J. Differential Equations 252 (2012) 137–167

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle

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ARTICLE INFO

Article history:
Received 30 June 2010
Revised 28 August 2011
Available online 1 October 2011

Keywords:
Degenerate parabolic equation
Anisotropic
Dirichlet boundary problem
Kinetic formulation
Comparison theorem
Uniqueness and existence

ABSTRACT

We study the comparison principle for anisotropic degenerate parabolic–hyperbolic equations with initial and nonhomogeneous boundary conditions. We prove a comparison theorem for any entropy sub- and super-solution, which immediately deduces the L^1 contractivity and therefore, uniqueness of entropy solutions. The method used here is based upon the kinetic formulation and the kinetic techniques developed by Lions, Perthame and Tadmor. By adapting and modifying those methods to the case of Dirichlet boundary problems for degenerate parabolic equations we can establish a comparison property. Moreover, in the quasi-isotropic case the existence of entropy solutions is proved.

1. Introduction

Let Ω be an open and bounded rectangle of $\mathbb{R}^d$ and $T > 0$. Let Q denote the set $(0, T) \times \Omega$, $\partial\Omega$ the boundary of Ω and Σ the set $(0, T) \times \partial\Omega$. We deal with the uniqueness and existence of solutions of anisotropic degenerate parabolic equation

$$\partial_t u + \operatorname{div} A(u) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j} \beta_{ij}(u) = g \quad \text{in } Q \tag{1.1}$$

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.09.008

with the initial condition

$$u(0,x) = u_0(x) \quad \text{in } \Omega \tag{1.2}$$

and the boundary condition

$$u(t,x) = u_b(t,x) \quad \text{on } \Sigma, \tag{1.3}$$

where $u(t,x) : Q \to \mathbb{R}$ is the unknown function and $u_0(x) : \Omega \to \mathbb{R}$ and $u_b(t,x) : \Sigma \to \mathbb{R}$ are given functions. $A(u) = (A_1(u), \ldots, A_d(u)(u))$ is the flux and $B(u) = (\beta_{ij}(u))$ is the diffusion matrix. It is assumed that $A_i(u)$ and $\beta_{ij}(u)$ are functions in $W^{1,\infty}_{loc}(\mathbb{R})$. The precise assumption on data u_0, u_b and g will be stated later.

Since (1.1) is allowed to be completely degenerate, global solutions are in general discontinuous and some weak solutions must be considered. Moreover the boundary condition (1.3) is not necessarily satisfied in the classical sense that a trace of the solution exists and equals the datum u_b on Σ. In the completely degenerate case Eq. (1.1) becomes a first order hyperbolic equation and it is well known that a smooth solution of (1.1) is constant along the maximal segment of the characteristic line in Q. Now suppose that this segment intersects both $\{0\} \times \Omega$ and Σ. Then the problem (1.1)–(1.3) would be overdetermined if (1.3) were assumed in the classical sense. Thus one needs to work within a suitable framework of entropy solutions and entropy boundary conditions to obtain uniqueness and existence results. In the BV setting Bardos, LeRoux and Nédélec [4] first gave an interpretation of the boundary condition (1.3) as an "entropy" inequality on Σ, which is the so-called BLN condition. However, since the trace of solutions is involved in the formulation of the BLN condition, it makes no sense if the solution is merely in L^∞. Otto [25] extended the Dirichlet problem for hyperbolic equations to the L^∞ setting and proved a unique entropy solution by introducing an integral formulation of the boundary condition.

For degenerate parabolic equations (in which the diffusion matrix $B(u)$ is merely symmetric and nonnegative) the isotropic diffusion case first has been developed in recent years. The isotropic case means that B takes the form

$$B(u) = \beta(u)I$$

for some nondecreasing function $\beta(u)$, where I denotes the $d \times d$ identity matrix. In such a case Carrillo [7] succeeded in proving uniqueness and existence of entropy solutions under the homogeneous boundary condition $u_b \equiv 0$ by mainly using the doubling variable technique developed by Kružkov [20]. Mascia, Porretta and Terracina [23] and Michel and Vovelle [24] extended those results to the case of nonhomogeneous boundary condition by using also the doubling variable technique. On the other hand the uniqueness were proved in [19] by using the kinetic formulation which were introduced in [22] (also see [15]), without relying on the doubling variable technique. We also refer to [3,8,14,16,17] for the corresponding results on the isotropic case.

The anisotropic case was successfully treated by Chen and Perthame [12] for the Cauchy problem via the kinetic formulation and the regularization by convolution (see [26]). In their notions of solution the parabolic dissipative measure is explicitly included in the entropy inequality. In contract with anisotropic case a particular form of the parabolic dissipative measure is constructed in [7] (also see [19]) from the Kružkov entropy inequality. For the Cauchy problem in the anisotropic case we refer to [5,6,10–12,27]. The initial–boundary value problem of the anisotropic case is more delicate and has been treated in more recent years. Bendahmane and Karlsen treated in [6] (also see [1,2]) a class of doubly nonlinear degenerate parabolic equations with homogeneous Dirichlet boundary conditions. In particular, in [6] they proved the uniqueness of entropy solutions but did not give any proof of the existence. As far as the authors know, in the L^∞ setting there are few papers which treat nonhomogeneous Dirichlet problems for the anisotropic case and the existence of solutions seems to remain open even in the quasi-isotropic case (i.e. $\beta_{ij}(u) = 0$ whenever $i \neq j$).

In this paper we shall consider the nonhomogeneous Dirichlet problem for anisotropic equations only on rectangular domains. Motivated by [24] and [12], we introduce a notion of entropy solution of (1.1)–(1.3) and prove the uniqueness of the entropy solution via the kinetic techniques extended to initial–boundary value problems. The reason why restricting to rectangular domains is as follows: In the isotropic case the diffusion matrix $B(u)$ is invariant under changes of coordinates represented by orthogonal matrices. Hence, by such a change of coordinates we may consider an epigraph in $\mathbb{R}^d$ of a function defined on an appropriate open set in $\mathbb{R}^{d-1}$ as a neighborhood in Ω of a point of the boundary $\partial\Omega$. This fact, together with a partition of unity, enables us to treat more general domains than rectangular domains (see [19]). However, in the anisotropic case, in fact even in the quasi-isotropic case (i.e. $\beta_{ij}(u) = 0$ whenever $i \neq j$), a change of coordinates would lead to a violation of the conditions imposed on $B(u)$ in the definition of entropy solutions, more precisely, conditions (i) and (iii) in Definition 2.1 below. Thus we could not "rectify" the boundary of more general domains by local charts.

For existence of entropy solutions it has been proved by Wu and Zhao [29] that a generalized solution in a space BV exists for anisotropic equations with homogeneous Dirichlet problems. We see that the generalized solution u coincides with our entropy solution introduced herein except for the condition that $\partial_{x_{x_i}} \beta_{ii}(u) \in L^2(Q)$. But, we will use this condition to ensure a trace of $\beta_{ii}(u)$ on Σ for solutions in a space L^∞. In order to obtain the condition we will restrict ourselves to the quasi-isotropic case in the existence result. Finally, it would be interesting to prove the uniqueness result (Theorem 2.2 stated below) via the doubling variable techniques by Kružkov as was done in [24]. Unfortunately, to the best of our knowledge, we do not know whether those techniques can be adapted to the problem (1.1)–(1.3). It would be also interesting to remove the "additional" condition $\partial_{x_i} \beta_{ii}(u) \in L^2(Q)$.

The paper is organized as follows. In Section 2 we will give some notations and the notions of entropy solutions and state the main comparison theorem (Theorem 2.2) for entropy solutions. Section 3 is devoted to the proof of the theorem. In Section 4 the existence of entropy solution will be proved in the quasi-isotropic case.

2. Notions of solutions and a comparison theorem

We now give some notations and the notion of weak entropy solutions. Define

$$\operatorname{sgn}^+(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leqslant 0, \end{cases} \quad \text{and} \quad \operatorname{sgn}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \geqslant 0, \end{cases}$$

and $r^+ = r \vee 0$, $r^- = -(r \wedge 0)$ with $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. The semi-Kružkov entropies $\eta_k^\pm$ are the convex functions defined by

$$\eta_k^\pm(r) = (r-k)^\pm, \quad k \in \mathbb{R},$$

while the corresponding entropy fluxes are functions defined by

$$\mathcal{F}^\pm(r,k) = \operatorname{sgn}^\pm(r-k)\big(A(r) - A(k)\big).$$

We assume that $\Omega = \prod_{i=1}^d (a_i^-, a_i^+)$ is an open bounded rectangle of $\mathbb{R}^d$ with $2d$ faces

$$(\partial\Omega)_{i^*} = \big\{\big(x_1, \ldots, x_{i-1}, a_i^*, x_{i+1}, \ldots x_d\big);\ a_j^- < x_j < a_j^+ \text{ for } j = 1, 2, \ldots, d,\ j \neq i\big\}$$

and the outward normals $\mathbf{n}_{i^*}$ to Ω along $(\partial\Omega)_{i^*}$ for $i \in \{1, 2, \ldots, d\}$, where the super-index $*$ denotes the symbol $+$ or $-$. We set $\Sigma_{i^*} = (0, T) \times (\partial\Omega)_{i^*}$. Set $J = \{1^+, \ldots, d^+, 1^-, \ldots, d^-\}$ and $J_0 = \{0\} \cup J$. For $\nu > 0$ and $i^* \in J$ we set $U_{i^*}^\nu$, $(\partial\Omega)_{i^*}^\nu$, $\Omega_{i^*}^\nu$, $\tilde{\Omega}_{i^*}^\nu$ and $\Delta_{i^*}^\nu$ as follows: $U_{i^*}^\nu$ is the open subset of all $x \in \Omega$ such that $\operatorname{dist}(x, (\partial\Omega)_{i^*}) < \nu$ and $\operatorname{dist}(x, (\partial\Omega)_{i^*}) < \operatorname{dist}(x, (\partial\Omega)_{j^*})$ for all $j \in \{1, 2, \ldots, d\}$ with

$j \neq i$. $(\partial\Omega)^{\nu}_{i*}$ is the subset of all $x \in (\partial\Omega)_{i*}$ such that $x - s\mathbf{n}_{i*} \in U^{\nu}_{i*}$ for all $s \in (0,\nu)$. $\Omega^{\nu}_{i*} = \{x - s\mathbf{n}_{i*}; x \in (\partial\Omega)^{\nu}_{i*}, s \in (0,\nu)\}$, the largest cylinder generated by $\mathbf{n}_{i*}$ included in U^{ν}_{i*}. $\tilde{\Omega}^{\nu}_{i*} = \{x - s\mathbf{n}^{\nu}_{i*}; x \in (\partial\Omega)^{\nu}_{i*}, s \in (-\nu,\nu)\}$. $\Delta^{\nu}_{i*} = U^{\nu}_{i*} \setminus \Omega^{\nu}_{i*}$. We have $\operatorname{meas}(\bigcup_{i*\in J} \Delta^{\nu}_{i*}) \leqslant Const.\ \nu^2$. Moreover, we set $i^* = 0$ if $i = 0$, $\Omega^{\nu}_0 = U^{\nu}_0 = \Omega \setminus \bigcup_{i*\in J} U^{\nu}_{i*}$ and $\Omega^{\nu} = \bigcup_{i*\in J_0} \Omega^{\nu}_{i*}$. Since the family $\{U_0^{\frac{\nu}{2}}, \tilde{\Omega}^{\nu}_{i+}, \tilde{\Omega}^{\nu}_{i-}\}^d_{i=1}$ is an open cover of $\overline{\Omega^{2\nu}}$, we can choose a partition $\{\lambda_0, \lambda_{i+}, \lambda_{i-}\}^d_{i=1}$ of unity on $\overline{\Omega}^{2\nu}$ subordinate to the open cover. For $x \in (x_1,\dots,x_d)$ we denote $\bar{x}_i = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)$ and write $(\bar{x}_i, x_i)$ for x. We also denote $Q^{\nu}_{i*} = (0,T) \times \Omega^{\nu}_{i*}$, $\Sigma^{\nu}_{i*} = (0,T) \times (\partial\Omega)^{\nu}_{i*}$, $\Pi^{\nu}_{i*} = \{\bar{x}_i; x \in \operatorname{supp}(\lambda_{i*}) \cap \Omega\}$ and $\Theta^{\nu}_{i*} = (0,T) \times \Pi^{\nu}_{i*}$, $Q^{\nu} = \bigcup_{i*\in J_0} Q^{\nu}_{i*}$ and $\Sigma^{\nu} = \bigcup_{i*\in J} \Sigma^{\nu}_{i*}$.

To regularize functions, for small $\rho, s > 0$ let us consider a smooth function $\theta_{\rho,s} : \mathbb{R} \to \mathbb{R}_+$ such that $\operatorname{supp}\theta_{\rho,s} \subset [\frac{\rho s}{2}, (1+\rho)s]$, $\theta_{\rho,s}(r) = s^{-1}$ for $r \in [\rho s, s]$ and $\int_{\mathbb{R}} \theta_{\rho,s}(r)\,dr = 1$. Then, for $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_d) \in \mathbb{R}^{d+1}_+$ we set $\gamma^0_{\rho,\epsilon}(x) = \prod^d_{i=1} \theta_{\rho,\epsilon_i}(x_i)$ and $\gamma_{\rho,\epsilon}(t,x) = \theta_{\rho,\epsilon_0}(t)\gamma^0_{\rho,\epsilon}(x)$.

We will make the following assumptions throughout the paper:

(A1) $\Omega = \prod^d_{i=1}(a^-_i, a^+_i)$ is an open bounded rectangle of $\mathbb{R}^d$.

(A2) For $i, j = 1,2,\dots,d$, $A_i(u)$ and $\beta_{ij}(u)$ are functions in $W^{1,\infty}_{loc}(\mathbb{R})$. The $d \times d$ matrix $DB(u) = (D\beta_{ij}(u))$ is symmetric and nonnegative so that we can always write

$$D\beta_{ij}(u) = \sum^K_{k=1} \sigma_{ik}(u)\sigma_{jk}(u), \quad \sigma_{ik} \in L^{\infty}_{loc}(\mathbb{R})$$

with some index K, where $D\beta_{ij}$ denotes the derivative of β_{ij} with respect to u.

(A3) $u_0 \in L^{\infty}(\Omega)$, $u_b \in L^{\infty}(\Sigma)$ with $\beta_{ij}(u_b) \in W^{1,1}(\Sigma)$ and $g \in L^{\infty}(Q)$.

According to [12,24] we introduce the definition of entropy sub- and super-solutions. To this end we use the notations $s_{ik}(u)$ and $s^{\psi}_{ik}(u)$ for $\psi \in C(\mathbb{R})$:

$$Ds_{ik}(u) = \sigma_{ik}(u), \qquad Ds^{\psi}_{ik}(u) = \psi(u)\sigma_{ik}(u).$$

Definition 2.1. Let $u \in L^{\infty}(Q)$ and set

$$M = \sup\{|DA(r)|; |r| \leqslant \|u\|_{L^{\infty}(Q)} \vee \|u_b\|_{L^{\infty}(\Sigma)}\}.$$

(1) u is said to be an entropy sub-solution of problem (1.1)–(1.3) if it satisfies:
  (i) $\sum^d_{i=1} \partial_{x_i} s_{ik}(u) \in L^2(Q)$ for $k = 1,2,\dots,K$.
  (ii) $\sum^d_{i=1} \partial_{x_i} s^{\psi}_{ik}(u) = \psi(u) \sum^d_{i=1} \partial_{x_i} s_{ik}(u)$ for any $\psi \in C(\mathbb{R})$ and $k = 1,2,\dots,K$.
  (iii) (*Parabolic boundary condition*) For $i = 1,2,\dots,d$, $\partial_{x_i}\beta_{ii}(u) \in L^2(Q)$ and $\beta_{ii}(u) = \beta_{ii}(u_b)$ on Σ in the sense that

$$\lim_{s\to 0+} \frac{1}{s} \int_0^s \int_{\Sigma_{i*}} \left|\beta_{ii}\big(u\big(t,\bar{x}_i,a^*_i - r^*\big)\big) - \beta_{ii}\big(u_b(t,\bar{x}_i)\big)\right|^2 dt\,d\bar{x}_i\,dr = 0,$$

  where $r^* = r$ if $* = +$ and $r^* = -r$ if $* = -$.

