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Unilateral global bifurcation phenomena and nodal solutions for p-Laplacian ☆

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In this paper, we establish a Dancer-type unilateral global bifurcation result for one-dimensional p-Laplacian problem

$$\begin{cases} -(\varphi_p(u'))' = \mu m(t)\varphi_p(u) + g(t,u;\mu), & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Under some natural hypotheses on the perturbation function $g : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we show that $\mu_k(p)$ is a bifurcation point of the above problem and there are two distinct unbounded continua, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$, consisting of the bifurcation branch $\mathcal{C}_k$ from $(\mu_k(p), 0)$, where $\mu_k(p)$ is the k-th eigenvalue of the linear problem corresponding to the above problem.

As the applications of the above result, we study the existence of nodal solutions for the following problem

$$\begin{cases} (\varphi_p(u'))' + f(t,u) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Moreover, based on the bifurcation result of Girg and Takáč (2008) [13], we prove that there exist at least a positive solution and a negative one for the following problem

$$\begin{cases} -\mathrm{div}(\varphi_p(\nabla u)) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

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1. Introduction

In the celebrated work [29], P.H. Rabinowitz established Rabinowitz's unilateral global bifurcation theory (Theorem 1.27 and Theorem 1.40 of [29]). However, as pointed out by Dancer [6,7] and López-Gómez [20], the proofs of these theorems contain gaps, the original statement of Theorem 1.40 of [29] is not correct, and the original statement of Theorem 1.27 of [29] is stronger than what one can actually prove so far. Although there exist some gaps in the proofs of Rabinowitz's Theorem 1.27 and Theorem 1.40, Theorem 1.27 and Theorem 1.40 have been used several times in the literature to analyze the global behavior of the component of nodal solutions emanating from $u=0$ in wide classes of elliptic boundary value problems for equations and systems [23,17,3,21]. In 2008, Girg and Takáč proved a Dancer-type bifurcation theorem (Theorem 3.7, [13]) in which the continua bifurcate from the principle eigenvalue for a high-dimensional p-Laplacian equation.

In this paper, we will also establish a Dancer-type bifurcation theorem for one-dimensional p-Laplacian problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = \mu m(t)\varphi_p(u) + g(t,u;\mu), & t \in (0,1), \\ u(0)=u(1)=0, \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$, μ is a positive parameter, $m(t) \geqslant 0$ and $m(t) \not\equiv 0$ for $t \in (0,1)$ is a continuous weight function, $g : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition in the first two variables and

$$\lim_{|s|\to 0} \frac{g(t,s;\mu)}{|s|^{p-1}} = 0 \tag{1.2}$$

uniformly for a.e. $t \in (0,1)$ and μ on bounded sets.

Under the condition of (1.2), we will show that $(\mu_k(p), 0)$ is a bifurcation point of (1.1) and there are two distinct unbounded continua, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$, consisting of the bifurcation branch $\mathcal{C}_k$ from $(\mu_k(p), 0)$, where $\mu_k(p)$ is the k-th eigenvalue of the linear problem corresponding to (1.1).

The proofs are based on the local properties of solutions of (1.1) bifurcating from $(\mu_k(p), 0)$ (see Lemma 3.1). Although the proof of the above result follows the same steps as do for the semilinear case from [6] and the high-dimensional p-Laplacian case from [13], their methods cannot be applied directly to obtain our result. Indeed, the proof of Lemma 3 of [6] strictly depends on the linear property of operator L and needs some smooth property, to guarantee $l(H)$ is well defined, of the perturbation function H. And the proof of Lemma 5.8 of [13] strictly depends on properties, such as Poincaré inequality holding, of the principle eigenvalue λ_1 of high-dimensional p-Laplacian eigenvalue problem, which are used to get local asymptotic analysis for λ near λ_1. Thus, we construct a new perturbation function which is different from those in [6] and [13]. Then, using the local properties of solutions of (1.1) bifurcating from $(\mu_k(p), 0)$ (see Lemma 3.1) and functional analysis method, we can obtain a Dancer-type bifurcation result for (1.1).

Based on the unilateral global bifurcation result (see Theorem 3.2), we investigate the existence of nodal solutions for the following one-dimensional p-Laplacian 0-Dirichlet problem

$$\begin{cases} \left(\varphi_p\left(u'\right)\right)' + f(t,u) = 0, & t \in (0,1), \\ u(0)=u(1)=0, \end{cases} \tag{1.3}$$

where $f \in C([0,1] \times \mathbb{R})$.

It is well known that when $f(t,u) = \lambda\varphi_p(u)$, problem (1.3) has a nontrivial solution if and only if λ is an eigenvalue of the 0-Dirichlet problem

$$\begin{cases} \left(\varphi_p\left(u'\right)\right)' + \lambda\varphi_p(u) = 0, & t \in (0,1), \\ u(0)=u(1)=0. \end{cases} \tag{1.4}$$

In particular, when $\lambda = \lambda_k(p) = (k\pi_p)^p$, there exists solution $u(t) = \frac{\alpha}{k\pi_p}\sin_p(k\pi_p t)$ of (1.4), where $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$, $\alpha = u'(0)$ and $\sin_p$ is the *p-sine* function (see [9] or [33]). It is obvious that $u_k^+ = \frac{\alpha}{k\pi_p}\sin_p(k\pi_p t)$ when $\alpha > 0$ and $u_k^- = \frac{\alpha}{k\pi_p}\sin_p(k\pi_p t)$ when $\alpha < 0$, such that u_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

When $p = 2$, Ma and Thompson [23] considered the interval of r, for which there exist nodal solutions of the boundary value problem

$$\begin{cases} u'' + rm(t)f(u) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.5}$$

under some suitable assumptions on f and m. Using the bifurcation theory of Rabinowitz [28,29], they proved that if $\frac{\widetilde{\lambda}_k}{f_\infty} < r < \frac{\widetilde{\lambda}_k}{f_0}$ or $\frac{\widetilde{\lambda}_k}{f_0} < r < \frac{\widetilde{\lambda}_k}{f_\infty}$, (1.5) has two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0, where $\widetilde{\lambda}_k$ is the k-th eigenvalue of linear problem of (1.5), $f_0 = \lim_{|s|\to 0}\frac{f(s)}{s}$, $f_\infty = \lim_{|s|\to+\infty}\frac{f(s)}{s}$. The idea of using bifurcation methods to study the solvability of nonlinear boundary value problems has been applied to study some two-point, three-point and periodic boundary value problems, see [22,24,25]. The results they obtained extended some well-known theorems of the existence of positive solutions for related problems [15,11].

For $p \neq 2$, M. Del Pino, M. Elgueta and R. Manásevich [9] investigated the existence of solutions for (1.3) using the Leray–Schauder degree by the deformation along p.

Of course, the natural question is that whether nodal solutions exist for (1.3) if $\frac{f(x,s)}{\varphi_p(s)}$ crosses $\lambda_k(p)$? In this paper, we will provide positive answer for this question (see Section 4).

In high-dimensional case, Girg and Takáč [13] proved a Dancer-type bifurcation theorem (Theorem 3.7, [13]) in which the continua bifurcate from the principle eigenvalue for a quasilinear elliptic eigenvalue problem. Based on their results, we will investigate the existence of constant sign solutions for the following p-Laplacian 0-Dirichlet problem

$$\begin{cases} -\mathrm{div}\big(\varphi_p(\nabla u)\big) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geqslant 2$ is a bounded smooth domain and $f \in C(\Omega \times \mathbb{R})$.

By a solution of (1.6) we understand $u \in W_0^{1,p}(\Omega)$ satisfying (1.6) in the weak sense, i.e., such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$. It is well known that when $f(x,u) = \lambda\varphi_p(u)$, problem (1.6) has a nontrivial solution if and only if λ is an eigenvalue of the 0-Dirichlet problem

$$\begin{cases} -\mathrm{div}\big(\varphi_p(\nabla u)\big) = \lambda\varphi_p(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

In particular, there exist a positive solution and a negative one when $\lambda = \lambda_1(p)$, where $\lambda_1(p)$ is the principle eigenvalue of (1.7).

Under the assumptions that $\frac{f(x,s)}{\varphi_p(s)}$ crosses $\lambda_1(p)$, X.L. Fan, Y.Z. Zhao and G.F. Huang [12] proved that problem (1.6) possesses two solutions including at least one nontrivial solution. In [16], using the topological degree argument, Y.S. Huang and H.S. Zhou obtained the existence of positive solutions for (1.6) in the N-dimensional case under the similar assumptions as in [12].

Again, one will ask the question that whether a positive solution and a negative one exist for (1.6) if $\frac{f(x,s)}{\varphi_p(s)}$ crosses $\lambda_1(p)$? In this paper, we also give a positive answer for this question (see Section 5).

The rest of this paper is arranged as follows. In Section 2, we establish the Rabinowitz-type global bifurcation theory for (1.1). In Section 3, we establish the unilateral global bifurcation theory for (1.1). In Section 4, we prove the existence of nodal solutions for (1.3) with crossing nonlinearity. In Section 5, we prove the existence of constant sign solutions for (1.6) with crossing nonlinearity.

