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Global existence for systems of wave equations with nonresonant nonlinearities and null forms

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Abstract

We consider the Cauchy problem for systems of nonlinear wave equations with different propagation speeds in three space dimensions. We prove global existence of small amplitude solutions for systems with some nonresonant nonlinearities which may depend on both of the unknowns and their derivatives. Our method here can be also adopted to treat the null forms.

Keywords: Nonlinear wave equation; Global existence; Null condition; Multiple speeds; Nonresonant nonlinearity

1. Introduction

This paper is devoted to the study of the Cauchy problem for systems of nonlinear wave equations in three space dimensions. We consider the system of nonlinear wave equations

$$\Box_i u_i = F_i(u, \partial u, \nabla_x \partial u) \text{ in } (0, \infty) \times \mathbb{R}^3 \quad (1 \leqslant i \leqslant m), \tag{1.1}$$

with initial data

$$u_i(0, x) = \varepsilon f_i(x), \ \ \partial_t u_i(0, x) = \varepsilon g_i(x) \text{ for } x \in \mathbb{R}^3 \quad (1 \leqslant i \leqslant m), \tag{1.2}$$

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where $u=(u_j)_{j=1}^m$, and $\Box_i=\partial_t^2-c_i^2\Delta_x$ with some positive given constants c_i. We use the notation $\partial_0=\partial_t$ and $\partial_k=\partial_{x_k}$ for $1\leqslant k\leqslant 3$ throughout this paper. In the above system (1.1), ∂u and $\nabla_x\partial u$ stand for the first and second derivatives of u, respectively. More precisely, $\partial u=(\partial_a u_j)$ with $0\leqslant a\leqslant 3$ and $1\leqslant j\leqslant m$, and $\nabla_x\partial u=(\partial_k\partial_a u_j)$ with $1\leqslant k\leqslant 3$, $0\leqslant a\leqslant 3$ and $1\leqslant j\leqslant m$.

We suppose that $F(u,v,w)=\big(F_j(u,v,w)\big)_{j=1}^m$ is a function of $(u,v,w)\in\mathbb{R}^m\times\mathbb{R}^{4m}\times\mathbb{R}^{12m}$ satisfying

$$F(u,v,w)=O(|u|^2+|v|^2+|w|^2)\ \text{near}\ (u,v,w)=(0,0,0). \tag{1.3}$$

We write the elements of the vectors $v\in\mathbb{R}^{4m}$ and $w\in\mathbb{R}^{12m}$ as $v_{j,a}$ and $w_{j,ka}$ with $1\leqslant j\leqslant m$, $1\leqslant k\leqslant 3$ and $0\leqslant a\leqslant 3$, respectively, where $v_{j,a}$ corresponds to $\partial_a u_j$, and $w_{j,ka}$ to $\partial_k\partial_a u_j$.

To assure the hyperbolicity of the system, we always assume

$$\frac{\partial F_i}{\partial w_{j,ka}}(u,v,w)=\frac{\partial F_j}{\partial w_{i,ka}}(u,v,w) \tag{1.4}$$

for any $i,\ j\in\{1,\dots,m\}$, $1\leqslant k\leqslant 3$ and $0\leqslant a\leqslant 3$. Because only classical solutions are considered in this paper, we may also assume

$$\frac{\partial F_i}{\partial w_{j,kl}}(u,v,w)=\frac{\partial F_i}{\partial w_{j,lk}}(u,v,w)\ \text{for any}\ 1\leqslant i,\ j\leqslant m\ \text{and}\ 1\leqslant k,l\leqslant 3. \tag{1.5}$$

For simplicity, we suppose that $f=(f_j)_{j=1}^m$ and $g=(g_j)_{j=1}^m$ in (1.2) belong to $C_0^\infty(\mathbb{R}^3;\mathbb{R}^m)$. ε in (1.2) is a positive parameter which is always supposed to be small. Without loss of generality, we may assume that the speed c_i in the definition of $\Box_i$ satisfies

$$0<c_1\leqslant c_2\leqslant\cdots\leqslant c_m. \tag{1.6}$$

We are interested in the condition to assure global existence of classical solutions for (1.1) and (1.2) with small ε. Since there are examples of quadratic nonlinearity for which some solution blows up in finite time no matter how small ε is, we need some restriction on the quadratic nonlinearity to get global solutions. Such a condition is known as the "Null Condition". The Null Condition was first introduced by Klainerman [14], for the single speed case, that is $c_1=\cdots=c_m$, to show the global existence of solutions for small data (see also Christodoulou [5]; for the corresponding results in two space dimensions, see [2], [3], [6], [7], [10] and [11]). The Null Condition is closely connected to the following null forms (see the discussion after Definition 1.1 below):

$$Q_0(\phi, \psi; c_i) = (\partial_t \phi)(\partial_t \psi) - c_i^2 \sum_{j=1}^{3} (\partial_j \phi)(\partial_j \psi), \tag{1.7}$$

$$Q_{ab}(\phi, \psi) = (\partial_a \phi)(\partial_b \psi) - (\partial_b \phi)(\partial_a \psi) \quad (0 \leqslant a < b \leqslant 3). \tag{1.8}$$

Now we consider the case where the propagation speeds c_i do not necessarily coincide with each other. We refer to this case as the multiple speeds case.

The global existence for the multiple speeds case with the nonlinear term F depending only on derivatives of u, i.e., $F = F(\partial u, \nabla_x \partial u)$ is studied in [16], [1], [21] and [20] (see [8] for the two space dimensional case). The multiple speeds case with F depending on both of u and its derivatives is considered in Kubota – Yokoyama [17] and the author [12] and [13].

Before we describe the results in [12] and [13], we introduce our Null Condition.

For non-negative integer p and a smooth function $G(u, v, w)$, we write $G^{(p)}(u, v, w)$ for the pth degree term of the Taylor expansion of G around the origin, that is

$$G^{(p)}(u, v, w) = \sum_{|\alpha|+|\beta|+|\gamma|=p} \frac{\partial_u^\alpha \partial_v^\beta \partial_w^\gamma G(0,0,0)}{\alpha! \beta! \gamma!} u^\alpha v^\beta w^\gamma. \tag{1.9}$$

Here we have used the standard notation of multi-indices; for example, $\partial_u^\alpha = \partial_{u_1}^{\alpha_1} \cdots \partial_{u_m}^{\alpha_m}$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $u^\alpha = u_1^{\alpha_1} \cdots u_m^{\alpha_m}$ and so on.

Given $c_1, \ldots, c_m$, we classify the indices by their corresponding speeds. We define

$$I(i) = \left\{ j \in \{1, \ldots, m\}; c_j = c_i \right\} \text{ for } 1 \leqslant i \leqslant m. \tag{1.10}$$

To state our Null Condition, we introduce

$$Y_i^m = \left\{ y = (y_1, \ldots, y_m) \in \mathbb{R}^m; \;\; y_j = 0 \text{ for all } j \notin I(i) \right\} \tag{1.11}$$

and

$$\mathcal{N}_i = \left\{ X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4; X_0^2 - c_i^2 \sum_{k=1}^{3} X_k^2 = 0 \right\} \tag{1.12}$$

for $i = 1, \dots, m$. For $y = (y_j)_{j=1}^m \in \mathbb{R}^m$ and $X = (X_a)_{a=0}^3 \in \mathbb{R}^4$, we define $V(y, X) \in \mathbb{R}^{4m}$ and $W(y, X) \in \mathbb{R}^{12m}$ by

$$V(y, X) = \big(V_{j,a}(y, X)\big)_{1 \leqslant j \leqslant m, 0 \leqslant a \leqslant 3} = (y_j X_a)_{1 \leqslant j \leqslant m, 0 \leqslant a \leqslant 3}, \tag{1.13}$$

$$\begin{aligned} W(y, X) &= \big(W_{j,ka}(y, X)\big)_{1 \leqslant j \leqslant m, 1 \leqslant k \leqslant 3, 0 \leqslant a \leqslant 3} \\ &= (y_j X_k X_a)_{1 \leqslant j \leqslant m, 1 \leqslant k \leqslant 3, 0 \leqslant a \leqslant 3}. \end{aligned} \tag{1.14}$$

Now we can state our Null Condition.

Definition 1.1. We say that $F(u, v, w) = \big(F_i(u, v, w)\big)_{i=1}^m$ satisfies the Null Condition (of degree 2), if both of the following two conditions are fulfilled:
(i) For each $i \in \{1, \dots, m\}$,

$$F_i^{(2)}\big(\lambda, V(\mu, X), W(\nu, X)\big) = 0 \tag{1.15}$$

holds for any $\lambda, \mu, \nu \in Y_i^m$ and any $X \in \mathcal{N}_i$, where V and W are given by (1.13) and (1.14), respectively.
(ii) $F^{(2)}(u, 0, 0) = 0$ holds for any $u \in \mathbb{R}^m$.

Remark. The above Null Condition coincides with that of Klainerman in [14] when $c_1 = \cdots = c_m$ (the above expression of the condition is motivated by [5]), and with the condition in [1], [21] and [20] when $F = F(\partial u, \nabla_x \partial u)$.

To simplify our exposition, we use the following notation throughout this paper: For a given function ϕ and a given family $\{\psi_\lambda\}_{\lambda \in \Lambda}$ of functions, we write $\phi = \sum\nolimits'_{\lambda \in \Lambda} \psi_\lambda$, if there exists a family $\{C_\lambda\}_{\lambda \in \Lambda}$ of constants such that $\phi = \sum_{\lambda \in \Lambda} C_\lambda \psi_\lambda$.

Now we want to derive the explicit representation of nonlinearities satisfying the Null Condition. We can easily check that F satisfies the Null Condition (of degree 2) if and only if $F_i^{(2)}$ has the form

$$F_i^{(2)} = N_i + R_i^1 + R_i^2, \tag{1.16}$$

where

$$N_i = \sideset{}{'}\sum_{\substack{j,k \in I(i) \\ |\alpha|,\ |\beta| = 0,1}} \Big\{ Q_0(\partial^\alpha u_j, \partial^\beta u_k; c_i) + \sideset{}{'}\sum_{0 \leqslant a < b \leqslant 3} Q_{ab}(\partial^\alpha u_j, \partial^\beta u_k) \Big\}, \tag{1.17}$$

$$R_i^1 = R_i^{11} + R_i^{12}, \quad R_i^2 = R_i^{21} + R_i^{22}, \tag{1.18}$$

$$R_i^{11} = \sum_{j=1}^{m} \sideset{}{'}\sum_{\substack{k \notin I(j)\\ |\alpha|,|\beta|=1,2}} (\partial^\alpha u_j)(\partial^\beta u_k), \quad R_i^{12} = \sum_{j=1}^{m} \sideset{}{'}\sum_{\substack{k \notin I(j)\\ |\alpha|=1,2}} u_j(\partial^\alpha u_k), \tag{1.19}$$

$$R_i^{21} = \sum_{j \notin I(i)} \sideset{}{'}\sum_{\substack{k \in I(j)\\ |\alpha|,|\beta|=1,2}} (\partial^\alpha u_j)(\partial^\beta u_k), \quad R_i^{22} = \sum_{j \notin I(i)} \sideset{}{'}\sum_{\substack{k \in I(j)\\ |\alpha|=1,2}} u_j(\partial^\alpha u_k). \tag{1.20}$$

Here Q_0 and Q_{ab} are null forms given by (1.7) and (1.8), respectively. We have used also the notation $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$. We refer to nonresonant terms like R_i^1 and R_i^2 as the "resonance forms", following the terminology in [1], [21] and [17].

