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Multiplicity of solutions for a class of elliptic problem in $\mathbb{R}^2$ with Neumann conditions☆

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Abstract

In this work, we study the existence and multiplicity of solutions for a class of elliptic problems in exterior domains of $\mathbb{R}^2$ with Neumann boundary conditions and nonlinearity with critical growth.

Keywords: Elliptic problem; Critical growth; Trudinger–Moser inequality

1. Introduction

In this paper, we are concerned with the existence and multiplicity of solutions for the following class of elliptic problem with Neumann conditions:

$$\begin{cases} -\Delta u + u = Q(x)f(u) & \text{in } \mathbb{R}^2 \setminus \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0 \quad \text{on} & \partial\Omega, \end{cases} \tag{P}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, Q is a continuous function satisfying

$$Q(x) > 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \Omega \quad \text{and} \quad \lim_{|x|\to\infty} Q(x) = \bar{Q} > 0, \tag{Q_1}$$

and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying the following hypotheses:

f has critical growth at both $+\infty$ and $-\infty$, that is, it behaves like $e^{\alpha_o s^2}$ as $|s| \to \infty$ for some $\alpha_o > 0$. More precisely,

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0 \quad \forall \alpha > \alpha_o, \qquad \lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty \quad \forall \alpha < \alpha_o.$$

Moreover, we assume that

$$|f(s)| \leqslant Ce^{4\pi s^2} \quad \text{for all } s \in \mathbb{R}. \tag{f_1}$$

There is $\theta > 2$ verifying

$$0 < \theta F(s) \leqslant sf(s) \quad \text{for all} \quad s \in \mathbb{R}. \tag{f_2}$$

There exists $q > 1$ such that

$$\limsup_{|s|\to 0} \frac{|f(s)|}{|s|^q} < \infty. \tag{f_3}$$

$$\text{The function } s \to \frac{f(s)}{s} \text{ is increasing in } (0, +\infty). \tag{f_4}$$

There are constants $p > 2$ and C_p such that

$$f(s) \geqslant C_p s^{p-1} \quad \text{for all } s \in [0, +\infty), \tag{f_5}$$

where

$$C_p > \left[\frac{2\xi^2\theta(p-2)}{p(\theta-2)} \right]^{(p-2)/2} S_p^p,$$

$$S_p = \inf_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{(\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)\, dx)^{1/2}}{(\int_{\mathbb{R}^2} \bar{Q} \mid u \mid^p \ dx)^{1/p}}$$

and $\xi > 0$ is a positive constant such that the extension operator $E : H^1(\mathbb{R}^2 \setminus \Omega) \to H^1(\mathbb{R}^2)$ satisfies

$$\|Eu\|_{H^1(\mathbb{R}^2)} \leqslant \xi \|u\|_{H^1(\mathbb{R}^2\setminus\Omega)} \quad \forall u \in H^1(\mathbb{R}^2 \setminus \Omega).$$

We recall that E exists because the set Ω has smooth bounded boundary (see [1]).

In [6], Benci and Cerami studied problem (P) assuming $N \geqslant 3$, $Q \equiv 1$ and $f(u) = |u|^{\eta-1}u$ with $1 < \eta < \frac{N+2}{N-2}$. They showed that (P), with Dirichlet condition, has not a *ground state solution*, that is, a solution of (P) with minima energy. However, Esteban in [10] proved that the same problem with Neumann condition has a *ground state solution*.

In [8], Cao also studied problem (P) for $N \geqslant 3$, $f(u) = |u|^{\eta-1}u$ and Q satisfying condition (Q_1). He showed that this problem has at least two solutions, a positive solution and a nodal solution, that is, a solution of (P) that changes of sign. In [3], Alves et al. showed that the results found in [8], also hold for the p-Laplacian operator and also for a larger class of nonlinearity.

Motivated by papers [3,8] and by some ideas developed in [4,7], we prove the existence of *ground state* and *nodal* solutions to (P). We used variational methods such as the Mountain Pass Theorem without Palais–Smale condition (see [5,14]) to obtain a positive ground state solution. In relation to nodal solution, we apply the implicit function Theorem. An important point in our work is that the nonlinearity has critical growth in $\mathbb{R}^2$, this fact implies that some estimates and arguments explored in [3,8] cannot be used. To overcome these difficulties, we used a version of a result due to Lions for the critical growth case in $\mathbb{R}^2$ proved by Alves et al. in [4].

Concerning the existence of *ground state solution*, we will prove the following result:

Theorem 1.1. *Suppose that f satisfies (f_1)–(f_5), Q satisfies (Q_1) and*

$$Q(x) \geqslant \bar{Q} - Ce^{-m|x|} \quad |x| \geqslant R_o, \tag{Q_2}$$

where C, R_o are positive constants and $m > 2$. Then (P) has a positive ground state solution.

In order to get nodal solution, it is necessary the following additional conditions on f: There exists $\sigma \geqslant 2$ such that

$$f'(s)s^2 - f(s)s \geqslant C|s|^{\sigma} \quad \forall s \in \mathbb{R} \tag{f_6}$$

and

$$|f'(s)s| \leqslant Ce^{4\pi s^2} \quad \forall s \in \mathbb{R}, \tag{f_7}$$

for some positive constant C.

