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Optimal time decay of the quantum Landau equation in the whole space

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A B S T R A C T

We are concerned with the Cauchy problem of the quantum Landau equation in the whole space. The existence of local in time nearby quantum Maxwellian solutions is proved by the iteration method and generalized maximum principle. Based on Kawashima's compensating function and nonlinear energy estimates, the global existence and the optimal time decay rate of those solutions are obtained under some conditions on initial data.

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1. Introduction and the statement of our main results

The classical Landau equation is given by

$$\partial_t F + v \cdot \nabla_x F = \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \psi(v-u)\big[F(u)\nabla_v F(v) - F(v)\nabla_u F(u)\big]\,du \right\},$$

where $F(t,x,v) \geqslant 0$ is the distribution function for the particles. The non-negative matrix ψ is defined as

$$\psi^{ij}(v) = \left\{ \delta^{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^{\gamma+2}. \tag{1.1}$$

It is well known in the physics literature that the classical Landau collision operator can be formally derived from the Boltzmann operator when the collision between particles become grazing. However, if we want to describe a gas of Fermi–Dirac particles, one has to modify the classical Boltzmann equation collision integral in order to take into account quantum effects. As a consequence of the Pauli exclusion principle, two particles are not uncorrelated any longer before and after collision. Thus the classical Boltzmann collision operator has to be replaced by the quantum Boltzmann collision operator. Using the quantum Boltzmann collision operator and passing to the grazing collision limit lead to the following quantum Landau equation

$$\partial_t F + v \cdot \nabla_x F = Q(F,F), \tag{1.2}$$

with

$$\begin{aligned} Q(F,G) &= \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \psi(v-u)\big[\big(1-F(u)\big)F(u)\nabla_v G(v) - \big(1-G(v)\big)G(v)\nabla_u F(u)\big]\,du \right\} \\ &= \sum_{i,j=1}^{3} \partial_{v_i} \int_{\mathbf{R}^3} \psi^{ij}(v-u)\big[\big(1-F(u)\big)F(u)\partial_{v_j} G(v) - \big(1-G(v)\big)G(v)\partial_{u_j} F(u)\big]\,du. \end{aligned} \tag{1.3}$$

The non-negative matrix ψ is given by (1.1), we recall that γ is a parameter leading to the standard classification of the hard potential ($\gamma > 0$), Maxwellian molecule ($\gamma = 0$) or soft potential ($\gamma < 0$), cf. [6]. In this paper, we consider the Cauchy problem of the quantum Landau equation (1.2) with initial data

$$F(0,x,v) = F_0(x,v), \quad x \in \mathbf{R}^3,\ v \in \mathbf{R}^3. \tag{1.4}$$

We denote a normalized global Maxwellian by

$$M(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\big(-|v|^2/2\big).$$

We also define the quantum Maxwellian $M_q = M_q(v)$ as

$$\frac{M(v)}{1+M(v)}.$$

Throughout this paper, we consider solutions which are perturbations near the quantum Maxwellian. We write the distribution $F(t,x,v)$ as a perturbation of M_q

$$F = M_q + \sqrt{\tilde{M}} f,$$

where a suitable choice of $\tilde{M}$ is

$$\tilde{M}(v) = \frac{M(v)}{(1+M(v))^2} = M_q(v)\big(1 - M_q(v)\big),$$

then the quantum Landau equation (1.2) for $f(t,x,v)$ takes the form

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f), \tag{1.5}$$

with $f(0,x,v) = f_0(x,v)$. The linearized collision operator L is defined as

$$\begin{aligned} Lf &= -\frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \psi(v-u)\big[\big(1 - M_q(u)\big) M_q(u) \nabla_v\big(\sqrt{\tilde{M}} f(v)\big) \right. \\ &\quad - \big(1 - M_q(v)\big) M_q(v) \nabla_u\big(\sqrt{\tilde{M}} f(u)\big) + \sqrt{\tilde{M}} f(u)\big(1 - 2M_q(u)\big) \nabla M_q(v) \\ &\quad \left. - \sqrt{\tilde{M}} f(v)\big(1 - 2M_q(v)\big) \nabla M_q(u)\big]\, du \right\} \\ &= -\frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \psi(v-u) \tilde{M}(u) \tilde{M}(v) \big\{ \nabla_v\big(\tilde{M}^{-\frac{1}{2}} f\big)(v) - \nabla_u\big(\tilde{M}^{-\frac{1}{2}} f\big)(u)\big\}\, du \right\} \\ &= Af + Kf, \end{aligned} \tag{1.6}$$

where

$$Af = -\frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \tilde{M}(v) \nabla_v\big(\tilde{M}^{-\frac{1}{2}} f\big)(v) \int_{\mathbf{R}^3} \psi(v-u) \tilde{M}(u)\, du \right\}, \tag{1.7}$$

$$Kf = \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \tilde{M}(v) \int_{\mathbf{R}^3} \psi(v-u) \tilde{M}(u) \nabla_u\big(\tilde{M}^{-\frac{1}{2}} f\big)(u)\, du \right\}, \tag{1.8}$$

and the collision operator $\Gamma(f,g)$ can be split into

$$\Gamma(f,g) = \Gamma_{bi}(f,g) + \Gamma_{non}(f,g),$$

where Γ_{bi} and Γ_{non} are given by

$$\begin{aligned} \Gamma_{bi}[f,g] &= \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \nabla_v(\sqrt{\tilde{M}} g)(v) \int_{\mathbf{R}^3} \psi(v-u)(\sqrt{\tilde{M}} f)(u)\, du \right\} \\ &\quad - \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \nabla_v(\sqrt{\tilde{M}} g)(v) \int_{\mathbf{R}^3} \psi(v-u)(2M_q \sqrt{\tilde{M}} f)(u)\, du \right\} \end{aligned}$$

$$- \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ (\sqrt{\tilde{M}} g)(v) \int_{\mathbf{R}^3} \psi(v-u) \nabla_u (\sqrt{\tilde{M}} f)(u)\, du \right\}$$

$$+ \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ (2M_q \sqrt{\tilde{M}} g)(v) \int_{\mathbf{R}^3} \psi(v-u) \nabla_u (\sqrt{\tilde{M}} f)(u)\, du \right\} \tag{1.9}$$

and

$$\begin{aligned} \Gamma_{non}[f,g] = &- \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \nabla_v (\sqrt{\tilde{M}} g)(v) \int_{\mathbf{R}^3} \psi(v-u) \big(\tilde{M} f^2\big)(u)\, du \right\} \\ &- \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \nabla_v M_q(v) \int_{\mathbf{R}^3} \psi(v-u) \big(\tilde{M} f^2\big)(u)\, du \right\} \\ &+ \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \big(\tilde{M} g^2\big)(v) \int_{\mathbf{R}^3} \psi(v-u) \nabla_u (\sqrt{\tilde{M}} f)(u)\, du \right\} \\ &+ \frac{1}{\sqrt{\tilde{M}}} \nabla_v \cdot \left\{ \big(\tilde{M} g^2\big)(v) \int_{\mathbf{R}^3} \psi(v-u) \nabla_u M_q(u)\, du \right\} \end{aligned} \tag{1.10}$$

respectively.