Remark. (i) For the single speed case $c_1 = \cdots = c_m$, only the null forms N_i appear in $F_i^{(2)}$, because $I(i) = \{1, \ldots, m\}$ for any $i \in \{1, \ldots, m\}$.
(ii) Global existence for R_i^{11} was proved by Kovalyov [16]. The resonance forms R_i^{21} were treated for the first time in Agemi–Yokoyama [1].

Now we can state the known results for systems, whose nonlinearity depends on both of the unknowns and their derivatives, with multiple speeds. In the results known so far, we need some assumption in addition to the Null Condition.

In [12], the author proved global existence of small solutions when the Null Condition and the following condition (H1) are fulfilled:

(H1) There exist some polynomials $G_{i,a}(u, v)$ $(1 \leqslant i \leqslant m,\ 0 \leqslant a \leqslant 3)$ of degree 2 such that

$$F_i^{(2)}(u, \partial u, \nabla_x \partial u) = \sum_{0 \leqslant a \leqslant 3} \partial_a G_{i,a}(u, \partial u) \ (1 \leqslant i \leqslant m) \tag{1.21}$$

holds for any $u \in C^2\big((0, \infty) \times \mathbb{R}^3\big)$.

Under the Null Condition and (H1), none of N_i, R_i^{11}, R_i^{12}, R_i^{21} and R_i^{22} have to vanish, but there is a strict restriction on the coefficients. For example, the coefficient for the term $Q_0(u_j, u_k; c_i)$ $(j, k \in I(i))$, which may appear in N_i, must be equal to 0, because we cannot write it in the form of (1.21), no matter what term we add to it. The coefficient of $Q_0(u_j, \partial_a u_k; c_i)$ must coincide with that of $Q_0(\partial_a u_j, u_k; c_i)$ (observe that $Q_0(u_j, \partial_a u_k) + Q_0(\partial_a u_j, u_k) = \partial_a Q_0(u_j, u_k)$). Similarly, the coefficients of $u_j \partial_a u_k$ and $u_k \partial_a u_j$ $(k \notin I(j))$, which may appear in R_i^{12}, must coincide with each other (observe $\partial_a(u_j u_k) = (\partial_a u_j)u_k + u_j(\partial_a u_k)$). Note that $F_i^{(2)}$ can depend on u explicitly when the Null Condition and (H1) are assumed.

In [13], the author proved global existence of small solutions under the Null Condition and another kind of additional assumption

(H2) $F(u, v, w) = O(|u|^3 + |v|^2 + |w|^2)$ near $(u, v, w) = (0, 0, 0)$. In other words, $F_i^{(2)}(u, v, w)$ does not depend on u for each $i \in \{1, \ldots, m\}$.

Under the Null Condition and (H2), R_i^{12} and R_i^{22} must vanish, but there is no further restriction on N_i, R_i^{11} and R_i^{21} because they do not depend on u from the beginning. This result is an extension of the result in Kubota–Yokoyama [17].

Observe that either of (H1) and (H2) puts restriction on R_i^{12} (and also on R_i^{22}). In this paper, we give a method to treat the nonlinearity of the form R_i^{12} without any further restriction on it. Our method here works also for the null forms N_i, the resonance forms R_i^{11} and higher nonlinearity of arbitrary forms, but not for R_i^{22} at all, and we need some restriction on R_i^{21} to apply the method. More precisely, we assume the following condition, instead of (H1) or (H2):

(H3) For each $i \in \{1, \dots, m\}$, (1.15) holds not only for any $(\lambda, \mu, \nu, X) \in Y_i^m \times Y_i^m \times Y_i^m \times \mathcal{N}_i$, but also for any

$$(\lambda, \mu, \nu, X) \in \bigcup_{\substack{j=1 \\ j \notin I(i)}}^{m} (Y_j^m \times Y_j^m \times Y_j^m \times \mathcal{N}_j).$$

The null condition and (H3) are satisfied if and only if each $F_i^{(2)}$ has the form (1.16) with the following special R_i^{21} and R_i^{22}:

$$R_i^{21} = \sum_{j \notin I(i)} \sum_{\substack{k \in I(j) \\ |\alpha|, |\beta| = 0,1}}' \left\{ Q_0(\partial^\alpha u_j, \partial^\beta u_k; c_j) + \sum_{0 \leqslant a < b \leqslant 3}' Q_{ab}(\partial^\alpha u_j, \partial^\alpha u_k) \right\}, \tag{1.22}$$

$$R_i^{22} = 0. \tag{1.23}$$

We emphasize again that (H3) places no further restriction on R_i^{11}, R_i^{12} and N_i. Our main result is the following:

Theorem 1.1. *Assume that* (1.4) *holds. Suppose that the Null Condition* (*of degree* 2) *and* (H3) *are fulfilled. Then, for any f, g* $\in C_0^\infty(\mathbb{R}^3; \mathbb{R}^m)$, *there exists a positive constant* ε_0 *such that, for any* $\varepsilon \in (0, \varepsilon_0]$, *the Cauchy problem* (1.1) *and* (1.2) *admits a unique global solution* $u \in C^\infty\big([0, \infty) \times \mathbb{R}^3; \mathbb{R}^m\big)$.

Remark. The assumption that f and g have compact support is not essential at all in our result. Since the constant ε_0 in the above theorem does not depend on the size of support of data explicitly in our proof, we can show the same result for more general data by using the standard approximation technique.

The proof of Theorem 1.1 will be given in Section 5. The main ingredient of our method lies in the estimate of the L^2 norms of u. To take advantage of the difference of speeds in nonlinear terms contained in R_i^{11} and R_i^{12}, we use the decay of the energy of u_i outside the light cone corresponding to the speed c_i (see Lemmas 3.5 and 3.9

below). We also need some decomposition of nonlinearity to treat R_i^{11} and R_i^{12}(see Lemma 5.4 below).

The usage of the decay of the energy outside the light cone in this paper is motivated by the methods of [15], [19] and [20] in some sense. Their main purpose of using the decay of the energy is to avoid the direct estimation of the fundamental solution, however we need the direct estimation here because decay estimates much better than theirs are needed in our proof to treat the nonlinearity depending on u itself.

2. Notations

In this section, we introduce some notations which will be used throughout this paper.

For each $i \in \{1, \ldots, m\}$, we write $U_i^*[f, g]$ for the solution to the Cauchy Problem

$$\begin{cases} \Box_i U_i^*[f,g](t,x) = 0 & \text{in } (0,\infty)\times\mathbb{R}^3, \\ U_i^*[f,g](0,x) = f(x), \ \ \partial_t U_i^*[f,g](0,x) = g(x) & \text{for } x \in \mathbb{R}^3. \end{cases}$$

Similarly, $U_i[\Phi]$ stands for the solution to the Cauchy problem

$$\begin{cases} \Box_i U_i[\Phi](t,x) = \Phi(t,x) & \text{in } (0,\infty)\times\mathbb{R}^3, \\ U_i[\Phi](0,x) = \partial_t U_i[\Phi](0,x) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases}$$

We introduce vector fields

$$S = t\partial_t + \sum_{j=1}^{3} x_j\partial_j \ \text{ and } \ \Omega_{jk} = x_j\partial_k - x_k\partial_j \ \text{ for } 1\leqslant j < k\leqslant 3. \tag{2.1}$$

We define also a family Γ of vector fields by

$$\Gamma_0 = S,\ \ \Gamma_1 = \Omega_{12},\ \ \Gamma_2 = \Omega_{13},\ \ \Gamma_3 = \Omega_{23},\ \ \Gamma_k = \partial_{k-4}\ \ (4\leqslant k\leqslant 7). \tag{2.2}$$

Using a multi-index $\alpha = (\alpha_0, \ldots, \alpha_7)$, we write Γ^α for the product $\Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1}\cdots\Gamma_7^{\alpha_7}$. We also use the notation $\partial^\beta = \partial_0^{\beta_0}\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$ and $\partial_x^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}\partial_3^{\gamma_3}$.

The following property can be easily checked by direct calculation:

$$\begin{aligned} &[S, \partial_a] = -\partial_a,\ \ [S, \Omega_{jk}] = 0, \\ &[\Omega_{jk}, \partial_a] = -\delta_{aj}\partial_k + \delta_{ak}\partial_j, \\ &[\Omega_{jk}, \Omega_{pq}] = \delta_{jp}\Omega_{qk} + \delta_{jq}\Omega_{kp} - \delta_{kp}\Omega_{qj} - \delta_{kq}\Omega_{jp}, \end{aligned}$$

for $0 \leqslant a \leqslant 3$, $1 \leqslant j < k \leqslant 3$ and $1 \leqslant p < q \leqslant 3$, where δ_{ab} is the Kronecker delta, and Ω_{jk} for $j > k$ is given by $\Omega_{jk} = -\Omega_{kj}$. From these identities we obtain

$$\Gamma^\alpha \Gamma^\beta \phi = \Gamma^{\alpha+\beta}\phi + \sum_{|\gamma| \leqslant |\alpha|+|\beta|-1}{}' \Gamma^\gamma \phi, \tag{2.3}$$

$$\partial_a \Gamma^\alpha \phi = \Gamma^\alpha \partial_a \phi + \sum_{\substack{0 \leqslant b \leqslant 3 \\ |\beta| \leqslant |\alpha|-1}}{}' \Gamma^\beta \partial_b \phi, \quad \Gamma^\alpha \partial_a \phi = \partial_a \Gamma^\alpha \phi + \sum_{\substack{0 \leqslant b \leqslant 3 \\ |\beta| \leqslant |\alpha|-1}}{}' \partial_b \Gamma^\beta \phi \tag{2.4}$$

for any smooth function ϕ. We have also $[\Box_i, \Gamma_0] = 2\Box_i$ and $[\Box_i, \Gamma_j] = 0$ for $1 \leqslant j \leqslant 7$, which lead to

$$\Box_i(\Gamma^\alpha \phi) = \Gamma^\alpha(\Box_i \phi) + \sum_{|\beta| \leqslant |\alpha|-1}{}' \Gamma^\beta(\Box_i \phi). \tag{2.5}$$

For a non-negative integer s, $1 \leqslant p \leqslant \infty$ and a smooth function $\phi(t,x)$, we define

$$|\phi(t,x)|_s = \sum_{|\alpha| \leqslant s} |\Gamma^\alpha \phi(t,x)|, \text{ and } \|\phi(t,\cdot)\|_{s,p} = \big\| |\phi(t,\cdot)|_s \big\|_{L^p(\mathbb{R}^3)}. \tag{2.6}$$

For a non-negative integer s, and a smooth function $f(x)$, we define

$$\|f\|^2_{H^{s,1}} = \sum_{|\alpha| \leqslant s} \int_{\mathbb{R}^3} (1+|x|)^2 |\partial_x^\alpha f(x)|^2 \, dx. \tag{2.7}$$

For $i = 1, \ldots, m$, we also introduce

$$L_{i,k} = \frac{x_k}{c_i}\partial_t + c_i t \partial_k \ (k = 1,2,3). \tag{2.8}$$

$L_{i,k}$ together with the vector fields belonging to Γ played an important role in the study of the single speed case, but the usage of $L_{i,k}$ will be restricted here, because $[\Box_j, L_{i,k}] = 0$ holds if and only if $c_j = c_i$.

3. Energy inequalities and decay of the energy

We start this section with the standard energy inequalities.