2. Global bifurcation phenomena for (1.1)

Let E be the Banach space $C_0^1[0,1]$ with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

Let $Y = L^1(0,1)$ with its usual normal $\|\cdot\|_{L^1}$.

We start by considering the following auxiliary problem

$$\begin{cases} (\varphi_p(u'))' = h, & \text{a.e. } t \in (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{2.1}$$

for a given $h \in L^1(0,1)$. By a solution of problem (2.1), we understand a function $u \in E$ with $\varphi_p(u')$ absolutely continuous which satisfies (2.1). Problem (2.1) is equivalently written as

$$u(t) = G_p(h)(t) := \int_0^t \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds,$$

where $a : Y \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds = 0.$$

It is known that $G_p : Y \to E$ is continuous and maps equi-integrable sets of Y into relatively compacts of E. One may refer to Lee and Sim [17] and Manásevich and Mawhin [26] for detail.

The bifurcation points of (1.1) are related to the eigenvalues of the problem

$$\begin{cases} (\varphi_p(u'(t)))' + \mu m(t)\varphi_p(u(t)) = 0, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.2}$$

It is well known that the set of all eigenvalues of problem (2.2) is an infinite sequence of simple eigenvalues

$$0 < \mu_1(p) < \mu_2(p) < \cdots < \mu_k(p) < \cdots, \qquad \lim_{k\to+\infty} \mu_k(p) = +\infty,$$

and the eigenfunction φ_k corresponding to $\mu_k(p)$ has exactly $k-1$ simple zeros in $(0,1)$ (see, e.g., [17,2]).

We define the operator $T_\mu^p : E \to E$ by

$$T_\mu^p(u)(t) = \int_0^t \varphi_p^{-1}\left(a\big(-\mu m\varphi_p\big(u(s)\big)\big) - \int_0^s \mu m(\tau)\varphi_p\big(u(\tau)\big)\,d\tau\right) ds =: G_p\big(-\mu m\varphi_p(u)\big)(t).$$

Then $T_\mu^p : E \to E$ is compact and problem (2.2) is equivalent to

$$u = T_\mu^p(u).$$

It is very known that $I - T_\mu^p$ is completely continuous vector field in $C^1[0,1]$. Thus the Leray–Schauder degree $\mathrm{d}_{\mathrm{LS}}(I - T_\mu^p, B_r(0), 0)$ is well defined for arbitrary r-ball $B_r(0)$ and $\mu \neq \mu_k$, $k \in \mathbb{N}$.

Lemma 2.1. *(See [17].) Let $\{\mu_k(p)\}_{k\in\mathbb{N}}$ be the sequence of eigenvalues of (2.2). Let μ be a constant with $\mu \neq \mu_k(p)$ for all $k \in \mathbb{N}$. Then for arbitrary $r > 0$,*

$$\deg\big(I - T_\mu^p, B_r(0), 0\big) = (-1)^\beta,$$

where β is the number of eigenvalues $\mu_k(p)$ of problem (2.2) less than μ.

Define the Nemitskii operators $H : \mathbb{R} \times E \to Y$ by

$$H(\mu, u)(t) := -\mu m(t)\varphi_p\big(u(t)\big) - g\big(t, u(t); \mu\big).$$

Then it is clear that H is continuous operator which sends bounded sets of $\mathbb{R} \times E$ into the equi-integrable sets of Y and problem (1.1) can be equivalently written as

$$u = G_p \circ H(\mu, u) := F(\mu, u).$$

F is completely continuous in $\mathbb{R} \times E \to E$ and $F(\mu, 0) = 0$, $\forall \mu \in \mathbb{R}$.

Theorem 2.1. *Assume (1.2) holds. Then for $p > 1$, $\mu_k(p)$ is a bifurcation point of (1.1) and the associated bifurcation branch $\mathcal{C}_k$ in $\mathbb{R} \times E$ whose closure contains $(\mu_k(p), 0)$ is either unbounded or contains a pair $(\overline{\mu}, 0)$ where $\overline{\mu}$ is an eigenvalue of (2.2) and $\overline{\mu} \neq \mu_k(p)$.*

Proof. From now on, for simplicity, we write $\mu_k = \mu_k(p)$. Suppose that $(\mu_k, 0)$ is not a bifurcation point of problem (1.1). Then there exist $\varepsilon > 0$, $\rho_0 > 0$ such that for $|\mu - \mu_k| \leqslant \varepsilon$ and $0 < \rho < \rho_0$ there is no nontrivial solution of the equation

$$u - F(\mu, u) = 0$$

with $\|u\| = \rho$. From the invariance of the degree under a compact homotopy we obtain that

$$\deg\big(I - F(\mu, \cdot), B_\rho(0), 0\big) \equiv \text{constant} \tag{2.3}$$

for $\mu \in [\mu_k - \varepsilon, \mu_k + \varepsilon]$.

By taking ε smaller if necessary, we can assume that there is no eigenvalue of (2.2) in $(\mu_k, \mu_k + \varepsilon]$. Fix $\mu \in (\mu_k, \mu_k + \varepsilon]$. We claim that the equation

$$u - G_p\big(-\mu m(t)\varphi_p\big(u(t)\big) - sg\big(t, u(t); \mu\big)\big) = 0 \tag{2.4}$$

has no solution u with $\|u\| = \rho$ for every $s \in [0, 1]$ and ρ sufficiently small. Suppose on the contrary, let $\{u_n\}$ be the solution of (2.4) with $\|u_n\| \to 0$ as $n \to +\infty$.

Let $v_n := \frac{u_n}{\|u_n\|}$, then v_n should be a solution of the problem

$$v_n(t) = G_p\left(-\mu m(t)\varphi_p\big(v_n(t)\big) - s\frac{g(t, u_n(t); \mu)}{\|u_n\|^{p-1}}\right). \tag{2.5}$$

Let

$$\widetilde{g}(t, u; \mu) = \max_{0 \leqslant |s| \leqslant u} \big|g(t, s; \mu)\big| \quad \text{for a.e. } t \in (0, 1) \text{ and } \mu \text{ on bounded sets,}$$

then $\widetilde{g}$ is nondecreasing with respect to u and

$$\lim_{u \to 0^+} \frac{\widetilde{g}(t, u; \mu)}{|u|^{p-1}} = 0. \tag{2.6}$$

Further it follows from (2.6) that

$$\frac{g(t, u; \mu)}{\|u\|^{p-1}} \leqslant \frac{\widetilde{g}(t, |u|; \mu)}{\|u\|^{p-1}} \leqslant \frac{\widetilde{g}(t, \|u\|_\infty; \mu)}{\|u\|^{p-1}} \leqslant \frac{\widetilde{g}(t, \|u\|; \mu)}{\|u\|^{p-1}} \to 0, \quad \text{as } \|u\| \to 0 \tag{2.7}$$

uniformly for a.e. $t \in (0, 1)$ and μ on bounded sets.

By (2.5), (2.7) and the compactness of G_p, we obtain that for some convenient subsequence $v_n \to v_0 \neq 0$ as $n \to +\infty$. Now v_0 verifies the equation

$$-\big(\varphi_p\big(v_0'\big)\big)' = \mu m(t)\varphi_p(v_0)$$

and $\|v_0\| = 1$. This implies that μ is an eigenvalue of (2.2). This is a contradiction. From the invariance of the degree under homotopies and Lemma 2.1 we then obtain

$$\deg\big(I - F(\mu, \cdot), B_r(0), 0\big) = \deg\big(I - T_\mu^p, B_r(0), 0\big) = (-1)^k. \tag{2.8}$$

Similarly, for $\mu \in [\mu_k - \varepsilon, \mu_k)$ we find that

$$\deg\big(I - F(\mu, \cdot), B_r(0), 0\big) = (-1)^{k-1}. \tag{2.9}$$

Relations (2.8) and (2.9) contradict (2.3) and hence $(\mu_k, 0)$ is a bifurcation point of problem (1.1).

By standard arguments in global bifurcation theory (see [29]), we can show the existence of a global branch of solutions of (1.1) emanating from $(\mu_k, 0)$. $\square$

We finally prove that the first choice of the alternative of Theorem 2.1 is the only possibility. In what follows, we use the terminology of Rabinowitz [30]. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in $(0, 1)$ and are positive near $t = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E (also see [27]). Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$ under the product topology.

Lemma 2.2. *If* (μ, u) *is a solution of* (1.1) *and* u *has a double zero, then* $u \equiv 0$.

