}_i - \delta, \bar{\lambda}_i + \delta] \subset \mathscr{G}_a,$$

so from Theorems 5 and 12 we conclude that there exist curves $\mathscr{C}_i = \mathscr{C}_i(a) \subset [\bar{\lambda}_i - \delta, \bar{\lambda}_i + \delta] \times X$ $(1 \leqslant i \leqslant n)$, as in Theorem 12, such that

$$\Sigma \cap (I \times X) = (\mathscr{S} \cap (I \times X)) \cup \bigcup_{i=1}^{n} \mathscr{C}_i.$$

Note that if we define

$$I_i(a) = (I - \mathscr{G}_a) \cap [\bar{\lambda}_i - \delta, \bar{\lambda}_i + \delta],$$

then, since by Theorem 5 each of the curves $\mathscr{C}_i$ must be contained in $(I - \mathscr{G}_a) \times X$, we shall have

$$\mathscr{C}_i(a) \subset I_i(a) \times X, \qquad 1 \leqslant i \leqslant n,$$

so that the λ-width of the curve $\mathscr{C}_i(a)$ approaches 0 as $a \to 0$.

Thus for a sufficiently small the bifurcation diagram for $\lambda \in I$ looks as in Fig. 7 (with the qualifications made with regard to Fig. 6 – complete justification of Fig. 7 follows from the results of the next section).

Proof of Theorem 12. Parts (i) and (ii) follow directly from Lemma 10. Part (iv) follows from (iii) and result (1) of Ortega quoted in Section 4. We now prove part (iii). It suffices to consider the cases $k = 0, 1$. We assume $h(\lambda_*) > 0$ (in the case $h(\lambda_*) < 0$ the proof is analogous), so that by Theorem 9

$$\gamma_T(u_0(a, \lambda_*)) = 1. \tag{84}$$

We claim that (84) remains true when λ_* is replaced by $\lambda \in (\lambda_*, \lambda_0)$. Indeed, this follows from the homotopy invariance of the index and the fact that, by part (ii) of our theorem, λ_0 is the unique point of bifurcation from $\mathscr{S}_0$ in $[\lambda_*, \lambda^*]$, so that $u_0(a, \lambda)$ is an isolated periodic solution of (1) when $\lambda \in [\lambda_*, \lambda_0)$.

Similarly, from Theorem 9 we have

$$\gamma_T(u_1(a, \lambda_*)) = -1$$

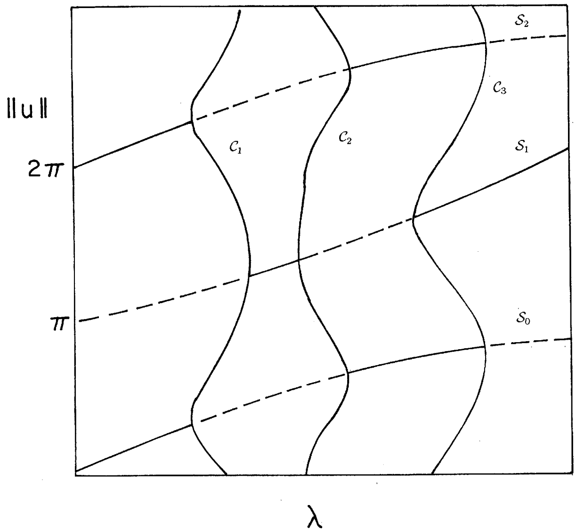

FIG. 7. Choosing a compact interval $I \subset \mathbf{R}$, then for $a > 0$ sufficiently small the bifurcation diagram for $\lambda \in I$ looks like this.

and the same argument as above implies that $\gamma_T(u_1(a,\lambda) = -1$ for all $\lambda \in [\lambda_*, \lambda_1)$.

A similar argument shows that $\gamma_T(u_0(a,\lambda)) = -1$ for $\lambda \in (\lambda_0, \lambda^*]$ and that $\gamma_T(u_1(a,\lambda)) = 1$ for $\lambda \in (\lambda_1, \lambda^*]$.