Lemma 3.1. *Let $f \in H^1(\mathbb{R}^3)$, $g \in L^2(\mathbb{R}^3)$ and $\Phi \in L^1\big([0,T); L^2(\mathbb{R}^3)\big)$. Then we have*

$$\|\partial U_i^*[f,g](t,\cdot)\|_{L^2(\mathbb{R}^3)} \leqslant C\left(\|\nabla_x f\|_{L^2(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)}\right), \tag{3.1}$$

$$\|\partial U_i[\Phi](t,\cdot)\|_{L^2(\mathbb{R}^3)} \leqslant C\int_0^t \|\Phi(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}\,d\tau \tag{3.2}$$

for any $t \in [0,T)$, where C is a constant independent of T.

From the classical theory for symmetric hyperbolic systems, we also have

Lemma 3.2. *Let $\phi = (\phi_i)_{i=1}^m$ be a smooth solution to*

$$\partial_t^2\phi_i(t,x) - \sum_{j=1}^m \sum_{\substack{1\leqslant k\leqslant 3\\ 0\leqslant a\leqslant 3}} G_{k,a}^{i,j}(t,x)\partial_k\partial_a\phi_j(t,x) = \Phi_i(t,x) \ \ in\ \ (0,T)\times\mathbb{R}^3$$

for $i=1,\dots,m$. We suppose that $G_{k,a}^{i,j} = G_{k,a}^{j,i}$ and $G_{k,l}^{i,j} = G_{l,k}^{i,j}$ hold for any $i,j \in \{1,\dots,m\}$, $1\leqslant k,l\leqslant 3$ and $0\leqslant a\leqslant 3$. We assume also that there exists a positive and uniform constant M such that

$$M^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^m \sum_{k,l=1}^3 G_{k,l}^{i,j}(t,x)\xi_{i,k}\xi_{j,l} \leqslant M|\xi|^2$$

holds for any $\xi = (\xi_{i,k})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant k\leqslant 3}} \in \mathbb{R}^{3m}$, where $|\xi|^2 = \sum_{i=1}^m\sum_{k=1}^3 \xi_{i,k}^2$.

Then we have

$$\begin{aligned}\|\partial\phi(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leqslant{}& C\|\partial\phi(0,\cdot)\|_{L^2(\mathbb{R}^3)} + C\int_0^t \|\partial G(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)}\|\partial\phi(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}\,d\tau\\ &+ C\int_0^t \|\Phi(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}\,d\tau,\end{aligned} \tag{3.3}$$

where $G = (G_{k,a}^{i,j})$ and $\Phi = (\Phi_i)$. Here the constant C depends only on the above constant M.

The following conformal energy, which was used also in Klainerman [14], plays an extremely important role in our proof, since it also provides us with the control for decay of the energy (see Lemma 3.5 below). Notice that following Lemma 3.3 and the proof of Lemma 3.5 are the only points where the vector fields $L_{i,j}$ enter in our proof.

Lemma 3.3. *Let $1\leqslant i\leqslant m$. Suppose ϕ to be a smooth solution of*

$$(\partial_t^2 - c_i^2\Delta_x)\phi(t,x) = \Phi(t,x) \quad in\ \ (0,T)\times\mathbb{R}^3 \tag{3.4}$$

with initial data $\phi = f$ and $\partial_t\phi = g$ at $t=0$.

Assume that ϕ vanishes sufficiently fast at spatial infinity. Then we have

$$\sum_{|\alpha|\leqslant 1} \|\Gamma^{\alpha}\phi(t,\cdot)\|_{L^2} + \sum_{j=1}^{3} \|L_{i,j}\phi(t,\cdot)\|_{L^2}$$
$$\leqslant C\left(\|f\|_{H^{1,1}} + \|g\|_{H^{0,1}}\right) + C\int_0^t \|w_+(\tau,|\cdot|)\Phi(\tau,\cdot)\|_{L^2}\,d\tau, \tag{3.5}$$

where $w_+(t,|x|) = 1+t+|x|$.

Proof. Using a certain change of variables, we may assume $c_i = 1$. For simplicity of exposition, we write L_j for $L_{i,j}$ with $c_i = 1$, i.e., $L_j = x_j\partial_t + t\partial_j$. We introduce

$$|\phi(t,x)|^2_{\Gamma,L,1} = \sum_{|\alpha|\leqslant 1} |\Gamma^{\alpha}\phi(t,x)|^2 + \sum_{j=1}^{3} |L_j\phi(t,x)|^2.$$

We also define

$$K = (1+t^2+|x|^2)\partial_t + 2tx\cdot\nabla_x + 2t = \partial_t + t(S+2) + \sum_{j=1}^{3} x_j L_j.$$

Multiplying (3.4) by $K\phi$, and then doing integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3} E[\phi](t,x)\,dx = \int_{\mathbb{R}^3} (K\phi)(t,x)\Phi(t,x)\,dx, \tag{3.6}$$

where

$$E[\phi](t,x) = \frac{1}{2}(1+t^2+|x|^2)\left\{(\partial_t\phi)^2 + \sum_{j=1}^{3}(\partial_j\phi)^2\right\} + \sum_{j=1}^{3} 2tx_j(\partial_j\phi)(\partial_t\phi)$$
$$+ 2t\phi(\partial_t\phi) - \phi^2 \tag{3.7}$$

(see Klainerman [14] for the details).

We see that there exists a constant C such that we have

$$\frac{1}{C}\int_{\mathbb{R}^3} |\phi(t,x)|^2_{\Gamma,L,1}\,dx \leqslant \int_{\mathbb{R}^3} E[\phi](t,x)\,dx \leqslant C\int_{\mathbb{R}^3} |\phi(t,x)|^2_{\Gamma,L,1}\,dx \tag{3.8}$$

for any smooth function v. In fact, using vector fields belonging to Γ, we can rewrite $E[\phi]$ as

$$2E[\phi] = (\partial_t\phi)^2 + \sum_{j=1}^{3}(\partial_j\phi)^2 + (S\phi)^2 + \sum_{j=1}^{3}(L_j\phi)^2 + \sum_{1\leqslant j<k\leqslant 3}(\Omega_{jk}\phi)^2 + 4t\phi(\partial_t\phi) - 2\phi^2. \tag{3.9}$$

By writing $t\partial_t\phi = S\phi - \sum_{j=1}^{3} x_j(\partial_j\phi)$, and then using integration by parts, we obtain

$$\int_{\mathbb{R}^3} t\phi(\partial_t\phi)\,dx = \int_{\mathbb{R}^3}\phi(S\phi)\,dx + \frac{3}{2}\int_{\mathbb{R}^3}\phi^2\,dx. \tag{3.10}$$

Set $\partial_r = \sum_{j=1}^{3}(x_j/|x|)\partial_j$ and $L_r = \sum_{j=1}^{3}(x_j/|x|)L_j = |x|\partial_t + t\partial_r$. By writing $t\partial_t\phi = (t/|x|)L_r\phi - (t^2/|x|)\partial_r\phi$ and integrating by parts, we also get

$$\int_{\mathbb{R}^3} t\phi(\partial_t\phi)\,dx = \int_{\mathbb{R}^3}\frac{t}{|x|}\phi(L_r\phi)\,dx + \frac{1}{2}\int_{\mathbb{R}^3}\frac{t^2}{|x|^2}\phi^2\,dx. \tag{3.11}$$

By (3.9) and (3.10), we can easily show the second half of (3.8).

From (3.10) and (3.11), we see

$$\begin{aligned}\int\{4t\phi(\partial_t\phi) - 2\phi^2\}\,dx &= 3\int\phi(S\phi)\,dx + \frac{5}{2}\int\phi^2\,dx \\ &\quad + \int\frac{t}{|x|}\phi(L_r\phi)\,dx + \frac{1}{2}\int\frac{t^2}{|x|^2}\phi^2\,dx \\ &\geqslant -\delta^2\int(S\phi)^2\,dx + \left(\frac{5}{2} - \frac{9}{4\delta^2}\right)\int\phi^2\,dx \\ &\quad -\frac{1}{2}\int(L_r\phi)^2\,dx \end{aligned} \tag{3.12}$$

for any $\delta > 0$. Since we have $(L_r\phi)^2 \leqslant \sum_{j=1}^{3}(L_j\phi)^2$, we obtain the first half of (3.8) from (3.9) and (3.12) by choosing δ close to 1 so that we have $\frac{9}{10} < \delta^2 < 1$.

Now, we define $\|\phi(t)\|_E^2 = \int E[\phi](t,x)\,dx$. Since we have

$$|K\phi(t,x)| \leqslant C(1+t+|x|)|\phi(t,x)|_{\Gamma,L,1},$$

from (3.6) and (3.8) we see

$$\begin{aligned} \frac{d}{dt}\|\phi(t)\|_E^2 &\leqslant C\int w_+(t,|x|)|\Phi(t,x)|\,|\phi(t,x)|_{\Gamma,L,1}\,dx \\ &\leqslant C\|w_+(t,|\cdot|)\Phi(t,\cdot)\|_{L^2}\|\phi(t)\|_E. \end{aligned} \tag{3.13}$$

Gronwall's lemma applied to (3.13) implies

$$\|\phi(t)\|_E \leqslant \|\phi(0)\|_E + C\int_0^t \|w_+(\tau,|\cdot|)\Phi(\tau,\cdot)\|_{L^2}\,d\tau. \tag{3.14}$$

In view of (3.8), this completes the proof. □

As an apparent consequence of Lemma 3.3, we have the following:

Corollary 3.4. *Let* $i \in \{1,\dots,m\}$. *Then we have*

$$\|U_i[\Phi](t,\cdot)\|_{1,2} \leqslant C\int_0^t \|w_+(\tau,|\cdot|)\Phi(\tau,\cdot)\|_{L^2(\mathbb{R}^3)}\,d\tau, \tag{3.15}$$

$$\left\|U_i^*[f,g](t,\cdot)\right\|_{1,2} \leqslant C\big(\|f\|_{H^{1,1}} + \|g\|_{H^{0,1}}\big). \tag{3.16}$$

where $w_+(t,|x|) = 1+t+|x|$ *for* $t>0$ *and* $x \in \mathbb{R}^3$.

To treat R_i^{11} and R_i^{12}, we need to estimate the decay of the energy outside the corresponding light cone. This also can be done through Lemma 3.3.

Lemma 3.5. *Define* $w_+(t,r) = 1+t+r$ *and* $w_i(t,r) = 1+|c_i t - r|$. *Then we have*

$$\left\|w_i(t,|\cdot|)\partial U_i[\Phi](t,\cdot)\right\|_{L^2} \leqslant C\int_0^t \|w_+(\tau,|\cdot|)\Phi(\tau,\cdot)\|_{L^2}\,d\tau, \tag{3.17}$$

$$\left\|w_i(t,|\cdot|)\partial U_i^*[f,g](t,\cdot)\right\|_{L^2} \leqslant C\big(\|f\|_{H^{1,1}} + \|g\|_{H^{0,1}}\big) \tag{3.18}$$

for $1 \leqslant i \leqslant m$.

Proof. Eqs. (3.17) and (3.18) are immediate consequences of Lemma 3.3 and the following inequality which is essentially due to Lindblad [18]:

$$w_i(t,|x|)|\partial\phi(t,x)| \leqslant C\left(\sum_{|\alpha|=1}|\Gamma^\alpha\phi(t,x)| + \sum_{j=1}^3 |L_{i,j}\phi(t,x)|\right) \tag{3.19}$$

for any sufficiently smooth function ϕ. Here we give a proof of (3.19).