Proof. Let u be a solution of (1.1) and $t^* \in [0,1]$ be a double zero. We note that

$$u(t) = \int_t^{t^*} \varphi_p^{-1}\left(\int_s^{t^*} \big(-\mu m(\tau)\varphi_p\big(u(\tau)\big) - g\big(\tau, u(\tau); \mu\big)\big)\, d\tau\right) ds.$$

First, we consider $t \in [0, t^*]$. Then

$$\begin{aligned}\big|u(t)\big| &\leqslant \int_t^{t^*} \varphi_p^{-1}\left(\left|\int_s^{t^*} \big(-\mu m(\tau)\varphi_p\big(u(\tau)\big) - g\big(\tau, u(\tau); \mu\big)\big)\, d\tau\right|\right) ds \\ &\leqslant \varphi_p^{-1}\left(\int_t^{t^*} \big|\mu m(\tau)\varphi_p\big(u(\tau)\big) + g\big(\tau, u(\tau); \mu\big)\big|\, d\tau\right),\end{aligned}$$

furthermore,

$$\begin{aligned}\varphi_p\big(\big|u(t)\big|\big) &\leqslant \int_t^{t^*} \big|\mu m(\tau)\varphi_p\big(u(\tau)\big) + g\big(\tau, u(\tau); \mu\big)\big|\, d\tau \\ &\leqslant \int_t^{t^*} \left|\mu m(\tau) + \frac{g(\tau, u(\tau); \mu)}{\varphi_p(u(\tau))}\right| \varphi_p\big(u(\tau)\big)\, d\tau \\ &\leqslant \int_t^{t^*} \left(\mu\big|m(\tau)\big| + \left|\frac{g(\tau, u(\tau); \mu)}{\varphi_p(u(\tau))}\right|\right) \varphi_p\big(\big|u(\tau)\big|\big)\, d\tau.\end{aligned}$$

In view of (1.2), for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$\big|g(t, s; \mu)\big| \leqslant \varepsilon\varphi_p\big(|s|\big)$$

uniformly with respect to a.e. $t \in (0,1)$ and fixed μ when $|s| \in [0, \delta]$. Hence,

$$\varphi_p\big(\big|u(t)\big|\big) \leqslant \int_t^{t^*} \left(|\mu| \max_{t\in[0,1]} m(t) + \varepsilon + \max_{s\in[\delta, \|u\|_\infty]} \left|\frac{g(\tau, s; \mu)}{\varphi_p(s)}\right|\right) \varphi_p\big(\big|u(\tau)\big|\big)\, d\tau.$$

By the Gronwall–Bellman inequality [4], we get $u \equiv 0$ on $[0, t^*]$. Similarly, using a modification of Gronwall–Bellman inequality [14, Lemma 2.2], we can get $u \equiv 0$ on $[t^*, 1]$ and the proof is complete. □

Lemma 2.3. *The last alternative of Theorem* 2.1 *is impossible if* $\mathcal{C}_k \subset \Phi_k \cup \{(\mu_k, 0)\}$.

Proof. Suppose on the contrary, if there exists $(\mu_m, u_m) \to (\mu_j, 0)$ when $m \to +\infty$ with $(\mu_m, u_m) \in \mathcal{C}_k$, $u_m \not\equiv 0$ and $j \neq k$. Let $v_m := \frac{u_m}{\|u_m\|}$, then v_m should be a solution of the problem

$$v_m = G_p\left(-\mu_m m(t)\varphi_p\big(v_m(t)\big) - \frac{g(t, u_m(t); \mu_m)}{\|u_m(t)\|^{p-1}}\right). \tag{2.10}$$

By (2.7), (2.10) and the compactness of G_p we obtain that for some convenient subsequence $v_m \to v_0 \neq 0$ as $m \to +\infty$. Now v_0 verifies the equation

$$-\left(\varphi_p\left(v_0'\right)\right)' = \mu_j m(t)\varphi_p(v_0)$$

and $\|v_0\| = 1$. Hence $v_0 \in S_j$ which is an open set in E, and as a consequence for some m large enough, $v_m \in S_j$, and this is a contradiction. □

Theorem 2.2. *Let* (1.2) *hold, then from each* $(\mu_k(p), 0)$ *it bifurcates an unbounded continuum* $\mathcal{C}_k$ *of solutions to problem* (1.1)*, with exactly* $k-1$ *simple zeros, where* μ_k *is the eigenvalue of problem* (2.2)*.*

Proof. Taking into account Theorem 2.1 and Lemma 2.3, we only need to prove that $\mathcal{C}_k \subset \Phi_k \cup \{(\mu_k, 0)\}$.

Suppose $\mathcal{C}_k \not\subset \Phi_k \cup \{(\mu_k, 0)\}$. Then there exists $(\mu, u) \in \mathcal{C}_k \cap (\mathbb{R} \times \partial S_k)$ such that $(\mu, u) \neq (\mu_k, 0)$, $u \notin S_k$, and $(\mu_n, u_n) \to (\mu, u)$ with $(\mu_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R} \times S_k)$. Since $u \in \partial S_k$, by Lemma 2.2, $u \equiv 0$. Let $w_n := \frac{u_n}{\|u_n\|}$, then w_n should be a solution of the problem

$$w_n = G_p\left(\mu_n m(t)\varphi_p\left(w_n(t)\right) + \frac{g(t, u_n(t); \mu_n)}{\|u_n(t)\|^{p-1}}\right). \tag{2.11}$$

By (2.7), (2.11) and the compactness of G_p we obtain that for some convenient subsequence $w_n \to w_0 \neq 0$ as $n \to +\infty$. Now w_0 verifies the equation

$$-\left(\varphi_p\left(w_0'\right)\right)' = \mu m(t)\varphi_p(w_0)$$

and $\|w_0\| = 1$. Hence $\mu = \mu_j$, for some $j \neq k$. Therefore, $(\mu_n, u_n) \to (\mu_j, 0)$ with $(\mu_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R} \times S_k)$. This contradicts Lemma 2.3. □

3. Unilateral global bifurcation phenomena for (1.1)

In this section, we will prove more details about the bifurcation from Theorem 2.2. Let $\mathbb{E} = \mathbb{R} \times E$, $\Phi(\mu, u) := u - F(\mu, u)$ and

$$\mathcal{S} := \overline{\left\{(\mu, u) \in \mathbb{E}\colon\ \Phi(\mu, u) = 0,\ u \neq 0\right\}}^{\mathbb{E}}.$$

In order to formulate and prove main results of this section, it is convenient to introduce Dancer's [6] and López-Gómez's notations [20]. Given any $\mu \in \mathbb{R}$ and $0 < s < +\infty$, we consider an open neighborhood of $(\mu, 0)$ in $\mathbb{E}$ defined by

$$\mathbb{B}_s(\mu_k, 0) := \left\{(\mu, u) \in \mathbb{E}\colon\ \|u\| + |\mu - \mu_k| < s\right\}.$$

And $B_s(0)$ denotes $\{u \in E\colon\ \|u\| < s\}$. Let E_0 be a closed subspace of E such that

$$E = \operatorname{span}\{\varphi_k\} \oplus E_0.$$

According to the Hahn–Banach theorem, there exists a linear functional $l \in E^*$, here E^* denotes the dual space of E, such that

$$l(\varphi_k) = 1 \quad \text{and} \quad E_0 = \left\{u \in E\colon\ l(u) = 0\right\}.$$

Finally, for any $0 < \eta < 1$, we define

$$K_\eta := \{(\mu, u) \in \mathbb{E}\colon\ |l(u)| > \eta\|u\|\}.$$

Since

$$u \mapsto |l(u)| - \|u\|$$

is continuous, K_η is an open subset of $\mathbb{E}$ consisting of two disjoint components K_η^+ and K_η^-, where

$$K_\eta^+ := \{(\mu, u) \in \mathbb{E}\colon\ l(u) > \eta\|u\|\},$$
$$K_\eta^- := \{(\mu, u) \in \mathbb{E}\colon\ l(u) < -\eta\|u\|\}.$$

In particular, both K_η^+ and K_η^- are convex cones, $K_\eta^- = -K_\eta^+$, and $\nu t\varphi_k \in K_\eta^\nu$ for every $t > 0$, where $\nu \in \{+, -\}$.

Applying the similar method to prove [20, Lemma 6.4.1] with obvious changes, we may obtain the following result, which localizes the possible solutions of (1.1) bifurcating from $(\mu_k, 0)$.

Lemma 3.1. *For every $\eta \in (0, 1)$ there exists a number $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$,*

$$\big(\big(\mathcal{S} \setminus \{(\mu_k, 0)\}\big) \cap \overline{\mathbb{B}}_\delta(\mu_k, 0)\big) \subset K_\eta.$$

Moreover, for each

$$(\mu, u) \in \big(\mathcal{S} \setminus \{(\mu_k, 0)\}\big) \cap \big(\overline{\mathbb{B}}_\delta(\mu_k, 0)\big),$$

there are $s \in \mathbb{R}$ and unique $y \in E_0$ such that

$$u = s\varphi_k + y \quad \text{and} \quad |s| > \eta\|u\|.$$

Furthermore, for these solutions (μ, u),

$$\mu = \mu_k + o(1) \quad \text{and} \quad y = o(s)$$

as $s \to 0$.

Remark 3.1. From the proof of Lemma 6.4.1 of [20], we can see that if $g(t, u; \mu)$ is replaced by $g_n(t, u; \mu)$ which satisfies

$$\lim_{\|u\| \to 0} \frac{g_n(t, u; \mu)}{\|u\|^{p-1}} = 0$$

uniformly for all $n \in \mathbb{N}$, then δ_0 can be chosen uniformly with respect to n.