To prove part (v) of the theorem we note that the fact that $\lambda_a(s)$ is even implies that $s = 0$ is either a local minimum or a local maximum point of $\lambda_a(s)$, which implies that the bifurcation point $(\lambda_a(0), w_a(0))$ from $\mathcal{S}_0$ is either subcritical or supercritical. Similarly, using the fact that $\lambda_a(s)$ is even and 2π-periodic, we have $\lambda_a(\pi - s) = \lambda_a(s - \pi) = \lambda_a(\pi + s)$ so that $s = \pi$ is either a local maximum or a local minimum point of $\lambda_a(s)$, which implies that the bifurcation point $(\lambda_a(\pi), w_a(\pi))$ from $\mathcal{S}_1$ is either subcritical or supercritical. ∎

We now proceed to the proof of Lemma 10. To prove the existence of a real-analytic function $\lambda_a(s)$ satisfying

$$\lambda_a(s) \in (\lambda_*, \lambda^*) \qquad \text{for all } s \in \mathbf{R}, \tag{85}$$

$$F_a(\lambda_a(s), s) = 0 \qquad \text{for all } s \in \mathbf{R}, \tag{86}$$

we first need to 'desingularize' the problem by getting rid of the known zeroes of F_a, the lines $s = k\pi$ $(k \in \mathbf{Z})$. We therefore define $\tilde{F}_a\colon \mathbf{R} \to \mathbf{R}$ by

$$\tilde{F}_a(\lambda, s) = \begin{cases} \frac{F_a(\lambda,s)}{\sin(s)} & \text{if } \sin(s) \neq 0, \\ (-1)^k D_s F_a(\lambda, s) & \text{if } \sin(s) = 0. \end{cases}$$

$\tilde{F}_a$ is real analytic, and $F_a^{-1}(0)$ is the union of $\tilde{F}_a^{-1}(0)$ with the lines $s = k\pi$ $(k \in \mathbf{Z})$, hence it suffices to prove the existence of a function $\lambda_a(s)$ satisfying (85) and

$$\tilde{F}_a(\lambda_a(s), s) = 0. \tag{87}$$

We assume that $h(\lambda_*) > 0$ (the proof in the case $h(\lambda_*) < 0$ is analogous), which by (72) implies that $h(\lambda^*) < 0$, which by (83) implies

$$h'(\lambda) < 0 \qquad \text{for all } \lambda \in [\lambda_*, \lambda^*]. \tag{88}$$

We now assume that a is sufficiently small so that $\lambda_*, \lambda^* \in \mathcal{G}_a$. We then have, by part (iv) of Lemma 7, $(-1)^k F_a(\lambda_*, s)$ is positive for $s \in (k\pi, (k+1)\pi)$. Therefore, $\tilde{F}_a(\lambda, s)$ is positive for all $s \neq k\pi$. In fact, we claim that $\tilde{F}_a$ is *positive* on the whole line $\lambda = \lambda_*$. Indeed, by the above, the only points s at which $\tilde{F}_a(\lambda_*, s)$ might vanish are $s = k\pi$ $(k \in \mathbf{Z})$. However, this would imply that $F_a(\lambda_*, s)$ has a *degenerate* zero at $s = k\pi$, but it follows from Lemma 3 and the fact that λ_* satisfies (74) that the zeroes of $F_a(\lambda_*, \cdot)$ are nondegenerate when $\lambda \in \mathcal{G}_a$. By the same argument $\tilde{F}_a$ is negative on the line $\lambda = \lambda^*$.

The existence of a function $\lambda_a(s)$ satisfying (85) and (86) will then follow from the implicit function theorem if we can show that, for $a > 0$ sufficiently small,

$$D_\lambda \tilde{F}_a(\lambda, s) < 0 \qquad \text{for all } (\lambda, s) \in [\lambda_*, \lambda^*] \times \mathbf{R}. \tag{89}$$

Equation (89) also implies that, for each fixed s, $\lambda = \lambda_a(s)$ is the only solution of $F_a(\lambda, s) = 0$ in $[\lambda_*, \lambda^*]$. We note that by (75) we have $\tilde{F}_0(\lambda, s) = h(\lambda)$, hence

$$D_\lambda \tilde{F}_0(\lambda, s) = h'(\lambda).$$

Therefore, by (88), we have (89) for $a = 0$. But since $\tilde{F}_a$ is continuous with respect to a, λ, s, and since we are dealing with an inequality on a compact set ($\lambda \in [\lambda_*, \lambda^*]$, $\tilde{F}_a$ is periodic with respect to s), we have (89) for all $a > 0$ sufficiently small, as we needed.