  (iv)

$$\int_Q (u-\kappa)^+ \partial_t \varphi + \mathcal{F}^+(u,\kappa) \cdot \nabla\varphi$$

$$- \sum_{i,j=1}^{d} \partial_{x_j}\big(\operatorname{sgn}^+(u-\kappa)\big(\beta_{ij}(u)-\beta_{ij}(\kappa)\big)\big)\partial_{x_i}\varphi + \operatorname{sgn}^+(u-\kappa) g\varphi\, dx\, dt$$

$$+ \int_{\Omega} (u_0-\kappa)^+ \varphi(0,x)\, dx + M \int_{\Sigma} (u_b-\kappa)^+ \varphi\, d\sigma\, dt$$

$$\geqslant \int_{Q} \delta(\kappa-u) \sum_{k=1}^{K} \left(\sum_{i=1}^{d} \partial_{x_i} s_{ik}(u) \right)^2 \varphi\, dx\, dt \tag{2.1}$$

in $\mathcal{D}'(\mathbb{R}_\kappa)$ for any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$ with $\varphi \geqslant 0$ such that $\sum_{i=1}^{d} \operatorname{sgn}^+(\beta_{ii}(u_b)-\beta_{ii}(\kappa))\varphi = 0$ a.e. on Σ. Here $d\sigma$ denotes the $(d-1)$-dimensional area element in $\partial\Omega$ and $\delta(\kappa)$ the Dirac measure concentrated at $\kappa = 0$.

(2) u is said to be an entropy super-solution of (1.1)–(1.3) if (2.1) is replaced by

$$\int_{Q} (u-\kappa)^- \partial_t\varphi + \mathcal{F}^-(u,\kappa)\cdot\nabla\varphi$$

$$- \sum_{i,j=1}^{d} \partial_{x_j}\big(\operatorname{sgn}^-(u-\kappa)\big(\beta_{ij}(u)-\beta_{ij}(\kappa)\big)\big)\partial_{x_i}\varphi + \operatorname{sgn}^-(u-\kappa) g\varphi\, dx\, dt$$

$$+ \int_{\Omega} (u_0-\kappa)^- \varphi(0,x)\, dx + M \int_{\Sigma} (u_b-\kappa)^- \varphi\, d\sigma\, dt$$

$$\geqslant \int_{Q} \delta(\kappa-u) \sum_{k=1}^{K} \left(\sum_{i=1}^{d} \partial_{x_i} s_{ik}(u) \right)^2 \varphi\, dx\, dt \tag{2.2}$$

in $\mathcal{D}'(\mathbb{R}_\kappa)$.

(3) The function u is said to be an entropy solution of (1.1)–(1.3) if it is both an entropy weak sub- and super-solution.

Remark 2.1. In (2.1) and (2.2) we notice that the equality $\sum_{j=1}^{d} \partial_{x_j}(\operatorname{sgn}^\pm(u-\kappa)(\beta_{ij}(u)-\beta_{ij}(\kappa))) = \operatorname{sgn}^\pm(u-\kappa)\sum_{j=1}^{d} \partial_{x_j}(\beta_{ij}(u)-\beta_{ij}(\kappa))$ holds and it belongs to $L^2(Q)$ under the assumptions (i), (ii) and (iii) in Definition 2.1. Indeed, since $D\beta_{ij} = \sum_{k=1}^{K} \sigma_{ik}\sigma_{jk}$, it follows that $D\beta_{ii} \geqslant 0$ and that $D\beta_{ii} = 0$ implies $D\beta_{ij} = 0$. Therefore, if $u > \kappa$ and $\beta_{jj}(u) - \beta_{jj}(\kappa) = 0$, then the monotonicity of β_{jj} implies that $D\beta_{jj}$ vanishes on the interval (κ, u) and so does $D\beta_{ij}$, and hence $\beta_{ij}(u)-\beta_{ij}(\kappa) = 0$. Thus,

$$\sum_{j=1}^{d} \partial_{x_j}\big(\operatorname{sgn}^+(u-\kappa)\big(\beta_{ij}(u)-\beta_{ij}(\kappa)\big)\big)$$

$$= \sum_{j=1}^{d} \partial_{x_j}\big(\operatorname{sgn}^+\big(\beta_{jj}(u)-\beta_{jj}(\kappa)\big)\big(\beta_{ij}(u)-\beta_{ij}(\kappa)\big)\big)$$

$$= \sum_{j=1}^{d} \operatorname{sgn}^+\big(\beta_{jj}(u)-\beta_{jj}(\kappa)\big)\partial_{x_j}\big(\beta_{ij}(u)-\beta_{ij}(\kappa)\big)$$

and

$$\begin{aligned}\sum_{j=1}^{d} \partial_{x_j} \beta_{ij}(u) &= \sum_{j=1}^{d} \partial_{x_j} \int_{-\|u\|_\infty}^{\infty} D\beta_{ij}(\xi)\, \mathrm{sgn}^+(u-\xi)\, d\xi \\ &= \sum_{k=1}^{K} \sum_{j=1}^{d} \partial_{x_j} \int_{-\|u\|_\infty}^{\infty} Ds_{jk}^{\sigma_{ik}}(\xi)\, \mathrm{sgn}^+(u-\xi)\, d\xi \\ &= \sum_{k=1}^{K} \sum_{j=1}^{d} \partial_{x_j} s_{jk}^{\sigma_{ik}}(u) = \sum_{k=1}^{K} \sigma_{ik}(u) \sum_{j=1}^{d} \partial_{x_j} s_{jk}(u).\end{aligned}$$

Therefore we obtain the assertion.

Remark 2.2. If u is an entropy solution of (1.1)–(1.3), then as will be seen in the proof of Lemma 3.2 below we have for every $i, j = 1, 2, \dots, d$,

$$\lim_{s\to 0+} \frac{1}{s} \int_0^s \beta_{ij}\bigl(u\bigl(t, \bar{x}_i, a_i^* - r^*\bigr)\bigr)\, dr = \beta_{ij}\bigl(u_b(t, \bar{x}_i)\bigr) \quad \text{in } L^1\bigl((0,T) \times (\partial\Omega)_{i^*}\bigr).$$

Therefore if in addition to (A1)–(A3) the assumptions that $\partial_t u_b \in L^1(\Sigma)$, $\nabla u_b \in L^2(\Sigma)$ and $\sum_{i,j=1}^{d} \partial^2_{x_i x_j} \beta_{ij}(u_b) \in L^2(\Sigma)$ are further assumed, then by a slight modification the proof of Proposition 4.1 of [24] still works well in our anisotropic case and obtains for all $\kappa \in \mathbb{R}$, all nonnegative $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$, all $i = 1, 2, \dots, d$,

$$\lim_{s\to 0+} \frac{1}{s} \int_0^s \int_{(\partial\Omega)_{i^*}} \mathcal{F}_i\bigl(u\bigl(t, \bar{x}_i, a_i^* - r^*\bigr), \kappa, u_b(t, \bar{x}_i)\bigr) \psi\bigl(t, \bar{x}_i, a_i^*\bigr)\, dt\, d\bar{x}_i\, dr \leqslant 0,$$

where

$$\mathcal{F}_i(u, \kappa, \omega) = g_i(u, \kappa) + g_i(u, \omega) - g_i(\kappa, \omega)$$

and

$$g_i(u, \kappa) = \mathrm{sgn}(u-\kappa)\bigl(A_i(u) - A_i(\kappa)\bigr) - \sum_{j=1}^{d} \partial_{x_j} \bigl|\beta_{ij}(u) - \beta_{ij}(\kappa)\bigr|.$$

This inequality is a generalization of the boundary condition formulated by Otto [25] to the case of degenerate parabolic equations. In this way a boundary condition is included in the entropy solution defined by Definition 2.1.

In the definition of entropy solutions we have assumed the existence of the trace of $\beta_{ii}(u)$ on the boundary $(\partial\Omega)_i$ in the sense of the condition (iii), which is assured from the following lemma.

Lemma 2.1. *Let $u(t,x)$ be an entropy sub-/super-solution and let $a_i^- \leqslant x_i^0 \leqslant a_i^+$. For a.e. $t \in (0,T)$ we have*

$$\left\| \left(x_i - x_i^0\right)^{-1} \left(\beta_{ii}\left(u(t,x)\right) - \beta_{ii}\left(u\left(t,x^0\right)\right)\right) \right\|_{L^2(\Omega)} \leqslant 2 \left\| \partial_{x_i} \beta_{ii}\left(u(t,x)\right) \right\|_{L^2(\Omega)},$$

where $x = (\bar{x}_i, x_i)$ and $x^0 = (\bar{x}_i, x_i^0)$.

Proof. By Hardy's inequality (see [28, Lemma 13.5]) we have

$$\begin{aligned} \left\| \left(x_i - x_i^0\right)^{-1} \left(\beta_{ii}\left(u(t,x)\right) - \beta_{ii}\left(u\left(t,x^0\right)\right)\right) \right\|_{L^2(\Omega)} &= \left\| \left(x_i - x_i^0\right)^{-1} \int_{x_i^0}^{x_i} \partial_{x_i} \beta_{ii}\left(u(t,\bar{x}_i,r)\right) dr \right\|_{L^2(\Omega)} \\ &\leqslant 2 \left\| \partial_{x_i} \beta_{ii}\left(u(t,x)\right) \right\|_{L^2(\Omega)}. \end{aligned}$$

Since $\partial_{x_i} \beta_{ii}(u) \in L^2(Q)$ by the condition (iii), the desired inequality holds for a.e. $t \in (0,T)$. □

We are now in a position to state the comparison theorem for entropy solutions.

Theorem 2.2. *Assume that* (A1), (A2) *and* (A3) *hold. Let u be an entropy sub-solution of* (1.1)–(1.3) *associated to data (u_0, u_b, g) and $\tilde{u}$ an entropy super-solution of* (1.1)–(1.3) *associated to data $(\tilde{u}_0, \tilde{u}_b, \tilde{g})$. Then we have for a.e. $t \in (0,T)$*

$$\begin{aligned} &\int_\Omega \left(u(t,x) - \tilde{u}(t,x)\right)^+ dx \\ &\quad \leqslant \int_\Omega \left(u_0(x) - \tilde{u}_0(x)\right)^+ dx - \sum_{i \neq j} \int_{\Sigma_t} \partial_{x_j}\left(\operatorname{sgn}^+(u_b - \tilde{u}_b)\left(\beta_{ij}(u_b) - \beta_{ij}(\tilde{u}_b)\right)\right) ds\, d\sigma \\ &\qquad + M \int_{\Sigma_t} \left(u_b(s,x) - \tilde{u}_b(s,x)\right)^+ ds\, d\sigma + \int_{Q_t} \left(g(s,x) - \tilde{g}(s,x)\right)^+ ds\, dx, \end{aligned} \tag{2.3}$$

where $\Sigma_t = (0,t) \times \partial\Omega$, $Q_t = (0,t) \times \Omega$, $M = \sup\{|DA(r)|;\, |r| \leqslant L\}$ with L the maximum of $\|u\|_{L^\infty(Q)}$, $\|\tilde{u}\|_{L^\infty(Q)}$, $\|u_b\|_{L^\infty(\Sigma)}$, $\|\tilde{u}_b\|_{L^\infty(\Sigma)}$, $\|g\|_{L^\infty(Q)}$, $\|\tilde{g}\|_{L^\infty(Q)}$, and $\sum_{i \neq j}$ denotes the summation over $i, j \in \{1, 2, \ldots, d\}$ with $i \neq j$.

Remark 2.3. The diagonal boundary terms $\int_{\Sigma_t} (\beta_{ii}(u_b) - \beta_{ii}(\tilde{u}_b))^+ \, dt\, d\sigma$ disappear on the right-hand side of (2.3). This is due to the flatness of the boundaries $(\partial\Omega)_{i^*}$. By contrast, the boundary term $\frac{L}{2} \int_\Sigma (\beta(u_b) - \beta(\tilde{u}_b))^+ \, dt\, d\sigma$ appears in the comparison inequality obtained in [19] which discusses the Dirichlet problem on a general C^2 bounded open subset Ω of $\mathbb{R}^d$ in the isotropic case $B(u) = \beta(u)I$, where L is the maximum of the mean curvatures on the boundary $\partial\Omega$.

3. Proof of the comparison theorem

To prove Theorem 2.2 let $u(t,x)$ be an entropy sub-solution of (1.1)–(1.3) with data (u_0, u_b, g). The parabolic defect measure $n(t,x,\xi)$ on $Q \times \mathbb{R}$ is given by

$$\left\langle n(t,x,\xi), \varphi \right\rangle = \int_{Q \times \mathbb{R}} \delta(\xi - u) \sum_{k=1}^K \left(\sum_{i=1}^d \partial_{x_i} s_{ik}(u) \right)^2 \varphi \, dt\, dx\, d\xi \tag{3.1}$$

for $\varphi \in C_c^\infty(Q \times \mathbb{R})$. The entropy defect measures $m_+(t,x,\xi)$ and $m_-(t,x,\xi)$ on $Q \times \mathbb{R}$ are defined by

$$\begin{aligned}
&\big\langle m_\pm(t,x,\xi), \varphi\big\rangle \\
&\quad = \int\limits_{Q \times \mathbb{R}} \Bigg\{ (u-\xi)^\pm \partial_t \varphi + \mathcal{F}^\pm(u,\xi) \cdot \nabla\varphi \\
&\qquad - \sum_{i,j=1}^d \partial_{x_j}\big(\operatorname{sgn}^\pm(u-\xi)\big(\beta_{ij}(u) - \beta_{ij}(\xi)\big)\big)\partial_{x_i}\varphi + \operatorname{sgn}^\pm(u-\xi) g\varphi \Bigg\}\, dt\, dx\, d\xi \\
&\qquad - \big\langle n(t,x,\xi), \varphi\big\rangle
\end{aligned} \tag{3.2}$$

for $\varphi \in C_c^\infty(Q \times \mathbb{R})$. Indeed, $m_\pm$ belong to $\mathcal{M}^+(Q \times \mathbb{R})$, the nonnegative Radon measures on $Q \times \mathbb{R}$, since (2.1) and (2.2) give $\langle m_\pm, \varphi\rangle \geqslant 0$ for any $\varphi \in C_c^\infty(Q \times \mathbb{R})$ with $\varphi \geqslant 0$. We also see that

$$m_\pm + n \in C\big(\mathbb{R}_\xi;\ w\text{-}\mathcal{M}^+(Q)\big), \tag{3.3}$$

$$\lim_{\xi \to \pm\infty} \big(m_\pm(\cdot,\xi) + n(\cdot,\xi)\big) = 0 \quad \text{in } w\text{-}\mathcal{M}^+(Q), \tag{3.4}$$

where w-$\mathcal{M}^+(Q)$ denotes the space $\mathcal{M}^+(Q)$ equipped with weak topology. We define the semi-equilibrium functions f_+ associated to an entropy sub-solution u and f_- associated to an entropy super-solution u by

$$f_\pm(t,x,\xi) = \operatorname{sgn}^\pm\big(u(t,x) - \xi\big).$$

The functions $f_\pm$ satisfy: For any $\phi \in C_c^\infty(Q \times \mathbb{R})$

$$\begin{aligned}
&\int\limits_{Q \times \mathbb{R}} f_\pm\Bigg(\partial_t + a(\xi)\cdot\nabla + \sum_{i,j=1}^d D\beta_{ij}(\xi)\partial_{x_i}\partial_{x_j}\Bigg)\phi + \delta(u-\xi) g\phi\, dt\, dx\, d\xi \\
&\quad = \pm \int\limits_{Q \times \mathbb{R}} \partial_\xi \phi\, d(m_\pm + n).
\end{aligned} \tag{3.5}$$

Lemma 3.1. *Let u be an entropy sub- (resp. super-)solution. Then there exists a function $f_+^{\tau_0}$ (resp. $f_-^{\tau_0}$) $\in L^\infty(\Omega \times \mathbb{R})$ such that*

$$\lim_{s \to 0+} \int\limits_{\Omega \times \mathbb{R}} \Bigg(\frac{1}{s}\int_0^s f_\pm(t,x,\xi)\, dt\Bigg)\phi\, dx\, d\xi = \int\limits_{\Omega \times \mathbb{R}} f_\pm^{\tau_0}(x,\xi)\phi\, dx\, d\xi \tag{3.6}$$

for any $\phi \in C_c^\infty(\Omega \times \mathbb{R})$.