For any fixed (t,x), the null space of L is the five dimensional space generated by

$$\mathscr{N} = span\big\{ \sqrt{\tilde{M}}, v_i \sqrt{\tilde{M}}, |v|^2 \sqrt{\tilde{M}} \big\}, \tag{1.11}$$

where $1 \leqslant i \leqslant 3$, cf. [17]. For any fixed (t,x), and any function $f(t,x,v)$, we define $\mathbf{P}$ as its v-projection in L^2_v, to the null space $\mathscr{N}$. We then decompose $f(t,x,v)$ uniquely as

$$f = \mathbf{P} f + (\mathbf{I} - \mathbf{P}) f. \tag{1.12}$$

As in [3,11,12,17], $\mathbf{P} f$ is the hydrodynamic, and $(\mathbf{I} - \mathbf{P}) f$ the microscopic part. We can further denote

$$\mathbf{P} f = \big\{ a_f(t,x) + v \cdot b_f(t,x) + |v|^2 c_f(t,x) \big\} \sqrt{\tilde{M}}. \tag{1.13}$$

For notational simplicity, we use $\langle \cdot, \cdot \rangle$ to denote the L^2 inner product in $\mathbf{R}^3_v$, with its L^2 norm given by $|\cdot|_2$, and we use $|\cdot|_\infty$ to denote the L^∞ norm in $\mathbf{R}^3_v$, moreover $(\cdot,\cdot)$ is L^2 inner product either in $\mathbf{R}^3_x \times \mathbf{R}^3_v$ or in $\mathbf{R}^3_x$ with corresponding L^2 norm $\|\cdot\|$, we also use $\|\cdot\|_\infty$ to denote the L^∞ norm in $\mathbf{R}^3_x \times \mathbf{R}^3_v$ or in $\mathbf{R}^3_x$. We use the standard notation H^s or $W^{k,p}$ to denote the Sobolev space. For an integrable function $g : \mathbf{R}^3 \to \mathbf{R}$, its Fourier transform $\hat{g}$ is defined by

$$\hat{g}(\xi) = \int_{\mathbf{R}^3} e^{-2\pi \mathrm{i} x \cdot \xi} g(x)\, dx,$$

for $\xi \in \mathbf{R}^3$, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

In order to obtain the optimal time decay of the global solution to (1.5), we introduce the space $Z_q = L^2(\mathbf{R}^3_v; L^q(\mathbf{R}^3_x))$. Its norm is defined by

$$\|f\|_{Z_q} = \left(\int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} \left|f(x,v)\right|^q dx \right)^{\frac{2}{q}} dv \right)^{\frac{1}{2}}.$$

Let

$$\sigma^{ij} = \int_{\mathbf{R}^3} \psi^{ij}(v-u)\tilde{M}(u)\,du,$$

and

$$\sigma^i = v_j \int_{\mathbf{R}^3} \psi^{ij}(v-u)\tilde{M}(u)\,du = \int_{\mathbf{R}^3} \psi^{ij}(v-u)u_j\tilde{M}(u)\,du.$$

We introduce a weight function as

$$w = w(v) = \left(1+|v|^2\right)^{\frac{\gamma+2}{2}}.$$

We denote the weighted L^2-norm

$$|g|^2_{2,\theta} = \int_{\mathbf{R}^3} w^{2\theta} g^2\,dv, \qquad \|g\|^2_\theta = \int_{\mathbf{R}^3\times\mathbf{R}^3} w^{2\theta} g^2\,dx\,dv.$$

And it is natural to define the following weighted σ-norm to characterize the dissipation rate

$$|g|^2_{\sigma,\theta} = \sum_{i,j=1}^3 \int_{\mathbf{R}^3} w^{2\theta} \left\{ \sigma^{ij}\partial_i g \partial_j g + \frac{1}{4}\left(\frac{M(v)-1}{M(v)+1}\right)^2 \sigma^{ij} v_i v_j g^2 \right\} dv,$$

$$\|g\|^2_{\sigma,\theta} = \sum_{i,j=1}^3 \int_{\mathbf{R}^3\times\mathbf{R}^3} w^{2\theta} \left\{ \sigma^{ij}\partial_i g \partial_j g + \frac{1}{4}\left(\frac{M(v)-1}{M(v)+1}\right)^2 \sigma^{ij} v_i v_j g^2 \right\} dx\,dv.$$

We note that

$$\|g\|^2_{\sigma,\theta} \sim \sum_{i,j=1}^3 \int_{\mathbf{R}^3\times\mathbf{R}^3} w^{2\theta} \left\{ \sigma^{ij}\partial_i g \partial_j g + \frac{1}{4}\sigma^{ij} v_i v_j g^2 \right\} dx\,dv,$$

since $0 \leqslant M(v) \leqslant (2\pi)^{-\frac{3}{2}}$.

Let $\alpha = [\alpha_1, \alpha_2, \alpha_3]$, $\beta = [\beta_1, \beta_2, \beta_3]$ denote multi-indices with length $|\alpha|$ and $|\beta|$ respectively, and let

$$\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

Furthermore, define $\beta_1 \leqslant \beta$ if no component of β_1 is greater than the component of β, and $\beta_1 < \beta$ if $\beta_1 \leqslant \beta$ and $|\beta_1| < |\beta|$. Besides we use $C_\beta^{\beta_1}$ to denote the usual binomial coefficient $\binom{\beta}{\beta_1}$. From now on, we use C or c to denote a generic positive constant may be different from line to line. $A \sim B$, means $cA \leqslant B \leqslant \frac{B}{c}$ for a generic constant $0 < c < 1$.

For any function g, let $\theta \geqslant 0$, we define the instant energy functional $\mathcal{E}_{N,\theta}(t)$ and the instant high-order energy functional $\mathcal{E}^h_{N,\theta}(t)$, respectively to be any functional which satisfies the equivalent relation

$$\mathcal{E}_{N,\theta}(f)(t) \sim \begin{cases} \sum_{|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_\theta, & \gamma \geqslant -2, \\ \sum_{|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_{\theta+|\beta|}, & -3 \leqslant \gamma < -2, \end{cases} \tag{1.14}$$

$$\mathcal{E}^h_{N,\theta}(f)(t) \sim \begin{cases} \sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_\theta, & \gamma \geqslant -2, \\ \sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_{\theta+|\beta|}, & -3 \leqslant \gamma < -2. \end{cases} \tag{1.15}$$

And define the dissipation rate $\mathcal{D}_{N,\theta}(g)$ as

$$\mathcal{D}_{N,\theta}(g)(t) \sim \begin{cases} \sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_{\sigma,\theta}, & \gamma \geqslant -2, \\ \sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_{\sigma,\theta+|\beta|}, & -3 \leqslant \gamma < -2. \end{cases} \tag{1.16}$$

Beside, we denote $\mathcal{E}_{N,0} = \mathcal{E}_N, \mathcal{D}_{N,0} = \mathcal{D}_N$.

Throughout this paper, we shall assume $N \geqslant 8$. Our main results are as follows.

Theorem 1.1. *Let $1 \geqslant F(0,x,v) = M_q(v) + \sqrt{\tilde{M}(v)} f_0(x,v) \geqslant 0$, $\theta \geqslant 0$, $\gamma \geqslant -3$, and assume that $\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f_0)$ is small enough, then there exists a unique global solution $f(t,x,v)$ to the Cauchy problem* (1.5) *such that $1 \geqslant F(t,x,v) = M_q(v) + \sqrt{\tilde{M}(v)} f(t,x,v) \geqslant 0$ and*

$$\mathcal{E}_{N,\theta}(t) + c\int_0^t \mathcal{D}_{N,\theta}(s)\,ds \leqslant C\mathcal{E}_{N,\theta}(0), \tag{1.17}$$

for any $t \geqslant 0$. Moreover, there is $\mathcal{E}^h_{N,\theta}(t)$ such that

$$\frac{d}{dt}\mathcal{E}^h_{N,\theta}(t) + c\mathcal{D}_{N,\theta}(t) \leqslant C\|\nabla_x \mathbf{P} f\|^2, \tag{1.18}$$

holds for any $t \geqslant 0$.

From the proof later on, the above nonlinear energy estimates together with the time decay estimates on the linearized homogeneous system indeed lead to optimal time decay rates of the instant energy functionals $\mathcal{E}_{N,\theta}(t)$ ($\theta = 0,1,2,\ldots$) and $\mathcal{E}^h_{N,\theta}(t)$. Precisely, our another main result in this paper is stated as follows. Set $\epsilon_{N,\theta}$ as

$$\epsilon_{N,\theta} = \mathcal{E}_{N,\theta}(0) + \|f_0\|^2_{Z^1}. \tag{1.19}$$

Theorem 1.2. *Let $f(t,x,v)$ be the solution to the Cauchy problem* (1.5) *obtained in Theorem* 1.1. *For any fixed $\theta = 0,1,2,\ldots$, if $\gamma \geqslant -2$ and $\epsilon_{N,\theta\vee 1}$ is sufficiently small where $\theta \vee 1 = \max\{\theta,1\}$, then*

$$\|f(t)\|^2 \leqslant C\epsilon_{N,\theta\vee 1}(1+t)^{-\frac{3}{2}}, \tag{1.20}$$

$$\mathcal{E}^h_{N,\theta}(t) \leqslant C\epsilon_{N,\theta\vee 1}(1+t)^{-\frac{5}{2}}, \tag{1.21}$$

hold for any $t \geqslant 0$.