Let $\delta > 0$ be the constant from Lemma 3.1. For $0 < \varepsilon \leqslant \delta$ we define $\mathcal{D}^\nu_{\mu_k,\varepsilon}$ to be the component of $\{(\mu_k, 0)\} \cup (\mathcal{S} \cap \overline{\mathbb{B}}_\varepsilon \cap K_\eta^\nu)$ containing $(\mu_k, 0)$, $\mathcal{C}^\nu_{\mu_k,\varepsilon}$ to be the component of $\overline{\mathcal{C}_k \setminus \mathcal{D}^{-\nu}_{\mu_k,\varepsilon}}$ containing $(\mu_k, 0)$, and $\mathcal{C}_k^\nu$ to be the closure of $\bigcup_{0<\varepsilon\leqslant\delta} \mathcal{C}^\nu_{\mu_k,\varepsilon}$. Clearly, $\mathcal{C}_k^\nu$ is connected. Thanks to Lemma 3.1, the definition of $\mathcal{C}_k^\nu$ is independent from the choice of η and $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$.

The following unilateral global bifurcation result is a close analogue of Dancer's result [6, Theorem 2] shown originally for abstract semilinear equations.

Theorem 3.1. *Either $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are both unbounded, or else $\mathcal{C}_k^+ \cap \mathcal{C}_k^- \neq \{(\mu_k, 0)\}$.*

As in the semilinear case in Dancer [6, Theorem 2], our proof of Theorem 3.1 is based on the following three lemmata.

Lemma 3.2. *Suppose $\delta_1, \delta_2 > 0$ such that $0 < \delta_1 + \delta_2 < \delta$ and $\Phi(\mu, u) \neq 0$ if $\|u\| = \delta_1$ and $|\mu - \mu_k| \leqslant \delta_2$. If $0 < \sigma < \delta_2$ and $\beta(\sigma)$ is sufficiently small and positive, then*

$$\deg\big(\Phi(\mu_k + \sigma, \cdot), W^\nu, 0\big) - \deg\big(\Phi(\mu_k - \sigma, \cdot), W^\nu, 0\big) = (-1)^{k-1},$$

where $W^\nu = \{u \in E\colon (\mu, u) \in K_\eta^\nu,\ \beta(\sigma) < \|u\| < \delta_1\}$.

Proof. Recall that $u = l(u)\varphi_k + y$. We define

$$\widehat{g}(t, u; \mu) = \begin{cases} g(t, u; \mu) & \text{if } l(u) \leqslant -\eta\|u\|; \\ \frac{-l(u)}{\eta\|u\|} g(t, -\eta\|u\|\varphi_k + y; \mu) & \text{if } -\eta\|u\| < l(u) \leqslant 0; \\ -g(t, -u; \mu) & \text{if } l(u) > 0 \end{cases}$$

and

$$\widehat{\Phi}(\mu, u) = u - G_p\big(-\mu m(t)\varphi_p\big(u(t)\big) - \widehat{g}\big(t, u(t); \mu\big)\big).$$

Then the mapping $\widehat{\Phi}(\mu, u)$ is odd with respect to u.

By our hypothesis and Lemma 3.1, the equation $\Phi(\mu_k + \sigma, u) = 0$ has no solution in

$$B_{\delta_1} \setminus \big(W^+ \cup W^- \cup B_\beta\big).$$

By Lemma 3.1, $\widehat{\Phi}(\mu_k + \sigma, u) = \Phi(\mu_k + \sigma, u)$ on $\partial B_{\delta_1} \cup \partial B_\beta$. It follows that

$$\begin{aligned} \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), B_{\delta_1}, 0\big) &= \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), B_\beta, 0\big) \\ &\quad + \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), W^+, 0\big) + \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), W^-, 0\big). \end{aligned}$$

The oddness of $\widehat{\Phi}(\mu_k + \sigma, \cdot)$ and the definition of the degree in Schwartz [31] ensure that

$$\deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), W^+, 0\big) = \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), W^-, 0\big).$$

And the definition of $\widehat{\Phi}$ ensures that

$$\deg\big(\Phi(\mu_k + \sigma, \cdot), W^-, 0\big) = \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), W^-, 0\big).$$

Thus

$$2\deg\big(\Phi(\mu_k + \sigma, \cdot), W^-, 0\big) = \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), B_{\delta_1}, 0\big) - \deg\big(\widehat{\Phi}(\mu_k + \sigma, \cdot), B_\beta, 0\big). \quad (3.1)$$

Analogously,

$$2\deg\big(\Phi(\mu_k - \sigma, \cdot), W^-, 0\big) = \deg\big(\widehat{\Phi}(\mu_k - \sigma, \cdot), B_{\delta_1}, 0\big) - \deg\big(\widehat{\Phi}(\mu_k - \sigma, \cdot), B_\beta, 0\big). \quad (3.2)$$

By Lemma 3.1 and the definition of $\widehat{\Phi}$, we have

$$\deg\big(\widehat{\Phi}(\mu_k-\sigma,\cdot),B_\beta,0\big)=\deg\big(\Phi(\mu_k-\sigma,\cdot),B_\beta,0\big)$$

and

$$\deg\big(\widehat{\Phi}(\mu_k+\sigma,\cdot),B_\beta,0\big)=\deg\big(\Phi(\mu_k+\sigma,\cdot),B_\beta,0\big).$$

As in the proof of Theorem 2.1 one can show that

$$\deg\big(\Phi(\mu_k-\sigma,\cdot),B_\beta,0\big)=(-1)^{k-1}\quad\text{and}\quad\deg\big(\Phi(\mu_k+\sigma,\cdot),B_\beta,0\big)=(-1)^{k}.\tag{3.3}$$

From Lemma 3.1 and the definition of $\widehat{\Phi}$, we can see that

$$\deg\big(\widehat{\Phi}(\mu_k+\sigma,\cdot),B_{\delta_1},0\big)=\deg\big(\Phi(\mu_k+\sigma,\cdot),B_{\delta_1},0\big)$$

and

$$\deg\big(\widehat{\Phi}(\mu_k-\sigma,\cdot),B_{\delta_1},0\big)=\deg\big(\Phi(\mu_k-\sigma,\cdot),B_{\delta_1},0\big).$$

By our assumptions, for $\mu\in[\mu_k-\sigma,\mu_k+\sigma]$ the homotopy $\Phi(\mu,\cdot)$ is admissible on B_{δ_1}. The homotopy invariance of the degree ensures that

$$\deg\big(\Phi(\mu_k+\sigma,\cdot),B_{\delta_1},0\big)=\deg\big(\Phi(\mu_k-\sigma,\cdot),B_{\delta_1},0\big).$$

Subtracting (3.1) from (3.2) and using (3.3), we arrive at

$$\deg\big(\Phi(\mu_k+\sigma,\cdot),W^-,0\big)-\deg\big(\Phi(\mu_k-\sigma,\cdot),W^-,0\big)=(-1)^{k-1}.\qquad\square$$

Define $T^-_{\mu_k,\varepsilon}$ to be the component of $\mathcal{C}_k\setminus(\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta)$ containing $(\mu_k,0)$.

Lemma 3.3. *If $0<\varepsilon<\delta$, zero is an isolated solution of $\Phi(\mu_k,u)=0$, and $T^-_{\mu_k,\varepsilon}$ is bounded in $\mathbb{E}$, then*

$$\partial\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\cap T^-_{\mu_k,\varepsilon}\neq\emptyset.$$

Proof. Proof of Lemma 2 of [6] is also valid for the quasilinear case, and therefore the proof is omitted. $\square$

Lemma 3.4. *The statement of Lemma* 3.3 *holds without the assumption that zero is an isolated solution of $\Phi(\mu_k,u)=0$.*

Proof. For any $n\in\mathbb{N}$, choose continuous functions $f_n:[0,+\infty)\to[0,1]$ such that $f_n(s)=\varphi_p(s)$ for $0\leqslant|s|\leqslant\frac{1}{2n}$ and $f_n(s)=0$ for $|s|\geqslant\frac{1}{n}$. Define

$$\Phi_n(\mu,u):=u-G_p\big(-\mu m(t)\varphi_p\big(u(t)\big)-g\big(t,u(t);\mu\big)-f_n\big(l\big(u(t)\big)\big)\big\|u(t)\big\|^p\big).$$

Since $\lim_{\|u\|\to 0}\frac{g(t,u;\mu)}{\|u\|^{p-1}}=0$ and the definition of f_n, we have

$$\lim_{\|u\|\to 0} \frac{g(t,u;\mu) + f_n(l(u))\|u\|^p}{\|u\|^{p-1}} = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Let

$$\mathcal{S}_n := \overline{\{(\mu,u) \in \mathbb{E} : \Phi_n(\mu,u) = 0, u \neq 0\}}^{\mathbb{E}},$$

using Remark 3.1 and (3.4), we can show that

$$\big(\big(S_n \setminus \{(\mu_k,0)\}\big) \cap \overline{\mathbb{B}}_\delta(\mu_k,0)\big) \subset K_\eta.$$

Moreover, for each

$$(\mu,u) \in \big(S_n \setminus \{(\mu_k,0)\}\big) \cap \overline{\mathbb{B}}_\delta(\mu_k,0),$$

there exist $s \in \mathbb{R}$ and unique $y \in E_0$ such that

$$u = s\varphi_k + y \quad \text{and} \quad |s| > \eta\|u\|.$$

We claim that zero is an isolated solution of $\Phi_n(\mu_k,u) = 0$ for each positive integer n.