Proof. By weak* compactness there exist a sequence $\{s_k\} \downarrow 0$ and a function $f_+^{\tau_0} \in L^\infty(\Omega \times \mathbb{R})$ such that

$$\frac{1}{s_k}\int_0^{s_k} f_+(t,x,\xi)\, dt \rightharpoonup f_+^{\tau_0} \quad \text{in } w^*\text{-}L^\infty(\Omega \times \mathbb{R}).$$

One has to show that $f_+^{\tau_0}$ does not depend on the sequence $\{s_k\}$. In order to do so, let us consider the vector-valued function $F_\zeta = (F_\zeta^1, F_\zeta^2)$ defined on Q with $\zeta \in C_c^\infty(\mathbb{R})$,

$$F_\zeta^1(t,x) = \int_{\mathbb{R}} f_+(t,x,\xi)\zeta(\xi)\,d\xi,$$

$$F_\zeta^2(t,x) = \int_{\mathbb{R}} \big(a(\xi) - DB(\xi)\nabla_x\big) f_+(t,x,\xi)\zeta(\xi)\,d\xi.$$

Notice that $F_\zeta^2(t,x)$ exists by (3.5). Since $\int_{\mathbb{R}} a(\xi) f_+(t,x,\xi)\zeta(\xi)\,d\xi$ is finite, so is $\int_{\mathbb{R}} DB(\xi)\partial_x f_+(t,x,\xi) \times \zeta(\xi)\,d\xi$. It follows from (3.5) that

$$\begin{aligned} \operatorname{div}_{(t,x)} F_\zeta(t,x) &= \int_{\mathbb{R}} \big\{\partial_t f_+ + \operatorname{div}_x\big(a(\xi) - DB(\xi)\nabla_x\big) f_+\big\}\zeta(\xi)\,d\xi \\ &= \int_{\mathbb{R}} \delta(u-\xi) g\zeta(\xi)\,d\xi - \int_{\mathbb{R}} D\zeta(\xi)(m_+ + n)\,d\xi. \end{aligned} \tag{3.7}$$

Let $h = \sum_{i^* \in J_0} \lambda_{i^*}$ and let Ω' be a C^2 open subset of $\mathbb{R}^d$ such that $\Omega \cap \operatorname{supp}(h) \subset \Omega' \subset \Omega$. Note that h vanishes on a subset of the boundary $\partial\Omega'$ except the set $\bigcup_{i=1}^d (\Pi_{i^*}^\nu \times \{a_i^*\})$. By applying the result of Chen and Frid [9] to (3.7) in the domain $Q' = (0,T) \times \Omega'$, there exists $\mathcal{T} \in W^{-\frac{1}{2},2}(\partial_p Q') + \mathcal{M}(\partial_p Q')$ such that

$$\begin{aligned} \langle \mathcal{T}, \bar\psi\rangle = -\lim_{s\to 0+}\Bigg[&\int_{\Omega'\times\mathbb{R}} \left(\frac{1}{s}\int_0^s f_+\,dt\right)\zeta\psi h\,dx\,d\xi \\ &+ \sum_{i^*\in J} \int_{\Theta_{i^*}^\nu \times \mathbb{R}} \Bigg(\frac{1}{s}\int_0^s \mathbf{n}_{i^*}(x_0)\cdot\big(a(\xi) - DB(\xi)\nabla\big) \\ &\cdot f_+(t,x_r,\xi)\psi(t,x_r)\lambda_{i^*}(x_r)\,dr\Bigg)\,dt\,d\bar{x}_i\,d\xi\Bigg] \end{aligned} \tag{3.8}$$

for any $\psi \in C_c^\infty(\mathbb{R}^{d+1})$, where x_r stands for the point $(\bar{x}_i, a_i^* - r^*)$, $\partial_p Q'$ denotes the parabolic boundary of Q' and $\bar\psi$ is the restriction of ψ to $\partial_p Q'$. In particular, choosing a test function ψ satisfying $\bar\psi(0,x) = \phi(x) \in C_c^\infty(\Omega_{2\nu})$ and $\bar\psi = 0$ on $(0,T) \times \partial Q'$, one has

$$\langle \mathcal{T}, \bar\psi\rangle = -\int_{\Omega'\times\mathbb{R}} f_+^{\tau_0}\zeta\phi\,dx\,d\xi.$$

This means that $f_+^{\tau_0}$ is independent of the sequences $\{s_k\}$ and proves the lemma for the case of an entropy sub-solution. For an entropy super-solution the lemma is similarly proved. □

In what follows $\psi^{\lambda_{i^*}}$ stands for $\psi\lambda_{i^*}$ with a function $\psi(t,x,\xi)$ defined on $\mathbb{R}^{d+2}$ and an element λ_{i^*} of the partition of unity $\{\lambda_{i^*}\}_{i^*\in J_0}$. We sometimes denote λ_{i^*} by λ if there is no confusion.

Lemma 3.2. *For any $\psi \in C_c^\infty(\mathbb{R}^{d+2})$, any $\nu > 0$ and any $i^* \in J$ we have*

$$\begin{aligned}
&\int\limits_{Q_{i^*}^\nu \times \mathbb{R}} f_\pm\big(\partial_t + a(\xi)\cdot\nabla\big)\psi^{\lambda_{i^*}} - \sum_{k,j=1}^{d} D\beta_{kj}(\xi)\partial_{x_j} f_\pm \partial_{x_k}\psi^{\lambda_{i^*}} + \delta(u-\xi)g\psi^{\lambda_{i^*}}\,dt\,dx\,d\xi \\
&\quad - \int\limits_{\Theta_{i^*}^\nu \times \mathbb{R}} \sum_{j\neq i} D\beta_{ij}(\xi)\partial_{x_j} f_\pm^b \psi^{\lambda_{i^*}}\,dt\,d\bar{x}_i\,d\xi + \int\limits_{\Omega\times\mathbb{R}} f_\pm^{\tau_0}(x,\xi)\psi^{\lambda_{i^*}}(0,x,\xi)\,dx\,d\xi \\
&\quad + \lim_{s\to 0+} \int\limits_{\Theta_{i^*}^\nu \times \mathbb{R}} \left[\frac{1}{s}\int_0^s \big(-\mathbf{n}_{i^*}(x_0)\cdot a(\xi) - D\beta_{ii}(\xi)\partial_{x_i}\big) f_\pm \psi^{\lambda_{i^*}}(t,x_r,\xi)\,dr\right] dt\,d\bar{x}_i\,d\xi \\
&= \int\limits_{Q_{i^*}^\nu \times \mathbb{R}} \partial_\xi \psi^{\lambda_{i^*}}\,d(m_\pm + n),
\end{aligned} \tag{3.9}$$

where $x_r = (\bar{x}_i, a_i^* - r^*)$, $f_\pm^b(t,y,\xi) = \operatorname{sgn}^\pm(u_b(t,y)-\xi)$ for $(t,y)\in\Sigma$, $\xi\in\mathbb{R}$ and $\sum_{j\neq i}$ stands for the summation over $j\in\{1,2,\ldots,d\}$ with $j\neq i$.

Remark 3.1. Notice that $f_\pm$, $f_\pm^b$ and $m_\pm + n$ vanish as $\xi \to \pm\infty$. Therefore (3.9) for f_+ (resp. f_-) is still valid for each bounded test function ψ such that the support of ψ with respect to ξ is contained only in $[\kappa,\infty)$ (resp. $(-\infty,\kappa]$) for some $\kappa\in\mathbb{R}$.

Proof of Lemma 3.2. By the similarity we may prove the case of an entropy sub-solution and $i^* = i^-$. To simplify the notation we will drop the super-index ν. For $\tilde{\psi} \in C_c^\infty((0,T)\times\mathbb{R}^d)$ and $\zeta \in C_c^\infty(\mathbb{R})$

$$\begin{aligned}
&\int\limits_{\Theta_{i^-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s \mathbf{n}_{i^-}\cdot DB(\xi)\nabla f_+\big(t,\bar{x}_i,a_i^- + r,\xi\big)\tilde{\psi}^\lambda\big(t,\bar{x}_i,a_i^- + r\big)\,dr\right]\zeta(\xi)\,dt\,d\bar{x}_i\,d\xi \\
&= -\sum_{j=1}^{d} \int\limits_{\Theta_{i^-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s D\beta_{ij}(\xi)\partial_{x_j} f_+\tilde{\psi}^\lambda\,dr\right]\zeta\,dt\,d\bar{x}_i\,d\xi \\
&= -\int\limits_{\Theta_{i^-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s D\beta_{ii}(\xi)\partial_{x_i} f_+\tilde{\psi}^\lambda\,dr\right]\zeta\,dt\,d\bar{x}_i\,d\xi \\
&\quad - \sum_{j\neq i} \int\limits_{\Theta_{i^-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s D\beta_{ij}(\xi)\partial_{x_j} f_+\tilde{\psi}^\lambda\,dr\right]\zeta\,dt\,d\bar{x}_i\,d\xi.
\end{aligned} \tag{3.10}$$

Notice that $D\beta_{ii} \geqslant 0$ and $|D\beta_{ij}|^2 \leqslant D\beta_{ii}D\beta_{jj}$. If $j\neq i$, then we have

$$\left|\int\limits_{Q_{i^-}\times\mathbb{R}} D\beta_{ij}(\xi)\left[\frac{1}{s}\int_0^s \partial_{x_j} f_+\tilde{\psi}^\lambda(\cdot,r)\,dr - \partial_{x_j} f_+^b\tilde{\psi}^\lambda(\cdot)\right]\zeta(\xi)\,d\xi\,dt\,d\bar{x}_i\right|$$

$$= \left| \int_{Q_{i-}\times\mathbb{R}} D\beta_{ij}(\xi)\left[\frac{1}{s}\int_0^s f_+\partial_{x_j}\tilde{\psi}^\lambda(\cdot,r)\,dr - f_+^b\partial_{x_j}\tilde{\psi}^\lambda(\cdot)\right]\zeta(\xi)\,d\xi\,dt\,d\bar{x}_i\right|$$

$$\leqslant \frac{1}{s}\int_0^s \int_{Q_{i-}\times\mathbb{R}} D\beta_{ii}(\xi)^{\frac{1}{2}} D\beta_{jj}(\xi)^{\frac{1}{2}}\left\{\left|f_+ - f_+^b\right|\left|\partial_{x_j}\tilde{\psi}^\lambda(\cdot,r)\right|\right.$$

$$\left. + f_+^b\left|\partial_{x_j}\tilde{\psi}^\lambda(\cdot,r) - \partial_{x_j}\tilde{\psi}^\lambda(\cdot)\right|\right\}\left|\zeta(\xi)\right|dr\,d\xi\,dt\,d\bar{x}_i$$

$$\leqslant C\left(\frac{1}{s}\int_0^s \int_{Q_{i-}\times\mathbb{R}} D\beta_{ii}(\xi)\left|f_+ - f_+^b\right|^2\left|\zeta(\xi)\right|dr\,d\xi\,dt\,d\bar{x}_i\right)^{\frac{1}{2}}$$

$$+ \frac{C}{s}\int_0^s \int_{Q_{i-}} \left|\partial_{x_j}\tilde{\psi}^\lambda(\cdot,r) - \partial_{x_j}\tilde{\psi}^\lambda(\cdot)\right|dr\,dt\,d\bar{x}_i$$

$$\leqslant C\left(\frac{1}{s}\int_0^s \int_{Q_{i-}} \left|\beta_{ii}(u) - \beta_{ii}(u_b)\right|dr\,dt\,d\bar{x}_i\right)^{\frac{1}{2}}$$

$$+ \frac{C}{s}\int_0^s \int_{Q_{i-}} \left|\partial_{x_j}\tilde{\psi}^\lambda(\cdot,r) - \partial_{x_j}\tilde{\psi}^\lambda(\cdot)\right|dr\,dt\,d\bar{x}_i,$$

which tends to 0 as $s \to 0+$ by (iii) of Definition 2.1. Thus (3.8) with Ω' replaced by Ω_{i-}^{ν} together with (3.10) and Lemma 3.1 ensure that the following limits exist:

$$\lim_{s\to 0+}\int_{\Theta_{i-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s \mathbf{n}_{i^-}\cdot\big(-a(\xi) + DB(\xi)\nabla\big)f_+\psi^\lambda\big(t,\bar{x}_i,a_i^- + r,\xi\big)\,dr\right]dt\,d\bar{x}_i\,d\xi$$

$$= \lim_{s\to 0+}\int_{\Theta_{i-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s \big(-\mathbf{n}_{i^-}\cdot a(\xi) - D\beta_{ii}(\xi)\partial_{x_i}f_+\big)\psi^\lambda\big(t,\bar{x}_i,a_i^- + r,\xi\big)\,dr\right]dt\,d\bar{x}_i\,d\xi$$

$$- \sum_{j\neq i}\int_{\Theta_{i-}\times\mathbb{R}} D\beta_{ij}(\xi)\partial_{x_j}f_+^b\psi^\lambda\,dt\,d\bar{x}_i\,d\xi \tag{3.11}$$

for any $\psi \in C_c^\infty((0,T)\times\mathbb{R}^{d+1})$.

We set

$$W_{\rho,s}(r) = \int_0^r \theta_{\rho,s}(\tau)\,d\tau$$

for $r \in \mathbb{R}$ and small $\rho, s > 0$. Let $\tilde{\phi} \in C_c^\infty([0,T)\times\Omega\times\mathbb{R})$ and take $W_{\rho,s}(t)\tilde{\phi}^{\lambda_{i^-}}(t,x,\xi)$ as a test function in (3.5). Then

$$\int\limits_{Q_i^- \times \mathbb{R}} W_{\rho,s}(t) f_+ \big(\partial_t + a(\xi)\cdot\nabla\big)\tilde{\phi}^\lambda + \theta_{\rho,s}(t) f_+ \tilde{\phi}^\lambda \, dt\, dx\, d\xi$$

$$- \int\limits_{Q_{i^-} \times \mathbb{R}} W_{\rho,s}(t) \sum_{k,j=1}^{d} D\beta_{kj}(\xi)\partial_{x_j} f_+ \partial_{x_k}\tilde{\phi}^\lambda \, dt\, dx\, d\xi$$

$$+ \int\limits_{Q_{i^-} \times \mathbb{R}} W_{\rho,s}(t)\delta(u-\xi) g\tilde{\phi}^\lambda \, dt\, dx\, d\xi$$

$$= \int\limits_{Q_{i^-} \times \mathbb{R}} W_{\rho,s}(t)\partial_\xi \tilde{\phi}^\lambda d(m_+ + n).$$

Passing $\rho \to 0+$ and then $s \to 0+$, by Lemma 3.1 and the Lebesgue convergence theorem we have

$$\int\limits_{Q_{i^-} \times \mathbb{R}} f_+ \big(\partial_t + a(\xi)\cdot\nabla\big)\tilde{\phi}^\lambda - \sum_{k,j=1}^{d} D\beta_{kj}(\xi)\partial_{x_j} f_+ \partial_{x_k}\tilde{\phi}^\lambda + \delta(u-\xi) g\tilde{\phi}^\lambda \, dt\, dx\, d\xi$$

$$+ \int\limits_{\Omega \times \mathbb{R}} f_+^{\tau_0}(x,\xi)\tilde{\phi}^\lambda(0,x,\xi)\, dx\, d\xi$$

$$= \int\limits_{Q_{i^-} \times \mathbb{R}} \partial_\xi \tilde{\phi}^\lambda d(m_+ + n). \tag{3.12}$$

Next we set

$$\bar{W}_{\rho,s}(x) = \int\limits_0^{x_i - a_i^-} \theta_{\rho,s}(\tau)\, d\tau \quad \text{for } x = (x_1, \dots, x_d) \in \Omega.$$

Let $\psi(t,x,\xi) \in C_c^\infty([0,T) \times \mathbb{R}^{d+1})$ and put $\tilde{\phi} = \bar{W}_{\rho,s}\psi$ in (3.12). Noting that $\nabla \bar{W}_{\rho,s} = -\theta_{\rho,s}(x_i - a_i^-)\mathbf{n}_{i^-}$, we obtain

$$\int\limits_{Q_{i^-} \times \mathbb{R}} \bar{W}_{\rho,s} f_+ \big(\partial_t + \big(a(\xi) - DB(\xi)\nabla\big) f_+ \cdot \nabla\big)\psi^\lambda + \delta(u-\xi) g\psi^\lambda \, dt\, dx\, d\xi$$

$$- \int\limits_{Q_{i^-} \times \mathbb{R}} \theta_{\rho,s}\big(x_i - a_i^-\big)\big(a(\xi) - DB(\xi)\nabla\big) f_+ \cdot \mathbf{n}_{i^-}\psi^\lambda \, dt\, dx\, d\xi$$

$$+ \int\limits_{\Omega \times \mathbb{R}} f_+^{\tau_0}\psi^\lambda(0,x,\xi)\, dx\, d\xi$$

$$= \int\limits_{Q_{i^-} \times \mathbb{R}} \bar{W}_{\rho,s}\partial_\xi \psi^\lambda d(m_+ + n).$$

Hence, letting $\rho \to 0+$ and $s \to 0+$ and using (3.11), one immediately obtains (3.9). □