Now, we give a brief review of previous work on related topics. There has been extensive investigations on the Landau equation or related models, cf. [5–10,12–15,19–21,23–29] and references therein. In what follows let us mention some of them. The global existence of renormalized solutions to the inhomogeneous equation was established by Lions [18] and Villani [26] for the classical Landau equation. In the celebrated paper [6], Degond and Lemou analyzed the spectral properties and dispersion relations for the linearized Fokker–Planck operator in the case of hard potentials. In the context of perturbations of equilibrium, Guo [12] constructed the global classical solution of the Landau equation in a period box by a refined energy method which has been widely used in the study of the Boltzmann equation and related models. Chen, Desvillettes and He [5] studied the smooth effect of the classic solutions of the full Landau equation. Recently, Strain and Guo [23] developed a weighted energy method to get the exponential rate of convergence for the Boltzmann equation and Landau equation with soft potentials on the torus. Duan, Ukai, Yang and Zhao [10] got the optimal time decay rate of the Boltzmann equation with time-periodic external forcing by using the energy-spectrum method. And very recently, Yang and Yu [27–29] introduced a method by combining Kawashima's compensating function approach and the macro–micro decomposition to get the optimal convergence rate of some kinds of kinetic equations (the relativistic Boltzmann equation and Landau equation, the Vlasov–Maxwell–Fokker–Planck system, etc.) in the whole space. We also mention that Duan and Strain [8,9] studied the hypocoercivity property of nonlinear Vlasov–Poisson–Boltzmann system and Vlasov–Maxwell–Boltzmann system by the method of Fourier analysis and nonlinear energy estimates.

Although the classical Landau equation has been extensively investigated, the quantum Landau equation has received scant attention. Lemou [17] extended the results of [6] to the relativistic and quantum version. In the case of hard and Maxwellian potentials, Bagland [2] established existence and uniqueness of a weak solution of spatial homogeneous quantum Landau equation, and recently Chen [4] proved that the weak solution obtained by Bagland [2] becomes immediately smooth if all the moments for the initial datum are finite. In this paper, we consider the Cauchy problem for the full quantum Landau equation (1.2)–(1.4) with hard potentials as well as soft potentials. We mainly use Kawashima's compensating function to deduce the uniform energy estimate, from which the time-decay properties of solutions to the linearized equation can be obtained, then combing the careful estimates of the nonlinear collision operator, we derived the optimal convergence rate $O(1)(1+t)^{-3/4}$ and $O(1)(1+t)^{-5/4}$ in the norm of $\sqrt{\mathcal{E}_{N,\theta}}$ and $\sqrt{\mathcal{E}^h_N}$ respectively. We shall point out that the term $F(1-F)$ is quadratic and the estimates for the nonlinear collision operator Γ_{non} defined in (1.10) is different from the bilinear term Γ of [15,27]. In [15,27], Guo, Yang and Yu mainly used the splitting

$$\Gamma(f,f) = \Gamma(\mathbf{P}f,\mathbf{P}f) + \Gamma\big(\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\big) + \Gamma\big(\{\mathbf{I}-\mathbf{P}\}f,\mathbf{P}f\big) + \Gamma\big(\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\big) \tag{1.22}$$

to obtain the dissipation norm of f, however, for Γ_{non}, (1.22) is not true. As a compensation, we should firstly deduce Lemma 2.5 in which the $x-$ integrations are not considered, then apply the decomposition $f = \mathbf{P}f + \{\mathbf{I}-\mathbf{P}\}f$ as well as the Sobolev embedding inequality (2.46) to estimate the quadratic terms. We also remark that by the comparison principle, one can find that the solutions of (1.2) should satisfy $0 \leqslant F \leqslant 1$ which is coincide with the Pauli exclusion principle as mentioned above.

The rest of this paper is arranged as follows. In Section 2, we give some basic estimates for later use. Section 3 is devoted to the estimates for the hydrodynamic part $\mathbf{P}f$ and the optimal time decay rates of the linearized quantum Landau equation are also obtained in this section. In Section 4, we establish the local existence of the quantum Landau equation, and the global existence as well as the optimal decay rates is given in Section 5 and Section 6, respectively.

2. Preliminaries

In this section, we give some preliminary lemmas which will be used in the proof of Theorem 1.1. The following is the positivity of L.

Lemma 2.1. *For the linear collision operator L, it holds that $\langle Lg, g\rangle = 0$ if and only if $g(v) = \{a + \mathbf{b}\cdot v + c|v|^2\}\tilde{M}^{\frac{1}{2}}$, where $a, c \in \mathbf{R}$ and $\mathbf{b} \in \mathbf{R}^3$. Moreover, there exists a $\delta > 0$ such that*

$$\langle Lg, g\rangle \geqslant \delta\big|(\mathbf{I}-\mathbf{P})g\big|_\sigma^2. \tag{2.1}$$

Proof. By the definitions of (1.6), we integrate by parts over v variable and change the variables $u \to v$ to obtain

$$\begin{aligned}
\langle Lg, g\rangle &= \int\limits_{\mathbf{R}^3\times\mathbf{R}^3} \psi(v-u)\tilde{M}(u)\tilde{M}(v)\big\{\nabla_v\big(\tilde{M}^{-\frac{1}{2}}g\big)(v) - \nabla_u\big(\tilde{M}^{-\frac{1}{2}}g\big)(u)\big\} \\
&\quad \times \big\{\nabla_v\big(\tilde{M}^{-\frac{1}{2}}g\big)(v)\big\}\,du\,dv \\
&= \int\limits_{\mathbf{R}^3\times\mathbf{R}^3} \psi(v-u)\tilde{M}(u)\tilde{M}(v)\big\{\nabla_v\big(\tilde{M}^{-\frac{1}{2}}g\big)(v) - \nabla_u\big(\tilde{M}^{-\frac{1}{2}}g\big)(u)\big\} \\
&\quad \times \big\{-\nabla_u\big(\tilde{M}^{-\frac{1}{2}}g\big)(u)\big\}\,du\,dv \\
&= \frac{1}{2}\int\limits_{\mathbf{R}^3\times\mathbf{R}^3} \psi(v-u)\tilde{M}(u)\tilde{M}(v)\big\{\nabla_v\big(\tilde{M}^{-\frac{1}{2}}g\big)(v) - \nabla_u\big(\tilde{M}^{-\frac{1}{2}}g\big)(u)\big\} \\
&\quad \times \big\{\nabla_v\big(\tilde{M}^{-\frac{1}{2}}g\big)(v) - \nabla_u\big(\tilde{M}^{-\frac{1}{2}}g\big)(u)\big\}\,du\,dv \geqslant 0.
\end{aligned} \tag{2.2}$$

If $\langle Lg, g\rangle = 0$, performing the similar calculations as Lemma 4 in [12], we get that $g(v) = \{a + \mathbf{b}\cdot v + c|v|^2\}\tilde{M}^{\frac{1}{2}}$.