Theorem 1.2. *Suppose that f satisfies* (f_1)–(f_7), *Q satisfies* (Q_1) *and*

$$Q(x) \geqslant \bar{Q} + Ce^{-\mu|x|} \quad \forall x \in \mathbb{R}^N, \tag{Q_3}$$

where C is a positive constant and $\mu < \frac{1}{q+1}$. *Then* (P) *has a nodal solution.*

This paper is organized as follows: in Section 2, we recall some results involving the limit problem. In Section 3, we state some lemmas and propositions used in the proof of the main results. In Section 4, we prove the main results. In Section 5 we prove some technical lemmas and propositions stated in the Section 3.

To finish this section, we would like to cite also the papers of Adimurthi and Yadava [2] and de Figueiredo et al. [9] and the references therein, where elliptic problems in $\mathbb{R}^2$ have being considered.

2. The limit problem

In this work, we need to recall some results involving the limit problem

$$\begin{cases} -\Delta u + u = \bar{Q} f(u) & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2). \end{cases} \tag{P_∞}$$

Hereafter, if h is a Lebesgue integrable function and B is a measurable set, we write $\int_B h$ for $\int_B h\, dx$. Moreover, if $h \in H^1(\mathbb{R}^2 \setminus \Omega)$ we denote by $\|h\|$ its usual norm.

The energy functional $I_\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ associated to problem (P_∞) is given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) - \int_{\mathbb{R}^2} \bar{Q} F(u^+),$$

where $F(u) = \int_0^u f(t)\, dt$ and $u^+(x) = \max\{u(x), 0\}$. Using the hypotheses on function f, we have that $I_\infty \in C^1(H^1(\mathbb{R}^2), \mathbb{R})$ and the weak solutions of (P_∞) are nontrivial critical points of I_∞.

Repeating the same arguments explored by Cao [7] and Alves et al. [4], it is possible to check that I_∞ verifies the Mountain Pass Geometry and that there exists a positive function $\bar{u} \in B_1(0) \setminus \{0\} \subset H^1(\mathbb{R}^2)$ verifying

$$I_\infty(\bar{u}) = c_\infty \quad \text{and} \quad I'_\infty(\bar{u}) = 0,$$

where c_∞ is the minimax level of the Mountain Pass Theorem applied to I_∞. In this case, the function $\bar{u}$ is a ground state solution to (P_∞). Moreover, we have the following result.

Theorem 2.1. *Assume that* (f_1) *and* (f_3) *hold. Then, any positive solution* $\bar{u}$ *of problem* (P_∞) *with* $\|\bar{u}\|_{H^1(\mathbb{R}^2)} < 1$ *satisfies*:

$$\text{(I)} \qquad \lim_{|x|\to\infty} \bar{u}(x) = 0$$

and

$$\text{(II)} \quad C_1 e^{-a|x|} \leqslant \bar{u}(x) \leqslant C_2 e^{-b|x|} \quad in \ \mathbb{R}^2,$$

where $C_1, C_2 > 0$ *are positive constants and* $0 < b < 1 < a$. *Moreover, we can be chosen* $a = 1 + \delta, b = 1 - \delta$ *for* $\delta > 0$.

Proof. Using conditions (f_1) and (f_3), for each $\tau > 1$ and $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(s)s|, |F(s)| \leqslant \varepsilon |s|^2 + C_\varepsilon (e^{4\pi\tau s^2} - 1)|s| \quad \forall s \in \mathbb{R}. \tag{2.1}$$

Using the fact that $\|\bar{u}\|_{H^1(\mathbb{R}^2)} < 1$ and arguments found in [4,7], there exists q near 1, $q > 1$ such that

$$h(x) = f(\bar{u}(x)) \in L^q(\mathbb{R}^2).$$

By bootstrap arguments, for $x \in \mathbb{R}^2$ and $R > 0$, it follows that $\bar{u} \in W^{2,q}(B_R(x))$ with

$$\|\bar{u}\|_{W^{2,q}(B_R(x))} \leqslant C\{|h|_{L^q(B_{2R}(x))} + |\bar{u}|_{L^q(B_{2R}(x))}\}$$

which implies,

$$\|\bar{u}\|_{W^{2,q}(B_R(x))} \leqslant C\{|h|_{L^q(B_{2R}(x))} + |\bar{u}|_{L^2(B_{2R}(x))}\}.$$

Since the imbedding $W^{2,q}(B_R(x)) \hookrightarrow C(\bar{B}_R(x))$ is continuous,

$$\|\bar{u}\|_{L^\infty(B_R(x))} \leqslant C\{|h|_{L^q(B_{2R}(x))} + |\bar{u}|_{L^2(B_{2R}(x))}\}.$$

The last inequality implies that $\bar{u} \in L^\infty(\mathbb{R}^2)$ and $\lim_{|x|\to\infty} \bar{u}(x) = 0$.

The inequalities in (II) involving the exponential functions follow with the same arguments found in Li and Yan [11]. □

Remark 2.1. (i) Theorem 2.1 completes the result proved in [11], because our nonlinearity has a different behavior at infinity.

(ii) With the same arguments used in the proof of Theorem 2.1, we can show that all positive weak solutions u_1 of (P), with $\|u_1\| < \frac{1}{\xi}$, has exponential decaying.

3. Statement of lemmas and propositions

In this section we state some necessary results to prove Theorems 1.1 and 1.2. The proofs of some of them are in Section 5.

3.1. Technical results to get ground state solution

The first lemma can be found in Alves et al. [3].