Lemma 3.3. *Let $u(t,x)$ be an entropy sub- or super-solution of* (1.1)–(1.3). *Then, for any $\phi \in C_c^\infty([0,T)\times\mathbb{R}^d)$ with $\phi \geqslant 0$, any $\zeta \in C_c^\infty(\mathbb{R})$ with $\zeta \geqslant 0$, any $\nu > 0$ and any $i \in \{1,2,\dots,d\}$,*

$$\liminf_{s\to 0+} \int_{\Theta_{i^*}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s D\beta_{ii}(\xi)\partial_{x_i} f_\pm\big(t,\bar{x}_i,a_i^*-r^*,\xi\big)\cdot\varphi^{\lambda_{i^*}}\big(t,\bar{x}_i,a_i^*-r^*\big)\zeta(\xi)\,dr\right]dt\,d\bar{x}_i\,d\xi \geqslant 0.$$

Proof. We will prove only the case where u is an entropy sub-solution and $a_i^*-r^* = a_i^- + r$. Define the function $\zeta_-(\tau)$ by $\zeta_-(\tau) = \zeta(\xi_-)$ with $\xi_- = \inf\{\xi;\ \tau = \beta_{ii}(\xi)\}$. We notice that $\zeta_-(\tau)$ is left continuous and $\operatorname{sgn}^+(u-\xi) = \operatorname{sgn}^+(\beta_{ii}(u)-\beta_{ii}(\xi))$ whenever $D\beta_{ii}(\xi) > 0$, because β_{ii} is nondecreasing. We set $\operatorname{Sgn}^+(r) = 1-\operatorname{sgn}^+(-r)$, $\operatorname{sgn}_\epsilon^+(r) = \sin(\frac{\pi}{2\epsilon}(r^+\wedge\epsilon))$ and $\operatorname{Sgn}_\epsilon^+(r) = \sin(\frac{\pi}{2\epsilon}((r+\epsilon)^+\wedge\epsilon))$ for $r\in\mathbb{R}$ and $\epsilon > 0$. Clearly, $\operatorname{Sgn}_\epsilon^+(r)$ and $\operatorname{sgn}_\epsilon^+(r)$ converge to $\operatorname{Sgn}^+(r)$ and $\operatorname{sgn}^+(r)$, respectively, as $\epsilon \to 0+$. Setting $I_i = (\beta_{ii}(-\infty),\beta_{ii}(\infty))$ and using the condition $\partial_{x_i}\beta_{ii}(u) \in L^2(Q)$ in Definition 2.1 we have for a.e. (t,x)

$$\begin{aligned}
&\int_{\mathbb{R}} D\beta_{ii}(\xi)\partial_{x_i} f_+\big(t,\bar{x}_i,a_i^-+r,\xi\big)\zeta(\xi)\phi^\lambda\,d\xi \\
&\quad= \int_{\mathbb{R}} \partial_{x_i}\operatorname{sgn}^+\big(\beta_{ii}(u)-\beta_{ii}(\xi)\big)\zeta(\xi)\phi^\lambda\,d\beta_{ii}(\xi) \\
&\quad= \lim_{\epsilon\to 0+}\int_{I_i} \partial_{x_i}\operatorname{sgn}^+\big(\beta_{ii}(u)-\tau\big)\operatorname{Sgn}_\epsilon^+\big(\tau-\beta_{ii}(u_b)\big)\zeta_-(\tau)\phi^\lambda\,d\tau \\
&\qquad+ \lim_{\epsilon\to 0+}\int_{I_i} \partial_{x_i}\operatorname{sgn}^+\big(\beta_{ii}(u)-\tau\big)\operatorname{sgn}_\epsilon^+\big(\beta_{ii}(u_b)-\tau\big)\zeta_-(\tau)\phi^\lambda\,d\tau \\
&\quad= \lim_{\epsilon\to 0+}\int_{I_i} \partial_{x_i}\operatorname{sgn}^+\big(\beta_{ii}(u)-\tau\big)\operatorname{Sgn}_\epsilon^+\big(\tau-\beta_{ii}(u_b)\big)\zeta_-(\tau)\phi^\lambda\,d\tau \\
&\qquad+ \lim_{\epsilon\to 0+}\int_{I_i} \partial_{x_i}\operatorname{sgn}^+\big(\beta_{ii}(u)-\tau\big)\operatorname{sgn}_\epsilon^+\big(\beta_{ii}(u)\wedge\beta_{ii}(u_b)-\tau\big)\zeta_-(\tau)\phi^\lambda\,d\tau \\
&\quad= \lim_{\epsilon\to 0+}\int_{I_i} \delta\big(\beta_{ii}(u)-\tau\big)\partial_{x_i}\beta_{ii}(u)\operatorname{Sgn}_\epsilon^+\big(\tau-\beta_{ii}(u_b)\big)\zeta_-(\tau)\phi^\lambda\,d\tau \\
&\qquad+ \lim_{\epsilon\to 0+}\int_{I_i} \delta\big(\beta_{ii}(u)-\tau\big)\partial_{x_i}\beta_{ii}(u)\operatorname{sgn}_\epsilon^+\big(\beta_{ii}(u)\wedge\beta_{ii}(u_b)-\tau\big)\zeta_-(\tau)\phi^\lambda\,d\tau \\
&\quad= \lim_{\epsilon\to 0+}\partial_{x_i}\beta_{ii}(u)\operatorname{Sgn}_\epsilon^+\big(\beta_{ii}(u)-\beta_{ii}(u_b)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda \\
&\qquad+ \lim_{\epsilon\to 0+}\partial_{x_i}\beta_{ii}(u)\operatorname{sgn}_\epsilon^+\big(\beta_{ii}(u)\wedge\beta_{ii}(u_b)-\beta_{ii}(u)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda \\
&\quad= \partial_{x_i}\beta_{ii}(u)\operatorname{Sgn}^+\big(\beta_{ii}(u)-\beta_{ii}(u_b)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda \\
&\qquad+ \partial_{x_i}\beta_{ii}(u)\operatorname{sgn}^+\big(\beta_{ii}(u)\wedge\beta_{ii}(u_b)-\beta_{ii}(u)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda \\
&\quad= \partial_{x_i}\beta_{ii}(u)\operatorname{Sgn}^+\big(\beta_{ii}(u)-\beta_{ii}(u_b)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda \\
&\quad\geqslant \big(\partial_{x_i}\beta_{ii}(u)\wedge 0\big)\operatorname{sgn}^+\big(\beta_{ii}(u)-\beta_{ii}(u_b)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda.
\end{aligned}$$

Here we used the fact that $\partial_{x_i}\beta_{ii}(u) = 0$ a.e. on the set $\{(t,x) \in Q;\, \beta_{ii}(u(t,x)) = \beta_{ii}(u_b(t,\bar{x}_i))\}$ (see e.g. [18, p. 53]). Moreover, the left continuity of the functions sgn^+ and ζ_- ensures that on every point (t,x) where $\partial_{x_i}\beta_{ii}(u(t,x)) < 0$ we have

$$\partial_{x_i}\beta_{ii}(u)\,\mathrm{sgn}^+\big(\beta_{ii}(u) - \beta_{ii}(u_b)\big)\zeta_-\big(\beta_{ii}(u)\big)\phi^\lambda = \partial_{x_i} \int_{\beta_{ii}(u_b)}^{\beta_{ii}(u)} \mathrm{sgn}^+\big(\tau - \beta_{ii}(u_b)\big)\zeta_-(\tau)\,d\tau\,\phi^\lambda.$$

Therefore, integration by parts yields the estimate

$$\begin{aligned}
&\frac{1}{s}\int_{\Theta_{i-}\times\mathbb{R}}\int_0^s D\beta_{ii}(\xi)\partial_{x_i} f_+\big(t,\bar{x}_i,a_i^- + r,\xi\big)\zeta(\xi)\phi^\lambda\,d\xi\,dr\,dt\,d\bar{x}_i \\
&\quad\geqslant \frac{1}{s}\int_{\Theta_{i-}\cap\{\partial_{x_i}\beta_{ii}(u)<0\}}\int_{\beta_{ii}(u_b)}^{\beta_{ii}(u)} \mathrm{sgn}^+\big(\tau - \beta_{ii}(u_b)\big)\zeta_-(\tau)\,d\tau\,\phi^\lambda\,dt\,d\bar{x}_i \\
&\qquad - \frac{1}{s}\int_{\Theta_{i-}\cap\{\partial_{x_i}\beta_{ii}(u)<0\}}\int_0^s\int_{\beta_{ii}(u_b)}^{\beta_{ii}(u)} \mathrm{sgn}^+\big(\tau - \beta_{ii}(u_b)\big)\zeta_-(\tau)\,d\tau\,\partial_{x_i}\phi^\lambda\,dr\,dt\,d\bar{x}_i \\
&\quad\geqslant -\frac{1}{s}\int_{\Theta_{i-}}\int_0^s \|\zeta_-\|_\infty\big(\beta_{ii}(u) - \beta_{ii}(u_b)\big)^+\big|\partial_{x_i}\phi^\lambda\big|\,dr\,dt\,d\bar{x}_i.
\end{aligned}$$

By virtue of condition (iii) of Definition 2.1 the above estimates immediately deduce the desired inequality. □

Lemma 3.4. *Under the assumptions of Theorem 2.2 there exist families of Young measures $\{\nu_x^{\tau_0}\}_{x\in\Omega}$ and $\{\tilde{\nu}_x^{\tau_0}\}_{x\in\Omega}$ supported in $(-\infty, \|u\|_{L^\infty}]$ and $[-\|\tilde{u}\|_{L^\infty}, \infty)$, respectively, and nonnegative functions $m_+^0(x,\xi)$ and $\tilde{m}_-^0(x,\xi)$ defined on $\Omega\times\mathbb{R}_\xi$ such that*

$$\begin{aligned}
&m_+^0, \tilde{m}_-^0 \in C\big(\mathbb{R}_\xi; w\text{-}\mathcal{M}^+(\Omega)\big), \\
&\lim_{\xi\to+\infty} m_+^0(x,\xi) = \lim_{\xi\to-\infty}\tilde{m}_-^0(x,\xi) = 0 \quad a.e.\ x\in\Omega, \\
&f_+^{\tau_0}(x,\xi) = \nu_x^{\tau_0}\big([\xi,\infty)\big) = \partial_\xi m_+^0(x,\xi) + \mathrm{sgn}^+\big(u_0(x) - \xi\big), \\
&f_-^{\tau_0}(x,\xi) = -\tilde{\nu}_x^{\tau_0}\big((-\infty,\xi]\big) = \partial_\xi \tilde{m}_-^0(x,\xi) + \mathrm{sgn}^-\big(\tilde{u}_0(x) - \xi\big),
\end{aligned} \tag{3.13}$$

where u_0 and $\tilde{u}_0$ are the initial data associated with entropy solutions u and $\tilde{u}$, respectively.

Proof. For $s > 0$ we set

$$\nu_x^s(\xi) = \frac{1}{s}\int_0^s \delta\big(\xi - u(t,x)\big)\,dt.$$

Note that $\nu_x^s(\mathbb{R}) = 1$. Hence, for each $x \in \Omega$ there exist a positive sequence $\{s_j^x\} \downarrow 0$ as $j \to \infty$ and a Radon measure $\nu_x^{\tau_0}$ on $\mathbb{R}$ such that $\nu_x^{\tau_0}(\mathbb{R}) = 1$ and

$$\langle \nu_x^{\tau_0}, \zeta \rangle = \lim_{j\to\infty} \langle \nu_x^{s_j^x}, \zeta \rangle \quad \text{for any } \zeta \in C_c^\infty(\mathbb{R}).$$

(See [13, p. 54].) Since $\nu_x^s(\xi) = -\partial_\xi(\frac{1}{s}\int_0^s f_+(t,x,\xi)\,dt)$, it follows from Lemma 3.1 that

$$\nu_x^{\tau_0}(\xi) = -\partial_\xi f_+^{\tau_0}(x,\xi) \quad \text{in } \mathcal{M}(\mathbb{R}).$$

This implies that $\nu_x^{\tau_0}$ does not depend on the choice of subsequences. Integrating over $[\xi,\infty)$ and noting that $\lim_{\xi\to\infty} f_+^{\tau_0}(x,\xi) = 0$, we obtain $\nu_x^{\tau_0}([\xi,\infty)) = f_+^{\tau_0}(x,\xi)$.

Next, use (3.9) with the test function $\psi = \operatorname{sgn}^+(\xi-\kappa)\varphi(t,x)$, where $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$ and $\kappa \in \mathbb{R}$. Then

$$\begin{aligned}
&\int_{Q_{i^*}} (u-\kappa)^+\partial_t\varphi^\lambda + \mathcal{F}^+(u,\kappa)\cdot\nabla\varphi^\lambda - \sum_{k,j=1}^d \operatorname{sgn}^+(u-\kappa)\big(\beta_{kj}(u)-\beta_{kj}(\kappa)\big)\partial_{x_k}\varphi^\lambda \\
&\qquad + \operatorname{sgn}^+(u-\kappa)g\varphi^\lambda\,dt\,dx + \int_{\Theta_{i^*}} \sum_{j\neq i^*} \partial_{x_j}\operatorname{sgn}^+(u_b-\kappa)\big(\beta_{ij}(u_b)-\beta_{ij}(\kappa)\big)\varphi^\lambda\,dt\,d\bar{x}_i \\
&\qquad + \int_\Omega \bigg(\int_{\mathbb{R}} f_+^{\tau_0}(x,\xi)\operatorname{sgn}^+(\xi-\kappa)\,d\xi\bigg)\varphi^\lambda(0,x)\,dx \\
&\qquad + \lim_{s\to 0+}\int_{\Theta_{i^*}\times\mathbb{R}} \Bigg[\frac{1}{s}\int_0^s \big(-\mathbf{n}_{i^*}a(\xi) - D\beta_{ii}(\xi)\partial_{x_i}\big)f_+\varphi^\lambda\,dr\Bigg]dt\,d\bar{x}_i\,d\xi \\
&= \int_{Q_{i^*}} \varphi^\lambda d\big(m_+(\kappa)+n(\kappa)\big).
\end{aligned} \tag{3.14}$$

Since u is an entropy sub-solution, (3.14) implies

$$-\int_\Omega (u_0-\kappa)^+\psi^\lambda\,dx + \int_\Omega\bigg(\int_{\mathbb{R}} f_+^{\tau_0}(x,\xi)\operatorname{sgn}^+(\xi-\kappa)\,d\xi\bigg)\psi^\lambda\,dx \leqslant \int_{(0,T)\times\Omega_{i^*}} \psi^\lambda\,dm_+(\kappa)$$

for any $\psi \in C_c^\infty(\Omega)$ with $\psi \geqslant 0$, any $\kappa \in \mathbb{R}$ and any $T > 0$. Thus, the arbitrariness of T yields that for a.e. $x \in \Omega$,

$$\int_\kappa^\infty f_+^{\tau_0}(x,\xi) \leqslant \big(u_0(x)-\kappa\big)^+.$$

Define the function $m_+^0(x,\xi)$ defined on $\Omega\times\mathbb{R}$ by

$$m_+^0(x,\xi) = \big(u_0(x)-\xi\big)^+ - \int_\xi f_+^{\tau_0}(x,\eta)\,d\eta.$$

It is easy to see that $\lim_{\xi\to\infty} m_+^0 = 0$ and (3.13) holds. $\square$

Lemma 3.5. *Let $1 \leqslant i \leqslant d$. Let $f_+^\tau(t,\bar{x}_i,\xi)$ and $\tilde{f}_-(t,\bar{x}_i,\xi)$ be weak* cluster points of*

$$\frac{1}{s}\int_0^s f_+\big(t,\bar{x}_i,a_i^*-r^*,\xi\big)\,dr \quad \text{and} \quad \frac{1}{s}\int_0^s \tilde{f}_-\big(t,\bar{x}_i,a_i^*-r^*,\xi\big)\,dr,$$

respectively, as $s \to 0+$ in $L^\infty(\Theta_{i^*}\times\mathbb{R})$. There exist Young measures $\{\nu_{t,y}^\tau\}_{(t,y)\in\Sigma}$ and $\{\tilde{\nu}_{t,y}^\tau\}_{(t,y)\in\Sigma}$, supported in $(-\infty,\|u\|_{L^\infty}]$ and $[-\|\tilde{u}\|_{L^\infty},\infty)$, respectively, and functions $m_+^b(t,y,\xi)$ and $\tilde{m}_-^b(t,y,\xi)$ defined on $\Sigma\times\mathbb{R}$ such that

$$\lim_{\xi\to\infty} m_+^b(t,y,\xi) = \lim_{\xi\to-\infty} m_-^b(t,y,\xi) = 0 \quad \text{a.e. } (t,y)\in\Sigma$$

and for each $\xi\in\mathbb{R}$ it holds that

$$\begin{aligned}
&f_+^\tau(t,y,\xi) = \nu_{t,y}^\tau\big([\xi,\infty)\big), \qquad \tilde{f}_-^\tau(t,y,\xi) = -\tilde{\nu}_{t,y}^\tau\big((-\infty,\xi]\big),\\
&-a(\xi)\cdot\mathbf{n}_{i^*} f_+^\tau = \partial_\xi m_+^b + M\,\mathrm{sgn}^+(u_b-\xi),\\
&-a(\xi)\cdot\mathbf{n}_{i^*} \tilde{f}_-^\tau = \partial_\xi \tilde{m}_-^b + M\,\mathrm{sgn}^-(\tilde{u}_b-\xi),\\
&\int_{\Theta_{i^*}} m_+^b\big(t,\bar{x}_i,a_i^*,\xi\big)\varphi^{\lambda_{i^*}}\big(t,\bar{x}_i,a_i^*\big)\,dt\,d\bar{x}_i \geqslant 0
\end{aligned} \tag{3.15}$$

for any $\varphi\in C(\Sigma)$, $\varphi\geqslant 0$, satisfying $\mathrm{sgn}^+(\beta_{ii}(u_b)-\beta_{ii}(\xi))\varphi = 0$ a.e. on Σ, and

$$\int_{\Theta_{i^*}} \tilde{m}_-^b\big(t,\bar{x}_i,a_i^*,\xi\big)\tilde{\varphi}^{\lambda_{i^*}}\big(t,\bar{x}_i,a_i^*\big)\,dt\,d\bar{x}_i \geqslant 0$$

for any $\tilde{\varphi}\in C(\Sigma)$, $\tilde{\varphi}\geqslant 0$, satisfying $\mathrm{sgn}^-(\beta_{ii}(\tilde{u}_b)-\beta_{ii}(\xi))\tilde{\varphi} = 0$ a.e. on Σ.