To see that λ_a is even, we recall that by Lemma 7(ii), F_a is odd with respect to s, which implies that $\tilde{F}_a$ is even with respect to s, hence, using (87),

$$\tilde{F}_a(\lambda_a(-s), s) = \tilde{F}_a(\lambda_a(-s), -s) = 0,$$

but since, fixing s, $\lambda = \lambda_a(s)$ is the *unique* solution of the equation $\tilde{F}_a(\lambda, s) = 0$ in the interval $[\lambda_*, \lambda^*]$, we must have $\lambda_a(-s) = \lambda_a(s)$. A similar argument, using Lemma 7(i), shows the 2π-periodicity of λ_a.

7. ANALYZING THE BIFURCATING CURVES

We now assume that the hypotheses of Theorem 12 hold, and wish to study the form of the function $\lambda_a(s)$ (hence of the curve $\mathscr{C}(a)$) for $a > 0$ sufficiently small. This will enable us to obtain information on the maximal number of periodic solutions for $\lambda \in (\lambda_*, \lambda^*)$, on the nature of the bifurcations from the curves $\mathscr{S}_k$ (subcritical or supercritical), and on the stability of the periodic solutions lying on the bifurcation curve.

The basic idea now is to expand $\lambda_a(s)$ with respect to a. It will now be convenient to denote

$$\mu(a, s) = \lambda_a(s).$$

By the Taylor formula, we have

$$\mu(a, s) = \mu(0, s) + aD_a\mu(0, s) + O(a^2) \tag{90}$$

(by the periodicity of μ with respect to s this holds uniformly with respect to $s \in \mathbf{R}$).

Substituting $a = 0$ into the identity

$$B(\mu(a, s), a, s) = 0 \tag{91}$$

and using (40), we have

$$h(\mu(0, s)) \sin(s) = 0 \qquad \text{for all } s \in \mathbf{R},$$

which implies $h(\mu(0, s)) = 0$ for all $s \in \mathbf{R}$, and since (72) and (83) imply that h has a unique zero in $[\lambda_*, \lambda^*]$, which we denote by $\bar{\lambda}$, we conclude that

$$\mu(0, s) = \bar{\lambda} \qquad \text{for all } s \in \mathbf{R}. \tag{92}$$

Our aim now is to compute the derivative $D_a\mu(0,s)$. To do so we differentiate identity (91) with respect to a, obtaining

$$D_\lambda B(\mu(a,s),a,s)D_a\mu(a,s) + D_aB(\mu(a,s),a,s) = 0.$$

Setting $a = 0$ and using (92), we get

$$D_\lambda B(\bar{\lambda},0,s)D_a\mu(0,s) + D_aB(\bar{\lambda},0,s) = 0, \tag{93}$$

so to compute $D_a\mu(0,s)$ we must compute $D_\lambda B(\bar{\lambda},0,s)$ and $D_aB(\bar{\lambda},0,s)$. By (40), we obtain

$$D_\lambda B(\bar{\lambda},0,s) = \sin(s)h'(\bar{\lambda}). \tag{94}$$

To compute $D_aB(\bar{\lambda},0,s)$, we differentiate the definition of B (see (34)) with respect to a,

$$D_aB(\bar{\lambda},a,s) = (I-Q)\circ D_xN(\lambda,s+S(\bar{\lambda},a,s))(D_aS(\bar{\lambda},a,s)).$$

Substituting $a = 0$ and using (37), we get

$$D_aB(\bar{\lambda},0,s) = (I-Q)\circ D_xN(\bar{\lambda},s)(D_aS(\bar{\lambda},0,s)). \tag{95}$$