Lemma 3.1. *Let $F \in C^2(\mathbb{R}, \mathbb{R}_+)$ be a convex and even function such that $F(0) = 0$ and $f(s) = F'(s) \geqslant 0$ $\forall s \in [0, \infty)$. Then, for all $u, v \geqslant 0$*

$$|F(u - v) - F(u) - F(v)| \leqslant 2(f(u)v + f(v)u).$$

The next lemma is related to the Mountain Pass Geometry and we do not make its proof because it is well known. See for example Alves et al. [4]. Hereafter, let us denote by $I : H^1(\mathbb{R}^2 \setminus \Omega) \to \mathbb{R}$ the energy functional related to (P), that is,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2\setminus\Omega}(|\nabla u|^2 + u^2) - \int_{\mathbb{R}^2\setminus\Omega} Q(x)F(u).$$

Lemma 3.2. *The functional I verifies the Mountain Pass Geometry, that is,*

(i) *There exist $r, \rho > 0$ such that $I(u) \geqslant r$, $\quad \|u\| = \rho$.*
(ii) *There exists $e \in B^c_\rho(0)$ such that $I(e) < 0$.*

Using a version of Mountain Pass Theorem without Palais–Smale condition (see [14, Theorem 1.15]) and (f_4), there exists $u_n \in H^1(\mathbb{R}^2 \setminus \Omega)$ satisfying

$$I(u_n) \to c_1 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty,$$

where

$$c_1 = \inf\left\{\sup_{t\geqslant 0} I(tu); u \in H^1(\mathbb{R}^2 \setminus \Omega) \setminus \{0\}\right\}. \tag{3.1}$$

The next result establishes a relation between the levels c_1 and c_∞.

Proposition 3.1. *Assume that Q satisfies (Q_1)–(Q_2). Then*

$$0 < c_1 < c_\infty.$$

Proof. See Section 5.

The following result may be proved in much the same way as in Lions [12].

Lemma 3.3. *Let* $\{u_n\} \subset H^1(\mathbb{R}^2 \setminus \Omega)$ *be a bounded sequence such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{U_{R,y}} |u_n|^2 = 0,$$

for some $R > 0$ *and* $U_{R,y} = B_R(y) \cap (\mathbb{R}^2 \setminus \Omega)$ *with* $U_{R,y} \neq \emptyset$. *Then,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^2\setminus\Omega} |u_n|^{q+1} = 0 \quad for\ all\ q > 1.$$

Proposition 3.2. *Let* $\{u_n\} \subset H^1(\mathbb{R}^2 \setminus \Omega)$ *be a sequence with* $u_n \rightharpoonup 0$ *and*

$$\limsup_{n\to\infty} \|u_n\|^2 \leqslant m < \frac{1}{2\xi^2}.$$

If there exists $R > 0$ *such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{U_{R,y}} |u_n|^2 = 0,$$

and (f_1)–(f_5) *hold, we have*

$$\int_{\mathbb{R}^2\setminus\Omega} F(u_n), \int_{\mathbb{R}^2\setminus\Omega} f(u_n)u_n \to 0 \quad as\ n \to \infty.$$

Proof. See Section 5.

Proposition 3.3. *If* $\{u_n\} \subset H^1(\mathbb{R}^2 \setminus \Omega)$ *satisfies*

$$I(u_n) \to c_1 \quad and \quad I'(u_n) \to 0,$$

we have that $\limsup_{n\to\infty} \|u_n\|_{\mathbb{R}^2\setminus\Omega} < \frac{1}{\sqrt{2}\xi}$. *Moreover, the weak limit* u_1 *of* $\{u_n\}$ *in* $H^1(\mathbb{R}^2 \setminus \Omega)$ *is a nontrivial critical point of* I *with* $I(u_1) = c_1$.

Proof. See Section 5.

3.2. Technical results to get nodal solutions

Consider the closed set

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^2 \setminus \Omega) \mid u^{\pm} \not\equiv 0,\ \ I'(u^{\pm})u^{\pm} = 0\}$$

and $\widehat{c}$ the following real number

$$\widehat{c} = \inf_{u \in \mathcal{M}} I(u).$$

The proof of the next lemma follows by similar arguments explored in [3] and we omit it.

Lemma 3.4. *Assume that* $(f_1), (f_3), (f_6)$ *and* (f_7) *hold. Then, there exists a sequence* $(u_n) \subset \mathcal{M}$ *satisfying*

$$I(u_n) \to \widehat{c} \quad \textit{and} \quad I'(u_n) \to 0.$$

The next proposition is a key point in our arguments to find nodal solution, because it gives a good estimate to $\widehat{c}$.

Proposition 3.4. *Suppose that* Q *satisfies* (Q_1)–(Q_3). *Then*

$$0 < \widehat{c} < c_1 + c_\infty. \tag{3.2}$$

Proof. See Section 5.