Proof. We will again prove only the case of an entropy sub-solution and $i^* = i^-$. By a similar argument as in the proof of Lemma 3.4 we can obtain a family of Young measures $\{\nu_{t,y}^\tau\}_{(t,y)\in\Sigma}$ on $\mathbb{R}$ such that $f_+^\tau(t,y,\xi) = \nu_{t,y}^\tau([\xi,\infty))$.

We show the following inequality holds:

$$\begin{aligned}
&-M\int_{\Theta_{i^-}} (u_b-\kappa)^+\varphi^\lambda\,dt\,d\bar{x}_i + \int_{\Theta_{i^-}\times\mathbb{R}} \big(-a(\xi)\cdot\mathbf{n}_{i^-}\big) f_+^\tau\,\mathrm{sgn}^+(\xi-\kappa)\varphi^\lambda\,dt\,d\bar{x}_i\,d\xi\\
&\quad\leqslant \int_{\Theta_{i^-}} \varphi^\lambda d\big(m_+(\kappa)+n(\kappa)\big)
\end{aligned} \tag{3.16}$$

for any $\kappa\in\mathbb{R}$, any $\varphi\in C_c^\infty((0,T)\times\mathbb{R}^d)$ satisfying $\mathrm{sgn}^+(\beta_{ii}(u_b)-\beta_{ii}(\kappa))\varphi = 0$ a.e. on Σ and for any weak* cluster point $f_+^\tau(t,\bar{x}_i,\xi)$ of $\frac{1}{s}\int_0^s f_+(t,\bar{x}_i,a_i^-+r,\xi)\,dr$ as $s\to 0+$. To this end let $\kappa_+ = \sup\beta_{ii}^{-1}(\beta_{ii}(\kappa))$ and $\kappa_- = \inf\beta_{ii}^{-1}(\beta_{ii}(\kappa))$. If $\kappa_+<\infty$, then the function ρ on $[0,\infty)$ defined by $\rho(\xi) = \beta_{ii}(\xi+\kappa_+)-\beta_{ii}(\kappa_+)$ is strictly increasing on $[0,\epsilon_0]$ for some $\epsilon_0>0$. For each $\epsilon\in(0,\epsilon_0]$ we set

$$s_\epsilon^+(\xi,\kappa_+) = \begin{cases} \big(\frac{\rho(\xi)}{\rho(\epsilon)}\big)\wedge 1 & \text{if } \xi\geqslant 0,\\ 0 & \text{if } \xi<0.\end{cases}$$

Then we define the function $S_\epsilon^+(\xi,\kappa)$ by $S_\epsilon^+(\xi,\kappa) = s_\epsilon^+(\xi-\kappa,\kappa_+)s_\epsilon^+(|\xi-\kappa_+|,\kappa_+)$ if $\kappa_+ < \infty$ and $S_\epsilon^+(\xi,\kappa) = (\frac{(\xi-\kappa)^+}{\epsilon}) \wedge 1$ if $\kappa_+ = \infty$. We use (3.9) with $\psi(t,x,\xi)$ replaced by $S_\epsilon^+(\xi,\kappa)\varphi^\lambda$ (see Remark 3.1) to obtain

$$\int_{Q_{i-}\times\mathbb{R}} f_+ S_\epsilon^+(\xi,\kappa)\big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda - S_\epsilon^+(\xi,\kappa)\sum_{k,j=1}^d D\beta_{k_j}(\xi)\partial_{x_j} f_+ \partial_{x_k}\varphi^\lambda$$
$$+ \delta(u-\xi) g S_\epsilon^+(\xi,\kappa)\varphi^\lambda \,dt\,dx\,d\xi + \int_{\Theta_{i-}\times\mathbb{R}} \sum_{j\neq i} D\beta_{ij}(\xi)\partial_{x_j} f_+^b S_\epsilon^+(\xi,\kappa)\varphi^\lambda \,dt\,d\bar{x}_i\,d\xi$$
$$+ \lim_{s\to+0}\int_{\Theta_{i-}\times\mathbb{R}} \left[\frac{1}{s}\int_0^s \big(-\mathbf{n}_{i^-}\cdot a(\xi) - D\beta_{ii}(\xi)\partial_{x_i}\big) f_+ S_\epsilon^+(\xi,\kappa)\varphi^\lambda \,dr\right] dt\,d\bar{x}_i\,d\xi$$
$$= \int_{Q_{i-}\times\mathbb{R}} \partial_\xi S_\epsilon^+(\xi,\kappa)\varphi^\lambda d(m_+ + n). \tag{3.17}$$

Since $S_\epsilon^+(\xi,\kappa)$ tends to $\operatorname{sgn}^+(\xi-\kappa)(1-\mathbf{1}_{\{\kappa_+\}}(\xi))$ as $\epsilon \to 0$ where $\mathbf{1}_{\{a\}}$ is the indicator function of the set $\{a\}$ (with the notational convention $\mathbf{1}_{\{\infty\}} = 0$ when $a = \infty$), the Lebesgue convergence theorem assures the existence of the limit of each term, as $\epsilon \to 0+$, on the left-hand side of (3.17) except the diffusion term

$$\frac{1}{s}\int_0^s \int_{\Theta_{i-}\times\mathbb{R}} D\beta_{ii}(\xi) S_\epsilon^+(\xi,\kappa)\partial_{x_i} f_+ \varphi^\lambda\big(t,\bar{x}_i,a_i^- + r\big)\,dr\,dt\,d\bar{x}_i\,d\xi. \tag{3.18}$$

The right hand of (3.17) equals

$$\rho(\epsilon)^{-1}\int_{Q_{i-}}\left(\int_\kappa^{\kappa+\epsilon} D\beta_{ii}(\xi-\kappa+\kappa_+)\,d\xi - \int_{\kappa_+-\epsilon}^{\kappa_+} D\beta_{ii}(2\kappa_+-\xi)\,d\xi + \int_{\kappa_+}^{\kappa_++\epsilon} D\beta_{ii}(\xi)\,d\xi\right)\varphi^\lambda d(m_+ + n) \tag{3.19}$$

provided $\kappa_+ < \infty$ and

$$\epsilon^{-1}\int_{Q_{i-}}\int_\kappa^{\kappa+\epsilon} \varphi^\lambda d(m_+ + n)$$

provided $\kappa_+ = \infty$. By change of variable (3.19) becomes

$$\rho(\epsilon)^{-1}\int_{Q_{i-}}\int_{\kappa_+}^{\kappa_++\epsilon} D\beta_{ii}(\xi)\varphi^\lambda d(m_+ + n)(\cdot,\xi+\kappa-\kappa_+)$$
$$+ \rho(\epsilon)^{-1}\int_{Q_{i-}}\int_{\kappa_++\epsilon}^{\kappa_+} D\beta_{ii}(\xi)\varphi^\lambda d(m_+ + n)(\cdot,2\kappa_+-\xi)$$

$$+\rho(\epsilon)^{-1}\int_{Q_{i-}}\int_{\kappa_+}^{\kappa_++\epsilon} D\beta_{ii}(\xi)\varphi^\lambda d(m_++n)(\cdot,\kappa)$$

$$=\rho(\epsilon)^{-1}\int_{Q_{i-}}\int_{\beta_{ii}(\kappa_+)}^{\beta_{ii}(\kappa_++\epsilon)} \varphi^\lambda d(m_++n)\big(\cdot,\beta_{ii}^{-1}(\tau)+\kappa-\kappa_+\big)$$

$$-\rho(\epsilon)^{-1}\int_{Q_{i-}}\int_{\beta_{ii}(\kappa_+)}^{\beta_{ii}(\kappa_++\epsilon)} \varphi^\lambda d(m_++n)\big(\cdot,2\kappa_+-\beta_{ii}^{-1}(\tau)\big)$$

$$+\rho(\epsilon)^{-1}\int_{Q_{i-}}\int_{\beta_{ii}(\kappa_+)}^{\beta_{ii}(\kappa_++\epsilon)} \varphi^\lambda d(m_++n)\big(\cdot,\beta_{ii}^{-1}(\tau)\big),$$

which tends as $\epsilon\to 0+$ to $\int_{Q_{i-}}\varphi^\lambda d(m_+(\kappa)+n(\kappa))$. In the case of $\kappa_+=\infty$ the right hand of (3.17) also tends as $\epsilon\to 0+$ to the same limit just as above.

Next we calculate the diffusion term (3.18). Since $D\beta_{ii}=0$ on (ξ_-,ξ_+), we can rewrite (3.18) as follows:

$$\frac{1}{s}\int_0^s\int_{\Theta_{i-}\times\mathbb{R}} D\beta_{ii}(\xi)s_\epsilon^+(\xi-\kappa_+,\kappa_+)\partial_{x_i}f_+\varphi^\lambda\,dr\,dt\,d\bar{x}_i\,d\xi$$

$$=\frac{1}{s}\int_0^s\int_{\Theta_{i-}}\partial_r\int_{-\infty}^{u(t,\bar{x}_i,a_i+r,\alpha)} D\beta_{ii}^{s_\epsilon^+(\cdot-\kappa_+,\kappa_+)}(\xi)\,d\xi\,\varphi^\lambda\,dr\,dt\,d\bar{x}_i$$

$$=\frac{1}{s}\int_0^s\int_{\Theta_{i-}}\partial_r\beta_{ii}^{s_\epsilon^+(\cdot-\kappa_+,\kappa_+)}(u)\varphi^\lambda\,dr\,dt\,d\bar{x}_i$$

$$=\frac{1}{s}\int_0^s\int_{\Theta_{i-}} s_\epsilon^+(u-\kappa_+,\kappa_+)\partial_r\beta_{ii}(u)\varphi^\lambda\,dr\,dt\,d\bar{x}_i, \tag{3.20}$$

where $D\beta_{ii}^{s_\epsilon^+}=s_\epsilon^+D\beta_{ii}$. By virtue of (3.8) with suitable choices of the test function $\zeta\in C_c^\infty(\mathbb{R})$ we have

$$\limsup_{s\to+0}\left|\frac{1}{s}\int_0^s\int_{\Theta_{i-}}\partial_r\beta_{ii}(u)\tilde{\psi}^\lambda\,dr\,dt\,d\bar{x}_i\right| = \limsup_{s\to+0}\left|\frac{1}{s}\int_0^s\int_{\Theta_{i-}}\partial_r\big(\beta_{ii}(u)-\beta_{ii}(\kappa)\big)\tilde{\psi}^\lambda\,dr\,dt\,d\bar{x}_i\right|$$
$$\leqslant\big|\langle\mathcal{T},\tilde{\psi}|_\Sigma\rangle\big|+2M\big\|\tilde{\psi}^\lambda\big\|_{L^\infty} \tag{3.21}$$

for any $\tilde{\psi}\in W^{1,2}(Q)\cap C(Q)\cap L^\infty(Q)$ such that $\tilde{\psi}(0,x)=0$, where $\tilde{\psi}|_\Sigma$ denotes the trace of $\tilde{\psi}$ on Σ.

On the other hand, by virtue of Lemma 2.1 and the property (iii) of Definition 2.1, Egorov's theorem ensures that for each $\eta>0$ there exist a closed subset $F_1\subset\Sigma$ and a positive sequence $s_n\downarrow 0$ such that $\mathcal{H}^d(\Sigma\setminus F_1)<\eta$ and $\sup_{0\leqslant r\leqslant s_n}|\beta_{ii}(u(t,\bar{x}_i,a_i^-+r))-\beta_{ii}(u_b(t,\bar{x}_i))|$ tends to 0 as $n\to\infty$ uniformly for $(t,\bar{x}_i)\in F_1$, where $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure on $\mathbb{R}^{d+1}$. Also, by Lusin's theorem for Radon measures there exists a closed subset $F_2\subset Q$ such that $(m_+(\kappa)+n(\kappa))(Q\setminus F_2)<\eta$ and $\beta_{ii}(u(t,x))$ is continuous on F_2. Let $\psi_\eta^j\in C(\mathbb{R}^{d+1})$, $j=1,2$, be the functions which satisfy $0\leqslant\psi_\eta^j\leqslant 1$, $\psi_\delta^j=0$ on $\mathbb{R}^{d+1}\setminus F_j$ and $\psi_\eta^j=1$ on the set $\{(t,x)\in F_j;\operatorname{dist}((t,x),\mathbb{R}^{d+1}\setminus F_j)>\eta\}$. We notice that $s_\epsilon^+(u-\kappa_+,\kappa_+)\psi_\eta^1\psi_\eta^2\varphi\in W^{1,2}(Q)\cap C(\bar{Q})\cap L^\infty(Q)$ and $s_\epsilon^+(u-\kappa_+,\kappa_+)\psi_\eta^1\psi_\eta^2\varphi=0$ on Σ. We can use (3.21) with $\tilde{\psi}=s_\epsilon^+(u-\kappa_+,\kappa_+)\psi_\eta^1\psi_\eta^2\varphi$ and obtain from (3.20)

$$\lim_{s\to 0+}\frac{1}{s}\int_0^s\int_{\Theta_{i-}\times\mathbb{R}} D\beta_{ii}(\xi)s_\epsilon^+(\xi-\kappa_+,\kappa_+)\partial_{x_i}f_+\psi_\eta^1\psi_\eta^2\varphi^\lambda\,dr\,dt\,d\bar{x}_i\,d\xi=0.$$

We pass to the limit as $s=s_n$ tends to 0 and then ϵ to 0 in (3.17) with φ replaced by $\psi_\eta^1\psi_\eta^2\varphi$. Noting also that $\nabla_{\bar{x}_i}(\beta_{ii}(u_b)-\beta_{ii}(\kappa))^+\psi_\eta^1\psi_\eta^2\varphi^\lambda=\nabla_{\bar{x}_i}\beta_{ii}(u_b)\operatorname{sgn}^+(\beta_{ii}(u_b)-\beta_{ii}(\kappa))\psi_\eta^1\psi_\eta^2\varphi^\lambda=0$ on Σ, one obtains

$$\begin{aligned}&\int_{Q_{i-}}(u-\kappa)^+\partial_t\big(\psi_\eta^1\psi_\eta^2\varphi^\lambda\big)+\mathcal{F}^+(u,\kappa)\cdot\nabla\big(\psi_\eta^1\psi_\eta^2\varphi^\lambda\big)\\&\quad-\sum_{k,j=1}^d\operatorname{sgn}^+(u-\kappa)\partial_{x_j}\big(\beta_{kj}(u)-\beta_{kj}(\kappa)\big)\partial_{x_k}\big(\psi_\eta^1\psi_\eta^2\varphi^\lambda\big)+\operatorname{sgn}^+(u-\kappa)g\psi_\eta^1\psi_\eta^2\varphi^\lambda\,dt\,dx\\&\quad+\int_{\Theta_{i-}}\sum_{j\neq i^-}\operatorname{sgn}^+(u_b-\kappa)\big(\beta_{ij}(u_b)-\beta_{ij}(\kappa)\big)\psi_\eta^1\psi_\eta^2\varphi^\lambda\,dt\,d\bar{x}_i+\int_{\Theta_{i-}}\int_\kappa^\infty-\mathbf{n}_{i^-}\cdot a(\xi)f_+^\tau\,dt\,d\bar{x}_i\,d\xi\\&=\int_{Q_{i-}}\psi_\eta^1\psi_\eta^2\varphi^\lambda d\big(m_+(\kappa)+n(\kappa)\big)\end{aligned}$$

for any weak* cluster point $f_+^\tau=f_+^\tau(t,\bar{x}_i,\xi)$. Since u is an entropy sub-solution and $\operatorname{sgn}^+(\beta_{ij}(u_b)-\beta_{ij}(\kappa))\varphi^\lambda=0$ on Σ, we obtain

$$\begin{aligned}&\int_{\Theta_{i-}}-M(u_b-\kappa)^+\psi_\eta^1\psi_\eta^2\varphi^\lambda\,dt\,d\bar{x}_i-\int_{\Theta_{i-}}\int_\kappa^\infty a(\xi)\cdot\mathbf{n}_{i^-}f_+^\tau\psi_\eta^1\psi_\eta^2\varphi^\lambda\,dt\,d\bar{x}_i\,d\xi\\&\quad\leqslant\int_{\Theta_{i-}}\psi_\eta^1\psi_\eta^2\varphi^\lambda\,dm_+(\kappa).\end{aligned}$$

However, since $\lim_{\eta\to 0+}\psi_\eta^1\psi_\eta^2=1$ $\mathcal{H}^d$-a.e. on Σ as well as $m_+(\kappa)$-a.e. on Q, passing $\eta\to 0+$ yields (3.16).