To compute $D_aS(\bar{\lambda},0,s)$, we differentiate relation (32) with respect to a and set $a = 0$, obtaining

$$D_aS(\bar{\lambda},0,s) = -[L|_{\tilde{X}}]^{-1}\circ Q(N(\bar{\lambda},s)).$$

Returning to (95), we have

$$D_aB(\bar{\lambda},0,s) = -(I-Q)\circ D_xN(\bar{\lambda},s)\circ[L|_{\tilde{X}}]^{-1}\circ Q(N(\bar{\lambda},s)). \tag{96}$$

We now compute the right-hand side of (96) explicitly. We have

$$N(\bar{\lambda},s) = \sin(\bar{\lambda}x_0(t)+s) = \sin(s)\cos(\bar{\lambda}x_0(t)) + \cos(s)\sin(\bar{\lambda}x_0(t)),$$

hence

$$\begin{aligned} Q(N(\bar{\lambda},s)) &= \sin(s)\cos(\bar{\lambda}x_0(t)) + \cos(s)\sin(\bar{\lambda}x_0(t)) \\ &\quad - \sin(s)\frac{1}{T}\int_0^T \cos(\bar{\lambda}x_0(t))\,dt - \cos(s)\frac{1}{T}\int_0^T \sin(\bar{\lambda}x_0(t))\,dt. \end{aligned} \tag{97}$$

Since $x_0 \in X_*$, the second integral on the right-hand side of (97) vanishes, and the first integral is equal to $Th(\bar{\lambda})$, and, since we have $h(\bar{\lambda}) = 0$, it vanishes too, so we have

$$Q(N(\bar{\lambda}, s)) = \sin(s)\cos(\bar{\lambda}x_0(t)) + \cos(s)\sin(\bar{\lambda}x_0(t)).$$

We now denote

$$z_1 = [L|_{\tilde{X}}]^{-1}[\cos(\bar{\lambda}x_0(t))], \qquad z_2 = [L|_{\tilde{X}}]^{-1}[\sin(\bar{\lambda}x_0(t))]$$

(by the vanishing of the integrals discussed above, we indeed have that $\cos(\bar{\lambda}x_0(t)), \sin(\bar{\lambda}x_0(t)) \in \tilde{Y}$, so z_1, z_2 are well defined), so that we have

$$[L|_{\tilde{X}}]^{-1} \circ Q(N(\bar{\lambda}, s)) = \sin(s)z_1 + \cos(s)z_2. \tag{98}$$

More explicitly, z_1, z_2 are the unique T-periodic solutions of the linear equations

$$z_1'' + cz_1' = \cos(\bar{\lambda}x_0(t)),$$

$$z_2'' + cz_2' = \sin(\bar{\lambda}x_0(t))$$

satisfying

$$\int_0^T z_1(t)\,dt = \int_0^T z_2(t)\,dt = 0.$$

Using (51) and (98), we have

$$\begin{aligned} D_xN(\bar{\lambda}, s) \circ [L|_{\tilde{X}}]^{-1} \circ Q(N(\bar{\lambda}, s)) &= \cos(\bar{\lambda}x_0(t) + s)[\sin(s)z_1 + \cos(s)z_2] \\ &= \cos(s)\sin(s)[\cos(\bar{\lambda}x_0(t))z_1 - \sin(\bar{\lambda}x_0(t))z_2] \\ &\quad + \cos^2(s)\cos(\bar{\lambda}x_0(t))z_2 - \sin^2(s)\sin(\bar{\lambda}x_0(t))z_1, \end{aligned}$$

hence, using (96),

$$\begin{aligned} &D_aB(\bar{\lambda}, 0, s) \\ &= -(I - Q) \circ D_xN(\bar{\lambda}, s) \circ [L|_{\tilde{X}}]^{-1} \circ Q(N(\bar{\lambda}, s)) \\ &= \cos(s)\sin(s)\frac{1}{T}\int_0^T [\sin(\bar{\lambda}x_0(t))z_2(t) - \cos(\bar{\lambda}x_0(t))z_1(t)]\,dt \\ &\quad + \sin^2(s)\frac{1}{T}\int_0^T \sin(\bar{\lambda}x_0(t))z_1(t)\,dt - \cos^2(s)\frac{1}{T}\int_0^T \cos(\bar{\lambda}x_0(t))z_2(t)\,dt. \end{aligned} \tag{99}$$