If $\big|c_i t - |x|\big| \geqslant 1$, (3.19) is shown by the following identities which can be checked easily:

$$\big(c_i^2 t^2 - |x|^2\big)\partial_t \phi = c_i^2 t(S\phi) - c_i \sum_{j=1}^{3} x_j L_{i,j}\phi,$$

$$\big(c_i^2 t^2 - |x|^2\big)\partial_j \phi = c_i t(L_{i,j}\phi) - x_j(S\phi) + \sum_{k\neq j} x_k(\Omega_{jk}\phi) \quad (j = 1, 2, 3).$$

On the other hand, if $\big|c_i t - |x|\big| \leqslant 1$, (3.19) is a triviality because the operators ∂_a $(0 \leqslant a \leqslant 3)$ are included in Γ. □

We turn our attention to the inhomogeneous wave equations with force terms written in the form of divergence.

Lemma 3.6. *For* $0 \leqslant a \leqslant 3$, *we have*

$$\|U_i[\partial_a \Psi](t, \cdot)\|_{1,2} \leqslant C \int_0^t \|\Psi(\tau, \cdot)\|_{1,2}\, d\tau + C\|\Psi(0, \cdot)\|_{H^{0,1}}. \tag{3.20}$$

Proof. The following identity can be checked easily:

$$U_i[\partial_a \Psi] = \partial_a U_i[\Psi] - \delta_{0a} U_i^*[0, \Psi(0, \cdot)], \tag{3.21}$$

where δ_{0a} is the Kronecker delta.

By Corollary 3.4, we have

$$\|U_i^*[0, \Psi(0, \cdot)](t, \cdot)\|_{1,2} \leqslant C\|\Psi(0, \cdot)\|_{H^{0,1}}. \tag{3.22}$$

Set $\phi = U_i[\Psi]$. By (2.5), we get $\Box_i(\Gamma^\alpha \phi) = \sum'_{|\beta|\leqslant 1} \Gamma^\beta \Psi$ for $|\alpha| \leqslant 1$. It is easy to see $\Gamma^\alpha \phi(0, x) = 0$, while $\partial_t \Gamma^\alpha \phi(0, x)$ is equal to either 0 or $\Psi(0, x)$. Therefore the standard energy inequality implies

$$\|\partial(\Gamma^\alpha \phi)(t, \cdot)\|_{L^2} \leqslant C\left(\|\Psi(0, \cdot)\|_{L^2} + \int_0^t \|\Psi(\tau, \cdot)\|_{1,2}\, d\tau\right) \quad \text{for } |\alpha| \leqslant 1. \tag{3.23}$$

Since we have $\|\partial\phi(t, \cdot)\|_{1,2} \leqslant \sum_{|\alpha|\leqslant 1} \|\partial\Gamma^\alpha \phi(t, \cdot)\|_{L^2}$ from (2.4), we obtain the result from (3.21), (3.22) and (3.23). □

For $w_i \partial U_i[\partial_a \Psi]$, we have a better estimate than Lemma 3.5. We begin with two lemmas which can be found in Klainerman–Sideris [15].

Lemma 3.7. *Define* $\sigma_i(t,r) = \sqrt{1+|c_i t - r|^2}$. *Let* $\phi \in C^2\big((0,\infty)\times\mathbb{R}^3\big)$. *Then we have*

$$\sigma_i(t,|x|)|\Delta\phi(t,x)| \leqslant C\Bigg(\sum_{|\alpha|\leqslant 1}|\partial\Gamma^\alpha\phi(t,x)| + t\,|\Box_i\phi(t,x)|\Bigg), \tag{3.24}$$

$$\sigma_i(t,|x|)|\partial_t^2\phi(t,x)| \leqslant C\Bigg(\sum_{|\alpha|\leqslant 1}|\partial\Gamma^\alpha\phi(t,x)| + |x|\,|\Box_i\phi(t,x)|\Bigg), \tag{3.25}$$

$$\sigma_i(t,|x|)|\nabla_x\partial_t\phi(t,x)| \leqslant C\Bigg(\sum_{|\alpha|\leqslant 1}|\partial\Gamma^\alpha\phi(t,x)| + t\,|\Box_i\phi(t,x)|\Bigg). \tag{3.26}$$

Proof. By using an appropriate change of variables, we can assume $c_i = 1$, and the above result for $c_i = 1$ is exactly Lemma 2.3 of [15]. □

Lemma 3.8. *Let* σ_i *be defined as before. Let* ϕ *be a smooth function decaying sufficiently fast at spatial infinity. Then we have*

$$\|\sigma_i\big(t,|\cdot|\big)\nabla_x\partial\phi(t,\cdot)\|_{L^2}\leqslant C\Bigg(\sum_{|\alpha|\leqslant 1}\|\partial\Gamma^\alpha\phi(t,\cdot)\|_{L^2} + t\|\Box_i\phi(t,\cdot)\|_{L^2}\Bigg), \tag{3.27}$$

$$\|\sigma_i(t,|\cdot|)\partial_t^2\phi(t,\cdot)\|_{L^2}\leqslant C\Bigg(\sum_{|\alpha|\leqslant 1}\|\partial\Gamma^\alpha\phi(t,\cdot)\|_{L^2} + \|r(\cdot)\Box_i\phi(t,\cdot)\|_{L^2}\Bigg), \tag{3.28}$$

where $r(x) = |x|$.

Proof. Eq. (3.28) is a direct consequence of (3.25). Similarly, from (3.24) and (3.26), we can easily see that $\|\sigma_i\Delta\phi\|_{L^2}$ and $\|\sigma_i\nabla_x\partial_t\phi\|_{L^2}$ are bounded by the right-hand side of (3.27). On the other hand, with the help of integration by parts, we get

$$\sum_{j,k=1}^{3}\|\sigma_i\partial_j\partial_k\phi\|_{L^2}^2\leqslant C\left(\|\partial\sigma_i\|_{L^\infty}^2\|\partial\phi\|_{L^2}^2 + \|\sigma_i\Delta\phi\|_{L^2}^2\right),$$

and $\sum_{j,k=1}^3\|\sigma_i\partial_j\partial_k\phi\|_{L^2}$ turns out to be dominated by the right-hand side of (3.27). See the proof of Lemma 3.1 in [15] for the details. □

Now we get the decay estimate of the energy.

Lemma 3.9. *For* $0 \leqslant a \leqslant 3$, *we have*

$$\begin{aligned}&\left\| w_i(t, |\cdot|)\partial U_i[\partial_a \Psi](t,\cdot)\right\|_{L^2}\\&\quad\leqslant C\int_0^t \|\Psi(\tau,\cdot)\|_{1,2}\, d\tau + C\left\{\|w_+(t,|\cdot|)\Psi(t,\cdot)\|_{L^2} + \|\Psi(0,\cdot)\|_{H^{0,1}}\right\}.\end{aligned} \tag{3.29}$$

Proof. Set $\phi = U_i[\Psi]$ and $\phi_0 = U_i^*[0, \Psi(0)]$. Then, as in the proof of Lemma 3.6, we have $U_i[\partial_a\Psi] = \partial_a\phi - \delta_{0a}\phi_0$. Therefore we get

$$\|w_i\partial U_i[\partial_a\Psi]\|_{L^2} \leqslant \|w_i\partial^2\phi\|_{L^2} + \|w_i\partial\phi_0\|_{L^2}. \tag{3.30}$$

Since $w_i \leqslant C\sigma_i$, Lemma 3.8 yields

$$\|w_i\partial^2\phi\|_{L^2} \leqslant \sum_{|\alpha|\leqslant 1}\|\partial\Gamma^\alpha\phi\|_{L^2} + \|w_+(t,|\cdot|)\Psi(t,\cdot)\|_{L^2}. \tag{3.31}$$

Now, using (3.23) to estimate $\|\partial\Gamma^\alpha\phi\|_{L^2}$ in (3.31), and (3.18) to estimate $\|w_i\partial\phi_0\|_{L^2}$ in (3.30), we obtain the result. □

4. Weighted $L^\infty - L^\infty$ decay estimates

In this section, we give a brief description of $L^\infty - L^\infty$ decay estimates.

Recall the definitions of w_+ and w_i: $w_+(t,r) = 1+t+r$ and $w_i(t,r) = 1 + |c_i t - r|$ for $i = 1,\dots,m$. First we consider homogeneous wave equations.

Lemma 4.1. *For a smooth function h on* $\mathbb{R}^3$, *a non-negative integer s and a positive constant* ν, *we define*

$$M_{s,\nu}[h] = \sup_{x\in\mathbb{R}^3}\sum_{|\alpha|\leqslant s}\left|(1+|x|)^{2+\nu}\partial_x^\alpha h(x)\right|.$$

Then, for $\nu > 0$, *we have*

$$w_+(t,|x|)w_i(t,|x|)^\nu |U_i^*[f,g](t,x)| \leqslant C\left(M_{1,\nu}[f] + M_{0,\nu}[g]\right), \tag{4.1}$$

$$w_+(t,|x|)w_i(t,|x|)^\nu |\partial U_i^*[f,g](t,x)| \leqslant C\left(M_{2,\nu}[f] + M_{1,\nu}[g]\right) \tag{4.2}$$

for any $(t,x) \in (0,\infty)\times\mathbb{R}^3$, *where the constant C depends only on* ν.

Proof. See Asakura [4] (see also Proposition 3.3 and the subsequent remark in Kubota–Yokoyama [17]). $\square$

To describe the result for inhomogeneous wave equations, we introduce several notations. For given constants c_i $(1 \leqslant i \leqslant m)$ satisfying $c_1 \leqslant \cdots \leqslant c_m$, we set c_0 and c_{m+1} by

$$c_0 = 0, \tag{4.3}$$

$$c_{m+1} = \frac{1}{3} \min_{j \in I}(c_j - c_{j-1}) \text{ with } I = \left\{ j \in \{1, \dots, m\}; c_j - c_{j-1} \neq 0 \right\}. \tag{4.4}$$

For $(t, r) \in [0, \infty) \times [0, \infty)$, we define

$$w_0(t, r) = 1 + |c_0 t - r| = 1 + r. \tag{4.5}$$

We also introduce subsets Λ_i $(1 \leqslant i \leqslant m)$ of $[0, \infty) \times [0, \infty)$ by

$$\Lambda_i = \left\{ (\tau, \lambda) \in [1, \infty) \times [1, \infty); |c_i \tau - \lambda| \leqslant c_{m+1} \tau \right\} \tag{4.6}$$

and Λ_0 by

$$\Lambda_0 = \{[0, \infty) \times [0, \infty)\} \setminus \bigcup_{i=1}^{m} \Lambda_i. \tag{4.7}$$

For a non-negative integer s, $v \geqslant 0$ and $i \in \{1, \dots, m\}$, we define

$$\langle \phi(t, x) \rangle_{v,s}^{(i)} = \begin{cases} w_+(t, |x|) \left(1 + \log \frac{w_+(c_i t, |x|)}{w_i(t, |x|)}\right)^{-1} |\phi(t, x)|_s, & \text{if } v = 0, \\ w_+(t, |x|) w_i(t, |x|)^v |\phi(t, x)|_s, & \text{if } v > 0, \end{cases} \tag{4.8}$$

$$[\phi(t, x)]_{v,s}^{(i)} = w_0(t, |x|) w_i(t, |x|)^v |\phi(t, x)|_s. \tag{4.9}$$

The weight functions $z_\mu(\tau, \lambda)$ and $\widetilde{z}_\mu(\tau, \lambda)$ on $[0, \infty) \times [0, \infty)$ are defined by

$$z_\mu(\tau, \lambda) = w_+(\tau, \lambda)^{1+\mu} w_i(\tau, \lambda)^{1-\mu} \quad \text{if } (\tau, \lambda) \in \Lambda_i \ (0 \leqslant i \leqslant m), \tag{4.10}$$

$$\widetilde{z}_\mu(\tau, \lambda) = w_+(\tau, \lambda) w_i(\tau, \lambda)^{1+\mu} \quad \text{if } (\tau, \lambda) \in \Lambda_i \ (0 \leqslant i \leqslant m). \tag{4.11}$$

Now we are in a position to describe the result due to Kubota–Yokoyama [17].