To prove (2.1), we use the method of contradiction. Assuming the contrary, we have a sequence of normalized functions $g_n(v)$ such that $|g_n|_\sigma = 1$, which also satisfy

$$g_n \in \mathscr{N}^\perp, \tag{2.3}$$

$$\langle Lg_n, g_n\rangle = \langle Ag_n, g_n\rangle + \langle Kg_n, g_n\rangle \leqslant \frac{1}{n}. \tag{2.4}$$

We denote the weak limit, with respect to the inner product $\langle\cdot\rangle$, of g_n (up to a subsequence) by g_0. Hence

$$|g_0|_\sigma \leqslant 1.$$

On the other hand, one get from (1.7), (1.8) that

$$Ag = \frac{v_iv_j}{4}\frac{(M(v)-1)^2}{(M(v)+1)^2}\sigma^{ij}g - \partial_j\big(\partial_i g\sigma^{ij}\big) - \frac{v_iv_jM(v)}{(M(v)+1)^2}\sigma^{ij}g - \frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right)g, \tag{2.5}$$

$$\langle Kg_1, g_2\rangle = \int\limits_{\mathbf{R}^3\times\mathbf{R}^3} \psi(v-u)\tilde{M}(u)\tilde{M}(v)\nabla_u\big(\tilde{M}^{-\frac{1}{2}}g_1\big)(u)\nabla_v\big(\tilde{M}^{-\frac{1}{2}}g_2\big)(v)\,du\,dv, \tag{2.6}$$

then we claim that

$$\left\langle \frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g, g\right\rangle + \left\langle \frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right) g, g\right\rangle + \langle Kg, g\rangle \leqslant \frac{C}{m}|g|_\sigma^2 + C(m)\int\limits_{|v|\leqslant m} g^2\,dv, \tag{2.7}$$

for any $m>1$. To verify (2.7), we only compute the first term in the left hand side of (2.7), since the second and third term can be estimated by following the same way as Lemma 4 in [12]. We split

$$\left\langle \frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g, g\right\rangle = \left(\int\limits_{|v|>m} + \int\limits_{|v|\leqslant m}\right)\frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g^2\,dv.$$

It suffice to consider the first integral over $|v|>m$ $(m>1)$, since

$$M(v) \leqslant \frac{(M(v)-1)^2}{4m} \quad \big(|v|>m>1\big),$$

hence

$$\int\limits_{|v|>m} \frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g^2\,dv \leqslant \frac{1}{m}\int\limits_{|v|>m} \frac{v_i v_j}{4}\frac{(M(v)-1)^2}{(M(v)+1)^2}\sigma^{ij} g^2\,dv.$$

With (2.7) in our hands, now we turn to check

$$\begin{aligned}&\left\langle \frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g_n, g_n\right\rangle + \left\langle \frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right) g_n, g_n\right\rangle \\ &\quad\to \left\langle \frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g_0, g_0\right\rangle + \left\langle \frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right) g_0, g_0\right\rangle, \quad n\to\infty,\end{aligned} \tag{2.8}$$

and

$$\langle Kg_n, g_n\rangle \to \langle Kg_0, g_0\rangle, \quad n\to\infty. \tag{2.9}$$

Since $\partial_i g_n$ is bound in $L^2(|v|\leqslant m)$ from $|g_n|_\sigma = 1$, Theorem 2.32 in [1] yields the following strong convergence

$$\begin{aligned}&\left\langle \chi_m(v)\frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g_n, g_n\right\rangle + \left\langle \chi_m(v)\frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right) g_n, g_n\right\rangle \\ &\quad\to \left\langle \chi_m(v)\frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g_0, g_0\right\rangle + \left\langle \chi_m(v)\frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right) g_0, g_0\right\rangle,\end{aligned} \tag{2.10}$$

where $\chi_m(v)$ is defined as

$$\chi_m(v) = \begin{cases} 1, & |v|\leqslant m, \\ 0, & |v|>m. \end{cases}$$

Furthermore, by first choosing m sufficiently large and then letting $n \to \infty$, we can see that (2.8) is true. On the other hand, in view of Lemma 5 in [12], one can get that (2.9) is also valid.

Now we return to (2.4), letting $n \to \infty$, we have $\langle Lg_0, g_0\rangle = 0$, or equivalently

$$\begin{aligned} 0 &= 1 - \left\langle \frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij} g_0, g_0 \right\rangle - \left\langle \frac{M(v)-1}{M(v)+1}\partial_j\left(\frac{\sigma^i}{2}\right) g_0, g_0 \right\rangle + \langle K g_0, g_0\rangle \\ &= 1 - |g_0|_\sigma^2 + \langle L g_0, g_0\rangle, \end{aligned}$$

which yields the contradiction

$$|g_0|_\sigma^2 = 1, \qquad \langle L g_0, g_0\rangle = 0, \quad g_0 \in \mathcal{N}^\perp.$$

Therefore (2.1) is true. This completes the proof of Lemma 2.1. $\square$

The following lemma is devoted to the estimates of the σ-norm.

Lemma 2.2. *(See [6,12].) Let $\gamma \geqslant -3$, there exists $C > 0$, such that*

$$|g|_\sigma^2 \geqslant C\left\{\left|\big(1+|v|\big)^{\frac{\gamma}{2}} \mathbf{P}_v \partial_i g\right|_2^2 + \left|\big(1+|v|\big)^{\frac{\gamma+2}{2}} (\mathbf{I}-\mathbf{P}_v)\partial_i g\right|_2^2 + \left|\big(1+|v|\big)^{\frac{\gamma+2}{2}} g\right|_2^2\right\}, \tag{2.11}$$

where $\mathbf{P}_v$ is the projection defined as

$$\mathbf{P}_v h_i = \sum h_j v_j \frac{v_i}{|v|^2}, \quad 1 \leqslant i \leqslant 3,$$

for any vector-valued function $h(v) = [h_1(v), h_2(v), h_3(v)]$.

We now summarize some estimates for x, v derivatives of the collision operators L and Γ.

Lemma 2.3. *For linear operators* (1.7) *and* (1.8)*, if $\gamma \geqslant -3$, let $\beta > 0$, $\theta \geqslant 0$, for any $\eta > 0$, there exists $C_{\eta,|\beta|} > 0$ such that*

$$\big(\partial_\beta^\alpha A g, w^{2\theta}\partial_\beta^\alpha g\big) \geqslant \big\|\partial_\beta^\alpha g\big\|_{\sigma,\theta}^2 - \eta \sum_{\beta_1 \leqslant \beta} \big\|\partial_{\beta_1}^\alpha g\big\|_{\sigma,\theta}^2 - C_{\eta,|\beta|}\big\|\tilde{M}^{\frac{1}{16}}\partial^\alpha g\big\|^2, \tag{2.12}$$

and

$$\big(\partial_\beta^\alpha K g_1, w^{2\theta}\partial_\beta^\alpha g_2\big) \leqslant \left\{\eta \sum_{\beta_1 \leqslant \beta} \big\|\partial_{\beta_1}^\alpha g_1\big\|_{\sigma,\theta} + C_{\eta,|\beta|}\big\|\tilde{M}^{\frac{1}{16}}\partial^\alpha g_1\big\|\right\}\big\|\partial_\beta^\alpha g_2\big\|_{\sigma,\theta}. \tag{2.13}$$

Proof. Recalling (2.5), we have

$$\begin{aligned} \big(\partial_\beta^\alpha A g, w^{2\theta}\partial_\beta^\alpha g\big) &= \big\|\partial_\beta^\alpha g\big\|_{\sigma,\theta}^2 + \underbrace{\sum_{|\beta_1| \geqslant 1} C_\beta^{\beta_1}\left(\partial_{\beta_1}\left\{\frac{v_i v_j}{4}\frac{(M(v)-1)^2}{(M(v)+1)^2}\sigma^{ij}\right\}\partial_{\beta-\beta_1}^\alpha g, w^{2\theta}\partial_\beta^\alpha g\right)}_{I_1} \\ &\quad + \underbrace{\sum_{|\beta_1| \geqslant 1} C_\beta^{\beta_1}\big(\partial_{\beta_1}\sigma^{ij}\partial_{\beta-\beta_1}^\alpha \partial_i g, w^{2\theta}\partial_j \partial_\beta^\alpha g\big)}_{I_2} \end{aligned}$$

$$
\begin{aligned}
&+\underbrace{\sum_{|\beta_2|\geqslant 0} C_\beta^{\beta_2}\big(\partial_{\beta_2}\sigma^{ij}\partial^\alpha_{\beta-\beta_2}\partial_i g, \partial_j w^{2\theta}\partial^\alpha_\beta g\big)}_{I_3}\\
&-\underbrace{\sum_{|\beta_2|\geqslant 0} C_\beta^{\beta_2}\left(\partial_{\beta_2}\left\{\frac{v_i v_j M(v)}{(M(v)+1)^2}\sigma^{ij}\right\}\partial^\alpha_{\beta-\beta_2} g, w^{2\theta}\partial^\alpha_\beta g\right)}_{I_4}\\
&-\underbrace{\sum_{|\beta_2|\geqslant 0} C_\beta^{\beta_2}\left(\partial_{\beta_2}\left\{\frac{M(v)-1}{M(v)+1}\partial_j\frac{\sigma^i}{2}\right\}\partial^\alpha_{\beta-\beta_2} g, w^{2\theta}\partial^\alpha_\beta g\right)}_{I_5}.
\end{aligned}
\tag{2.14}
$$