We now note that, since $x_0 \in X_*$, we have $\sin(\bar{\lambda} x_0(t)) \in X_*$, hence $z_2 \in X_*$. On the other hand, $\cos(\bar{\lambda} x_0(t))$ is $\frac{T}{2}$-periodic, hence so is z_1. These facts imply that

$$\int_0^T \cos(\bar{\lambda} x_0(t)) z_2(t)\, dt = 0, \quad \int_0^T \sin(\bar{\lambda} x_0(t)) z_1(t)\, dt = 0,$$

so (99) reduces to

$$D_a B(\bar{\lambda}, 0, s) = \cos(s)\sin(s)\frac{1}{T}\int_0^T [\sin(\bar{\lambda} x_0(t)) z_2(t) - \cos(\bar{\lambda} x_0(t)) z_1(t)]\, dt.$$

We also note that, using the definition of z_1, z_2, and integrating by parts, taking into account the periodicity, we have

$$\int_0^T \sin(\bar{\lambda} x_0(t)) z_2(t)\, dt = \int_0^T (z_2''(t) + c z_2'(t)) z_2(t)\, dt = -\int_0^T (z_2'(t))^2\, dt,$$

$$\int_0^T \cos(\bar{\lambda} x_0(t)) z_1(t)\, dt = \int_0^T (z_1''(t) + c z_1'(t)) z_1(t)\, dt = -\int_0^T (z_1'(t))^2\, dt.$$

Hence,

$$D_a B(\bar{\lambda}, 0, s) = \cos(s)\sin(s)\frac{1}{T}\int_0^T [(z_1'(t))^2 - (z_2'(t))^2]\, dt. \tag{100}$$

We define

$$\Lambda = \frac{1}{T}\int_0^T [(z_2'(t))^2 - (z_1'(t))^2]\, dt,$$

so we may write (100) as

$$D_a B(\bar{\lambda}, 0, s) = -\Lambda \cos(s)\sin(s). \tag{101}$$

From (93), (94), and (101), we obtain

$$D_a \mu(0, s) = \frac{\Lambda}{h'(\bar{\lambda})}\cos(s); \tag{102}$$

hence, from (90)

$$\lambda_a(s) = \mu(a, s) = \bar{\lambda} + a\frac{\Lambda}{h'(\bar{\lambda})}\cos(s) + O(a^2). \tag{103}$$

We now proceed to derive some important qualitative consequences with the aid of (103).

LEMMA 11. *Assume f satisfies* (2) *and $\lambda_* < \lambda^*$ satisfies* (72) *and* (83), *and $\Lambda \neq 0$. Then for $a > 0$ sufficiently small we have*

(i) $D_s\lambda_a(s) = 0$ *if and only if* $s = k\pi$ ($k \in \mathbf{Z}$).

(ii) *If* $\Lambda h(\lambda_*) > 0$ (<0) *then* $s = k\pi$ *is a minimum* (*maximum*) *point of* $\lambda_a(s)$ *when* k *is even, and a maximum* (*minimum*) *point of* $\lambda_a(s)$ *when* k *is odd.*

Proof. (i) Since $\lambda_a(s)$ is even and 2π-periodic, $D_s\lambda_a(s)$ is odd and 2π-periodic, which implies that $D_s\mu(a, k\pi) = 0$ for all $k \in \mathbf{Z}$. We now wish to show that (103) implies that, if $\Lambda \neq 0$, then, for $a > 0$ sufficiently small, the *only* values of s at which $D_s\lambda_a(s) = 0$ are $s = k\pi$. To do so we use Lemma 4, defining

$$g_1(s) = -a\frac{\Lambda}{h'(\bar{\lambda})}\sin(s), \qquad g_2(s) = D_s\lambda_a(s).$$

Differentiating (103) with respect to s (which is valid due to analyticity), we have

$$g_2(s) = D_s\lambda_a(s) = -a\frac{\Lambda}{h'(\bar{\lambda})}\sin(s) + O(a^2) = g_1(s) + O(a^2).$$

It is easy to see that this implies that for $a > 0$ sufficiently small, the hypotheses of Lemma 4 will hold for g_1, g_2 defined as above, enabling us to conclude that g_1, g_2 have the same number of zeroes in $[-\pi, \pi)$, and since g_1 has two zeroes in $[-\pi, \pi)$, so does $D_s\lambda_a(s)$. But since we already know that $s = -\pi$ and 0 are zeroes of $D_s\lambda_a(s)$, we conclude that $s = -\pi, 0$ are the only zeroes of $D_s\lambda_a(a, s)$ in $[-\pi, \pi)$ when a is sufficiently small, from which our claim follows.