Lemma 4.2. (i) *For* $\mu > 0$ *and* $\nu, \rho \geqslant 0$, *we have*

$$w_+(t,|x|)^{-\rho}\langle U_i[\Phi](t,x)\rangle_{\nu,s}^{(i)} \leqslant C \sup_{\substack{\tau\in[0,t]\\ y\in\mathbb{R}^3}} |y|w_+(\tau,|y|)^{\nu-\rho} z_\mu(\tau,|y|)|\Phi(\tau,y)|_s + C\sum_{|\alpha|\leqslant s-1} M_{0,\nu}[\Gamma^\alpha\Phi(0,\cdot)], \tag{4.12}$$

$$w_+(t,|x|)^{-\rho}\left[\partial U_i[\Phi](t,x)\right]_{1+\nu,s}^{(i)} \leqslant C \sup_{\substack{\tau\in[0,t]\\ y\in\mathbb{R}^3}} |y|w_+(\tau,|y|)^{\nu-\rho} z_\mu(\tau,|y|)|\Phi(\tau,y)|_{s+1} + C\sum_{|\alpha|\leqslant s-1} M_{1,\nu}[\Gamma^\alpha\Phi(0,\cdot)], \tag{4.13}$$

where $M_{s,\nu}$ *is defined in Lemma* 4.1.
(ii) *For* $\mu > 0$ *and* $\nu > 0$, *we have*

$$\langle U_i[\Phi](t,x)\rangle_{\nu,s}^{(i)} \leqslant C \sup_{\substack{\tau\in[0,t]\\ y\in\mathbb{R}^3}} |y|w_+(\tau,|y|)^{\nu}\widetilde{z}_\mu(\tau,|y|)|\Phi(\tau,y)|_s + C\sum_{|\alpha|\leqslant s-1} M_{0,\nu}[\Gamma^\alpha\Phi(0,\cdot)]. \tag{4.14}$$

Proof. Part (i) of the above lemma is nothing but Corollary 3.6 of [17]. Hence we only give a sketch of the proof for (4.14), which in fact is also proved implicitly in the proof of Theorem 3.7 in [17]. It suffices to prove the case $s = 0$, because (4.14) for general s can be obtained by applying Lemma 4.1 and (4.14) with $s = 0$ to the equation of $\Gamma^\alpha U_i[\Phi]$ with $|\alpha| \leqslant s$.

Set $r = |x|$. Without loss of generality, we may assume $c_i = 1$. By the explicit representation formula of the solution due to John [9] (see also [17] for instance), we have

$$|U_i[\Phi](t,x)| \leqslant C I(t,r) \sup_{\substack{\tau\in[0,t]\\ y\in\mathbb{R}^3}} |y|w_+(\tau,|y|)^{\nu}\widetilde{z}_\mu(\tau,|y|)|\Phi(\tau,y)|, \tag{4.15}$$

where

$$I(t,r) = \frac{1}{r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} w_+(\tau,\lambda)^{-\nu}\widetilde{z}_\mu(\tau,\lambda)^{-1}\,d\lambda\,d\tau.$$

From the definition of $\widetilde{z}_\mu$, we get

$$I(t,r)\leqslant \sum_{j=0}^{m}\frac{1}{r}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau} w_+(\tau,\lambda)^{-1-\nu}w_j(\tau,\lambda)^{-1-\mu}\,d\lambda\,d\tau \equiv \sum_{j=0}^{m} I_j(t,r). \tag{4.16}$$

For each I_j, introducing new variables by $p=\tau+\lambda$ and $q=\lambda-c_j\tau$, we obtain

$$I_j(t,r)=\frac{1}{(c_j+1)r}\int_{|t-r|}^{t+r}(1+p)^{-1-\nu}\,dp\int_{p_j}^{p}(1+|q|)^{-1-\mu}\,dq, \tag{4.17}$$

where $2p_j=(1-c_j)p+(1+c_j)(r-t)$. Since we have $\mu>0$, we get

$$\int_{p_j}^{p}(1+|q|)^{-1-\mu}dq\leqslant\int_{-\infty}^{\infty}(1+|q|)^{-1-\mu}dq=\frac{2}{\mu},$$

which implies

$$I_j(t,r)\leqslant\frac{C}{r}\int_{|t-r|}^{t+r}(1+p)^{-1-\nu}\,dp. \tag{4.18}$$

By explicit calculation, (4.18) leads to

$$I_j(t,r)\leqslant Cr^{-1}(1+|t-r|)^{-\nu}. \tag{4.19}$$

If $r\geqslant t/2$ and $r\geqslant 1/2$, then we have $r\geqslant(1+t+r)/5$ and (4.19) implies $I_j(t,r)\leqslant C(1+t+r)^{-1}(1+|t-r|)^{-\nu}$.

On the other hand, since the integrand in (4.18) is less than $(1+|t-r|)^{-\nu-1}$, (4.18) leads to

$$I_j(t,r)\leqslant C\frac{(t+r)-|t-r|}{r}(1+|t-r|)^{-\nu-1}\leqslant 2C(1+|t-r|)^{-\nu-1}. \tag{4.20}$$

If we have either $r\leqslant t/2$ or $r\leqslant 1/2$, then we get $1+|t-r|\geqslant(1+t+r)/4$. Therefore (4.20) implies $I_j(t,r)\leqslant C(1+t+r)^{-1}(1+|t-r|)^{-\nu}$ for such t and r. Summing up, we have proved $I(t,r)\leqslant C(1+t+r)^{-1}(1+|t-r|)^{-\nu}$ for all $(t,r)\in[0,\infty)\times[0,\infty)$, and in view of (4.15), this completes the proof of (4.14) for $s=0$. $\square$

5. Proof of the main theorem

First of all, we recall the estimates for the null forms.

Lemma 5.1. *Let $i \in \{1, \dots, m\}$, and s be a positive integer. Then we have*

$$|Q_0(\phi_1, \phi_2; c_i)(t,x)|_s \leqslant C w_+(t,|x|)^{-1} w_i(t,|x|) |\partial\phi|_{[\frac{s}{2}]} |\partial\phi|_s + w_+(t,|x|)^{-1}\big(|\partial\phi|_{[\frac{s}{2}]}|\phi|_{s+1} + |\phi|_{[\frac{s}{2}]+1}|\partial\phi|_s\big), \tag{5.1}$$

$$|Q_{ab}(\phi_1, \phi_2)(t,x)|_s \leqslant w_+(t,|x|)^{-1}\big(|\partial\phi|_{[\frac{s}{2}]}|\phi|_{s+1} + |\phi|_{[\frac{s}{2}]+1}|\partial\phi|_s\big) \tag{5.2}$$

for any (t,x) with $(t,|x|) \in \Lambda_i$, and for any smooth function

$$\phi(t,x) = \big(\phi_1(t,x), \phi_2(t,x)\big).$$

For the proof, see the author [13] (see also [21] and [20]).

From now on, we suppose that the assumptions in Theorem 1.1 are fulfilled. Let $u(t,x)$ be a local solution of (1.1) – (1.2) for $0 \leqslant t < T$ with some $T > 0$. We fix some integer K and three positive constants ρ, v_1 and v_2 satisfying

$$K \geqslant 9, \ \ 0 < \rho < 1/2, \ \ 0 < v_1 < v_2 < 1 - 2\rho.$$

We define

$$E(T) = \sup_{0 \leqslant t < T} \sum_{k=1}^{8} e_k(t), \tag{5.3}$$

where

$$e_1(t) = \sup_{x \in \mathbb{R}^3} \sum_{i=1}^{m} \langle u_i(t,x)\rangle^{(i)}_{1,K+2},$$

$$e_2(t) = \sup_{x \in \mathbb{R}^3} \sum_{i=1}^{m} [\partial u_i(t,x)]^{(i)}_{1+v_1,2K-6}, \quad e_3(t) = \sup_{x \in \mathbb{R}^3} \sum_{i=1}^{m} \langle u_i(t,x)\rangle^{(i)}_{v_2,2K-5},$$

$$e_4(t) = \sup_{x \in \mathbb{R}^3} \sum_{i=1}^{m} w_+(t,|x|)^{-2\rho} [\partial u_i(t,x)]^{(i)}_{1,2K-4},$$

$$e_5(t) = \sup_{x \in \mathbb{R}^3} \sum_{i=1}^{m} w_+(t,|x|)^{-2\rho} \langle u_i(t,x)\rangle^{(i)}_{0,2K-3},$$

$$e_6(t) = (1+t)^{-\rho} \sum_{i=1}^{m} \big\| w_i(t,|\cdot|) |\partial u_i(t,\cdot)|_{2K-1} \big\|_{L^2},$$

$$e_7(t) = (1+t)^{-\rho} \|u(t,\cdot)\|_{2K,2}, \quad e_8(t) = (1+t)^{-\rho} \|\partial u(t,\cdot)\|_{2K,2}.$$

Here, as before, w_+ and w_j for $0 \leqslant j \leqslant m$ are given by $w_+(t,r) = 1+t+r$ and $w_j(t,r) = 1 + |c_j t - r|$, respectively. Also remember that we have set $c_0 = 0$.

Let $j \in \{0, 1, \ldots, m\}$. Then, from the definition of Λ_j given by (4.6) and (4.7), we get

$$C^{-1} w_+(t,r) \leqslant w_j(t,r) \leqslant C w_+(t,r) \text{ for any } (t,r) \notin \Lambda_j. \tag{5.4}$$

Consequently we also have

$$w_k(t,r) \leqslant C w_j(t,r) \text{ for any } (t,r) \in \Lambda_k, \tag{5.5}$$

where $j, k \in \{0, 1, \ldots, m\}$.

The main result in this section is the following proposition:

Proposition 5.2. *Let $u \in C^\infty([0,T); \mathbb{R}^m)$ be a solution to the Cauchy problem* (1.1) *and* (1.2) *with some $T > 0$, and $E(T)$ be given by* (5.3). *Suppose that the assumptions in Theorem* 1.1 *are fulfilled. Then there exist positive constants B, C_0 and ε_1, which are independent of T, such that $E(T) \leqslant B$ implies*

$$E(T) \leqslant C_0 \big(\varepsilon + E(T)^2\big), \tag{5.6}$$

provided $\varepsilon \leqslant \varepsilon_1$.

From Proposition 5.2, using the standard bootstrap argument, we see that there exist positive constants ε_0 and M, which are independent of T, such that $E(T) \leqslant M\varepsilon$ holds for $\varepsilon \leqslant \varepsilon_0$. Theorem 1.1 follows immediately from this *a priori* bound for $E(T)$ in view of the local existence theorem. Therefore the remainder of this section will be devoted to the proof of Proposition 5.2. We suppose $\varepsilon \ll 1$ and $E(T) \ll 1$ in the following.