4. Proof of the main theorems

Proof of Theorem 1.1. First of all, to find a positive ground state solution we will assume that

$$f(t) = 0 \quad \forall t \leqslant 0.$$

By Proposition 3.3 and the Mountain Pass Theorem (see [5,14]), I has a critical point u_1 at the level c_1. We claim that u_1 is nonnegative. Indeed, we know that $I'(u_1){u_1}^- = 0$, thus $\|u^-\| = 0$ and $u_1^- = 0$. Using the maximum principle, we have $u_1 > 0$ in $\mathbb{R}^2 \setminus \Omega$. $\square$

Proof of Theorem 1.2. Let $(u_n) \subset \mathcal{M}$ be the sequence obtained in Lemma 3.4. Then, the weak limit u of $\{u_n\}$ in $H^1(\mathbb{R}^2 \setminus \Omega)$ is a nontrivial critical point of I and $u^{\pm} \neq 0$. To check the above claim, remember that

$$I(u_n) \to \widehat{c} \quad \text{and} \quad I'(u_n) \to 0.$$

Then

$$\frac{(\theta-2)}{2\theta} \limsup_{n\to\infty} \|u_n\|^2 \leqslant c_1 + c_\infty \leqslant 2c_\infty$$

which gives

$$\limsup_{n\to\infty} \|u_n\|^2 \leqslant \alpha_* = \frac{4\theta c_\infty}{\theta-2}.$$

From (f_1)–(f_5) (see [4]), it follows that

$$c_\infty < \frac{(\theta-2)}{4\xi^2\theta},$$

then

$$\limsup_{n\to\infty} \|v_n\|_{H^1(\mathbb{R}^2)} \leqslant \sqrt{\alpha_*}\,\xi < 1, \quad \text{for } v_n = Eu_n.$$

Using an inequality of Trudinger–Moser type showed by Cao in [7] and repeating the same arguments used in the proof of Proposition 3.3, we can conclude that u is a critical point of I. Now, we will prove that $u^{\pm} \neq 0$.

We have three cases to consider:

(I) $u^+ = u^- = 0$.
(II) $u^+ \neq 0$ and $u^- = 0$.
(III) $u^+ = 0$ and $u^- \neq 0$.

We will prove that the above cases do not hold, therefore $u^{\pm} \neq 0$. In what follows, we will prove only (I) because the other cases follow with the same type of arguments.

Analysis of (I): Applying Proposition 3.2 to the sequences $\{u_n^+\}$ and $\{u_n^-\}$, there exist $\eta, R > 0$ and sequences $\{y_n^1\}$ and $\{y_n^2\}$ in $\mathbb{R}^2$ with $|y_n^1|, |y_n^2| \to \infty$ verifying

$$\liminf_{n\to\infty} \int_{U_{R,y_n^1}} |u_n^+|^2 \geqslant \eta$$

and

$$\liminf_{n\to\infty}\int_{U_{R,y_n^2}}|u_n^-|^2\geqslant\eta.$$

Defining $w_n(x)=u_n(x+y_n^1)$ and $z_n(x)=u_n(x+y_n^2)$, there exist $w,z\in H^1(\mathbb{R}^2)\setminus\{0\}$ such that $w_n\to w$ and $z_n\to z$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$, with $w^+\neq 0$ and $z^-\neq 0$. Since $I'_\infty(w)=I'_\infty(z)=0$, we have

$$I'_\infty(w^+)w^+=0\quad\text{and}\quad I'_\infty(z^-)z^-=0.$$

In this way,

$$2c_\infty\leqslant I_\infty(w^+)+I_\infty(z^-)=\left[I_\infty(w^+)-\frac{1}{\theta}I'_\infty(w^+)w^+\right]+\left[I_\infty(z^-)-\frac{1}{\theta}I'_\infty(z^-)z^-\right].$$

By Fatou's Lemma

$$\liminf_{n\to\infty}\left[\int_{\mathbb{R}^2\setminus\Omega_n^1}(|\nabla w_n^+|^2+(w_n^+)^2)+\frac{1}{\theta}\int_{\mathbb{R}^2\setminus\Omega_n^1}(f(w_n^+)w_n^+-F(w_n^+))\right]\geqslant I_1$$

and

$$\liminf_{n\to\infty}\left[\int_{\mathbb{R}^2\setminus\Omega_n^2}(|\nabla z_n^-|^2+(z_n^-)^2)+\frac{1}{\theta}\int_{\mathbb{R}^2\setminus\Omega_n^2}(f(z_n^-)z_n^--F(z_n^-))\right]\geqslant I_2,$$

where $\Omega_n^1=\Omega-y_n^1$, $\Omega_n^2=\Omega-y_n^2$, $I_1=I_\infty(w^+)-\frac{1}{\theta}I'_\infty(w^+)w^+$ and $I_2=I_\infty(z^-)-\frac{1}{\theta}I'_\infty(z^-)z^-$. Consequently

$$2c_\infty\leqslant\liminf_{n\to\infty}\{I(u_n^+)+I(u_n^-)\}=\lim_{n\to\infty}I(u_n)=\widehat{c}<c_1+c_\infty$$

which is an absurd. $\square$

5. Proof of lemmas and propositions

In this section, we will prove some lemmas and propositions used in Section 3.