(ii) Differentiating (103) twice with respect to s, we have

$$D_s^2\lambda_a(s) = -a\frac{\Lambda}{h'(\bar{\lambda})}\cos(s) + O(a^2). \tag{104}$$

Note now that, since the sign of $h'(\bar{\lambda})$ is opposite to the sign of $h(\lambda_*)$, the condition $\Lambda h(\lambda_*) > 0$ is equivalent to $\frac{\Lambda}{h'(\bar{\lambda})} < 0$, and the result follows at once from this and (104). ∎

The information we have derived is sufficient for obtaining a complete qualitative description of the bifurcating curve $\mathscr{C}(a)$ for $a > 0$ sufficiently small. The result depends on the sign of $\Lambda h(\lambda_*)$. Let us assume for now that $\Lambda h(\lambda_*) > 0$. In this case, by part (ii) of Lemma 11, $s = k\pi$ (k even) are the minimum points of $\lambda_a(s)$, while $s = k\pi$ (k odd) are the maximum points.

Since $\lambda_0 = \lambda_a(0)$ is the unique value of $\lambda \in [\lambda_*, \lambda^*]$ at which bifurcation from the curves $\mathscr{S}_k$ (k even) occurs and $\lambda_1 = \lambda_a(\pi)$ is the unique value at which bifurcation from the curves $\mathscr{S}_k$ (k odd) occurs, we conclude that the bifurcation from $\mathscr{S}_k$ is supercritical for k even and subcritical for k odd, and that $\lambda_1 > \lambda_0$. Additionally, by the fact that $\lambda_a(s)$ is monotone with respect to s in the intervals $[0, \pi]$ and $[\pi, 2\pi]$ we see that, for $\lambda \in (\lambda_0, \lambda_1)$, we have exactly four geometrically distinct periodic solutions. We summarize the above, and the analogous statement for the case $\Lambda h(\lambda_*) < 0$, in

THEOREM 13. *Assume f satisfies* (2) *and* $\lambda_* < \lambda^*$ *satisfy* (72) and (83). *If* $\Lambda h(\lambda_*) > 0$ (<0) *then there exists* $a_0 > 0$ *such that when* $0 < a < a_0$, *we have*

(i) *The bifurcation points* $(\lambda_0, u_k(a, \lambda_0))$ *are supercritical* (*subcritical*) *for k even and the bifurcation points* $(\lambda_1, u_k(a, \lambda_1))$ *is subcritical* (*supercritical*) *for k odd.*

(ii) $\lambda_1 > \lambda_0$ $(\lambda_0 > \lambda_1)$.

(iii) *For* $\lambda \in (\lambda_0, \lambda_1)$ $(\lambda \in (\lambda_1, \lambda_0))$, (1) *has precisely four geometrically distinct periodic solutions, and for all other* $\lambda \in [\lambda_*, \lambda^*]$, (1) *has precisely two geometrically distinct periodic solutions.*

We see that for $\lambda \in (\lambda_0, \lambda_1)$ in the case $\Lambda h(\lambda_*) > 0$ (and $\lambda \in (\lambda_1, \lambda_0)$ in the case $\Lambda h(\lambda_*) < 0$), we have exactly two geometrically distinct periodic solutions besides $u_0(a, \lambda), u_1(a, \lambda)$. We note that these solutions, which we denote by v_1, v_2, are nonsymmetric, since Theorem 3 ensures that u_k ($k \in \mathbf{Z}$) are the only symmetric solutions. We claim that the two are related to each other by the symmetry $\mathscr{T}$. Indeed, $\mathscr{T}(v_1)$ is also a periodic solution, and it cannot be equal to one of the u_k's since this would imply that v_1 is also one of the u_k's. It also cannot be geometrically equivalent to v_1, because that would make it symmetric. Hence $\mathscr{T}(v_1)$ must be geometrically equivalent to v_2.