5.1. Estimate for $e_8(t)$

Let $|\alpha| \leqslant 2K$, and set $g^{i,j}_{k,a} = \frac{\partial F_i}{\partial w_{j,ka}}(u, \partial u, \nabla_x \partial u)$. By (2.5), we have

$$\Box_i (\Gamma^\alpha u_i) - \sum_{j,k,a} g^{i,j}_{k,a} \partial_k \partial_a (\Gamma^\alpha u_j) = \widetilde{F}_{i,\alpha}, \tag{5.7}$$

where

$$\widetilde{F}_{i,\alpha} = \Gamma^\alpha F_i - \sum_{j,k,a} g^{i,j}_{k,a} \partial_k \partial_a (\Gamma^\alpha u_j) + {\sum_{|\beta| \leqslant 2K-1}}' \Gamma^\beta F.$$

By (2.4), we have

$$\left|\widetilde{F}_{i,\alpha}\right| \leqslant C|u|_{K+2}\left(|u|_{2K} + |\partial u|_{2K}\right). \tag{5.8}$$

It is easy to see $|\partial g_{k,a}^{i,j}| \leqslant C|u|_{K+2}$. Since we have

$$C^{-1}\|\partial u\|_{2K,2} \leqslant \sum_{|\alpha|\leqslant 2K} \|\partial \Gamma^{\alpha} u\|_{L^2} \leqslant C\|\partial u\|_{2K,2}$$

by virtue of (2.4), Lemma 3.2 applied to (5.7) yields

$$\begin{aligned}\|\partial u(t,\cdot)\|_{2K,2} \ &\leqslant \ C\varepsilon + C\int_0^t (1+\tau)^{\rho-1} e_1(\tau)\,\{e_7(\tau)+e_8(\tau)\}\, d\tau \\ &\leqslant \ C\left(\varepsilon + (1+t)^{\rho} E(T)^2\right) \text{ for } 0 \leqslant t < T.\end{aligned} \tag{5.9}$$

Here we have used also (5.8). As an immediate consequence of (5.9), we obtain

$$e_8(t) \leqslant C\left(\varepsilon + E(T)^2\right) \text{ for } 0 \leqslant t < T. \tag{5.10}$$

5.2. Estimate for $e_6(t)$ and $e_7(t)$

Set $H_i(u,v,w) = F_i(u,v,w) - F_i^{(2)}(u,v,w)$ for $1 \leqslant i \leqslant m$. Then we have

$$H_i(u,v,w) = O(|u|^3 + |v|^3 + |w|^3) \text{ near } (u,v,w) = (0,0,0). \tag{5.11}$$

Since F satisfies the Null Condition and (H3), using the decomposition (1.16) of $F_i^{(2)}$, we get

$$F_i = N_i + R_i^{11} + R_i^{12} + R_i^{21} + H_i, \tag{5.12}$$

where N_i is given by (1.17), R_i^{11} and R_i^{12} by (1.19), while R_i^{21} satisfies (1.22).

First we claim

Lemma 5.3. *Set $\widetilde{u}_i = (u_j)_{j\in I(i)}$. Then we have*

$$\begin{aligned}|N_i(t,x)|_s \ &\leqslant \ Cw_+(t,|x|)^{-1} w_i(t,|x|) \left|\partial\widetilde{u}_i\right|_{[\frac{s}{2}]+1} \left|\partial\widetilde{u}_i\right|_{s+1} \\ &\quad + Cw_+(t,|x|)^{-1}\left\{\left|\partial\widetilde{u}_i\right|_{[\frac{s}{2}]+1} |\widetilde{u}_i|_{s+1} + |\widetilde{u}_i|_{[\frac{s}{2}]+2}\left|\partial\widetilde{u}_i\right|_{s+1}\right\}\end{aligned} \tag{5.13}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}^3$ satisfying $(t,|x|) \in \Lambda_i$. We also have

$$|N_i(t,x)|_s \leqslant C\left|\partial\widetilde{u}_i\right|_{[\frac{s}{2}]+1} \left|\partial\widetilde{u}_i\right|_{s+1} \tag{5.14}$$

for any $(t,x) \in (0,\infty)\times\mathbb{R}^3$.

Proof. Eq. (5.13) is a consequence of Lemma 5.1. On the other hand, (5.14) follows from the fact that N_i is a quadratic function of $(\partial^\alpha \widetilde{u}_i)$ with $1 \leqslant |\alpha| \leqslant 2$. $\square$

By Lemma 5.3, we have

$$|N_i(t,x)|_{2K-1} \leqslant C w_+^{-2}(t,|x|) e_1(t) \big(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K} \big) \tag{5.15}$$

for any $(t,x) \in [0,T) \times \mathbb{R}^3$. In fact, if $(t,|x|) \in \Lambda_i$, (5.13) implies

$$|N_i|_{2K-1} \leqslant C w_+^{-1} w_i \, |\widetilde{u}_i|_{K+2} \left(|u|_{2K} + |\partial u|_{2K} \right) \leqslant w_+^{-2} e_1 \big(|u|_{2K} + |\partial u|_{2K} \big).$$

On the other hand, if $(t,|x|) \notin \Lambda_i$, (5.14) and (5.4) lead to

$$|N_i|_{2K-1} \leqslant C \, |\widetilde{u}_i|_{K+2} \, |\partial u|_{2K} \leqslant w_+^{-2} e_1(t) |\partial u|_{2K}.$$

Because R_i^{21} has the same structure as N_i, we also have

$$|R_i^{21}(t,x)|_{2K-1} \leqslant C w_+(t,|x|)^{-2} e_1(t) \big(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K} \big). \tag{5.16}$$

As for H_i, it is easy to see

$$|H_i(t,x)|_{2K-1} \leqslant C w_+(t,|x|)^{-2} e_1(t)^2 \big(|u(t,x)|_{2K-1} + |\partial u(t,x)|_{2K} \big). \tag{5.17}$$

To treat $R_i^1 \left(= R_i^{11} + R_i^{12}\right)$, we need some decomposition. This decomposition is one of the main idea in our proof.

Lemma 5.4. *Let* $|\alpha| \leqslant 2K-1$. *Define* $R = \{(j,k); 1 \leqslant j,k \leqslant m, c_j \neq c_k\}$. *Then, there exist some functions* $R^1_{i,\alpha,*}$ *and* $R^1_{i,\alpha,a}$ *such that*

$$\Gamma^\alpha R_i^1 = R^1_{i,\alpha,*} + \sum_{a=0}^{3} \partial_a R^1_{i,\alpha,a}, \tag{5.18}$$

$$|R^1_{i,\alpha,*}(t,x)| \leqslant C \sum_{(j,k)\in R} |u_j(t,x)|_{K+2} |\partial u_k(t,x)|_{2K-1}, \tag{5.19}$$

$$|R^1_{i,\alpha,a}(t,x)|_1 \leqslant C \sum_{(j,k)\in R} |u_j(t,x)|_{K+2} \big(|u_k(t,x)|_{2K} + |\partial u_k(t,x)|_{2K}\big). \tag{5.20}$$

Proof. From the definition of R^1_i, it suffices to prove the result for $R^1_i = (\partial_b u_j)(\partial_c u_k)$, $(\partial_b u_j)(\partial_c \partial_d u_k)$, $(\partial_b \partial_c u_j)(\partial_d \partial_e u_k)$, $u_j(\partial_b u_k)$ and $u_j(\partial_b \partial_c u_k)$ with $c_j \neq c_k$.

First we consider the case $R^1_i = u_j(\partial_b u_k)$. Since we have

$$\Gamma^\alpha R^1_i = \sideset{}{'}\sum_{|\beta|+|\gamma|=|\alpha|} (\Gamma^\beta u_j)(\Gamma^\gamma \partial_b u_k),$$

it suffices to find the decomposition for each $(\Gamma^\beta u_j)(\Gamma^\gamma \partial_b u_k)$. If $|\beta| \leqslant |\gamma|$, we have $|\beta| \leqslant K-1$ and $|\gamma| \leqslant 2K-1$. Therefore, $|(\Gamma^\beta u_j)(\Gamma^\gamma \partial_b u_k)|$ itself is dominated by the right-hand side of (5.19). If $|\beta| \geqslant |\gamma|$, using (2.4), we can write

$$\begin{aligned}
(\Gamma^\beta u_j)(\Gamma^\gamma \partial_b u_k) &= \sideset{}{'}\sum_{\substack{0\leqslant b'\leqslant 3\\ |\gamma'|\leqslant|\gamma|}} (\Gamma^\beta u_j)(\partial_{b'} \Gamma^{\gamma'} u_k) \\
&= \sideset{}{'}\sum_{\substack{0\leqslant b'\leqslant 3\\ |\gamma'|\leqslant|\gamma|}} \Big(\partial_{b'} \Big\{(\Gamma^\beta u_j)(\Gamma^{\gamma'} u_k)\Big\} - (\partial_{b'} \Gamma^\beta u_j)(\Gamma^{\gamma'} u_k)\Big) \qquad (5.21)\\
&= \sideset{}{'}\sum_{\substack{0\leqslant b'\leqslant 3\\ |\gamma'|\leqslant|\gamma|}} \partial_{b'} \Big\{(\Gamma^\beta u_j)(\Gamma^{\gamma'} u_k)\Big\} + \sideset{}{'}\sum_{\substack{0\leqslant b''\leqslant 3\\ |\beta'|\leqslant|\beta| \\ |\gamma'|\leqslant|\gamma|}} (\Gamma^{\beta'} \partial_{b''} u_j)(\Gamma^{\gamma'} u_k).
\end{aligned}$$

Since we have $|\beta'| \leqslant |\beta| \leqslant 2K-1$ and $|\gamma'| \leqslant |\gamma| \leqslant K-1$, we get

$$|(\Gamma^\beta u_j)(\Gamma^{\gamma'} u_k)|_1 \leqslant C|u_k|_K |u_j|_{2K}, \tag{5.22}$$

$$|(\Gamma^{\beta'} \partial_{b''} u_j)(\Gamma^{\gamma'} u_k)| \leqslant C|u_k|_{K-1} |\partial u_j|_{2K-1}. \tag{5.23}$$

The above two inequalities imply the desired result.

Next, we consider the case $R^1_i = u_j(\partial_b \partial_c u_k)$. As before, it suffices to consider $(\Gamma^\beta u_j)(\Gamma^\gamma \partial_b \partial_c u_k)$ with $|\beta| + |\gamma| \leqslant 2K-1$. If $|\beta| \leqslant |\gamma|$, we decompose it as

$$\begin{aligned}
(\Gamma^\beta u_j)(\Gamma^\gamma \partial_b \partial_c u_k) &= \Big((\Gamma^\beta u_j)(\Gamma^\gamma \partial_b \partial_c u_k) - \partial_b \Big\{(\Gamma^\beta u_j)(\Gamma^\gamma \partial_c u_k)\Big\}\Big) \\
&\quad + \partial_b \Big\{(\Gamma^\beta u_j)(\Gamma^\gamma \partial_c u_k)\Big\}. \qquad (5.24)
\end{aligned}$$

Noting that (2.4) implies

$$|\Gamma^\gamma \partial_b \partial_c u_k - \partial_b \Gamma^\gamma \partial_c u_k| \leqslant C|\partial^2 u_k|_{|\gamma|-1},$$

we can easily check the estimates corresponding to (5.19) and (5.20). Observing that (5.21), (5.22) and (5.23) are still valid if we replace u_k by $\partial_c u_k$, we obtain the desired result also for $|\beta| \geqslant |\gamma|$.