Suppose on the contrary, let u is a nontrivial solution of $\Phi_n(\mu_k,u) = 0$, such that

$$0 < \|u\| := b < \delta.$$

Thus, $u = s\varphi_k + y$ with $s = l(u)$, $y \in E_0$ and $y = o(s)$. Let $v = \frac{u}{s}$. It follows that $\lim_{s\to 0} v = \varphi_k$. Consequently, we have

$$v + \frac{y}{s} = G_p\left(-\mu_k m(t)\varphi_p\left(v(t) + \frac{y(t)}{s}\right) - \frac{g(t,u(t);\mu_k) + f_n(l(u(t)))\|u(t)\|^p}{\varphi_p(s)}\right). \tag{3.5}$$

Letting $s \to 0$ on the both sides of (3.5) and using the continuous property of G_p, we can obtain that

$$\varphi_k = G_p\left(-\mu_k m(t)\varphi_p\big(\varphi_k(t)\big) - \lim_{s\to 0} \frac{g(t,u(t);\mu_k) + f_n(l(u(t)))\|u(t)\|^p}{\varphi_p(s)}\right),$$

i.e.,

$$-\big(\varphi_p\big(\varphi_k'\big)\big)' = \mu_k m(t)\varphi_p(\varphi_k) + \lim_{s\to 0} \frac{g(t,u;\mu_k) + f_n(l(u))\|u\|^p}{\varphi_p(s)}.$$

Therefore,

$$\lim_{s\to 0} \frac{g(t,u;\mu_k) + f_n(l(u))\|u\|^p}{\varphi_p(s)} = 0. \tag{3.6}$$

While, in view of (1.2) and the definition of f_n, we obtain that

$$\begin{aligned}\lim_{s\to 0}\frac{g(t,u;\mu_k)+f_n(l(u))\|u\|^p}{\varphi_p(s)} &= \lim_{s\to 0}\frac{g(t,u;\mu_k)}{\varphi_p(s)}+\lim_{s\to 0}\frac{f_n(l(u))\|u\|^p}{\varphi_p(s)}\\ &= \lim_{s\to 0}\left(\frac{g(t,u;\mu_k)}{\varphi_p(u)}\cdot\frac{\varphi_p(u)}{\varphi_p(s)}\right)+b^p\\ &= \lim_{s\to 0}\frac{g(t,u;\mu_k)}{\varphi_p(u)}\cdot\varphi_p(\varphi_k)+b^p\\ &= b^p\neq 0.\end{aligned}$$

This contradicts (3.6).

Now let $0<\varepsilon<\delta$ and assume that $T^-_{\mu_k,\varepsilon}$ is bounded in $\mathbb{E}$. Let T_n be a component of $\mathcal{S}_n\setminus(\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta)$ containing $(\mu_k,0)$. Suppose that the conclusion of our lemma is false. This means that

$$\partial\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\cap T^-_{\mu_k,\varepsilon}=\emptyset.$$

The definition of $T^-_{\mu_k,\varepsilon}$ implies that

$$\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\cap T^-_{\mu_k,\varepsilon}=\emptyset.$$

Since $T^-_{\mu_k,\varepsilon}$ is bounded, we can find $R>0$ such that $T^-_{\mu_k,\varepsilon}\subset\mathbb{B}_R(\mu_k,0)$.

Combining these facts with a classical topological result from Whyburn [32, Chap. I, Statement (9.3)], we conclude that

$$K:=\big(\mathcal{S}\cap\overline{\mathbb{B}}_R(\mu_k,0)\big)\setminus\big(\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\big)=k_1\cup k_2,$$

where k_1, k_2 are disjoint compact subsets of K, such that $T^-_{\mu_k,\varepsilon}\subset k_1$ and

$$\big(\mathcal{S}\cap\partial\mathbb{B}_R(\mu_k,0)\big)\cup\big(\mathcal{S}\cap\partial\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\big)\subset k_2.$$

Consequently, there exists a bounded open set U in $\mathbb{E}$ such that $k_1\subset U$, $k_2\cap\overline{U}=\emptyset$, $(\mu_k,0)\in U$, $(\partial U\cap\mathcal{S})\subset(\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta)$ and $\partial\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\cap U=\emptyset$.

Applying Lemma 3.3 to Φ_n, we have

$$\partial\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\cap T_n\neq\emptyset\quad\text{for each } n\in\mathbb{N}.$$

By the connectedness of T_n, there exists an $(u_n,\mu_n)\in\partial U\cap T_n$. By choosing a subsequence if necessary, we may assume that $u_n\rightharpoonup u^*$ in E and $\mu_n\to\mu^*$ in $\mathbb{R}$. Letting $n\to+\infty$ on the both of $\Phi_n(\mu_n,u_n)=0$ and using the compact and continuous properties of G_p, we can show $\Phi(\mu^*,u^*)=0$. By the closed property of ∂U, we conclude that $(\mu^*,u^*)\in\partial U\cap T_n$. In view of the definition of T_n, we can obtain that

$$\big(\mu^*,u^*\big)\in\big(\mathcal{S}\cap\overline{\mathbb{B}}_R(\mu_k,0)\big)\setminus\big(\mathbb{B}_\varepsilon(\mu_k,0)\cap K^+_\eta\big)\subset k_2.$$

This contradicts $\partial U\cap k_2=\emptyset$. $\square$

Proof of Theorem 3.1. Define $T^-_{\mu_k}$ to be the closure of $\bigcup_{0<\varepsilon\leqslant\delta}T^-_{\mu_k,\varepsilon}$. Then $T^-_{\mu_k}\subseteq\mathcal{C}^-_k$. Suppose $\mathcal{C}^-_k$ is bounded. By Lemma 3.4, for any $0<\varepsilon\leqslant\delta$, we have

$$\partial\mathbb{B}_\varepsilon(\mu_k, 0) \cap K_\eta^+ \cap T_{\mu_k}^- \neq \emptyset.$$

It follows that

$$\big(T_{\mu_k}^- \setminus \big(\mathbb{B}_\delta(\mu_k, 0) \cap K_\eta^-\big)\big) \cap \partial\mathbb{B}_\varepsilon(\mu_k, 0) \neq \emptyset.$$

Furthermore, for every open set U of $(\mu_k, 0)$ and $U \subseteq \mathbb{B}_\delta(\mu_k, 0)$, the above implies that

$$\big(T_{\mu_k}^- \setminus \big(\mathbb{B}_\delta(\mu_k, 0) \cap K_\eta^-\big)\big) \cap \partial U \neq \emptyset. \tag{3.7}$$

Let $\mathfrak{L} = (T_{\mu_k}^- \setminus (\mathbb{B}_\delta(\mu_k, 0) \cap K_\eta^-))$ and T be the component of $T_{\mu_k}^- \setminus (\mathbb{B}_\delta(\mu_k, 0) \cap K_\eta^-)$ containing $(\mu_k, 0)$. Let $\mathcal{U}_\rho$ be a ρ-neighborhood of T and $K = \overline{\mathcal{U}_\rho} \cap \mathfrak{L}$.

We claim that $T \cap \partial\mathbb{B}_\delta(\mu_k, 0) \neq \emptyset$.

In fact, if $T \cap \partial\mathbb{B}_\delta(\mu_k, 0) = \emptyset$, Lemma 1.1 implies that $K = K_1 \cup K_2$, where K_1, K_2 are disjoint compact subsets of K containing T and $\partial\mathbb{B}_\delta(\mu_k, 0) \cap K$, respectively. Let $\mathcal{O}$ be any ε-neighborhood in $\mathbb{E}$ of K_1. It is obvious that $\mathcal{O} \subseteq \mathbb{B}_\delta(\mu_k, 0)$ and $\partial\mathcal{O} \cap \mathfrak{L} = \emptyset$. This is a contradiction.

By the definition of $\mathcal{C}_k^+$ and the fact $T \cap \partial\mathbb{B}_\delta(\mu_k, 0) \neq \emptyset$, we have

$$\mathcal{C}_{\mu_k}^+ \supseteq \mathcal{C}_{\mu_k,\delta}^+ \supseteq \overline{\mathcal{C}_{\mu_k} \setminus \big((\mu_k, 0) \cup \big(\mathcal{S} \cap \big(\mathbb{B}_\delta(\mu_k, 0) \cap K_\eta^-\big)\big)\big)} \supseteq T.$$

Therefore, $\mathcal{C}_k^+ \cap \mathcal{C}_k^- \neq \{(\mu_k, 0)\}$. Similar argument could be used for $\mathcal{C}_k^+$. The proof is complete. □

Remark 3.2. From the proof of Theorem 3.1, we can see that, if $\mathcal{C}_k^+$ or $\mathcal{C}_k^-$ is bounded, there exists a connected set $Q \subseteq (\mathcal{C}_k^+ \cap \mathcal{C}_k^-) \setminus \{(\mu_k, 0)\}$ such that $Q \cap (\mathbb{R} \times (E_0 \setminus \{0\})) \neq \emptyset$. Indeed, we can show that $T \cap \partial\mathbb{B}_\delta(\mu_k, 0)\,|_{E_0} \neq \emptyset$, where $\partial\mathbb{B}_\delta(\mu_k, 0)\,|_{E_0}$ denotes the restriction of $\partial\mathbb{B}_\delta(\mu_k, 0)$ on E_0. Suppose on the contrary, if $T \cap \partial\mathbb{B}_\delta(\mu_k, 0)\,|_{E_0} = \emptyset$, $T \cap \partial\mathbb{B}_\delta(\mu_k, 0) \neq \emptyset$ implies that $u = s\varphi_k$ for any $u \in T \cap \partial\mathbb{B}_\delta(\mu_k, 0)$. However, that is impossible if we take $|s|$ small enough. Therefore, the result of Theorem 3.1 is stronger than that of Theorem 6.4.3 of [20].