Proof of Proposition 3.1. Let $\bar{u}$ be a positive ground state solution of problem (P_∞) and define $u_n(x)=\bar{u}(x-x_n)$, $x_n=(0,\ldots,n)$. By the characterization of c_1

given in (3.1),

$$c_1 \leqslant \max_{t \geqslant 0} I(tu_n).$$

Let $\gamma_n \in (0, \infty)$ such that

$$I(\gamma_n u_n) = \max_{t \geqslant 0} I(tu_n),$$

then

$$\begin{aligned} c_1 &\leqslant I(\gamma_n u_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^2\setminus\Omega}(|\gamma_n \nabla u_n|^2 + |\gamma_n u_n|^2) - \int_{\mathbb{R}^2\setminus\Omega} Q(x)F(\gamma_n u_n) \\ &= I_\infty(\gamma_n u_n) - \frac{1}{2}t_n\gamma_n^2 + \int_\Omega \bar{Q}F(\gamma_n u_n) + \int_{\mathbb{R}^2\setminus\Omega}(\bar{Q} - Q)F(\gamma_n u_n), \end{aligned} \tag{5.1}$$

where

$$t_n = \int_\Omega (|\nabla u_n|^2 + |u_n|^2).$$

Now, notice that $I(\gamma_n u_n) = \max_{t \geqslant 0} I(tu_n)$ if and only if

$$\int_{\mathbb{R}^2\setminus\Omega}(|\nabla u_n|^2 + |u_n|^2) = \int_{\mathbb{R}^2\setminus\Omega} Q(x)\,\frac{f(\gamma_n u_n)}{(\gamma_n u_n)}u_n^2. \tag{5.2}$$

It is not difficult to see that the sequence (γ_n) is bounded and that $\gamma_n \to 1$, for some subsequence still denoted by (γ_n). By (2.1) and (5.1)

$$\begin{aligned} c_1 \leqslant{} & I_\infty(\bar{u}) - t_n\left(\frac{\gamma_n^2}{2} - O(\varepsilon)\right) + C_\varepsilon\int_\Omega \gamma_n u_n(e^{4\pi\gamma_n\tau u_n^2} - 1)\bar{Q}\,dx \\ & + \int_{\mathbb{R}^2\setminus\Omega}(\bar{Q} - Q)F(\gamma_n u_n)\,dx \end{aligned}$$

thus,

$$c_1 \leqslant I_\infty(\bar{u}) - Ct_n + s_n,$$

where C is a positive constant and

$$s_n = C\int_\Omega u_n(e^{4\pi\gamma_n\tau u_n^2}-1)\bar{Q}\,dx + \int_{\mathbb{R}^2\setminus\Omega}(\bar{Q}-Q)F(\gamma_n u_n).$$

We claim that

$$\frac{s_n}{t_n} \to 0. \tag{5.3}$$

Indeed, by Theorem 2.1

$$t_n = \int_\Omega(|\nabla u_n|^p + |u_n|^p) \geqslant \int_\Omega |u_n|^p \geqslant Ce^{-2an}.$$

Estimate of s_n:

Fix $R>0$ such that $\Omega\subset B_R(0)$ and observe that

$$\int_\Omega u_n(e^{4\pi\gamma_n\tau u_n^2}-1)\,dx = \int_{\Omega_n}\bar{u}(e^{4\pi\gamma_n\tau\bar{u}^2}-1)\,dx,$$

where $\Omega_n = \Omega + x_n$. Consequently,

$$\int_\Omega u_n(e^{4\pi\gamma_n\tau u_n^2}-1)\,dx \leqslant Ce^{-bn}(e^{4\pi\tau_1 e^{-2nb}}-1),$$

where $\gamma_n\tau\leqslant\tau_1\ \forall n\in\mathbb{N}$. Note that,

$$\begin{aligned}\int_{\mathbb{R}^2\setminus\Omega}(\bar{Q}-Q)F(\gamma_n u_n) &= \int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|>r_n\}}(\bar{Q}-Q)F(\gamma_n u_n)\\ &\quad+\int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|\leqslant r_n\}}(\bar{Q}-Q)F(\gamma_n u_n),\end{aligned}$$

where $r_n=(1-r)n$ with $r>0$ and r near 0. From (Q_2), it follows that

$$\int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|>r_n\}}(\bar{Q}-Q)F(\gamma_n u_n)\leqslant Ce^{-mr_n}.$$

From conditions (f_1) and (f_3), it follows that

$$|F(s)|\leqslant\varepsilon|s|^{q+1}+C_\varepsilon(e^{4\pi\tau s^2}-1)|s| \quad \forall s\in\mathbb{R},$$

then

$$\int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|\leqslant r_n\}}(\bar{Q}-Q)F(\gamma_n u_n) \leqslant C_1\varepsilon\int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|\leqslant r_n\}}|u_n|^{q+1} + C_2\int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|\leqslant r_n\}}u_n(e^{4\pi\tau_1 u_n^2}-1)\,dx.$$

Therefore,

$$\int_{(\mathbb{R}^2\setminus\Omega)\cap\{|x|\leqslant r_n\}}(\bar{Q}-Q)F(\gamma_n u_n)\leqslant Ce^{-b(q+1)r_n}n^2+Ce^{-br_n}(e^{4\pi\tau_1 e^{-2br_n}}-1)n^2.$$

By the estimates obtained above

$$\frac{s_n}{t_n}\leqslant\frac{Ce^{-bn}(e^{4\pi\tau_1 e^{-2nb}}-1)}{e^{-2na}}+\frac{Ce^{-mr_n}}{e^{-2an}}+\frac{Ce^{-b(q+1)r_n}n^2}{e^{-2na}}+\frac{Ce^{-br_n}(e^{4\pi\tau_1 e^{-2r_n b}}-1)n^2}{e^{-2na}},$$

and since $\frac{a}{b}\to 1$ as $\delta\to 0$ (see Theorem 2.1), we obtain

$$\frac{s_n}{t_n}\to 0. \quad \square$$

From (5.3), it follows that $c_1 < c_\infty$.