The remaining cases are easier. If $R_i^1 = (\partial_b u_j)(\partial_c u_k)$, then $\Gamma^\alpha R_i$ itself enjoys the estimate (5.19) by virtue of the Leibniz formula. For $R_i^1 = (\partial_b u_j)(\partial_c \partial_d u_k)$, the desired result can be obtained by using the following decomposition which is similar to (5.24):

$$\Gamma^\alpha R_i^1 = \left(\Gamma^\alpha R_i^1 - \partial_c \left\{(\partial_b u_j)(\Gamma^\alpha \partial_d u_k)\right\}\right) + \partial_c \left\{(\partial_b u_j)(\Gamma^\alpha \partial_d u_k)\right\} \equiv R^1_{i,\alpha,*} + \partial_c R^1_{i,\alpha,c}.$$

We can treat the case $R_i^1 = (\partial_b \partial_c u_j)(\partial_d \partial_e u_k)$ in a similar manner. This completes the proof. $\square$

If $(j,k) \in R$, since we have $c_j \neq c_k$, (5.4) implies

$$w_+(t,|x|)^{-1} w_j(t,|x|)^{-1} w_k(t,|x|)^{-1} \leqslant C w_+(t,|x|)^{-2}$$

for any $(t,x) \in [0,T) \times \mathbb{R}^3$. Therefore (5.19) leads to

$$|R^1_{i,\alpha,*}(t,x)| \leqslant C w_+(t,|x|)^{-2} e_1(t) \sum_{k=1}^m w_k(t,|x|) |\partial u_k(t,x)|_{2K-1} \tag{5.25}$$

for $|\alpha| \leqslant 2K-1$. By (5.20), we also obtain

$$|R^1_{i,\alpha,a}(t,x)|_1 \leqslant C w_+(t,|x|)^{-1} e_1(t) \left(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K}\right). \tag{5.26}$$

Now we are in a position to estimate $e_6(t)$ and $e_7(t)$. For $|\alpha| \leqslant 2K-1$, (2.5) and Lemma 5.4 imply

$$\begin{aligned}\square_i(\Gamma^\alpha u_i) &= \sum_{|\beta| \leqslant 2K-1}{}' \Gamma^\beta (N_i + R_i^1 + R_i^{21} + H_i) \\ &= \sum_{|\beta| \leqslant 2K-1}{}' \left\{\Gamma^\beta (N_i + R_i^{21} + H_i) + R^1_{i,\beta,*}\right\} + \sum_{|\beta| \leqslant 2K-1}{}' \sum_{a=0}^3 \partial_a R^1_{i,\beta,a}.\end{aligned}$$

We set $F_{i,\beta} = \Gamma^{\beta}(N_i + R_i^{21} + H_i) + R^1_{i,\beta,*}$. Then we have

$$\Gamma^{\alpha} u_i = U_i^*[f_{i,\alpha}, g_{i,\alpha}] + \sum_{|\beta| \leqslant 2K-1}{}' U_i[F_{i,\beta}] + \sum_{|\beta| \leqslant 2K-1}{}' \sum_{a=0}^{3} U_i[\partial_a (R^1_{i,\beta,a})], \tag{5.27}$$

where $f_{i,\alpha} = \Gamma^{\alpha} u_i(0)$ and $g_{i,\alpha} = \partial_t \Gamma^{\alpha} u_i(0)$.

From Corollary 3.4 and Lemma 3.5, we get

$$\|U_i^*[f_{i,\alpha}, g_{i,\alpha}](t,\cdot)\|_{1,2} + \|w_i(t,|\cdot|)\partial U_i^*[f_{i,\alpha}, g_{i,\alpha}](t,\cdot)\|_{L^2} \leqslant C\varepsilon. \tag{5.28}$$

(5.15), (5.16), (5.17) and (5.25) lead to

$$\begin{aligned} |F_{i,\beta}(t,x)| \leqslant\ & C w_+(t,|x|)^{-2} e_1(t) \Big(|u(t,x)|_{2K} + |\partial u(t,x)|_{2K} \\ & + \sum_{j=1}^{m} w_j(t,|x|) |\partial u_j(t,x)|_{2K-1} \Big) \end{aligned}$$

for any β with $|\beta| \leqslant 2K-1$. Therefore we obtain

$$\|w_+(t,|\cdot|) F_{i,\beta}(t,\cdot)\|_{L^2} \leqslant C(1+t)^{\rho-1} e_1(t) \big(e_6(t) + e_7(t) + e_8(t) \big). \tag{5.29}$$

By virtue of (5.29), Corollary 3.4 and Lemma 3.5 yield

$$\|U_i[F_{i,\beta}](t,\cdot)\|_{1,2} + \|w_i(t,|\cdot|)\partial U_i[F_{i,\beta}](t,\cdot)\|_{L^2} \leqslant C(1+t)^{\rho} E(T)^2 \tag{5.30}$$

for $0 \leqslant t < T$.

From (5.26), we have

$$\begin{aligned} &\|R^1_{i,\beta,a}(t,\cdot)\|_{1,2} \leqslant C(1+t)^{\rho-1} e_1(t) \big(e_7(t) + e_8(t) \big), \\ &\|w_+(t,\cdot) R_{i,\beta,a}(t,\cdot)\|_{L^2} \leqslant C(1+t)^{\rho} e_1(t) \big(e_7(t) + e_8(t) \big) \end{aligned}$$

and therefore Lemmas 3.6 and 3.9 lead to

$$\|U_i[\partial_a R^1_{i,\beta,a}](t,\cdot)\|_{1,2} + \|w_i \partial U_i[\partial_a R^1_{i,\beta,a}](t,\cdot)\|_{L^2} \leqslant C \big(\varepsilon^2 + (1+t)^{\rho} E(T)^2 \big) \tag{5.31}$$

for $0 \leqslant t < T$.

From (5.27), (5.28), (5.30) and (5.31), we obtain

$$\|\Gamma^{\alpha} u_i(t,\cdot)\|_{1,2} + \|w_i(t,|\cdot|)\partial \Gamma^{\alpha} u_i(t,\cdot)\|_{L^2} \leqslant C \big(\varepsilon + (1+t)^{\rho} E(T)^2 \big) \tag{5.32}$$

for $|\alpha| \leqslant 2K - 1$. Since we have $\|u_i(t,\cdot)\|_{2K,2} \leqslant C \sum_{|\alpha| \leqslant 2K-1} \|\Gamma^\alpha u_i(t,\cdot)\|_{1,2}$ and $\big\| w_i |\partial u_i(t,\cdot)|_{2K-1} \big\|_{L^2} \leqslant C \sum_{|\alpha| \leqslant 2K-1} \|w_i \partial \Gamma^\alpha u_i(t,\cdot)\|_{L^2}$, (5.32) implies

$$e_6(t) + e_7(t) \leqslant C\big(\varepsilon + E(T)^2\big) \text{ for } 0 \leqslant t < T. \tag{5.33}$$

5.3. Estimate for $e_4(t)$ and $e_5(t)$

Let $j \in \{0, 1, \ldots, m\}$. Then (5.5) implies

$$|u(t,x)|_{K+2} \leqslant C w_+(t,|x|)^{-1} w_j(t,|x|)^{-1} e_1(t) \tag{5.34}$$

for any (t,x) satisfying $(t,|x|) \in \Lambda_j$.

Since $F_i = O(|u|^2 + |\partial u|^2 + |\nabla_x \partial u|^2)$, using the Sobolev-type inequality

$$|x|\,|\phi(x)|_s \leqslant C_s \|\phi\|_{s+2,2}$$

which holds for any $\phi \in \mathscr{S}(\mathbb{R}^3)$ and a non-negative integer s (see Klainerman–Sideris [15] for the proof), we obtain

$$\begin{aligned} |y|\,|F_i(\tau,y)|_{2K-3} &\leqslant C|u(\tau,y)|_{K+2}|y|\big(|u(\tau,y)|_{2K-3} + |\partial u(\tau,y)|_{2K-2}\big) \\ &\leqslant C(1+\tau)^\rho |u(\tau,y)|_{K+2}\big(e_7(t) + e_8(t)\big). \end{aligned} \tag{5.35}$$

(5.34) and (5.35) lead to

$$\begin{aligned} &|y| w_+(\tau,|y|)^{-2\rho} z_\mu(\tau,|y|) |F_i(\tau,y)|_{2K-3} \\ &\quad \leqslant C w_+(\tau,|y|)^{\mu-\rho} w_j(\tau,|y|)^{-\mu} e_1(t)\big(e_7(t)+e_8(t)\big) \\ &\quad \leqslant C E(T)^2 \end{aligned} \tag{5.36}$$

for any (τ, y) satisfying $(\tau,|y|) \in \Lambda_j$ $(j = 0, 1, \ldots, m)$, provided $\mu \leqslant \rho$. Therefore Lemmas 4.1 and 4.2 (i) lead to

$$e_4(t) + e_5(t) \leqslant C\big(\varepsilon + E(T)^2\big) \text{ for } 0 \leqslant t < T. \tag{5.37}$$

5.4. Estimate for $e_1(t)$, $e_2(t)$ and $e_3(t)$

In this subsection, we will prove

$$e_1(t) + e_2(t) + e_3(t) \leqslant C\big(\varepsilon + E(T)^2\big) \text{ for } 0 \leqslant t < T, \tag{5.38}$$

which completes the proof of Proposition 5.2 together with (5.10), (5.33) and (5.37). We use Lemmas 4.1 and 4.2 (i) to estimate $e_2(t)$, and Lemma 4.2 (ii), instead of (i), to get control of $e_1(t)$ and $e_3(t)$. Then it turns out that our task is to show

$$\sup_{(\tau,y)\in[0,T)\times\mathbb{R}^3} \left\{ I_\mu[F_i](\tau, y) \right\} \leqslant C E(T)^2, \tag{5.39}$$

for sufficiently small μ, where, for a smooth function Φ, $I_\mu[\Phi]$ is given by

$$I_\mu[\Phi](\tau, y) = I_{1,\mu}[\Phi](\tau, y) + I_{2,\mu}[\Phi](\tau, y) + I_{3,\mu}[\Phi](\tau, y)$$

with

$$\begin{aligned} I_{1,\mu}[\Phi](\tau, y) &= |y| w_+(\tau, |y|) \widetilde{z}_\mu(\tau, |y|) |\Phi(\tau, y)|_{K+2}, \\ I_{2,\mu}[\Phi](\tau, y) &= |y| w_+(\tau, |y|)^{\nu_1} z_\mu(\tau, |y|) |\Phi(\tau, y)|_{2K-5} \end{aligned}$$

and

$$I_{3,\mu}[\Phi](\tau, y) = |y| w_+(\tau, |y|)^{\nu_2} \widetilde{z}_\mu(\tau, |y|) |\Phi(\tau, y)|_{2K-5}.$$

Observing that we can write

$$I_{2,\mu}[\Phi](\tau, y) = w_+(\tau, |y|)^{-(\nu_2-\nu_1)+\mu} w_j(\tau, |y|)^{-2\mu} I_{3,\mu}[\Phi](\tau, y)$$

for $(\tau, |y|) \in \Lambda_j$, we see

$$I_{2,\mu}[\Phi](\tau, y) \leqslant I_{3,\mu}[\Phi](\tau, y), \tag{5.40}$$

provided $0 < \mu \leqslant \nu_2 - \nu_1$. Therefore the bounds of $I_{1,\mu}[\Phi]$ and $I_{3,\mu}[\Phi]$ give the bound of $I_\mu[\Phi]$ for small μ.

Now we are going to estimate $I_\mu[N_i]$. To start with, we note that

$$|u_i(t, x)|_{2K-3} \leqslant C w_+(t, |x|)^{-1+2\rho+\delta} w_i(t, |x|)^{-\delta} e_5(t) \tag{5.41}$$

holds for any $\delta > 0$, since

$$w_+(t, |x|)^{1-\delta} w_i(t, |x|)^{\delta} |u_i(t, x)|_{2K-3} \leqslant C[u_i(t, \cdot)]^{(i)}_{2K-3,0}.$$

For simplicity of exposition, we abbreviate $w_+(\tau, |y|)$ and $w_j(\tau, |y|)$ as w_+ and w_j, respectively in what follows. Note that $2K - 6 \geqslant K + 3$ for $K \geqslant 9$.