Now we turn to estimate I_i $(1 \leqslant i \leqslant 5)$ term by term. First of all, by Lemma 3 in [12], we have

$$
\left|\partial_{\beta_1}\left\{\frac{v_i v_j}{4}\frac{(M(v)-1)^2}{(M(v)+1)^2}\sigma^{ij}\right\}\right| + \left|\partial_{\beta_1}\sigma^{ij}\right| + \left|\partial_j\sigma^i\right| \leqslant C\big[1+|v|\big]^{\gamma+1},
\tag{2.15}
$$

provided $|\beta_1| \geqslant 1$. Therefore I_1, I_5 are bounded by

$$
\begin{aligned}
&C\int_{\mathbf{R}^3\times\mathbf{R}^3}\big[1+|v|\big]^{\gamma+1}w^{2\theta}\big\{\big|\partial^\alpha_{\beta-\beta_1}g\big| + \big|\partial^\alpha_{\beta-\beta_2}g\big|\big\}|\partial^\alpha_\beta g|\,dx\,dv\\
&\quad = C\int_{\mathbf{R}^3\times\{|v|\geqslant m\}} + C\int_{\mathbf{R}^3\times\{|v|\leqslant m\}}\\
&\quad \leqslant C\int_{\mathbf{R}^3\times\{|v|\leqslant m\}} + \frac{C}{m}\sum_{\bar\beta\leqslant\beta}\big\|\partial^\alpha_{\bar\beta}g\big\|_{\sigma,\theta}\big\|\partial^\alpha_\beta g\big\|_{\sigma,\theta}.
\end{aligned}
\tag{2.16}
$$

For the part $|v| \leqslant m$, for any $\eta > 0$, we use the compact interpolation in the Sobolev space to get

$$
\begin{aligned}
&C\int_{\mathbf{R}^3\times\{|v|\leqslant m\}}\big[1+|v|\big]^{\gamma+1}w^{2\theta}\big\{\big|\partial^\alpha_{\beta-\beta_1}g\big| + \big|\partial^\alpha_{\beta-\beta_2}g\big|\big\}\big|\partial^\alpha_\beta g\big|\,dx\,dv\\
&\quad \leqslant C_\eta\int_{\mathbf{R}^3\times\{|v|\leqslant m\}}\big|w^\theta\partial^\alpha g\big|^2\,dv\,dx + \eta\sum_{|\bar\beta|=|\beta|+1}\int_{\mathbf{R}^3\times\{|v|\leqslant m\}}\big|w^\theta\partial^\alpha_{\bar\beta}g\big|^2\,dv\,dx\\
&\quad \leqslant \eta\big\|\partial^\alpha_\beta g\big\|^2_{\sigma,\theta} + C_\eta\big\|\tilde M^{\frac{1}{16}}\partial^\alpha g\big\|^2.
\end{aligned}
\tag{2.17}
$$

For I_2, if $|\beta_1| \geqslant 2$, in light of Lemma 3 in [12],

$$
\big|\partial_{\beta_1}\sigma^{ij}\big| \leqslant C\big(1+|v|\big)^\gamma.
$$

Hence

$$\begin{aligned}
&\sum_{|\beta_1|\geqslant 2} C_\beta^{\beta_1}\left|\left(\partial_{\beta_1}\sigma^{ij}\partial^\alpha_{\beta-\beta_1}\partial_i g, w^{2\theta}\partial_j\partial^\alpha_\beta g\right)\right| \\
&\quad\leqslant C\left\{\int\limits_{\mathbf{R}^3\times\{|v|\leqslant m\}} + \int\limits_{\mathbf{R}^3\times\{|v|\geqslant m\}}\right\}\left[1+|v|\right]^\gamma w^{2\theta}\left|\partial^\alpha_\beta\partial_i g\right|\left|\partial^\alpha_{\beta-\beta_1}\partial_j g\right| dx\,dv \\
&\quad\leqslant \eta\left\|\partial^\alpha_\beta g\right\|^2_{\sigma,\theta} + C_\eta\left\|\tilde{M}^{\frac{1}{16}}\partial^\alpha g\right\|^2 \\
&\qquad + \frac{C}{m}\int\limits_{\mathbf{R}^3\times\{|v|\geqslant m\}}\left|\left[1+|v|\right]^{\frac{\gamma+2}{2}} w^\theta\partial^\alpha_{\beta-\beta_1}\partial_i g\right|\left|\left[1+|v|\right]^{\frac{\gamma}{2}} w^\theta\partial^\alpha_\beta\partial_j g\right| dx\,dv \\
&\quad\leqslant \eta\left\|\partial^\alpha_\beta g\right\|^2_{\sigma,\theta} + C_\eta\left\|\tilde{M}^{\frac{1}{16}}\partial^\alpha g\right\|^2 + \frac{C}{m}\sum_{\bar\beta\leqslant\beta}\left\|\partial^\alpha_{\bar\beta} g\right\|_{\sigma,\theta}\left\|\partial^\alpha_\beta g\right\|_{\sigma,\theta}.
\end{aligned} \tag{2.18}$$

If $|\beta_1| = 1$, an integration by part of the v variable yields

$$\left(\partial_{\beta_1}\sigma^{ij}\partial^\alpha_{\beta-\beta_1}\partial_i g, w^{2\theta}\partial_j\partial^\alpha_\beta g\right) = -\frac{1}{2}\left(\partial_{\beta_1}\left\{w^{2\theta}\partial_{\beta_1}\sigma^{ij}\right\}\partial^\alpha_{\beta-\beta_1}\partial_i g, \partial_j\partial^\alpha_{\beta-\beta_1} g\right).$$

Since $|\beta_1| = 1$, $|\partial_{\beta_1}\{w^{2\theta}\partial_{\beta_1}\sigma^{ij}\}| \leqslant C(1+|v|)^\gamma w^{2\theta}$, by following the same splitting as (2.18) we obtain

$$|I_2| \leqslant \eta\left\|\partial^\alpha_\beta g\right\|^2_{\sigma,\theta} + C_\eta\left\|\tilde{M}^{\frac{1}{16}}\partial^\alpha g\right\|^2 + \frac{C}{m}\left\|\partial^\alpha_\beta g\right\|^2_{\sigma,\theta}.$$

We now consider I_3, if $\beta_2 = 0$, since $|\partial_i\{\sigma^{ij}\partial_j w^{2\theta}\}| \leqslant w^{2\theta}(1+|v|)^\gamma$, after an integration by part of the v variable, we have

$$\begin{aligned}
|I_3| &\leqslant C\int\limits_{\mathbf{R}^3\times\mathbf{R}^3}\left[1+|v|\right]^\gamma\left\{w^\theta\left|\partial^\alpha_\beta g\right|\right\}^2 dx\,dv \\
&\leqslant \left(\int\limits_{\mathbf{R}^3\times\{|v|\leqslant m\}} + \int\limits_{\mathbf{R}^3\times\{|v|\geqslant m\}}\right)\left[1+|v|\right]^\gamma\left\{w^\theta\left|\partial^\alpha_\beta g\right|\right\}^2 dx\,dv \\
&\leqslant C_\eta\int\limits_{\mathbf{R}^3\times\{|v|\leqslant m\}}|g|^2\,dv\,dx + \eta\sum_{|\bar\beta|=|\beta|+1}\int\limits_{\mathbf{R}^3\times\{|v|\leqslant m\}}\left|w^\theta\partial^\alpha_{\bar\beta} g\right|^2 dv\,dx + \frac{C}{m}\left\|\partial^\alpha_\beta g\right\|^2_{\sigma,\theta}.
\end{aligned} \tag{2.19}$$

If $|\beta_2| > 0$, $|\partial_{\beta_2}\sigma^{ij}\partial_j w^{2\theta}| \leqslant w^{2\theta}(1+|v|)^\gamma$, then following the same spirit of (2.19), the desired estimates for I_3 can be obtained.