Proof of Proposition 3.2. By hypotheses

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{U_{R,y}}|u_n|^2=0,$$

together with Lemma 2.4, we get

$$u_n\to 0 \quad \text{in } L^{q'}(\mathbb{R}^2\setminus\Omega) \quad \text{for all } q'\in(2,+\infty).$$

Denoting $v_n = Eu_n$, it follows that

$$\|v_n\|_{H^1(\mathbb{R}^2)}\leqslant\xi\|u_n\|<\frac{1}{\sqrt{2}}<1$$

and by an inequality of Trudinger–Moser type found in [7], there exist $\tau, q > 1$, sufficiently close to 1 such that the sequence

$$f_n(x)=e^{4\pi\tau v_n^2}-1 \quad \forall x\in\mathbb{R}^2$$

belongs to $L^q(\mathbb{R}^2)$ and there exists $C > 0$ such that $|f_n|_q \leqslant C$ for all $n \in \mathbb{N}$. Therefore, the sequence

$$h_n(x) = e^{4\pi\tau u_n^2 - 1} \quad x \in \mathbb{R}^2 \setminus \Omega$$

belongs to $L^q(\mathbb{R}^2 \setminus \Omega)$ and there exists $C > 0$ such that $|h_n|_q \leqslant C$ for all $n \in \mathbb{N}$. On the other hand, we have

$$\int_{\mathbb{R}^2\setminus\Omega} f(u_n)u_n \leqslant \varepsilon \int_{\mathbb{R}^2\setminus\Omega} u_n^2 + C_\varepsilon \int_{\mathbb{R}^2\setminus\Omega} u_n(e^{4\pi\tau u_n^2} - 1)$$

which implies that

$$\int_{\mathbb{R}^2\setminus\Omega} f(u_n)u_n \leqslant \varepsilon C + C\left\{\int_{\mathbb{R}^2\setminus\Omega} |u_n|^{q'}\right\}^{q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

From this, we infer that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2\setminus\Omega} f(u_n)u_n = 0.$$

By similar arguments,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2\setminus\Omega} F(u_n) = 0. \qquad \Box$$

Proof of Proposition 3.3. Using analogous arguments explored in [4], we get

$$c_\infty < \frac{(\theta - 2)}{4\xi^2\theta}.$$

On the other hand, using the hypotheses involving the sequence $\{u_n\}$, we have

$$\frac{(\theta - 2)}{2\theta} \limsup_{n\to\infty} \|u_n\|^2 \leqslant c_1.$$

Thus, there exists $n_o \in \mathbb{N}$ such that

$$\|u_n\| < \frac{1}{\sqrt{2}\xi} \quad \forall n \geqslant n_o.$$

Denoting $v_n = Eu_n$, we have that

$$\|v_n\|_{H^1(\mathbb{R}^2)} \leqslant \xi \|u_n\|$$

and then,

$$\|v_n\|_{H^1(\mathbb{R}^2)} < \frac{1}{\sqrt{2}} < 1 \quad \forall n \geqslant n_o.$$

Using similar arguments explored in [4], it follows that

$$\int_{\mathbb{R}^2 \setminus \Omega} f(u_n)v \to \int_{\mathbb{R}^2 \setminus \Omega} f(u)v \quad \forall v \in H^1(\mathbb{R}^2 \setminus \Omega),$$

where u is the weak limit of $\{u_n\}$. The last limit implies that u is a critical point of I. Now, let us show that u is nonzero. Assuming by contradiction that $u = 0$, we have two situations to consider:

$$\text{(I)} \quad \lim_{n\to\infty} \sup_{y \in R^2} \int_{U_{R,y}} |u_n|^2 = 0$$

or

$$\text{(II)} \quad \text{There exist } \eta > 0 \text{ and } y_n \in \mathbb{R}^2 \text{ such that } \liminf_{n\to\infty} \int_{U_{R,y_n}} |u_n|^2 \geqslant \eta.$$

We will show that the aforementioned cases (I) and (II) do not hold, thus we can conclude that $u \neq 0$.

Analysis of (I): If (I) holds, by Proposition 2.2, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^2 \setminus \Omega} f(u_n)u_n = 0.$$

This fact implies that $\|u_n\| \to 0$, which is an absurd, because $I(u_n) \to c_1 > 0$. Therefore, (I) does not hold.

Analysis of (II): Let $w_n(x) = u_n(x + y_n)$ for $x \in \mathbb{R}^2 \setminus \Omega_n$ where $\Omega_n = \Omega - y_n$. From Sobolev imbedding, we have that $|y_n| \to \infty$. Hence the limit set related to $\mathbb{R}^2 \setminus \Omega_n$ as n goes to infinity is $\mathbb{R}^2$. Notice also that $\{w_n\}$ is bounded in $H^1_{\text{loc}}(\mathbb{R}^2)$ and its weak limit w is different from zero. Denoting $\widehat{w}_n = Ew_n$, it follows that

$$\|\widehat{w}_n\|_{H^1(\mathbb{R}^2)} \leqslant \xi \|w_n\|_{\Theta_n},$$

where $\Theta_n = \mathbb{R}^2 \setminus \Omega_n$. Then,

$$\|\widehat{w}_n\|_{H^1(\mathbb{R}^2)} \leqslant \xi \|u_n\| < \frac{1}{\sqrt{2}} \quad \forall n \in \mathbb{N}.$$

Using similar arguments explored in the previous results, we conclude that w is a critical point of the functional I_∞ and $w_n \to w$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$. Thus, by Fatou's lemma

$$c_\infty \leqslant I_\infty(w) = I_\infty(w) - \frac{1}{\theta} I'_\infty(w)w \leqslant \liminf_{n\to\infty} I(u_n) = c_1 < c_\infty$$

which is an absurd, and (II) also does not hold.