Connecting Theorem 2.2 with Theorem 3.1, we can easily deduce the following unilateral global bifurcation result.

Theorem 3.2. *Let $\nu \in \{+, -\}$. Then $\mathcal{C}_k^\nu$ is unbounded in $\mathbb{R} \times E$ and*

$$\mathcal{C}_k^\nu \subset \big\{(\mu_k, 0)\big\} \cup \big(\mathbb{R} \times S_k^\nu\big) \quad \textit{or} \quad \mathcal{C}_k^\nu \subset \big\{(\mu_k, 0)\big\} \cup \big(\mathbb{R} \times S_k^{-\nu}\big). \tag{3.8}$$

Proof. By Theorem 2.2, we can get (3.8) easily. We only need to prove that both $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$ are unbounded. Suppose on the contrary, without loss of generality, we may suppose that $\mathcal{C}_k^-$ is bounded. By Theorem 3.1, we know that $(\mathcal{C}_k^- \cap \mathcal{C}_k^+) \setminus \{(\mu_k, 0)\} \neq \emptyset$. Therefore, in view of (3.8), there exists $(\mu_*, u_*) \in \mathcal{C}_k^- \cap \mathcal{C}_k^+$ such that $(\mu_*, u_*) \neq (\mu_k, 0)$ and $u_* \in S_k^+ \cap S_k^-$. This contradicts the definitions of S_k^+ and S_k^-. □

Remark 3.3. It is easy to see that the results of Theorem 3.2 remain true even if "$m(t)$ is a continuous weight function" is substituted by "$m(t)$ is a measurable weight function".

4. Nodal solutions of one-dimensional *p*-Laplacian with crossing nonlinearity

We use Theorem 3.2 to prove the existence of nodal solutions for problem (1.3) with crossing nonlinearity.

In this section, we suppose that

(H_1) $\lambda_k(p) \leqslant a(t) \equiv \lim_{|s|\to+\infty} \frac{f(t,s)}{\varphi_p(s)}$ *uniformly on* $[0,1]$, *and the inequality is strict on some subset of positive measure in* $(0,1)$;

(H_2) $0 \leqslant \lim_{|s|\to 0} \frac{f(t,s)}{\varphi_p(s)} \equiv c(t) \leqslant \lambda_k(p)$ *uniformly on* $[0,1]$, *and all the inequalities are strict on some subset of positive measure in* $(0,1)$;

(H_3) $f(t,s)\varphi_p(s) > 0$ *for a.e.* $t \in (0,1)$ *and* $s \neq 0$.

Remark 4.1. From (H_1)–(H_3), we can see that there exist a positive constant ϱ and a subinterval $[\alpha,\beta]$ of $(0,1)$ such that $\frac{f(t,s)}{\varphi_p(s)} \geqslant \varrho$ for all $t \in [\alpha,\beta]$ and $s \neq 0$.

The main results of this section are the following:

Theorem 4.1. *Suppose that* $f(t,u)$ *satisfies* (H_1), (H_2) *and* (H_3), *then problem* (1.3) *possesses two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0, *and* u_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Similarly, we also have the following:

Theorem 4.2. *Suppose that* $f(t,u)$ *satisfies* (H_3) *and*

(H_1') $\lambda_k(p) \geqslant a(x) \equiv \lim |s| \to +\infty \frac{f(t,s)}{\varphi_p(s)} \geqslant 0$ *uniformly on* $[0,1]$, *and all the inequalities are strict on some subset of positive measure in* $(0,1)$;

(H_2') $\lim_{|s|\to 0} \frac{f(t,s)}{\varphi_p(s)} \equiv c(x) \geqslant \lambda_k(p)$ *uniformly on* $[0,1]$, *and the inequality is strict on some subset of positive measure in* $(0,1)$, *then problem* (1.3) *possesses two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0, *and* u_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Remark 4.2. We would like to point out that even in the semilinear case, the assumptions (H_1) and (H_2) are weaker than the corresponding conditions of Theorem 1.1 of [23]. In fact, if we let $f(t,s) \equiv rm(t)f(s)$, then we can get $\lim_{|s|\to+\infty} \frac{f(t,s)}{\varphi_p(s)} \equiv rm(t)f_\infty := a(t)$ and $\lim_{|s|\to 0} \frac{f(t,s)}{\varphi_p(s)} \equiv rm(t)f_0 := c(t)$. By the strict decreasing of $\widetilde{\lambda}_k(f)$ with respect to weight function f (see [2]), we can show that our condition $c(t) \leqslant \lambda_k(p) \leqslant a(t)(\equiv a(t))$ is equivalent to the condition $\frac{\widetilde{\lambda}_k}{f_\infty} < r < \frac{\widetilde{\lambda}_k}{f_0}$. Similarly, our condition $c(t) \geqslant \lambda_k(p) \geqslant a(t)(\equiv a(t))$ is equivalent to the condition $\frac{\widetilde{\lambda}_k}{f_0} < r < \frac{\widetilde{\lambda}_k}{f_\infty}$. Therefore, Theorem 1.1 of [2] is the corollary of Theorems 4.1 and 4.2, even in the case of $p = 2$.

We only prove Theorem 4.1, since the proof of Theorem 4.2 is similar.

Proof of Theorem 4.1. Firstly, we study the bifurcation phenomena for the following p-Laplacian eigenvalue problem with crossing nonlinearity

$$\begin{cases} \left(\varphi_p(u')\right)' + \mu f(t,u) = 0, & \text{a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where $\mu > 0$ is a parameter.

Let $\zeta \in C((0,1)\times\mathbb{R})$ be such that

$$f(t,u) = c(t)\varphi_p(u) + \zeta(t,u)$$

with

$$\lim_{|u|\to 0} \frac{\zeta(t,u)}{\varphi_p(u)} = 0 \quad \text{uniformly on } [0,1]. \tag{4.2}$$

Hence, the condition (1.2) holds. Using Theorem 3.2, we have that there are two distinct unbounded continua, $\mathcal{C}_k^+$ and $\mathcal{C}_k^-$, consisting of the bifurcation branch $\mathcal{C}_k$ from $(\mu_k(p), 0)$, such that

$$\mathcal{C}_k^\nu \subset \{(\mu_k, 0)\} \cup (\mathbb{R} \times S_k^\nu) \quad \text{or} \quad \mathcal{C}_k^\nu \subset \{(\mu_k, 0)\} \cup (\mathbb{R} \times S_k^{-\nu}).$$

It is clear that any solution of (4.1) of the form $(1, u)$ yields a solution u of (1.3). We will show $\mathcal{C}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

By the strict decreasing of $\mu_k(c(t))$ with respect to $c(t)$ (see [2]), where $\mu_k(c(t))$ is the k-th eigenvalue of (2.2) corresponding to the weight function $c(t)$, we have $\mu_k(c(t)) > \mu_k(\lambda_k(p)) = 1$.

Let $(\mu_n, y_n) \in \mathcal{C}_k^\nu$ where $y_n \not\equiv 0$ satisfies

$$\mu_n + \|y_n\| \to +\infty.$$

We note that $\mu_n > 0$ for all $n \in \mathbb{N}$, since $(0, 0)$ is the only solution of (1.3) for $\mu = 0$ and $\mathcal{C}_k^\nu \cap (\{0\} \times E) = \emptyset$.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$\mu_n \subset (0, M]$$

for $n \in \mathbb{N}$ large enough, then $\mathcal{C}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

In this case it follows that

$$\|y_n\| \to +\infty.$$

Let $\xi \in C((0, 1) \times \mathbb{R})$ be such that

$$f(t, u) = a(t)\varphi_p(u) + \xi(t, u)$$

with

$$\lim_{|u| \to +\infty} \frac{\xi(t, u)}{\varphi_p(u)} = 0 \quad \text{uniformly on } [0, 1]. \tag{4.3}$$

We divide the equation

$$-\big(\varphi_p(y_n')\big)' - \mu_n a(t)\varphi_p(y_n) = \mu_n \xi(t, y_n)$$

by $\|y_n\|$ and set $\overline{y}_n = \frac{y_n}{\|y_n\|}$. Since $\overline{y}_n$ is bounded in E, after taking a subsequence if necessary, we have that $\overline{y}_n \rightharpoonup \overline{y}$ for some $\overline{y} \in E$ and $\overline{y}_n \to \overline{y}$ in Y with $\|\overline{y}\| = 1$. By (4.3), using the similar proof of (2.7), we have that

$$\lim_{n \to +\infty} \frac{\xi(t, y_n(t))}{\|y_n\|^{p-1}} = 0 \quad \text{in } Y.$$

By the compactness of G_p we obtain

$$-\big(\varphi_p(\overline{y}')\big)' - \overline{\mu} a(t)\varphi_p(\overline{y}) = 0,$$

where $\overline{\mu} = \lim_{n \to +\infty} \mu_n$, again choosing a subsequence and relabeling if necessary.