Now, define the function $m_+^b(t,y,\xi)$ by

$$m_+^b(t,y,\xi)=M\big(u_b(t,y)-\xi\big)^+-\int_\xi^\infty-\mathbf{n}_{i^-}\cdot a(\eta)f_+^\tau(t,y,\eta)\,d\eta$$

for $(t,y)\in\Sigma$, $\xi\in\mathbb{R}$. Clearly, we see that $\lim_{\xi\to\infty}m_+^b=0$ and (3.15) holds. □

We are now in a position to prove the comparison theorem.

Proof of Theorem 2.2. Let f_+, n and m_+ be defined for an entropy sub-solution u as above. $f_+^{\tau_0}$ denotes the time kinetic trace and f_+^τ a cluster point of the space kinetic trace associated with u. The corresponding ones associated with an entropy super-solution $\tilde{u}$ will be denoted by $\tilde{f}_-$, $\tilde{n}$, $\tilde{m}_-$, $\tilde{f}_-^{\tau_0}$ and $\tilde{f}_-^\tau$, respectively.

Let $i \in \{1, 2, \dots, d\}$. We set for $(t, \bar{x}_i, \xi) \in \Theta_{i^*} \times \mathbb{R}$,

$$F_+(t, \bar{x}_i, \xi) = -\mathbf{n}_{i^*} \cdot a(\xi) f_+^\tau\big(t, \bar{x}_i, a_i^*, \xi\big) - \sum_{j \neq i} D\beta_{ij}(\xi)\partial_{x_j} f_+^b,$$

$$\tilde{F}_-(t, \bar{x}_i, \xi) = -\mathbf{n}_{i^*} \cdot a(\xi) \tilde{f}_-^\tau\big(t, \bar{x}_i, a_i^*, \xi\big) - \sum_{j \neq i} D\beta_{ij}(\xi)\partial_{x_j} \tilde{f}_-^b,$$

where $\tilde{f}_-^b = \mathrm{sgn}^-(\tilde{u}_b - \xi)$. Notice that both F_+ and $\tilde{F}_-$ belong to $L^1(\Sigma \times \mathbb{R}_\xi)$ by the hypothesis (A3). We set $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_d) \in \mathbb{R}_+^{d+1}$ and define

$$\rho_\epsilon(t, x) = \theta_{\epsilon_0,\epsilon_0}(t) \prod_{j=1}^d \theta_{\epsilon_0,\epsilon_j}(x_j).$$

Then we set

$$f_+^\epsilon = (f_+ \times \mathbf{1}_{Q_{i^*}^\nu}) *_{(t,x)} \rho_\epsilon,$$

$$f_+^{\tau_0,\epsilon} = \big(f_+^{\tau_0} \times \mathbf{1}_{\Omega_{i^*}^\nu}\big) *_x \left(\prod_{j=1}^d \theta_{\epsilon_0,\epsilon_j}\right),$$

$$F_+^\epsilon = \big(F_+ \times \big(\mathbf{1}_{\Sigma_{i^*}^\nu}\, dt\, d\bar{x}_i \otimes \delta\big(x_i - a_i^*\big)\big)\big) *_{(t,x)} \rho_\epsilon,$$

$$m_+^\epsilon = (m_+ \times \mathbf{1}_{Q_{i^*}^\nu}) *_{(t,x)} \rho_\epsilon,$$

$$n^\epsilon = (n \times \mathbf{1}_{Q_{i^*}^\nu}) *_{(t,x)} \rho_\epsilon,$$

where $\mathbf{1}_{Q_{i^*}^\nu}$ denotes the characteristic function of $Q_{i^*}^\nu$ defined on $\mathbb{R}^{d+1}$, while $\mathbf{1}_{\Sigma_{i^*}^\nu}$ denotes the characteristic function of $\Sigma_{i^*}^\nu$ defined on $\mathbb{R}^d$ and $\mathbf{1}_{\Sigma_{i^*}^\nu}\, dt\, d\bar{x}_i \otimes \delta(x_i - a_i^*)$ is the product measure of $\mathbf{1}_{\Sigma_{i^*}^\nu}\, dt\, d\bar{x}_i$ and $\delta(x_i - a_i^*)$. Next, for the regularization in ξ we set for $\eta > 0$,

$$\theta_\eta(\xi) = \theta_{\eta,\eta}(\xi)$$

and define

$$f_+^{\epsilon,\eta} = f_+^\epsilon *_\xi \theta_\eta$$

and

$$\big[\delta(u - \xi)g\big]^{\epsilon,\eta} = \big(\delta(u - \xi)g\mathbf{1}_{Q_{i^*}^\nu}\big) *_{(t,x,\xi)} \rho_\epsilon \theta_\eta.$$

The functions $f_+^{\tau_0,\epsilon,\eta}$, $F_+^{\epsilon,\eta}$, $m_+^{\epsilon,\eta}$, $n^{\epsilon,\eta}$ are similarly defined. As for the functions $\tilde{f}_-$, $\tilde{f}_-^{\tau_0}$, $\tilde{F}_-$, etc., their regularizations $\tilde{f}_-^{\tilde{\epsilon}}$, $\tilde{f}_-^{\tilde{\epsilon},\eta}$, $\tilde{f}_-^{\tau_0,\tilde{\epsilon}}$, $\tilde{f}_-^{\tau_0,\tilde{\epsilon},\eta}$, $\tilde{F}_-^{\tilde{\epsilon}}$, $\tilde{F}_-^{\tilde{\epsilon},\eta}$, etc. are similarly defined in the same way as above, but with the different parameter $\tilde{\epsilon}$ instead of ϵ.

Let $\psi \in C_c^\infty([0,T)\times\mathbb{R}^{d+1})$, $\psi \geqslant 0$, and let $i^* = i^-$ again without loss of generality. Applying (3.9) to the test function $\psi^{\lambda_{i^-}} * (\check{\rho}_\epsilon \check{\theta}_\eta)$, where $\check{\rho}_\epsilon(t,x) = \rho_\epsilon(-t,-x)$ and $\check{\theta}_\eta(\xi) = \theta_\eta(-\xi)$, and using Lemma 3.3 one has

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} f_+^{\epsilon,\eta}\big(\partial_t + a(\xi)\nabla\big)\psi^\lambda - \sum_{k,j=1}^d \partial_{x_j}\big(\big(D\beta_{kj}(\xi) f_+^\epsilon\big) * \theta_\eta\big)\partial_{x_k}\psi^\lambda \\
&\qquad + \big(\big[\delta(u-\xi)g\big]^{\epsilon,\eta} + F_+^{\epsilon,\eta} + f_+^{\tau_0,\epsilon}\theta_{\epsilon_0,\epsilon_0}\big)\psi^\lambda\,dt\,dx\,d\xi \\
&\quad \geqslant \int_{\mathbb{R}^{d+2}} \partial_\xi \psi^\lambda d\big(m_+^{\epsilon,\eta} + n^{\epsilon,\eta}\big) + \int_{\mathbb{R}^{d+2}} R_1^u \psi^\lambda\,dt\,dx\,d\xi
\end{aligned} \tag{3.22}$$

with

$$R_1^u = \operatorname{div}\big\{a(\xi) f_+^{\epsilon,\eta} - \big(a(\xi) f_+^\epsilon\big) * \theta_\eta\big\}.$$

Similarly we can regularize the equation satisfied by $\tilde{f}_-$ by the same manner and obtain for the same ψ's

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} -\tilde{f}_-^{\tilde{\epsilon},\eta}\big(\partial_t + a(\xi)\cdot\nabla\big)\psi^\lambda + \sum_{k,j=1}^d \partial_{x_j}\big(\big(D\beta_{kj}(\xi)\tilde{f}_-^{\tilde{\epsilon}}\big) * \theta_\eta\big)\partial_{x_k}\psi^\lambda \\
&\qquad - \big(\big[\delta(\tilde{u}-\xi)\tilde{g}\big]^{\tilde{\epsilon},\eta} + \tilde{F}_-^{\tilde{\epsilon},\eta} + \tilde{f}_-^{\tau_0,\tilde{\epsilon}}\theta_{\tilde{\epsilon}_0,\tilde{\epsilon}_0}\big)\psi^\lambda\,dt\,dx\,d\xi \\
&\quad \geqslant -\int_{\mathbb{R}^{d+2}} \partial_\xi \psi^\lambda d\big(\tilde{m}_-^{\tilde{\epsilon},\eta} + \tilde{n}^{\tilde{\epsilon},\eta}\big) - \int_{\mathbb{R}^{d+2}} R_1^{\tilde{u}} \psi^\lambda\,dt\,dx\,d\xi
\end{aligned} \tag{3.23}$$

with

$$R_1^{\tilde{u}} = \operatorname{div}_x\big\{a(\xi)\tilde{f}_-^{\tilde{\epsilon},\eta} - \big(a(\xi)\tilde{f}_-^{\tilde{\epsilon}}\big) * \theta_\eta\big\}.$$

Now, let us fix a nonnegative test function $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$. We apply (3.22) to the test function $\psi = -\tilde{f}_-^{\tilde{\epsilon},\eta}(t,x,\xi)\varphi(t,x)$ (notice the test function is admissible by Remark 3.1), and apply (3.23) to the test function $\psi = f_+^{\epsilon,\eta}(t,x,\xi)\varphi(t,x)$. We sum the two resulting inequalities and use the formula for integration by parts on the left-hand side of the resultant inequality to get

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} -f_+^{\epsilon,\eta}\tilde{f}_-^{\tilde{\epsilon},\eta}\big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda + \tilde{f}_-^{\tilde{\epsilon},\eta}\sum_{k,j=1}^d \partial_{x_k}\partial_{x_j}\big(\big(D\beta_{kj}(\xi) f_+^\epsilon\big) * \theta_\eta\big)\varphi^\lambda \\
&\qquad - f_+^{\epsilon,\eta}\sum_{k,j=1}^d \partial_{x_k}\partial_{x_j}\big(\big(D\beta_{kj}(\xi)\tilde{f}_-^{\tilde{\epsilon}}\big) * \theta_\eta\big)\varphi^\lambda \\
&\qquad + \big(-f_+^{\tau_0,\epsilon}\theta_{\epsilon_0,\epsilon_0}\tilde{f}_-^{\tilde{\epsilon},\eta} - \tilde{f}_-^{\tau_0,\tilde{\epsilon}}\theta_{\tilde{\epsilon}_0,\tilde{\epsilon}_0} f_+^{\epsilon,\eta} - F_+^{\epsilon,\eta}\tilde{f}_-^{\tilde{\epsilon},\eta} - \tilde{F}_-^{\tilde{\epsilon},\eta} f_+^{\epsilon,\eta}\big)\varphi^\lambda \\
&\qquad - \big[\delta(u-\xi)g\big]^{\epsilon,\eta} f_-^{\tilde{\epsilon},\eta}\varphi^\lambda - \big[\delta(\tilde{u}-\xi)\tilde{g}\big]^{\tilde{\epsilon},\eta} f_+^{\epsilon,\eta}\varphi^\lambda\,dt\,dx\,d\xi
\end{aligned}$$

$$\geqslant -\int_{\mathbb{R}^{d+2}} \partial_\xi \tilde{f}_-^{\tilde{\epsilon},\eta}\varphi^\lambda d\big(m_+^{\epsilon,\eta}+n^{\epsilon,\eta}\big) - \int_{\mathbb{R}^{d+2}} \partial_\xi f_+^{\epsilon,\eta}\varphi^\lambda d\big(\tilde{m}_-^{\tilde{\epsilon},\eta}+\tilde{n}^{\tilde{\epsilon},\eta}\big)$$
$$-\int_{\mathbb{R}^{d+2}} \big(R_1^u \tilde{f}_-^{\tilde{\epsilon},\eta}+R_1^{\tilde{u}} f_+^{\epsilon,\eta}\big)\varphi^\lambda\,dt\,dx\,d\xi.$$

Then we can rewrite this inequality as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} -f_+^{\epsilon,\eta}\tilde{f}_-^{\tilde{\epsilon},\eta}\big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda \\
&\quad + \sum_{k,j=1}^{d}\big\{\tilde{f}_-^{\tilde{\epsilon},\eta}\partial_{x_j}\big(\big(D\beta_{kj}(\xi)f_+^{\epsilon}\big)*\theta_\eta\big) + f_+^{\epsilon,\eta}\partial_{x_j}\big(\big(D\beta_{kj}(\xi)\tilde{f}_-^{\tilde{\epsilon}}\big)*\theta_\eta\big)\big\}\partial_{x_k}\varphi^\lambda \\
&\quad - \big(\big[\delta(u-\xi)g\big]^{\epsilon,\eta}\tilde{f}_-^{\tilde{\epsilon},\eta} + \big[\delta(\tilde{u}-\xi)\tilde{g}\big]^{\tilde{\epsilon},\eta} f_+^{\epsilon,\eta}\big)\varphi^\lambda\,dt\,dx\,d\xi \\
&\geqslant I_1 + I_2 + I_3 + \tilde{I}_3 + I_4 + \tilde{I}_4 \\
&\quad + \int_{\mathbb{R}^{d+2}} \big\{f_+^{\tau_0,\epsilon}\theta_{\epsilon_0,\epsilon_0}\tilde{f}_-^{\tilde{\epsilon},\eta} + \tilde{f}_-^{\tau_0,\tilde{\epsilon}}\theta_{\tilde{\epsilon}_0,\tilde{\epsilon}_0} f_+^{\epsilon,\eta} + F_+^{\epsilon,\eta}\tilde{f}_-^{\tilde{\epsilon},\eta} + \tilde{F}_-^{\tilde{\epsilon},\eta} f_+^{\epsilon,\eta}\big\}\varphi^\lambda\,dt\,dx\,d\xi,
\end{aligned} \tag{3.24}$$

where

$$I_1 = -\int_{\mathbb{R}^{d+2}} \big(R_1^u \tilde{f}_-^{\tilde{\epsilon},\eta} + R_1^{\tilde{u}} f_+^{\epsilon,\eta}\big)\varphi^\lambda\,dt\,dx\,d\xi,$$

$$I_2 = -2\int_{\mathbb{R}^{d+2}} \sum_{l=1}^{K}\sum_{k,j=1}^{d} \partial_{x_k}\big(\big(\sigma_{kl}f_+^{\epsilon}\big)*\theta_\eta\big)\partial_{x_j}\big(\big(\sigma_{jl}\tilde{f}_-^{\tilde{\epsilon}}\big)*\theta_\eta\big)\varphi^\lambda\,dt\,dx\,d\xi,$$

$$I_3 = \int_{\mathbb{R}^{d+2}} \Bigg\{-\sum_{k,j=1}^{d}\partial_{x_k}\tilde{f}_-^{\tilde{\epsilon},\eta}\partial_{x_j}\big(\big(D\beta_{k_j}f_+^{\epsilon}\big)*\theta_\eta\big) + \sum_{l=1}^{K}\sum_{k,j=1}^{d}\partial_{x_k}\big(\big(\sigma_{kl}f_+^{\epsilon}\big)*\theta_\eta\big)\partial_{x_j}\big(\big(\sigma_{jl}\tilde{f}_-^{\tilde{\epsilon}}\big)*\theta_\eta\big)\Bigg\}\varphi^\lambda\,dt\,dx\,d\xi,$$

$$\tilde{I}_3 = \int_{\mathbb{R}^{d+2}} \Bigg\{-\sum_{k,j=1}^{d}\partial_{x_k}f_+^{\epsilon,\eta}\partial_{x_j}\big(\big(D\beta_{k_j}\tilde{f}_-^{\tilde{\epsilon}}\big)*\theta_\eta\big) + \sum_{l=1}^{K}\sum_{k,j=1}^{d}\partial_{x_k}\big(\big(\sigma_{kl}f_+^{\epsilon}\big)*\theta_\eta\big)\partial_{x_j}\big(\big(\sigma_{jl}\tilde{f}_-^{\tilde{\epsilon}}\big)*\theta_\eta\big)\Bigg\}\varphi^\lambda\,dt\,dx\,d\xi,$$

$$I_4 = \int_{\mathbb{R}^{d+2}} \delta(\xi-\tilde{u}) *_{(t,x,\xi)} (\rho_{\tilde{\epsilon}}\theta_\eta)n^{\epsilon,\eta}\varphi^\lambda\,dt\,dx\,d\xi,$$

$$\tilde{I}_4 = \int_{\mathbb{R}^{d+2}} \delta(\xi-u) *_{(t,x,\xi)} (\rho_{\epsilon}\theta_\eta)\tilde{n}^{\tilde{\epsilon},\eta}\varphi^\lambda\,dt\,dx\,d\xi.$$

Here we used the fact that m_+ and $\tilde{m}_-$ are nonnegative. Thanks to the lemmas [12, Lemmas 4.1 and 4.2] we have that I_1, I_3 and $\tilde{I}_3$ tend to 0 as $\eta \to 0+$ and $I_2 + I_4 + \tilde{I}_4 \geqslant 0$.