The equality $I(u_1) = c_1$ follows from definition of c_1 and of limit

$$\liminf_{n\to\infty} I(u_n) \leqslant c_1. \qquad \square$$

Proof of Proposition 3.4. Let $\bar{u}$ be a ground state solution of (P_∞) and u_1 is a positive ground state of (P). Let us define $\bar{u}_n(x) = \bar{u}(x - x_n)$, where $x_n = (0, \ldots, 0, n)$ and for $\alpha, \beta > 0$

$$\begin{aligned} h^\pm(\alpha, \beta, n) = & \int_{\mathbb{R}^2\setminus\Omega} |\nabla(\alpha u_1 - \beta \bar{u}_n)^\pm|^2 + |(\alpha u_1 - \beta \bar{u}_n)^\pm|^2 \\ & - \int_{\mathbb{R}^2\setminus\Omega} Qf((\alpha u_1 - \beta \bar{u}_n)^\pm)(\alpha u_1 - \beta \bar{u}_n)^\pm. \end{aligned}$$

Since

$$\int_{\mathbb{R}^2\setminus\Omega} (|\nabla u_1|^2 + u_1^2) - \int_{\mathbb{R}^2\setminus\Omega} Qf(u_1)u_1 = 0,$$

by (f_3) it yields that

$$\begin{aligned} & \int_{\mathbb{R}^2\setminus\Omega} \left(\left|\frac{1}{2}\nabla u_1\right|^2 + \left|\frac{1}{2}u_1\right|^2 \right) - \int_{\mathbb{R}^2\setminus\Omega} Qf\left(\frac{1}{2}u_1\right)\frac{1}{2}u_1 \\ & = \int_{\mathbb{R}^2\setminus\Omega} Q\left(\frac{f(u_1)}{(u_1)} - \frac{f(\frac{1}{2}u_1)}{(\frac{1}{2}u_1)} \right)\left(\frac{u_1}{2}\right)^2 > 0, \end{aligned}$$

and

$$\int_{\mathbb{R}^2\setminus\Omega}(|2\nabla u_1|^2+|2u_1|^2)-\int_{\mathbb{R}^2\setminus\Omega} Qf(2u_1)2u_1$$
$$=\int_{\mathbb{R}^2\setminus\Omega} Q\left(\frac{f(u_1)}{(u_1)}-\frac{f(2u_1)}{(2u_1)}\right)(2u_1)^2<0.$$

Thus, for n large enough we get

$$\int_{\mathbb{R}^2\setminus\Omega}\left(\left|\frac{1}{2}\nabla\bar{u}_n\right|^2+\left|\frac{1}{2}\bar{u}_n\right|^2\right)-\int_{\mathbb{R}^2\setminus\Omega} Q(x)f\left(\frac{1}{2}\bar{u}_n\right)\frac{1}{2}\bar{u}_n>0,$$

and

$$\int_{\mathbb{R}^2\setminus\Omega}(|2\nabla\bar{u}_n|^2+|2\bar{u}_n|^2)-\int_{\mathbb{R}^2\setminus\Omega} Q(x)f(2\bar{u}_n)2\bar{u}_n<0.$$

Since, $\bar{u}(x)\to 0$ as $|x|\to\infty$, there exists $n_o>0$ such that

$$\begin{cases} h^+(\frac{1}{2},\beta,n)>0,\\ h^+(2,\beta,n)<0,\end{cases} \tag{5.4}$$

for $n\geqslant n_o$ and $\beta\in[\frac{1}{2},2]$. Now, for all $\alpha\in[\frac{1}{2},2]$ we have

$$\begin{cases} h^+(\alpha,\frac{1}{2},n)>0,\\ h^+(\alpha,2,n)<0.\end{cases} \tag{5.5}$$

By the Mean Value Theorem (see [13]), there exist α^*,β^* such that $\frac{1}{2}\leqslant\alpha^*,\ \ \beta^*\leqslant 2$ and

$$h^{\pm}(\alpha^*,\beta^*,n)=0\quad\text{for } n\geqslant n_o,$$

that is

$$\alpha^*u_1-\beta^*\bar{u}_n\in\mathcal{B}\quad\text{for } n\geqslant n_o.$$

Hence, we only need to verify that

$$\sup_{\frac{1}{2}\leqslant\alpha,\beta\leqslant 2} I(\alpha u_1-\beta\bar{u}_n)<c_1+c_\infty\quad\text{for } n\geqslant n_o.$$

Indeed, since

$$I(\alpha u_1 - \beta \bar{u}_n) = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \alpha u_1 - \beta \nabla \bar{u}_n|^2 + |\alpha u_1 - \beta \bar{u}_n|^2 - \int_{\mathbb{R}^2 \setminus \Omega} Q(x) F(\alpha u_1 - \beta \bar{u}_n),$$

using Lemma 3.1, we get

$$I(\alpha u_1 - \beta \bar{u}_n) \leqslant \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\nabla(\alpha u_1) - \nabla(\beta \bar{u}_n)|^2 + \frac{1}{2} \int_{\mathbb{R}^2 \setminus \Omega} |\alpha u_1 - \beta \bar{u}_n|^2 - I_1,$$

where

$$I_1 = \int_{\mathbb{R}^2 \setminus \Omega} QF(\alpha u_1) + \int_{\mathbb{R}^2 \setminus \Omega} QF(\beta \bar{u}_n) - 2 \int_{\mathbb{R}^2 \setminus \Omega} f(\alpha u_1) \beta \bar{u}_n + \alpha u_1 f(\beta \bar{u}_n).$$