It is clear that $\bar{y} \in \overline{\mathcal{C}_k^\nu} \subseteq \mathcal{C}_k^\nu$ since $\mathcal{C}_k^\nu$ is closed in $\mathbb{R} \times E$. Therefore, $\overline{\mu}(a(t))$ is the k-th eigenvalue of

$$\begin{cases} \big(\varphi_p\big(u'(t)\big)\big)' + \mu a(t)\varphi_p\big(u(t)\big) = 0 \quad \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

By the strict decreasing of $\overline{\mu}(a(t))$ with respect to $a(t)$ (see [2]), where $\overline{\mu}(a(t))$ is the k-th eigenvalue corresponding to the weight function $a(t)$, we have $\overline{\mu}(a(t)) < \overline{\mu}(\lambda_k(p)) = 1$. Therefore, $\mathcal{C}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Step 2. We show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that

$$\lim_{n\to+\infty} \mu_n = +\infty.$$

On the other hand, we note that

$$-\big(\varphi_p\big(y_n'\big)\big)' = \mu_n \frac{f(t, y_n)}{\varphi_p(y_n)} \varphi_p(y_n).$$

In view of Remark 4.1, we have $\mu_n \frac{f(t,y_n)}{\varphi_p(y_n)} > \lambda_k(p)$ for n large enough and all $t \in [\alpha, \beta]$. By Lemma 2.5 of [18] on $[\alpha, \beta]$, we get y_n must change its sigh more than $k-1$ times in (α, β) for n large enough, which contradicts the fact that $y_n \in S_k^{\pm}$.

Therefore,

$$\mu_n \leqslant M.$$

for some constant number $M > 0$ and $n \in \mathbb{N}$ sufficiently large. □

5. Constant sign solutions for high-dimensional p-Laplacian with crossing nonlinearity

In this section, based on the bifurcation result of Girg and Takáč [13], we will study the existence of constant sign solutions for problem (1.6). From now on, for simplicity, we write $X := W_0^{1,p}(\Omega)$.

Similarly with the assumptions of Theorem 4.1, we suppose that

(f_1) $\lambda_1(p) \leqslant a_\pm(x) \equiv \lim_{s\to\pm\infty} \frac{f(x,s)}{\varphi_p(s)}$ *uniformly on* $\overline{\Omega}$, *and the inequality is strict on some subset of positive measure in* Ω;

(f_2) $0 \leqslant \lim_{|s|\to 0} \frac{f(x,s)}{\varphi_p(s)} \equiv c(x) \leqslant \lambda_1(p)$ *uniformly on* $\overline{\Omega}$, *and all the inequalities are strict on some subset of positive measure in* Ω;

(f_3) $f(x,s)\varphi_p(s) > 0$ *for a.e.* $x \in \Omega$ *and* $s \neq 0$.

The main results of this section are the following:

Theorem 5.1. *Suppose that* $f(x,u)$ *satisfies* (f_1), (f_2) *and* (f_3), *then problem* (1.6) *possesses at least a positive and a negative solution.*

Analogously, we also have the following:

Theorem 5.2. *Suppose that* $f(x,u)$ *satisfies* (f_3) *and*

(f_1') $\lambda_1(p) \geqslant a_\pm(x) \equiv \lim_{s\to\pm\infty} \frac{f(x,s)}{\varphi_p(s)} \geqslant 0$ *uniformly on* $\overline{\Omega}$, *and all the inequalities are strict on some subset of positive measure in* Ω;

(f_2') $\lim_{|s|\to 0} \frac{f(x,s)}{\varphi_p(s)} \equiv c(x) \geqslant \lambda_1(p)$ *uniformly on* $\overline{\Omega}$, *and the inequality is strict on some subset of positive measure in* Ω, *then problem* (1.6) *possesses at least a positive and a negative solution.*

Remark 5.1. We note that the assumption (f_1) is weaker than the condition (4) of [12] because we don't require $\lim_{s\to\pm\infty} \frac{f(x,s)}{\varphi_p(s)} \leqslant \lambda_2(p)$ which is essential in [12]. And the assumptions (f_1) and (f_2) are weaker than the conditions (F_1) and (F_2) of [16] even in the case of $f \geqslant 0$. Moreover, it is obvious that our results are better than the results of [12,16].

Remark 5.2. By the $C^{1,\alpha}$ $(0<\alpha<1)$ regularity results for quasilinear elliptic equations with p-growth condition [19], $u \in C^{1,\alpha}(\overline{\Omega})$ for any solution u of (1.6) since f is continuous and subcritical.

The existence of constant sign solutions of (1.6) is related to the following p-Laplacian eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(\varphi_p(\nabla u)\big) = \mu f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\mu > 0$ is a parameter. Therefore, we will study the bifurcation phenomena for (5.1) with crossing nonlinearity. Moreover, the bifurcation points of (5.1) are related to the eigenvalues of the problem

$$\begin{cases} \operatorname{div}\big(\varphi_p\big(\nabla u(x)\big)\big) + \mu c(x)\varphi_p\big(u(x)\big) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

It is well known that there exists a principle eigenvalue $\mu_1(p)$ of (5.2) (see [1] or [10]).

Lemma 5.1. *Assume* (f_1) *and* (f_2) *hold. Then for* $p > 1$, μ_1 *is a bifurcation point of* (5.1) *and the associated bifurcation branch* $\mathcal{C}_1$ *in* $\mathbb{R} \times X$ *whose closure contains* $(\mu_1, 0)$ *is either unbounded or contains a pair* $(\overline{\mu}, 0)$ *where* $\overline{\mu}$ *is a eigenvalue of* (5.2) *and* $\overline{\mu} \neq \mu_1$.

Proof. Let $\vartheta \in C(\Omega \times \mathbb{R})$ be such that

$$f(x,u) = c(x)\varphi_p(u) + \vartheta(x,u)$$

with

$$\lim_{|u|\to 0} \frac{\vartheta(x,u)}{\varphi_p(u)} = 0 \quad \text{and} \quad \lim_{u\to\pm\infty} \frac{\vartheta(x,u)}{\varphi_p(u)} = a_\pm(x) - c(x) \quad \text{uniformly on } \overline{\Omega}. \tag{5.3}$$

From (5.3), we can see that $\vartheta(x,u)$ satisfies the Hypotheses (H_0) of [13]. Now, Proposition 3.5 can be applied to get the results of this lemma. $\square$

Let $\mathcal{C}_1^\nu$ be the component from Theorem 3.7 of [13], we have known that $\mathcal{C}_1 = \mathcal{C}_1^+ \cup \mathcal{C}_1^-$ (see [13]). The following result plays a fundamental role in our study.

Lemma 5.2. *(See [13].) Either* $\mathcal{C}_1^+$ *and* $\mathcal{C}_1^-$ *are both unbounded, or else* $\mathcal{C}_1^+ \cup \mathcal{C}_1^- \neq \{(0, \mu_1)\}$.

Finally, let

$$\mathbb{S}_1^+ = \big\{u \in C^{1,\alpha}(\overline{\Omega}):\ u(x) > 0,\ \text{for all } x \in \Omega\big\}$$

and

$$\mathbb{S}_1^- = \{u \in C^{1,\alpha}(\overline{\Omega}):\ u(x) < 0,\ \text{for all } x \in \Omega\}.$$

Using the same method to prove [8, Lemma 3.1] with obvious changes, we may obtain the following result.

Lemma 5.3. *Let (f_1) and (f_2) hold. Then we have*

$$\mathcal{C}_1 \subset \{(\mu_1(p), 0)\} \cup (\mathbb{R} \times \mathbb{S}_1^{\pm})$$

and $\mathcal{C}_1$ is unbounded in $\mathbb{R} \times X$.

Connecting Lemma 5.2 with Lemma 5.3, then applying the similar method to prove Theorem 3.2 with obvious changes, we may obtain the following unilateral global bifurcation result.

Lemma 5.4. *Let $\nu \in \{+, -\}$. Then $\mathcal{C}_1^\nu$ is unbounded in $\mathbb{R} \times X$ and*

$$\mathcal{C}_1^\nu \subset \{(\mu_1(p), 0)\} \cup (\mathbb{R} \times \mathbb{S}_1^\nu) \quad \textit{or} \quad \mathcal{C}_1^\nu \subset \{(\mu_1(p), 0)\} \cup (\mathbb{R} \times \mathbb{S}_1^{-\nu}). \tag{5.4}$$

We use Lemma 5.4 to prove the main results of this section. We only prove Theorem 5.1, since the proof of Theorem 5.2 is similar.