We now wish to determine the dynamical stability of the periodic solutions on the bifurcating curve $\mathscr{C}$ in the case $c > 0$. We assume that $a > 0$ is sufficiently small so that the results of Theorem 13 hold, as well as (61). We first assume $\Lambda > 0$ and $h(\lambda_*) < 0$. Then, by Theorem 13, we have that $\lambda_0 > \lambda_1$, and the bifurcation is subcritical at $(\lambda_0, u_k(a, \lambda_0))$ (k even) and supercritical at $(\lambda_1, u_k(a, \lambda_0))$ (k odd). From Theorem 12(iii) it follows that, for all $k \in \mathbf{Z}$, $\gamma_T(u_k(a, \lambda)) = -1$ for $\lambda \in (\lambda_1, \lambda_0)$. Therefore, by result (2) of Ortega quoted in Section 4, when $\lambda \in (\lambda_1, \lambda_0)$ we have $\gamma_T(v_1) + \gamma_T(v_2) = 2$, where v_1, v_2 are the solutions on $\mathscr{C}$. But since v_1, v_2 are related by the symmetry $\mathscr{T}$ it is not hard to see that their indices must be equal, which implies $\gamma_T(v_1) = \gamma_T(v_2) = 1$, so that by result (1) of Ortega, both v_1 and v_2 are stable.

An analogous analysis can be carried out for the other three cases $\Lambda > 0$, $h(\lambda_*) > 0$, $\Lambda < 0, h(\lambda_*) > 0$, and $\Lambda < 0$, $h(\lambda_*) < 0$, and the results are summarized in

THEOREM 14. *Assume that* f *satisfies* (2) *and* $\lambda_* < \lambda^*$ satisfies (72) *and* (83), *that*

$$\Lambda > 0 \quad (< 0)$$

and that $c > 0$. *Then there exists* $a_0 > 0$ *such that when* $0 < a < a_0$ *the solutions on the bifurcating branch* $\mathscr{C}$ *are stable* (*unstable*).

Figs. 8–11 represent the information obtained from Theorems 13 and 14.

In conclusion, we see that the exchange of stability which occurs as λ varies in the interval $[\lambda_*, \lambda^*]$ can occur in two distinct ways dependent on the sign of Λ: if Λ is positive, then we observe a continuous transition, so that in some subinterval of $[\lambda_*, \lambda^*]$ the observable periodic solutions will be the nonsymmetric ones, while if $\Lambda < 0$ the transition will be a genuine jump.

To compute Λ corresponding to some zero $\bar{\lambda}$ of h, one needs to compute some integrals. Since it is the sign of Λ which determines the disposition of the corresponding curve $\mathscr{C}$ for $a > 0$ small, one may use finite-precision

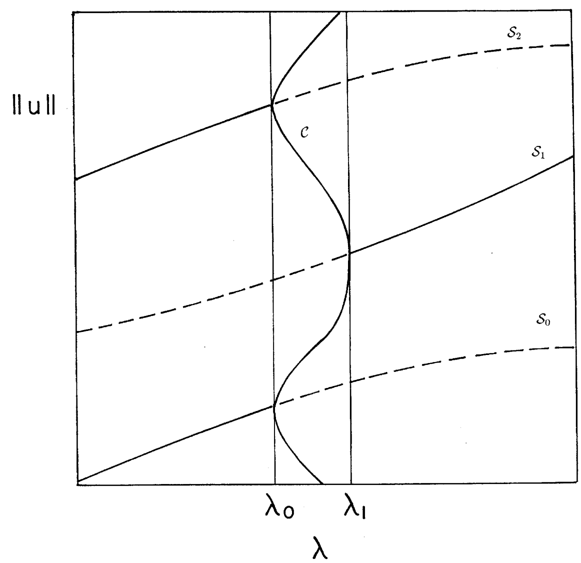

FIG. 8. Information given by Theorems 13 and 14: $h(\lambda_*) > 0$, $\Lambda > 0$.