Letting successively η, $\tilde{\epsilon}_0$, $(\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{i-1}, \tilde{\epsilon}_{i+1}, \dots, \tilde{\epsilon}_d)$ and then $\tilde{\epsilon}_i$ pass to 0 in (3.24) and noting that the regularized function f_+^ϵ vanishes at the parabolic boundary $\partial_p Q$, we get

$$\begin{aligned}
&\int\limits_{Q_{i-}\times\mathbb{R}} -f_+^\epsilon \tilde{f}_- \big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda + \sum_{k,j=1}^d \big\{\tilde{f}_- \partial_{x_j}\big(D\beta_{kj}(\xi) f_+^\epsilon\big) + f_+^\epsilon \partial_{x_j}\big(D\beta_{kj}(\xi)\tilde{f}_-\big)\big\}\partial_{x_k}\varphi^\lambda \\
&\quad + \big(\big[\delta(u-\xi)g\big]^\epsilon \tilde{f}_- + \delta(\tilde{u}-\xi)\tilde{g} f_+^\epsilon\big)\varphi^\lambda \, dt\, dx\, d\xi \\
&\geqslant \int\limits_{Q_{i-}\times\mathbb{R}} \big(f_+^{\tau_0,\epsilon}\theta_{\epsilon_0,\epsilon_0}\tilde{f} + F_+^\epsilon \tilde{f}_-\big)\varphi^\lambda \, dt\, dx\, d\xi.
\end{aligned} \tag{3.25}$$

Then we pass in turn ϵ_0, ϵ_i and $\epsilon^i = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \dots, \epsilon_d)$ to 0 in (3.25). The first term on the right-hand side of (3.25) tends to $\int_{\Omega_i\times\mathbb{R}} f_+^{\tau_0}(x,\xi)\tilde{f}_-^{\tau_0}(x,\xi)\varphi^\lambda(0,x)\,dx\,d\xi$ by Lemma 3.1 (also see Lemma 3.5). In order to treat the second term, noting Remark 2.1 we compute:

$$\begin{aligned}
&\int\limits_{Q_{i-}\times\mathbb{R}} D\beta_{ij}(\xi)\big(\partial_{x_j} f_+^b\big)^\epsilon \tilde{f}_- \varphi^\lambda \, dt\, dx\, d\xi \\
&\quad = \int\limits_{\mathbb{R}^{d+1}} D\beta_{ij}(\xi)\big(\partial_{x_j} f_+^b\big)^{(\epsilon_0,\epsilon^i)}\big(t,\bar{x}_i,a_i^-,\xi\big)\big(\tilde{f}_-\varphi^\lambda \mathbf{1}_{Q_{i-}}\big) * \check{\theta}_{\epsilon_0,\epsilon_i}\big(t,\bar{x}_i,a_i^-,\xi\big)\, dt\, d\bar{x}_i\, d\xi,
\end{aligned}$$

which tends as $\epsilon_0 \to 0+$ to

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} D\beta_{ij}(\xi)\big(\partial_{x_j} f_+^b\big)^{(0,\epsilon^i)}\big(t,\bar{x}_i,a_i^-,\xi\big) \\
&\quad \cdot \left(\frac{1}{\epsilon_i}\int\limits_0^{\epsilon_i} \operatorname{sgn}^-\big(\beta_{ii}\big(\tilde{u}\big(t,\bar{x}_i,a_i^-+r\big) - \beta_{ii}(\xi)\big)\varphi^\lambda\big(t,\bar{x}_i,a_i^-+r\big)\,dr\big)\right) dt\, d\bar{x}_i\, d\xi,
\end{aligned}$$

where

$$\big(\partial_{x_j} f_+^b\big)^{(\epsilon_0,\epsilon^i)}(t,\bar{x}_i,x_i,\xi) = \int\limits_{\mathbb{R}^d} \big(\partial_{x_j} f_+^b \mathbf{1}_{\Sigma_{i-}}\big)(s,\bar{y}_i,x_i,\xi)\cdot\theta_{\epsilon_0,\epsilon_0}(t-s)\prod_{j\neq i}^d \theta_{\epsilon_0,\epsilon_j}(x_j-y_j)\,ds\,d\bar{y}_i$$

with the notational convention that $\theta_{0,0} = \delta$. In a similar way as in the proof of Lemma 3.2 it follows from (iii) of Definition 2.1 that the above integral tends as $\epsilon_i \to 0+$ and then $\epsilon^i \to 0$ to

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} D\beta_{ij}(\xi)\big(\partial_{x_j} f_+^b \mathbf{1}_{\Sigma_{i-}}\big)\big(t,\bar{x}_i,a_i^-,\xi\big)\operatorname{sgn}^-\big(\beta_{ii}\big(\tilde{u}_b(t,\bar{x}_i)\big) - \beta_{ii}(\xi)\big)\varphi^\lambda\big(t,\bar{x}_i,a_i^-\big)\,dt\,d\bar{x}_i\,d\xi \\
&\quad = \int\limits_{\Sigma_{i-}\times\mathbb{R}} D\beta_{ij}(\xi)\partial_{x_j} f_+^b \tilde{f}_-^b \varphi^\lambda\, dt\, d\bar{x}_i\, d\xi.
\end{aligned}$$

Thus we obtain

$$\int\limits_{Q_{i-}\times\mathbb{R}} -f_+\tilde{f}_-\big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda$$
$$+\sum_{k,j=1}^{d}\partial_{x_j}\big(D\beta_{kj}(\xi)f_+\tilde{f}_-\big)\partial_{x_k}\varphi^\lambda\,dt\,dx\,d\xi + \int\limits_{Q_{i-}}\operatorname{sgn}^+(u-\tilde{u})(g-\tilde{g})\varphi^\lambda\,dt\,dx$$
$$\geqslant \int\limits_{\Omega_{i-}\times\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}\varphi^\lambda(0,x)\,dx\,d\xi$$
$$+\int\limits_{\Sigma_i^-\times\mathbb{R}}\bigg(-\mathbf{n}_{i^-}\cdot a(\xi)f_+^\tau\tilde{f}_-^\tau + \sum_{j\neq k}D\beta_{ij}(\xi)\partial_{x_j}f_+^b\tilde{f}_-^b\bigg)\varphi^\lambda\,dt\,d\bar{x}_i\,d\xi \tag{3.26}$$

for any weak* cluster point $f_+^\tau(t,y,\xi)$ and for some weak* cluster point $\tilde{f}_-^\tau(t,y,\xi)$.

On the other hand, we first let η, ϵ_0, $(\epsilon_1,\ldots,\epsilon_{i-1},\epsilon_{i+1},\ldots,\epsilon_d)$ and ϵ_i in turn pass to 0 in (3.24) and secondly we let $\tilde{\epsilon}_0$, $\tilde{\epsilon}_i$ and $(\tilde{\epsilon}_1,\ldots,\tilde{\epsilon}_{i-1},\tilde{\epsilon}_{i+1},\ldots,\tilde{\epsilon}_d)$ in turn pass to 0 to obtain

$$\int\limits_{Q_{i-}\times\mathbb{R}} -f_+\tilde{f}_-\big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda$$
$$+\sum_{k,j=1}^{d}\partial_{x_j}\big(D\beta_{kj}(\xi)f_+\tilde{f}_-\big)\partial_{x_k}\varphi^\lambda\,dt\,dx\,d\xi + \int\limits_{Q_{i-}}\operatorname{sgn}(u-\tilde{u})(g-\tilde{g})\varphi^\lambda\,dt\,dx$$
$$\geqslant \int\limits_{\Omega_{i-}\times\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}\varphi^\lambda(0,x)\,dx\,d\xi$$
$$+\int\limits_{\Sigma_{i-}\times\mathbb{R}}\bigg(-\mathbf{n}_{i^-}\cdot a(\xi)\tilde{f}_-^\tau f_+^\tau + \sum_{j\neq i}D\beta_{ij}(\xi)\partial_{x_j}\tilde{f}_-^b f_+^b\bigg)\varphi^\lambda\,dt\,d\bar{x}_i\,d\xi \tag{3.27}$$

for any weak* cluster point $\tilde{f}_-^\tau(t,y,\xi)$ and for some weak* cluster point $f_+^\tau(t,y,\xi)$. Summing (3.26) and (3.27) yields

$$\int\limits_{Q_{i-}\times\mathbb{R}} -f_+\tilde{f}_-\big(\partial_t + a(\xi)\cdot\nabla\big)\varphi^\lambda$$
$$+\sum_{k,j=1}^{d}\partial_{x_j}\big(D\beta_{kj}(\xi)f_+\tilde{f}_-\big)\partial_{x_k}\varphi^\lambda\,dt\,dx\,d\xi + \int\limits_{Q_{i-}}\operatorname{sgn}^+(u-\tilde{u})(g-\tilde{g})\varphi^\lambda\,dt\,dx$$
$$\geqslant \int\limits_{\Omega_{i-}\times\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}\varphi^\lambda(0,x)\,dx\,d\xi$$
$$+\int\limits_{\Sigma_{i-}\times\mathbb{R}}\bigg(-\mathbf{n}_{i^-}\cdot a(\xi)f_+^\tau\tilde{f}_-^\tau + \frac{1}{2}\sum_{j\neq i^*}D\beta_{ij}(\xi)\partial_{x_j}\big(f_+^b f_-^b\big)\bigg)\varphi^\lambda\,dt\,d\bar{x}_i\,d\xi \tag{3.28}$$

for some weak* cluster points f_+^τ and $\tilde{f}_-^\tau$. We compute each term of (3.28). Firstly, the left-hand side of (3.28) becomes

$$\int_{Q_{i-}} (u-\tilde{u})^+\partial_t\varphi^\lambda + \mathcal{F}^+(u,\tilde{u})\nabla\varphi^\lambda$$
$$+\sum_{k,j=1}^{d} \partial_{x_j}\big(\operatorname{sgn}^+(u-\tilde{u})\big(\beta_{kj}(u)-\beta_{kj}(\tilde{u})\big)\big)\partial_{x_k}\varphi^\lambda + \operatorname{sgn}^+(u-\tilde{u})(g-\tilde{g})\varphi^\lambda\,dt\,dx. \tag{3.29}$$

Secondly, by virtue of Lemma 3.5 and by using integration by parts one can calculate:

$$\begin{aligned}
\int_{\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}\,d\xi &= \int_{-\infty}^{\tilde{u}_0} \nu_x^{\tau_0}\big([\xi,\infty)\big)\big(\partial_\xi\tilde{m}_-^0+\operatorname{sgn}^-(\tilde{u}_0-\xi)\big)\,d\xi - \int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0} \nu_x^{\tau_0}\big([\xi,\infty)\big)\tilde{\nu}_x^{\tau_0}\big((-\infty,\xi]\big)\,d\xi\\
&\quad - \int_{u_0\vee\tilde{u}_0}^{\infty} \big(\partial_\xi m_+^0+\operatorname{sgn}^+(u_0-\xi)\big)\tilde{\nu}_x^{\tau_0}\big((-\infty,\xi]\big)\,d\xi\\
&= \int_{-\infty}^{u_0} \nu_x^{\tau_0}\big([\xi,\infty)\big)\partial_\xi\tilde{m}_-^0\,d\xi - \int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0} \nu_x^{\tau_0}\big([\xi,\infty)\big)\tilde{\nu}_x^{\tau_0}\big((-\infty,\xi]\big)\,d\xi\\
&\quad - \int_{u_0\vee\tilde{u}_0}^{\infty} \partial_\xi m_+^0\tilde{\nu}_x^{\tau_0}\big((-\infty,\xi]\big)\,d\xi\\
&= \nu_x^{\tau_0}\big([\tilde{u}_0,\infty)\big)\tilde{m}_-^0(\cdot,\tilde{u}_0) + \int_{-\infty}^{\tilde{u}_0} \tilde{m}_-^0\,d\nu_x^{\tau_0} - \int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0} \nu_x^{\tau_0}\big(\big([\xi,\infty)\big)\tilde{\nu}_x^{\tau_0}\big((-\infty,\xi]\big)\,d\xi\\
&\quad + m_+^0(\cdot,u_0\vee\tilde{u}_0)\tilde{\nu}_x^{\tau_0}\big((-\infty,u_0\vee\tilde{u}_0]\big) + \int_{u_0\vee\tilde{u}_0}^{\infty} m_+^0\,d\tilde{\nu}_-^{\tau_0}\\
&\geqslant -\int_{\tilde{u}_0}^{u_0\vee\tilde{u}_0} d\xi = -(u_0-\tilde{u}_0)^+.
\end{aligned}$$

Here we used the fact that $\frac{d\nu_x^{\tau_0}([\xi,\infty))}{d\xi} = -d\nu_x^{\tau_0}(\xi)$ and $\frac{d\tilde{\nu}_-^{\tau_0}((-\infty,\xi])}{d\xi} = d\tilde{\nu}_-^{\tau_0}(\xi)$. Thus we obtain

$$\int_{\Omega_{i-}\times\mathbb{R}} f_+^{\tau_0}\tilde{f}_-^{\tau_0}\varphi^\lambda\,dx\,d\xi \geqslant -\int_{\Omega_{i-}} (u_0-\tilde{u}_0)^+\varphi^\lambda(0,x)\,dx. \tag{3.30}$$

Finally, we analogously calculate the boundary term by using Lemma 3.5:

$$\begin{aligned}
\int_{\mathbb{R}} \mathbf{n}_{i^-} \cdot a(\xi) f_+^\tau \tilde{f}_-^\tau \, d\xi &= -\int_{-\infty}^{\tilde{u}_b} \partial_\xi \tilde{m}_-^b \, \nu_{t,y}^\tau\big([\xi,\infty)\big)\, d\xi \\
&\quad - \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} \mathbf{n}_{i^-} \cdot a(\xi) \nu_{t,y}^\tau\big([\xi,\infty)\big) \tilde{\nu}_{t,y}^\tau\big((-\infty,\xi]\big)\, d\xi \\
&\quad + \int_{u_b \vee \tilde{u}_b}^{\infty} \partial_\xi m_+^b \, \tilde{\nu}_{t,y}^\tau\big((-\infty,\xi]\big)\, d\xi \\
&\leqslant \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} \big|a(\xi)\big|\, d\xi \\
&\leqslant M(u_b - \tilde{u}_b)^+.
\end{aligned}$$

Here y stands for the point $(\bar{x}_i, a_i^-)$ and we used the fact that $\frac{d\nu_{t,y}^\tau([\xi,\infty))}{d\xi} = -d\nu_{t,y}^\tau(\xi)$ and $\frac{d\tilde{\nu}_{t,y}^\tau((-\infty,\xi])}{d\xi} = d\tilde{\nu}_{t,y}^\tau(\xi)$. We also note that $m_+^b \geqslant 0$ if $\xi \geqslant u_b$ and $\tilde{m}_-^b \geqslant 0$ if $\xi \leqslant \tilde{u}_b$ by virtue of (3.15) in Lemma 3.5. Hence we have

$$\int_{\Sigma_{i^-} \times \mathbb{R}} (\mathbf{n}_{i^-} \cdot a) f_+^\tau \tilde{f}_-^\tau \varphi^\lambda \, dt\, d\bar{x}_i\, d\xi \leqslant M \int_{\Sigma_{i^-}} (u_b - \tilde{u}_b)^+ \varphi^\lambda \, dt\, d\bar{x}_i. \tag{3.31}$$