As to I_4, since $M(v)$ decays exponentially, the computations are much easier, we omit the details for brevity. This completes estimates for A. By the same procedure of computing $\langle\partial^\alpha_\beta Kg, w^{2\theta}\partial^\alpha_\beta g\rangle$ in [12], the stated estimates for K follows easily. Thus the proof of Lemma 2.2 is completed. □

Next, by Lemma 2.3 and the standard interpolation formula in Sobolev space, one can get that

Lemma 2.4. *(See [19].) Let $\beta > 0$, $\theta \geqslant 0$, there exists positive constant $C_{|\beta|,\theta} > 0$ such that*

$$\left(w^{2\theta}\partial^\alpha_\beta Lg, \partial^\alpha_\beta g\right) \geqslant \frac{1}{2}\left\|w^\theta\partial^\alpha_\beta g\right\|^2_\sigma - C_{|\beta|,\theta}\left\|w^\theta\partial^\alpha g\right\|^2_\sigma. \tag{2.20}$$

Remark 2.1. If $|\beta|=0$, $\theta\geqslant 0$, for any $\eta>0$, there exists $C_{\eta,\theta}>0$ such that

$$\big(\partial^\alpha Ag, w^{2\theta}\partial^\alpha g\big)\geqslant(1-\eta)\big\|\partial^\alpha g\big\|_{\sigma,\theta}^2-C_{\eta,\theta}\big\|\tilde M^{\frac{1}{16}}\partial^\alpha g\big\|^2, \tag{2.21}$$

and

$$\big(\partial^\alpha Kg_1, w^{2\theta}\partial^\alpha g_2\big)\leqslant\big\{\eta\big\|\partial^\alpha g_1\big\|_{\sigma,\theta}+C_{\eta,\theta}\big\|\tilde M^{\frac{1}{16}}\partial^\alpha g_1\big\|\big\}\big\|\partial^\alpha g_2\big\|_{\sigma,\theta}. \tag{2.22}$$

Therefore (2.20) can be improved as

$$\big(w^{2\theta}\partial^\alpha Lg,\partial^\alpha g\big)\geqslant\frac{1}{2}\big\|w^\theta\partial^\alpha g\big\|_\sigma^2-C_\theta\big\|\tilde M^{\frac{1}{16}}\partial^\alpha g\big\|^2. \tag{2.23}$$

Lemma 2.5. *For the quantum Landau kernel* (1.9) *and* (1.10)*, if $\gamma\geqslant-3$, let $|\alpha|+|\beta|\leqslant N$, $\theta\geqslant 0$, there exists $C>0$, such that*

$$\big\langle\partial_\beta^\alpha\Gamma_{bi}(f,g), w^{2\theta}\partial_\beta^\alpha h\big\rangle\leqslant C\sum\big\{\big|\partial_{\bar\beta_2}^{\alpha_2}f\big|_{2,\theta}\big|\partial_{\bar\beta_1}^{\alpha_1}g\big|_{\sigma,\theta}+\big|\partial_{\bar\beta_2}^{\alpha_2}f\big|_{\sigma,\theta}\big|\partial_{\bar\beta_1}^{\alpha_1}g\big|_{2,\theta}\big\}\big|\partial_\beta^\alpha h\big|_{\sigma,\theta}, \tag{2.24}$$

and

$$\begin{aligned}\big\langle\partial_\beta^\alpha\Gamma_{non}(f,g), w^{2\theta}\partial_\beta^\alpha h\big\rangle\leqslant{}& C\Bigg(\sum_{|\bar\alpha_2|+|\bar\beta_2'|\leqslant\frac{|\bar\beta_2|+|\alpha_2|}{2}}\big|\tilde M^{\frac18}\partial_{\bar\beta_2'}^{\bar\alpha_2}f\big|_\infty\Bigg)\sum\big|\partial_{\bar\beta_2-\bar\beta_2'}^{\alpha_2-\bar\alpha_2}f\big|_{2,\theta}\big|\partial_{\bar\beta_1}^{\alpha_1}g\big|_{\sigma,\theta}\big|\partial_\beta^\alpha h\big|_{\sigma,\theta}\\&+C\Bigg(\sum_{|\alpha_1|+|\bar\beta_1|\leqslant|\bar\beta_2|+|\alpha_2|}\big|\tilde M^{\frac18}\partial_{\bar\beta_1}^{\alpha_1}f\big|_\infty\Bigg)\sum\big|\partial_{\bar\beta_2}^{\alpha_2}f\big|_{2,\theta}\big|\partial_\beta^\alpha h\big|_{\sigma,\theta}\\&+C\Bigg(\sum_{|\bar\alpha_1|+|\bar\beta_1'|\leqslant\frac{|\bar\beta_1|+|\alpha_1|}{2}}\big|\tilde M^{\frac18}\partial_{\bar\beta_1'}^{\bar\alpha_1}g\big|_\infty\Bigg)\sum\big|\partial_{\bar\beta_2}^{\alpha_2}f\big|_{\sigma,\theta}\big|\partial_{\bar\beta_1-\bar\beta_1'}^{\alpha_1-\bar\alpha_1}g\big|_{2,\theta}\big|\partial_\beta^\alpha h\big|_{\sigma,\theta}\\&+C\Bigg(\sum_{|\alpha_1|+|\bar\beta_1|\leqslant|\bar\beta_2|+|\alpha_2|}\big|\tilde M^{\frac18}\partial_{\bar\beta_1}^{\alpha_1}g\big|_\infty\Bigg)\sum\big|\partial_{\bar\beta_2}^{\alpha_2}g\big|_{2,\theta}\big|\partial_\beta^\alpha h\big|_{\sigma,\theta},\end{aligned} \tag{2.25}$$

where the summation $\sum$ is over $\alpha_2+\alpha_1=\alpha$, $\bar\beta_1'\leqslant\bar\beta_1$, $\bar\beta_2'\leqslant\bar\beta_2$, $\bar\beta_1+\bar\beta_2\leqslant\beta$.

Proof. For brevity, we only prove (2.25). By the definition (1.10) and the product rule, we expand

$$\big\langle\partial_\beta^\alpha\Gamma_{non}(f,g), w^{2\theta}\partial_\beta^\alpha h\big\rangle=\sum_{i=6}^{9}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}}C_\alpha^{\alpha_1}C_\beta^{\beta_1}I_i$$

where I_i $(6\leqslant i\leqslant 9)$ takes the form

$$I_6=\begin{cases}-\langle\partial_\beta\{[\frac{v_iv_j}{4}\frac{(M-1)^2}{(M+1)^2}\partial^{\alpha_1}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_i\partial^{\alpha_1}g(v)]\psi^{ij}*[\tilde M\partial^{\alpha_2}(f^2)]\}, w^{2\theta}\partial_\beta^\alpha h\rangle\\ \quad+\langle\partial_\beta\{[\frac{v_i}{2}\frac{M-1}{M+1}\partial^{\alpha_1}g(v)+\partial_i\partial^{\alpha_1}g(v)]\psi^{ij}*[\tilde M\partial^{\alpha_2}(f^2)]\}, w^{2\theta}\partial_\beta^\alpha\partial_jh\rangle\\ \quad+\langle\partial_{\beta_1}[\frac{v_i}{2}\frac{M-1}{M+1}\partial^{\alpha_1}g(v)+\partial_i\partial^{\alpha_1}g(v)]\psi^{ij}*\partial_{\beta_2}[\tilde M\partial^{\alpha_2}(f^2)], \partial_j[w^{2\theta}]\partial_\beta^\alpha h\rangle,\end{cases}$$

$$I_7=\begin{cases}-\langle\partial_\beta\{[\frac{v_iv_j}{2}\frac{M-1}{M+1}M^{\frac12}(v)]\psi^{ij}*[\tilde M\partial^\alpha(f^2)]\},w^{2\theta}\partial^\alpha_\beta h\rangle\\ \quad+\langle\partial_\beta\{[v_iM^{\frac12}(v)]\psi^{ij}*[\tilde M\partial^\alpha(f^2)]\},w^{2\theta}\partial^\alpha_\beta\partial_j h\rangle\\ \quad+\langle\partial_{\beta_1}[v_iM^{\frac12}(v)]\psi^{ij}*\partial_{\beta_2}[\tilde M\partial^\alpha(f^2)],\partial_j[w^{2\theta}]\partial^\alpha_\beta h\rangle,\end{cases}$$