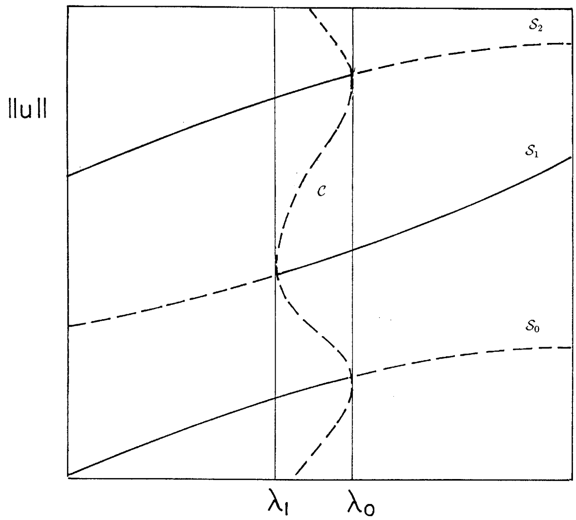

FIG. 9. Information given by Theorems 13 and 14: $h(\lambda_*) > 0$, $\Lambda < 0$.

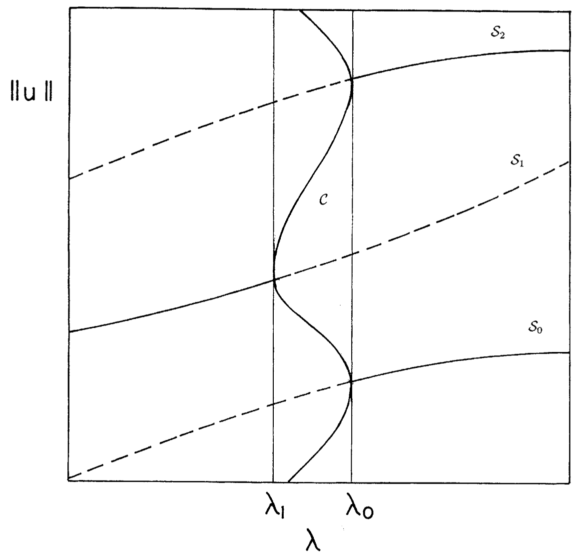

FIG. 10. Information given by Theorems 13 and 14: $h(\lambda_*) < 0$, $\Lambda > 0$.

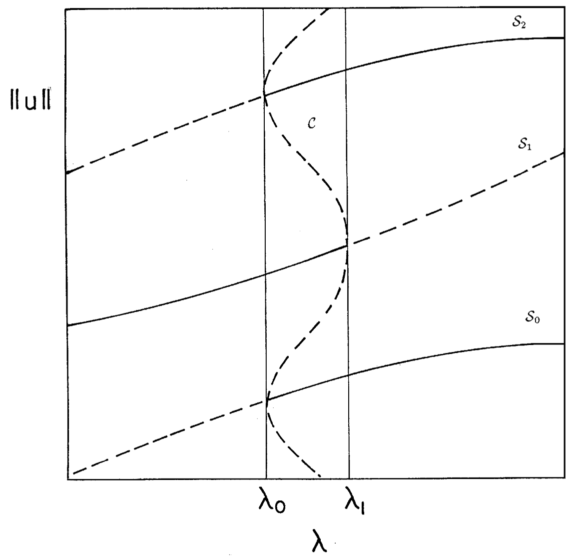

FIG. 11. Information given by Theorems 13 and 14: $h(\lambda_*) < 0$, $\Lambda < 0$.

numerical integration. We have computed the numbers Λ_i corresponding to several roots of h in the case of the sinusoidally forced pendulum with $c = 1, T = 2\pi$. We found that the Λ_i's are positive for the first 20 roots of h, so that for sufficiently small $a > 0$ the corresponding bifurcations are described by Figs. 9 and 11, which means that for some λ-intervals one will observe the *nonsymmetric oscillations*. This is indeed borne out by numerical simulations: for example in the simulation discussed in Section 4, with $f(t) = \sin(t)$, $a = \frac{1}{2}$, $c = 3$, for values of λ in the interval (7.5,7.8) one observes that the solution approaches a periodic motion whose mean value is *not* a multiple of π, that is a *nonsymmetric* periodic solution.

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