Moreover,

$$\begin{aligned}
&\int_{\Sigma_{i^-} \times \mathbb{R}} D\beta_{ij}(\xi) \partial_{x_j}\big(f_+^b \tilde{f}_-^b\big) \varphi^\lambda \, dt\, d\bar{x}_i\, d\xi \\
&\quad = \int_{\Sigma_{i^-}} \partial_{x_j}\big(\operatorname{sgn}^+(u_b - \tilde{u}_b)\big(\beta_{ij}(u_b) - \beta_{ij}(\tilde{u}_b)\big)\big) \varphi^\lambda \, dt\, d\bar{x}_i.
\end{aligned} \tag{3.32}$$

Combining (3.28) with (3.29) through (3.32) and choosing appropriate test functions φ's (and also noting $\nabla\varphi^\lambda = \lambda\nabla\varphi + \varphi\nabla\lambda$) we arrive at for a.e. $t \in (0,T)$

$$\begin{aligned}
&\int_{\Omega_{i^-}} \big(u(t,\cdot) - \tilde{u}(t,\cdot)\big)^+ \lambda_{i^-}\, dx \\
&\quad \leqslant \int_{\Omega_{i^-}} (u_0 - \tilde{u}_0)^+ \lambda_{i^-}\, dx + M \int_{(0,t)\times(\partial\Omega)_{i^-}} (u_b - \tilde{u}_b) \lambda_{i^-}\, ds\, d\bar{x}_i \\
&\qquad - \frac{1}{2} \sum_{j \neq i} \int_{(0,t)\times(\partial\Omega)_{i^-}} \partial_{x_j}\big(\operatorname{sgn}^+(u_b - \tilde{u}_b)\big(\beta_{ij}(u_b) - \beta_{ij}(\tilde{u}_b)\big)\big) \lambda_{i^-}\, ds\, d\bar{x}_i \\
&\qquad + \int_{(0,t)\times\Omega_{i^-}} (g - \tilde{g})^+ \lambda_{i^-}\, ds\, dx + E_{\nu_{i^-}}
\end{aligned} \tag{3.33}$$

for $1 \leqslant i \leqslant d$. Here

$$E_{\nu_{i*}} = \int_{(0,t)\times\Omega_{i*}} \mathcal{F}^+(u,\tilde{u})\cdot\nabla\lambda_{i*} - \sum_{k,j=1}^{d} \partial_{x_j}\big(\mathrm{sgn}^+(u-\tilde{u})\big(\beta_{kj}(u)-\beta_{kj}(\tilde{u})\big)\big)\partial_{x_k}\lambda_{i*}\,ds\,dx.$$

When $i=0$, we have $\lambda_0 \in C_c^\infty(\mathbb{R}^d)$ and $\mathrm{supp}(\lambda_0) \subset \Omega$. By extending u and $\tilde{u}$ to $\mathbb{R}^d$ by 0 outside of Ω we can use the result on the Cauchy problem obtained by Chen and Perthame [12] to obtain

$$\int_{\Omega_0} \big(u(t,\cdot)-\tilde{u}(t,\cdot)\big)^+\,dx \leqslant \int_{\Omega_0} (u_0-\tilde{u}_0)^+\lambda_0\,dx + \int_{(0,t)\times\Omega_0} (g-\tilde{g})^+\lambda_0\,ds\,dx + E_{\nu_0}.$$

Summing over $i \in \{0,1,\dots,d\}$ one has

$$\begin{aligned}
\int_{\Omega^\nu} \big(u(t,\cdot)-\tilde{u}(t,\cdot)\big)^+\,dx &\leqslant \int_{\Omega^\nu} (u_0-\tilde{u}_0)^+\,dx + M\sum_{i=1}^{d}\int_{(0,t)\times(\partial\Omega)_{i*}^\nu} \big(A_i(u_b)-A_i(\tilde{u}_b)\big)^+\lambda_{i*}\,ds\,d\bar{x}_i \\
&\quad - \sum_{i\neq j}\int_{(0,t)\times(\partial\Omega)_{i*}^\nu} \partial_{x_j}\big(\mathrm{sgn}^+(u_b-\tilde{u}_b)\big(\beta_{ij}(u_b)-\beta_{ij}(\tilde{u}_b)\big)\big)\,ds\,d\bar{x}_i \\
&\quad + \int_{(0,t)\times\Omega^\nu} (g-\tilde{g})^+\,ds\,dx + \sum_{i=0}^{d} E_{\nu_{i*}}.
\end{aligned} \tag{3.34}$$

The function h defined by

$$h = \big|\mathcal{F}^+(u,\tilde{u})\big| + \sum_{k,j=1}^{d} \big|\partial_{x_j}\big(\beta_{kj}(u)-\beta_{kj}(\tilde{u})\big)\big|$$

belongs to $L^2(Q)$ (see Remark 2.1). Since $\sum_{i=0}^d \lambda_{i*} = 1$ on $\Omega_{2\nu}$, we have with $D_\nu = (0,T)\times(\Omega\setminus\Omega_{2\nu})$

$$\begin{aligned}
\left|\sum_{i=0}^{d} E_{\nu_{i*}}\right| &\leqslant \|h\|_{L^2(D_\nu)} \left\|\nabla\left(\sum_{i=0}^{d}\lambda_{i*}\right)\right\|_{L^2(D_\nu)} \\
&\leqslant C\|h\|_{L^2(D_\nu)}\big(\mathrm{meas}(\Omega\setminus\Omega_{2\nu})\big)^{\frac{1}{2}}\nu^{-1} \\
&\leqslant C\|h\|_{L^2(D_\nu)}
\end{aligned}$$

and hence $\sum_{i=0}^d E_{\nu_{i*}}$ tends to 0 as $\nu \to 0+$. Consequently, passing ν to 0 in (3.34) yields (2.3). □

4. Existence of entropy solutions

As was mentioned in the introduction the existence of entropy solutions of (1.1)–(1.3) will be proved in the quasi-isotropic case under the additional smoothness condition on the flux $A(u)$ and the diffusion $\beta_{ii}(u)$.

Theorem 4.1. *Suppose that* (A1), (A2) *and* (A3) *are satisfied with $D\beta_{ij}=0$ when $i\neq j$. In addition suppose that A_i, $\beta_{ii}\in W^{2,\infty}_{loc}(\mathbb{R})$, $i=1,2,\dots,d$, and $u_b\in W^{1,\infty}(\Sigma)$. Then there exists a unique entropy solution of* (1.1)–(1.3).

To prove this theorem we consider the approximate problem

$$\partial_t u+\operatorname{div}A(u)-\sum_{i=1}^d\partial^2_{x_i}\bigl(\beta_{ii}(u)-\epsilon u\bigr)=g^\delta\quad\text{in }Q,$$
$$u|_{t=0}=u_0^\delta,\qquad u|_\Sigma=u_b^\delta,\tag{4.1}$$

where ϵ, $\delta>0$ and $u_0^\delta\in C^\infty(\Omega)$, $u_b^\delta\in C(\Sigma)$, $g^\delta\in C^\infty(Q)$ are functions such that u_0^δ and u_b^δ satisfy the compatibility condition on $\bar\Sigma\cap\bar\Omega$ and $u_0^\delta\to u_0$, $u_b^\delta\to u_b$, $g^\delta\to g$ in the L^1-norms as $\delta\to 0+$ and moreover $\|u_0^\delta\|_{L^\infty(\Omega)}\leqslant\|u_0\|_{L^\infty(\Omega)}$, $\|u_b^\delta\|_{L^\infty(\Sigma)}\leqslant\|u_b\|_{L^\infty(\Sigma)}$, $\|\nabla u_b^\delta\|_{L^\infty(\Sigma)}\leqslant\|\nabla u_b\|_{L^\infty(\Sigma)}$, $\|g^\delta\|_{L^\infty(Q)}\leqslant\|g\|_{L^\infty(Q)}$. The problem (4.1) has a unique smooth solution $u(t,x)=u^{\epsilon,\delta}(t,x)$ (see e.g. [21]). Multiply (4.1) by $(u-k)^+$ with sufficiently large $k>\|u_0\|_{L^\infty}\vee\|u_b\|_{L^\infty}\vee\|g\|_{L^\infty}$ and integrate over $(0,t)\times\Omega$ to obtain

$$\int_\Omega\bigl(u(t,\cdot)-k\bigr)^+dx\leqslant\int_\Omega\bigl(u_0^\delta-k\bigr)^+dx+2\bigl\|g^\delta\bigr\|_{L^\infty}\int_0^t\int_\Omega(u-k)^+\,dt\,dx,$$

which yields

$$\int_\Omega\bigl(u(t,\cdot)-k\bigr)^+dx\leqslant C\int_\Omega\bigl(u_0^\delta-k\bigr)^+dx=0$$

and so $u\leqslant k$. A similar estimate on $(u-k)^-$ gives the estimate

$$\bigl\|u^{\epsilon,\delta}\bigr\|_{L^\infty(Q)}\leqslant C\tag{4.2}$$

with some constant which does not depend on ϵ as well as δ. Then multiply (4.1) by u and integrate over Q to obtain

$$\begin{aligned}&\int_Q\sum_{i=1}^d\bigl(\bigl|\partial_{x_i}s_{ii}(u)\bigr|^2+\epsilon|\partial_{x_i}u|^2\bigr)\,dt\,dx\\
&\quad\leqslant\frac12\int_\Omega\bigl|u_0^\delta\bigr|^2dx-\int_\Sigma A\bigl(u_b^\delta\bigr)u_b^\delta\cdot\mathbf{n}\,dt\,d\sigma+\int_\Sigma\Lambda\bigl(u_b^\delta\bigr)\cdot\mathbf{n}\,dt\,d\sigma\\
&\qquad+\sum_{i=1}^d\int_{\Sigma_{i+}}\bigl(\partial_{x_i}\beta_{ii}\bigl(u_b^\delta\bigr)u_b^\delta+\epsilon u_b^\delta\partial_{x_i}u_b^\delta\bigr)\,dt\,d\bar x_i\\
&\qquad-\sum_{i=1}^d\int_{\Sigma_{i-}}\bigl(\partial_{x_i}\beta_{ii}\bigl(u_b^\delta\bigr)u_b^\delta+\epsilon u_b^\delta\partial_{x_i}u_b^\delta\bigr)\,dt\,d\bar x_i+\int_Q g^\delta u\,dt\,dx,\end{aligned}$$

where Λ is a primitive of A. Hence (4.2) and the assumption on u_b yield

$$\int_Q \sum_{i=1}^d \left(\left| \partial_{x_i} s_{ii}\left(u^{\epsilon,\delta}\right) \right|^2 + \epsilon \left| \partial_{x_i} u^{\epsilon,\delta} \right|^2 \right) dt\, dx \leqslant C \tag{4.3}$$

and then

$$\sum_{i=1}^d \int_Q \left| \partial_{x_i} \beta_{ii}\left(u^{\epsilon,\delta}\right) \right|^2 dt\, dx \leqslant C \tag{4.4}$$

with some constant C independent of ϵ and δ.

Next, let us denote $v = \partial_t u$. We have

$$\partial_t v + \operatorname{div}\big(a(u)v\big) - \sum_{i=1}^d \partial_{x_i}^2 \big(D\beta_{ii}(u)v + \epsilon v\big) = \partial_t g^\delta. \tag{4.5}$$

Multiply (4.5) by $D\varphi_\alpha(v)$, where $\varphi_\alpha(z) = (z^2 + \alpha^2)^{\frac{1}{2}}$, $\alpha > 0$, to obtain

$$\begin{aligned}
&\int_Q \partial_t \varphi_\alpha(v)\, dt\, dx + \int_\Sigma a\big(u_b^\delta\big)\partial_t u_b^\delta D\varphi_\alpha\big(\partial_t u_b^\delta\big) \cdot \mathbf{n}\, dt\, d\sigma \\
&\quad - \int_Q a(u) v D^2\varphi_\alpha(v) \cdot \nabla v\, dt\, dx - \int_\Sigma \sum_{i=1}^d \big(D\beta_{ii}\big(u_b^\delta\big)\partial_t u_b^\delta + \epsilon \partial_t u_b^\delta\big) D\varphi_\alpha\big(\partial_t u_b^\delta\big) n_i\, dt\, d\sigma \\
&\quad + \int_Q \sum_{i=1}^d \partial_{x_i} D\beta_{ii}(u) v D^2\varphi_\alpha(v) \partial_{x_i} v\, dt\, dx + \int_Q \sum_{i=1}^d \big(D\beta_{ii}(u) + \epsilon\big) D^2\varphi_\alpha(v) |\partial_{x_i} v|^2\, dt\, dx \\
&= \int_Q \partial_t g^\delta D\varphi_\delta(v)\, dt\, dx.
\end{aligned} \tag{4.6}$$

Note that $D^2\varphi_\alpha(z) \geqslant 0$, $|zD^2\varphi_\alpha(z)| \leqslant 1$ and $zS^2\varphi_\alpha(z) \to 0$ as $\alpha \to 0+$. Letting $\epsilon \to 0+$ and also noting $v(0,\cdot) = -\operatorname{div} A(u_0^\delta) + \sum_{i=1}^d \partial_{x_i}^2(\beta_{ii}(u_0^\delta) + \epsilon u_0^\delta) + g^\delta(0,\cdot)$, we obtain the estimate for $v = \partial_t u^{\epsilon,\delta}$:

$$\left\| \partial_t u^{\epsilon,\delta}(t,\cdot) \right\|_{L^1(\Omega)} \leqslant C_\delta \tag{4.7}$$

for some constant C_δ which may be depend on δ, but does not depend on ϵ.

Similarly, if we consider the function $w = \partial_{x_k} u^{\epsilon,\delta}$, then we can get the estimate

$$\left\| \partial_{x_k} u^{\epsilon,\delta}(t,\cdot) \right\|_{L^1(\Omega)} \leqslant C_\delta. \tag{4.8}$$

Let us fix $\delta > 0$ for the moment. By virtue of (4.2), (4.7), (4.8) and Kolmogorov's compactness theorem there is a subsequence (still denoted) $u^{\epsilon,\delta}(t,x)$ and $u^\delta \in C([0,T];\ L^1(\Omega))$ such that

$$\lim_{\epsilon \to 0+} u^{\epsilon,\delta} = u^\delta \quad \text{in } C\big([0,T]; L^1(\Omega)\big)$$

and thanks to (4.3) and (4.4),

$$w\text{-}\lim_{\epsilon\to 0+} \partial_{x_i}\beta_{ii}\big(u^{\epsilon,\delta}\big) = \partial_{x_i}\beta_{ii}\big(u^{\delta}\big) \quad \text{in } L^2(Q),$$
$$\big\|\partial_{x_i}\beta_{ii}\big(u^{\delta}\big)\big\|_{L^2(Q)} \leqslant C \tag{4.9}$$

and

$$\big\|\partial_{x_i}s_{ii}\big(u^{\delta}\big)\big\|_{L^2(Q)} \leqslant C. \tag{4.10}$$

Furthermore, since the smooth solution $u^{\epsilon,\delta}(t,x)$ satisfies inequality (2.1), so does the strong limit function $u^{\delta}(t,x)$. Thus we see that $u^{\delta}(t,x)$ is an entropy solution of (1.1)–(1.3) with data $(u_0^{\delta}, u_b^{\delta}, g^{\delta})$. Moreover, thanks to (2.3), for δ_1, $\delta_2 > 0$

$$\begin{aligned}
&\int_Q \big|u^{\delta_1}(t,x) - u^{\delta_2}(t,x)\big|\,dt\,dx \\
&\quad\leqslant \int_\Omega \big|u_0^{\delta_1}(x) - u_0^{\delta_2}(x)\big|\,dx + M\int_\Sigma \big|u_b^{\delta_1}(t,x) - u_b^{\delta_2}(t,x)\big|\,dt\,d\sigma \\
&\qquad + \int_Q \big|g^{\delta_1}(t,x) - g^{\delta_2}(t,x)\big|\,dt\,dx.
\end{aligned}$$

Therefore $\{u^{\delta}\}$ is a Cauchy sequence and there exists $u \in C([0,T];L^1(\Omega))$ such that

$$\lim_{\delta\to 0+} u^{\delta}(t,x) = u(t,x) \quad \text{in } C\big([0,T];L^1(\Omega)\big).$$

By virtue of (4.3) and (4.4), $\partial_{x_i}s_{ii}(u)$ and $\partial_{x_i}\beta_{ii}(u)$ belong to $L^2(Q)$. Since $u^{\delta}(t,x)$ is an entropy solution, so does $u(t,x)$. Thus the proof is complete. □

Acknowledgments

We would like to thank the referee for his careful reading as well as his useful and warm comments. The first author have been partially supported by Grant-in-Aid for Scientific Research (No. 22540235) the Japan Society for the Promotion of Science.

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