Proof of Theorem 5.1. Since the proof is similar to that of Theorem 4.1, we only give a rough sketch of the proof. It is clear that any solution of (5.1) of the form $(1, u)$ yields a solution u of (1.6). We will show $\mathcal{C}_1^\nu$ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$.

Firstly, for simplicity, we write $\mu_1 = \mu_1(p)$. By the strict decreasing of $\mu_1(f)$ with respect to f (see [5]), where $\mu_1(f)$ is the principal eigenvalue corresponding to the weight function f, we have $\mu_1(c(x)) > \mu_1(\lambda_1(p)) = 1$.

Let $(\mu_n, y_n) \in \mathcal{C}_1^\nu$ where $y_n \not\equiv 0$ satisfies

$$\mu_n + \|y_n\|_X \to +\infty.$$

We note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of (5.1) for $\lambda = 0$ and $\mathcal{C}_1^\nu \cap (\{0\} \times X) = \emptyset$.

Using a similar method as that of the proof of Theorem 4.1, we can show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough. It follows that

$$\|y_n\|_X \to +\infty.$$

Let $\varpi \in C(\Omega \times \mathbb{R})$ be such that

$$f(x, u) = a_\pm(x)\varphi_p(u) + \varpi(x, u)$$

with

$$\lim_{u \to \pm\infty} \frac{\varpi(x, u)}{\varphi_p(u)} = 0 \quad \text{and} \quad \lim_{|u| \to 0} \frac{\varpi(x, u)}{\varphi_p(u)} = c(x) - a_\pm(x) \quad \text{uniformly on } \overline{\Omega}. \tag{5.5}$$

We divide the equation

$$-\mathrm{div}\big(\varphi_p(\nabla y_n)\big) - \mu_n a_\pm(x)\varphi_p(y_n) = \mu_n \varpi(x, y_n)$$

by $\|y_n\|_X$ and set $\bar{y}_n = \frac{y_n}{\|y_n\|_X}$. Since $\bar{y}_n$ is bounded in X, after taking a subsequence if necessary, we have that $\bar{y}_n \rightharpoonup \bar{y}$ for some $\bar{y} \in X$ and $\bar{y}_n \to \bar{y}$ in $L^{p'}(\Omega)$ with $\|\bar{y}\|_X = 1$.

From (5.5), we can see that

$$\lim_{|u|\to 0} \frac{\varpi(x,u)}{\varphi_p(u)} = 0 \quad \text{and} \quad \lim_{|u|\to+\infty} \frac{\varpi(x,u)}{|u|^p} = 0 \quad \text{uniformly on } \overline{\Omega}.$$

It follows that for any $\varepsilon > 0$, there exists a constant C such that

$$\big|\varpi(x,u_n)\big| \leqslant \varepsilon |u_n|^{p-1} + C|u_n|^p. \tag{5.6}$$

By (5.6), we can easily show that

$$\lim_{n\to+\infty} \frac{\varpi(x, y_n(t))}{\|y_n\|_X^{p-1}} = 0 \quad \text{in } L^{p'}(\Omega),$$

where $p' = \frac{p}{p-1}$. By the compactness of $G_p : L^{p'}(\Omega) \to X$ (see [8]) we obtain

$$-\mathrm{div}\big(\varphi_p(\nabla \bar{y})\big) - \big(\bar{\mu} a_\pm(x)\varphi_p(\bar{y})\big) = 0,$$

where $\bar{\mu} = \lim_{n\to+\infty} \mu_n$, again choosing a subsequence and relabeling if necessary.

It is clear that $\bar{y} \in \overline{\mathcal{C}_1^\nu} \subseteq \mathcal{C}_1^\nu$ since $\mathcal{C}_1^\nu$ is closed in $\mathbb{R} \times X$. Therefore $\bar{y} \neq 0$, i.e., $\bar{\mu}(a_\pm(x))$ is the eigenvalue of (5.2). By the strict decreasing of $\bar{\mu}(f)$ with respect to f (see [5]), where $\bar{\mu}(f)$ is the principal eigenvalue corresponding to the weight function f, we have $\bar{\mu}(a_\pm(x)) < \bar{\mu}(\lambda_1(p)) = 1$. Therefore, $\mathcal{C}_1^\nu$ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$. □

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References

[1] A. Anane, Simplicité et isolation de la première valeur propre du p-Laplacien avec poids, C. R. Acad. Sci. Paris Sér. I 305 (1987) 725–728.
[2] A. Anane, O. Chakrone, M. Monssa, Spectrum of one dimensional p-Laplacian with indefinite weight, Electron. J. Qual. Theory Differ. Equ. 2002 (17) (2002), 11 pp.
[3] J. Blat, K.J. Brown, Bifurcation of steady state solutions in predator prey and competition systems, Proc. Roy. Soc. Edinburgh Sect. A 97 (1984) 21–34.
[4] H. Brezis, Operateurs Maximaux Monotone et Semigroup de Contractions dans les Espase de Hilbert, Math. Stud., vol. 5, North-Holland, Amsterdam, 1973.
[5] K.J. Brown, C. Cosner, J. Fleckinger, Principal eigenvalues for problems with indefinite weight function in $\mathbb{R}^N$, Proc. Amer. Math. Soc. 109 (1990) 147–155.
[6] E.N. Dancer, On the structure of solutions of non-linear eigenvalue problems, Indiana Univ. Math. J. 23 (1974) 1069–1076.
[7] E.N. Dancer, Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one, Bull. Lond. Math. Soc. 34 (2002) 533–538.
[8] M. Del Pino, R. Manásevich, Global bifurcation from the eigenvalues of the p-Laplacian, J. Differential Equations 92 (1991) 226–251.
[9] M. Del Pino, M. Elgueta, R. Manásevich, A homotopic deformation along p of a Leray–Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, J. Differential Equations 80 (1989) 1–13.
[10] P. Drábek, The least eigenvalue of nonhomogeneous degenerated quasilinear eigenvalue problems, Math. Bohemia 120 (2) (1995) 169–195.

[11] L.H. Erbe, H.Y. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (3) (1994) 743–748.
[12] X.L. Fan, Y.Z. Zhao, G.F. Huang, Existence of solutions for the p-Laplacian with crossing nonlinearity, Discrete Contin. Dyn. Syst. 5 (4) (2002) 1–24.
[13] P. Girg, P. Takáč, Bifurcations of positive and negative continua in quasilinear elliptic eigenvalue problems, Ann. Henri Poincaré 9 (2008) 275–327.
[14] B. Im, E. Lee, Y.H. Lee, A global bifurcation phenomena for second order singular boundary value problems, J. Math. Anal. Appl. 308 (2005) 61–78.
[15] J. Henderson, H.Y. Wang, Positive solutions for nonlinear eigenvalue problems, J. Math. Anal. Appl. 208 (1997) 252–259.
[16] Y.S. Huang, H.S. Zhou, Positive solution for $-\Delta_p u = f(x,u)$ with $f(x,u)$ growing as u^{p-1} at infinity, Appl. Math. Lett. 17 (2004) 881–887.
[17] Y.H. Lee, I. Sim, Global bifurcation phenomena for singular one-dimensional p-Laplacian, J. Differential Equations 229 (2006) 229–256.
[18] Y.H. Lee, I. Sim, Existence results of sign-changing solutions for singular one-dimensional p-Laplacian problems, Nonlinear Anal. 68 (2008) 1195–1209.
[19] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[20] J. López-Gómez, Spectral Theory and Nonlinear Functional Analysis, Chapman and Hall/CRC, Boca Raton, 2001.
[21] J. López-Gómez, Nonlinear eigenvalues and global bifurcation theory. Application to the search of positive solutions for general Lotka–Volterra reaction diffusion systems, Differential Integral Equations 7 (1984) 1427–1452.
[22] R. Ma, Bifurcation from infinity and multiple solutions for periodic boundary value problems, Nonlinear Anal. 42 (2000) 27–39.
[23] R. Ma, B. Thompson, Nodal solutions for nonlinear eigenvalue problems, Nonlinear Anal. 59 (2004) 707–718.
[24] R. Ma, B. Thompson, Multiplicity results for second-order two-point boundary value problems with superlinear or sublinear nonlinearities, J. Math. Anal. Appl. 303 (2005) 726–735.
[25] R. Ma, B. Thompson, Global behavior of positive solutions of nonlinear three-point boundary value problems, Nonlinear Anal. 60 (2005) 685–701.
[26] R. Manásevich, J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations 145 (2) (1998) 367–393.
[27] I. Peral, Multiplicity of solutions for the p-Laplacian, ICTP SMR 990/1, 1997.
[28] P.H. Rabinowitz, Nonlinear Sturm–Liouville problems for second order ordinary differential equations, Comm. Pure Appl. Math. 23 (1970) 939–961.
[29] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.
[30] P.H. Rabinowitz, Some aspects of nonlinear eigenvalue problems, Rocky Mountain J. Math. 3 (2) (1973) 161–202.
[31] J.T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach, New York, 1969.
[32] G.T. Whyburn, Topological Analysis, Princeton University Press, Princeton, NJ, 1964.
[33] M.R. Zhang, The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, J. Lond. Math. Soc. (2) 64 (2001) 125–143.