We define $\Lambda_{j,T}$ for $j \in \{0, \dots, m\}$ by

$$\Lambda_{j,T} = \left\{(\tau, y) \in [0, T) \times \mathbb{R}^3; (\tau, |y|) \in \Lambda_j\right\}.$$

Since we have $w_0^{-1} \leqslant C w_+^{-1}$ in $\Lambda_{i,T}$ by (5.4), Lemma 5.3 implies

$$\begin{aligned} |y|\,|N_i(\tau, y)|_{2K-5} &\leqslant C w_+^{2\rho-2} w_i^{-1-\nu_1} e_2 e_4 + C w_+^{2\rho-2+\delta} w_i^{-1-\nu_1-\delta} e_2 e_5 \\ &\quad + C w_+^{2\rho-2} w_i^{-2} e_1 e_4 \leqslant C w_+^{2\rho-2+\delta} w_i^{-1-\nu_1} E(T)^2 \end{aligned} \tag{5.42}$$

for any $(\tau, y) \in \Lambda_{i,T}$. Therefore we get

$$I_{3,\mu}[N_i](\tau, y) \leqslant C w_+^{-(1-2\rho)+\nu_2+\delta} w_i^{-\nu_1+\mu} E(T)^2 \leqslant C E(T)^2 \text{ in } \Lambda_{i,T}$$

for sufficiently small μ and δ, because we have $0 < \nu_1 < \nu_2 < 1 - 2\rho$.

Similarly to (5.42), from Lemma 5.3 we get

$$\begin{aligned} |y|\,|N_i(\tau, y)|_{K+2} &\leqslant C \left(w_+^{-2} w_i^{-1-2\nu_1} e_2 e_2 + w_+^{-2} w_i^{-1-\nu_1-\nu_2} e_2 e_3 + w_+^{-2} w_i^{-2-\nu_1} e_1 e_2 \right) \\ &\leqslant C w_+^{-2} w_i^{-1-2\nu_1} E(T)^2 \text{ in } \Lambda_{i,T}, \end{aligned}$$

which leads to

$$I_{1,\mu}[N_i](\tau, y) \leqslant C w_i^{\mu-2\nu_1} E(T)^2 \leqslant C E(T)^2 \text{ in } \Lambda_{i,T}$$

for $\mu < 2\nu_1$.

On the other hand, if $j \notin I(i)$, (5.14) implies

$$\begin{aligned} |y|\,|N_i(\tau, y)|_{K+2} &\leqslant C w_+^{-2-2\nu_1} w_0^{-1} e_2(t)^2 \leqslant C w_+^{-2-2\nu_1} w_j^{-1} E(T)^2, \\ |y|\,|N_i(\tau, y)|_{2K-5} &\leqslant C w_+^{2\rho-2-\nu_1} w_0^{-1} e_2(t) e_4(t) \leqslant C w_+^{2\rho-2-\nu_1} w_j^{-1} E(T)^2 \end{aligned}$$

for any $(\tau, y) \in \Lambda_{j,T}$, since we have $w_i \geqslant C w_+$ and also $w_0 \geqslant C w_j$ in Λ_j. Therefore we obtain

$$\begin{aligned} I_{1,\mu}[N_i](\tau, y) &\leqslant C w_+^{-2\nu_1} w_j^{\mu} E(T)^2 \leqslant C w_+^{-2\nu_1+\mu} E(T)^2 \leqslant C E(T)^2, \\ I_{3,\mu}[N_i](\tau, y) &\leqslant C w_+^{-(1-2\rho)+\nu_2-\nu_1} w_j^{\mu} E(T)^2 \leqslant C w_+^{-(1-2\rho)+\nu_2-\nu_1+\mu} E(T)^2 \leqslant C E(T)^2 \end{aligned}$$

for any $(\tau, y) \in \Lambda_{j,T}$ with $j \notin I(i)$, provided μ is small enough.

Summing up, we have proved

$$\sup_{(\tau,y)\in[0,T)\times\mathbb{R}^3} I_\mu[N_i](\tau,y)\leqslant CE(T)^2 \text{ for small } \mu. \tag{5.43}$$

Similarly we can get

$$\sup_{(\tau,y)\in[0,T)\times\mathbb{R}^3} I_\mu[R_i^{21}](\tau,y)\leqslant CE(T)^2 \text{ for small } \mu. \tag{5.44}$$

Next we consider $I_\mu[H_i]$. Assume $(\tau,y)\in\Lambda_{j,T}$ with some $j\in\{0,\dots,m\}$. Since we have $|u_i(\tau,y)|_{k+2}\leqslant Cw_+(\tau,|y|)^{-1}w_j(\tau,|y|)^{-1}e_1(\tau)$, we get

$$\begin{aligned}|H_i(\tau,y)|_s &\leqslant C|u(\tau,y)|_{K+2}^2\big(|u(\tau,y)|_s+|\partial u(\tau,y)|_{s+1}\big)\\ &\leqslant Cw_+^{-2}w_j^{-2}e_1(\tau)^2\big(|u(\tau,y)|_s+|\partial u(\tau,y)|_{s+1}\big)\end{aligned}\tag{5.45}$$

for $s\leqslant 2K$. (5.45) leads to

$$\begin{aligned}|y|\,|H_i(\tau,y)|_{K+2} &\leqslant Cw_+^{-2}w_j^{-2}e_1(\tau)^2\big(w_j^{-1}e_1(\tau)+w_j^{-1-\nu_1}e_2(\tau)\big),\\ |y|\,|H_i(\tau,y)|_{2K-5} &\leqslant Cw_+^{-2}w_j^{-2}e_1(\tau)^2\big(w_j^{-\nu_2}e_3(\tau)+w_+^{2\rho}w_j^{-1}e_4(\tau)\big)\end{aligned}$$

for any $(\tau,y)\in\Lambda_{j,T}$. Therefore, if μ is small enough, we obtain

$$\begin{aligned}I_{1,\mu}[H_i](\tau,y) &\leqslant Cw_j^{-2+\mu}E(T)^3\leqslant CE(T)^2,\\ I_{3,\mu}[H_i](\tau,y) &\leqslant Cw_+^{-(1-2\rho)+\nu_2}w_j^{-1-\nu_2+\mu}E(T)^3\leqslant CE(T)^2\end{aligned}$$

for $(\tau,y)\in\Lambda_{j,T}$. These estimates imply

$$\sup_{(\tau,y)\in[0,T)\times\mathbb{R}^3} I_\mu[H_i](\tau,y)\leqslant CE(T)^2 \text{ for small } \mu. \tag{5.46}$$

Finally we will prove

$$\sup_{(\tau,y)\in[0,T)\times\mathbb{R}^3} I_\mu[R_i^{1}](\tau,y)\leqslant CE(T)^2 \text{ for small } \mu. \tag{5.47}$$

It suffices to prove (5.47) for $R_i^1 = \partial^\alpha u_k \partial^\beta u_l$, where $0 \leqslant |\alpha| \leqslant 2$, $1 \leqslant |\beta| \leqslant 2$ and, most importantly, $c_k \neq c_l$. Suppose $(\tau, |y|) \in \Lambda_j$ for some $j \in \{0, \dots, m\}$. Then we have

$$\begin{aligned}
|y|\,|R_i^1(\tau, y)|_{K+2} &\leqslant C|y|(|u_k|_{K+2} + |\partial u_k|_{K+3})|\partial u_l|_{K+3} \\
&\leqslant C(w_+^{-1} w_k^{-1} e_1(t) + w_0^{-1} w_k^{-1-\nu_1} e_2(t)) w_l^{-1-\nu_1} e_2(t), \qquad (5.48)\\
|y|\,|R_i^1(\tau, y)|_{2K-5} &\leqslant C|y|\big(|\partial^\alpha u_k|_{K-3} |\partial^\beta u_l|_{2K-5} + |\partial^\beta u_l|_{K-3} |\partial^\alpha u_k|_{2K-5}\big) \\
&\leqslant C w_+^{2\rho-1} w_k^{-1} w_l^{-1} e_1(t) e_4(t) \\
&\quad + C w_l^{-1-\nu_1} e_2(t) \big(w_+^{-1} w_k^{-\nu_2} e_3(t) + w_+^{2\rho} w_0^{-1} w_k^{-1} e_4(t)\big).
\end{aligned} \qquad (5.49)$$

Since we have $w_0 \geqslant C w_+$ and $w_k \geqslant C w_+$ in Λ_l by (5.4), from (5.48) and (5.49) we get

$$\begin{aligned}
I_{1,\mu}[R_i^1](\tau, y) &\leqslant C w_l^{\mu-\nu_1} E(T)^2 \leqslant C E(T)^2, \\
I_{3,\mu}[R_i^1](\tau, y) &\leqslant C \left(w_+^{-(1-2\rho)+\nu_2} w_l^\mu + w_l^{-\nu_1+\mu} \right) E(T)^2 \\
&\leqslant C(w_+^{-(1-2\rho)+\nu_2+\mu} + 1) E(T)^2 \leqslant C E(T)^2
\end{aligned}$$

for any $(\tau, y) \in \Lambda_{l,T}$ and sufficiently small μ.

Similarly, for $(\tau, y) \in \Lambda_{k,T}$, we obtain

$$\begin{aligned}
I_{1,\mu}[R_i^1](\tau, y) &\leqslant C w_+^{-\nu_1} w_k^\mu E(T)^2 \leqslant C w_+^{\mu-\nu_1} E(T)^2 \leqslant C E(T)^2, \\
I_{3,\mu}[R_i^1](\tau, y) &\leqslant C \left(w_+^{-(1-2\rho)+\nu_2} w_k^\mu + w_+^{-1-\nu_1+\nu_2} w_k^{1-\nu_2+\mu} \right) E(T)^2 \\
&\leqslant C(w_+^{-(1-2\rho)+\nu_2+\mu} + w_+^{-\nu_1+\mu}) E(T)^2 \leqslant C E(T)^2,
\end{aligned}$$

provided μ is sufficiently small.

Now suppose $j \notin I(k) \cup I(l)$. Then (5.4) implies $w_k \geqslant C w_+$, $w_l \geqslant C w_+$ and $w_0 \geqslant C w_j$ in Λ_j. Hence (5.48) and (5.49) lead to

$$\begin{aligned}
I_{1,\mu}[R_i^1](\tau, y) &\leqslant C \left(w_+^{-1-\nu_1} w_j^{1+\mu} + w_+^{-2\nu_1} w_j^\mu \right) E(T)^2 \\
&\leqslant C(w_+^{\mu-\nu_1} + w_+^{\mu-2\nu_1}) E(T)^2 \leqslant C E(T)^2, \\
I_{3,\mu}[R_i^1](\tau, y) &\leqslant C \left(w_+^{-(1-2\rho)+\nu_2-1} w_j^{1+\mu} + w_+^{-1-\nu_1} w_j^{1+\mu} \right) E(T)^2 \\
&\quad + C w_+^{-(1-2\rho)+\nu_2-\nu_1} w_j^\mu E(T)^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(w_+^{-(1-2\rho)+\nu_2+\mu} + w_+^{-\nu_1+\mu} + w_+^{-(1-2\rho)+\nu_2-\nu_1+\mu})E(T)^2 \\
&\leqslant CE(T)^2
\end{aligned}
$$

for $(\tau, y) \in \Lambda_{j,T}$ with $j \notin I(k) \cup I(l)$, provided that μ is small enough.

Summing up, we have proved (5.47), and this completes the proof of (5.38). $\square$

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