$$I_8=\begin{cases}+\langle\partial_\beta\{[\frac{v_j}{2}\frac{M-1}{M+1}\tilde M^{\frac12}\partial^{\alpha_1}(g^2)]\psi^{ij}*[\frac{u_i}{2}\frac{M-1}{M+1}\tilde M^{\frac12}\partial^{\alpha_2}f(u)+\tilde M^{\frac12}\partial_i\partial^{\alpha_2}f(u)]\},w^{2\theta}\partial^\alpha_\beta h\rangle\\ \quad-\langle\partial^{\alpha_1}_{\beta_1}[\tilde M^{\frac12}g^2]\psi^{ij}*\partial_{\beta_2}[\frac{u_i}{2}\frac{M-1}{M+1}\tilde M^{\frac12}\partial^{\alpha_2}f(u)+\tilde M^{\frac12}\partial_i\partial^{\alpha_2}f(u)],w^{2\theta}\partial^\alpha_\beta\partial_j h\rangle\\ \quad-\langle\partial^{\alpha_1}_{\beta_1}[\tilde M^{\frac12}g^2]\psi^{ij}*\partial_{\beta_2}[\frac{u_i}{2}\frac{M(u)-1}{M(u)+1}\tilde M^{\frac12}\partial^{\alpha_2}f(u)+\tilde M^{\frac12}\partial_i\partial^{\alpha_2}f(u)],\partial_j[w^{2\theta}]\partial^\alpha_\beta h\rangle,\end{cases}$$

$$I_9=\begin{cases}+\langle\partial_\beta\{[\frac{v_j}{2}\frac{M-1}{M+1}\tilde M^{\frac12}\partial^{\alpha_1}(g^2)]\psi^{ij}*[u_i\tilde M]\},w^{2\theta}\partial^\alpha_\beta h\rangle\\ \quad-\langle\partial^{\alpha_1}_{\beta_1}[\tilde M^{\frac12}g^2]\psi^{ij}*\partial_{\beta_2}[u_i\tilde M],w^{2\theta}\partial^\alpha_\beta\partial_j h\rangle\\ \quad-\langle\partial^{\alpha_1}_{\beta_1}[\tilde M^{\frac12}(g^2)]\psi^{ij}*\partial_{\beta_2}[u_i\tilde M],\partial_j[w^{2\theta}]\partial^\alpha_\beta h\rangle.\end{cases}$$

Now we turn to compute I_i $(6\leqslant i\leqslant 9)$ term by term. First of all, we use I_i^k $(1\leqslant k\leqslant 3,\ 6\leqslant i\leqslant 9)$ to denote the corresponding terms in I_i for simplicity. For I_6^1 and I_6^2, we decompose the double integration region $[u,v]\in\mathbf{R}^3\times\mathbf{R}^3$ into three parts:

$$\big\{|v|\leqslant 1\big\},\qquad\big\{2|u|\geqslant|v|,|v|\geqslant 1\big\},\qquad\big\{2|u|\leqslant|v|,|v|\geqslant 1\big\}.$$

For the first part $\{|v|\leqslant 1\}$, letting $|\bar\beta_2'|+|\bar\alpha_2|\leqslant\frac{|\bar\beta_2|+|\alpha_2|}{2}$, noticing $\psi^{ij}=O(|v|^{\gamma+2})\in L^2_{loc}$, we have

$$\begin{aligned}&\bigg\langle\partial_\beta\bigg\{\bigg[\frac{v_iv_j}{4}\frac{(M-1)^2}{(M+1)^2}\partial^{\alpha_1}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_i\partial^{\alpha_1}g(v)\bigg]\psi^{ij}*\big[\tilde M\partial^{\alpha_2}\big(f^2\big)\big]\bigg\},\chi_1w^{2\theta}\partial^\alpha_\beta h\bigg\rangle\\&=C^{\beta_1}_\beta C^{\bar\beta_2}_{\beta_2}C^{\bar\beta_2'}_{\bar\beta_2}C^{\bar\alpha_2}_{\alpha_2}\int\limits_{\{|v|\leqslant1\}}\partial_{\beta_1}\bigg[\frac{v_iv_j}{4}\frac{(M-1)^2}{(M+1)^2}\partial^{\alpha_1}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_i\partial^{\alpha_1}g(v)\bigg]w^{2\theta}\partial^\alpha_\beta h\\&\quad\times\bigg\{\int\limits_{\mathbf{R}^3}\psi^{ij}(v-u)\partial_{\beta_2-\bar\beta_2}\tilde M\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\,\partial^{\bar\alpha_2}_{\bar\beta_2'}f\,du\bigg\}dv\\&\leqslant C\big|\tilde M^{\frac18}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\big|_\infty\big|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\big|_{2,\theta}\int\limits_{\{|v|\leqslant1\}}\big\{\big|\partial^{\alpha_1}_{\bar\beta_1}g(v)\big|+\big|\partial_i\partial^{\alpha_1}_{\bar\beta_1}g(v)\big|\big\}\big(1+|v|\big)^{\gamma+2}w^{2\theta}\big|\partial^\alpha_\beta h\big|\,dv\\&\leqslant C\big|\tilde M^{\frac18}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\big|_\infty\big|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar\beta_1}g\big|_{\sigma,\theta}\big|\partial^\alpha_\beta h\big|_{\sigma,\theta},\end{aligned}\tag{2.26}$$

and similarly

$$\begin{aligned}&\bigg\langle\partial_\beta\bigg\{\bigg[\frac{v_i}{2}\frac{M-1}{M+1}\partial^{\alpha_1}g(v)+\partial_i\partial^{\alpha_1}g(v)\bigg]\psi^{ij}*\big[\tilde M\partial^{\alpha_2}\big(f^2\big)\big]\bigg\},\chi_1w^{2\theta}\partial_j\partial^\alpha_\beta h\bigg\rangle\\&\quad\leqslant C\big|\tilde M^{\frac18}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\big|_\infty\big|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar\beta_1}g\big|_{\sigma,\theta}\big|\partial^\alpha_\beta h\big|_{\sigma,\theta},\end{aligned}\tag{2.27}$$

where we have used the fact that

$$\begin{aligned}
&\left|\int\limits_{\mathbf{R}^3} \psi^{ij}(v-u)\partial_{\beta_2-\bar{\beta}_2}\tilde{M}\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}_2'} f\,\partial^{\bar{\alpha}_2}_{\bar{\beta}_2'} f\,du\right| \\
&\quad\leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}_2'} f\big|_\infty\bigg(\int\limits_{\mathbf{R}^3}\big|\psi^{ij}(v-u)\big|^2\tilde{M}^{\frac{1}{4}}\,du\bigg)^{\frac{1}{2}}\bigg(\int\limits_{\mathbf{R}^3}\tilde{M}^{\frac{1}{4}}\big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}_2'} f\big|^2\,du\bigg)^{\frac{1}{2}} \\
&\quad\leqslant C\big(1+|v|\big)^{\gamma+2}\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}_2'} f\big|_\infty\big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}_2'} f\big|_{2,\theta}.
\end{aligned}\tag{2.28}$$

For the second part $\{2|u|\geqslant|v|,\ |v|\geqslant 1\}$, since

$$\big|\partial_{\beta_2-\bar{\beta}_2}\tilde{M}(u)\big|\leqslant C\tilde{M}^{\frac{1}{4}}(u)\tilde{M}^{\frac{1}{16}}(v),$$

by the same type of estimates in (2.28), the v-integrand in I_6^1 and I_6^2 are bounded by

$$\begin{aligned}
&C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}_2'} f\big|_\infty\big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}_2'} f\big|_{2,\theta}\int\limits_{\{|v|\geqslant 1\}}\tilde{M}^{\frac{1}{32}}(v)w^{2\theta}\big\{\big|\partial^{\alpha_1}_{\bar{\beta}_1} g(v)\big|+\big|\partial_i\partial^{\alpha_1}_{\bar{\beta}_1} g(v)\big|\big\}\big\{\big|\partial^\alpha_\beta h\big|+\big|\partial^\alpha_\beta\partial_j h\big|\big\}\,dv \\
&\quad\leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}_2'} f\big|_\infty\big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}_2'} f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{\sigma,\theta}\big|\partial^\alpha_\beta h\big|_{\sigma,\theta}.
\end{aligned}\tag{2.29}$$