Since u_1 is a solution of (P) and $\bar{u}_n$ depends of a ground state of (P_∞), we have

$$I(\alpha u_1 - \beta \bar{u}_n) \leqslant I(\alpha u_1) + I_\infty(\beta \bar{u}_n) - \int_{\mathbb{R}^2 \setminus \Omega} (Q - \bar{Q}) F(\beta \bar{u}_n) + C_1 \int_{\mathbb{R}^2 \setminus \Omega} (f(u_1) \bar{u}_n + u_1 f(\bar{u}_n)) + \int_{\Omega} \bar{Q} F(\beta \bar{u}_n).$$

Therefore, we conclude that

$$\begin{aligned} \sup_{\frac{1}{2} \leqslant \alpha, \beta \leqslant 2} I(\alpha u_1 - \beta \bar{u}_n) \leqslant\ & \sup_{\alpha \geqslant 0} I(\alpha u_1) + \sup_{\beta \geqslant 0} I_\infty(\beta \bar{u}_n) \\ & - \int_{\mathbb{R}^2 \setminus \Omega} (Q - \bar{Q}) F\left(\frac{1}{2} \bar{u}_n\right) \\ & + C_1 \int_{\mathbb{R}^2 \setminus \Omega} (f(\alpha u_1) \beta \bar{u}_n + \alpha u_1 f(\beta \bar{u}_n)) \\ & + \int_{\Omega} \bar{Q} F(2 \bar{u}_n). \end{aligned} \tag{5.6}$$

Now, by (Q_3), we obtain

$$\int_{\mathbb{R}^2 \setminus \Omega} (Q - \bar{Q}) F\left(\frac{1}{2} \bar{u}_n\right) \geqslant C e^{-\mu n}, \tag{5.7}$$

and by (f_1) we get

$$\int_{\Omega} \bar{Q} F(\bar{u}_n) \leqslant C_\varepsilon e^{-nb}\left(e^{4\pi\tau e^{-2nb}} - 1\right) + \varepsilon e^{-b(q+1)n}. \tag{5.8}$$

On the other hand, one has

$$\int_{\mathbb{R}^2\setminus\Omega} f(u_1)\bar{u}_n \leqslant \varepsilon \int_{\mathbb{R}^2\setminus\Omega} |u_1|^q |\bar{u}_n| + C_\varepsilon \int_{\mathbb{R}^2\setminus\Omega} \left(e^{4\pi\tau u_1^2} - 1\right)\bar{u}_n.$$

Notice that

$$\int_{\mathbb{R}^2\setminus\Omega} |u_1|^q \bar{u}_n = \int_{\Theta_n^1} |u_1|^q \bar{u}_n + \int_{\Theta_n^1} |u_1|^q |\bar{u}_n|$$

and

$$\int_{\mathbb{R}^2\setminus\Omega} \left(e^{4\pi\tau u_1^2} - 1\right)\bar{u}_n = \left(\int_{\Theta_n^1} + \int_{\Theta_n^2}\right)\left(e^{4\pi\tau u_1^2} - 1\right)\bar{u}_n,$$

where $\Theta_n^1 = (\mathbb{R}^2 \setminus \Omega) \cap \{|x| < \frac{1}{q+1}n\}$ and $\Theta_n^2 = (\mathbb{R}^2 \setminus \Omega) \cap \{|x| \geqslant \frac{1}{q+1}n\}$. Thus

$$\int_{\mathbb{R}^2\setminus\Omega} |u_1|^q \bar{u}_n \leqslant C_1 e^{-\frac{q}{q+1}bn} \tag{5.9}$$

and

$$\int_{\mathbb{R}^2\setminus\Omega} \left(e^{4\pi\alpha u_1^2} - 1\right)\bar{u}_n \leqslant C e^{-\frac{q}{q+1}nb} + C\left(e^{4\pi\tau e^{-\frac{2bn}{q+1}}} - 1\right),$$

hence

$$\int_{\mathbb{R}^2\setminus\Omega} f(u_1)\bar{u}_n \leqslant C_1 e^{-\frac{q}{q+1}bn} + C\left(e^{4\pi\tau e^{-\frac{2bn}{q+1}}} - 1\right). \tag{5.10}$$

Using similar arguments, we get

$$\int_{\mathbb{R}^2\setminus\Omega} u_1 f(\bar{u}_n) \leqslant C e^{-\frac{bn}{q+1}} + C\left(e^{4\pi\tau e^{-\frac{2bn}{q+1}}} - 1\right). \tag{5.11}$$

From (5.6)–(5.11), we have for n large enough

$$\sup_{\frac{1}{2}\leqslant\alpha,\beta\leqslant 2} I(\alpha u_1-\beta\bar{u}_n) < \sup_{\alpha\geqslant 0} I(\alpha u_1)+\sup_{\beta\geqslant 0} I_\infty(\beta\bar{u}_n) = c_1+c_\infty.$$

Consequently

$$\hat{c} < c_1+c_\infty,$$

which proves the proposition. □

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