Next, we consider the third part of $\{2|u|\leqslant|v|,\ |v|\geqslant 1\}$, for which we shall estimate I_6^1 and I_6^2 in slightly different ways. To estimate I_6^1, the key is to expand $\psi^{ij}(v-u)$ into

$$\psi^{ij}(v-u)=\psi^{ij}(v)-\partial_k\psi^{ij}(\bar{v})u_k,\tag{2.30}$$

where $\bar{v}$ is between $v-u$ and v. We plug (2.30) into the integrand of I_6^1, and noticing that

$$\sum_{i,j}\psi^{ij}(v)v_iv_j=0,\quad \sum_{i,j}\partial_k\psi^{ij}(v)v_iv_j=0,\quad \sum_j\psi^{ij}(v)v_j=0,\quad \sum_i\psi^{ij}(v)v_i=0,\tag{2.31}$$

we have

$$\begin{aligned}
&\bigg\langle\partial_\beta\bigg\{\bigg[\frac{v_iv_j}{4}\frac{(M-1)^2}{(M+1)^2}\partial^{\alpha_1}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_i\partial^{\alpha_1}g(v)\bigg]\psi^{ij}*\big[\chi_{\frac{|v|}{2}}\tilde{M}\partial^{\alpha_2}\big(f^2\big)\big]\bigg\}, \\
&\quad(1-\chi_1)w^{2\theta}\partial^\alpha_\beta h\bigg\rangle \\
&\quad=\int\limits_{\{|v|\geqslant 1\}}\partial_{\beta_1}\bigg[\frac{v_i}{2}\frac{(M-1)^2}{(M+1)^2}\partial^{\alpha_1}g(v)+\frac{M-1}{M+1}\partial_i\partial^{\alpha_1}g(v)\bigg]\partial_k\psi^{ij}(\bar{v})w^{2\theta}\partial^\alpha_\beta h\,dv \\
&\qquad\times\int\limits_{\{|2u|\leqslant|v|\}}\frac{1}{2}\partial_{\beta_2}\big(u_ju_k\tilde{M}\partial^{\alpha_2}\big(f^2\big)\big)\,du.
\end{aligned}\tag{2.32}$$

On the other hand,

$$\frac{1}{2}|v|\leqslant|v|-|u|\leqslant|\bar{v}|\leqslant|v|+|u|\leqslant\frac{3}{2}|v|,\tag{2.33}$$

thus

$$\partial_k \psi^{ij}(\bar{v}) \leqslant C\big(1+|v|\big)^{\gamma+1},$$

then one can see that (2.32) is bounded by

$$\begin{aligned}
&C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{\sigma,\theta}\big|\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta}\\
&\quad + C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\big(1+|v|\big)^{\frac{\gamma}{2}}\partial_i\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{2,\theta}\big|\big(1+|v|\big)^{\frac{\gamma+2}{2}}\partial^{\alpha}_{\beta} h\big|_{2,\theta}\\
&\quad \leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{\sigma,\theta}\big|\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta}.
\end{aligned}$$

For I_6^2, we again expand $\psi^{ij}(v-u)$ as

$$\psi^{ij}(v-u) = \psi^{ij}(v) - \partial_k\psi^{ij}(v)u_k + \frac{1}{2}\sum_{k,l}\partial_{k,l}\psi^{ij}(\bar{v})u_k u_l, \tag{2.34}$$

with $\bar{v}$ between v and $v-u$. We plug (2.34) into I_6^2 and we can decompose $\partial_i\partial^{\alpha_1}_{\beta_1} g$ and $\partial_j\partial^{\alpha}_{\beta} h$ into their $\mathbf{P}_v$ parts as well as $\mathbf{I}-\mathbf{P}_v$ parts. For the first term in the expansion (2.34), recalling (2.28), we obtain

$$\begin{aligned}
&\int\limits_{|v|\geqslant 1} w^{2\theta}\psi^{ij}\partial_i\partial^{\alpha_1}_{\beta_1} g\partial_j\partial^{\alpha}_{\beta} h\,dv \cdot \int\limits_{2|u|\leqslant|v|}\partial_{\beta_2}\big(\tilde{M}\partial^{\alpha_2}\big(f^2\big)\big)\,du\\
&\quad \leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\big(1+|v|\big)^{\frac{\gamma+2}{2}}\{\mathbf{I}-\mathbf{P}_v\}\partial_i\partial^{\alpha_1}_{\beta_1} g\big|_{2,\theta}\\
&\qquad \times\big|\big(1+|v|\big)^{\frac{\gamma+2}{2}}\{\mathbf{I}-\mathbf{P}_v\}\partial_j\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta},
\end{aligned} \tag{2.35}$$

where we have used (2.31) so that the sum of the terms with $\frac{v_i}{2}\frac{M-1}{M+1}\partial^{\alpha_1} g(v)$, $\mathbf{P}_v\partial_i\partial^{\alpha_1}_{\beta_1} g$ and $\mathbf{P}_v\partial_j\partial^{\alpha}_{\beta} h$ vanish. For the second term in the expansion (2.34), we have to compute

$$\begin{aligned}
&\int\limits_{\{|v|\geqslant 1\}}\partial_{\beta_1}\left[\frac{M-1}{M+1}\partial^{\alpha_1} g(v)\right]\partial_k\psi^{ij} w^{2\theta}\partial_j\partial^{\alpha}_{\beta} h\,dv \cdot \int\limits_{2|u|\leqslant|v|} -u_k\partial_{\beta_2-\bar{\beta}_2}\left\{\frac{u_i}{2}\tilde{M}\right\}\partial^{\alpha_2}_{\bar{\beta}_2}\big(f^2\big)\,du\\
&\quad + \int\limits_{\{|v|\geqslant 1\}}\big[\partial_i\partial^{\alpha_1}_{\beta_1} g(v)\big]\partial_k\psi^{ij} w^{2\theta}\partial_j\partial^{\alpha}_{\beta} h\,dv \cdot \int\limits_{2|u|\leqslant|v|} -u_k\partial_{\beta_2-\bar{\beta}_2}\tilde{M}\partial^{\alpha_2}_{\bar{\beta}_2}\big(f^2\big)\,du.
\end{aligned} \tag{2.36}$$

By expanding $\partial_i\partial^{\alpha_1}_{\beta_1} g$ and $\partial_j\partial^{\alpha}_{\beta} h$ into their $\mathbf{P}_v$ and $\mathbf{I}-\mathbf{P}_v$ parts yields

$$\begin{aligned}
&\partial_k\psi^{ij}\partial_i\partial^{\alpha_1}_{\beta_1} g(v)\partial_j\partial^{\alpha}_{\beta} h\\
&\quad = \partial_k\psi^{ij}\big\{\{\mathbf{I}-\mathbf{P}_v\}\partial_i\partial^{\alpha_1}_{\beta_1} g\{\mathbf{I}-\mathbf{P}_v\}\partial_j\partial^{\alpha}_{\beta} h + \{\mathbf{I}-\mathbf{P}_v\}\partial_i\partial^{\alpha_1}_{\beta_1} g\mathbf{P}_v\partial_j\partial^{\alpha}_{\beta} h\\
&\qquad + \mathbf{P}_v\partial_i\partial^{\alpha_1}_{\beta_1} g\{\mathbf{I}-\mathbf{P}_v\}\partial_j\partial^{\alpha}_{\beta} h\big\}.
\end{aligned} \tag{2.37}$$

Noticing that $\partial_k \psi^{ij}(v) \leqslant C(1+|v|)^{\gamma+1}$, one can get that (2.36) is no more than

$$C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{\sigma,\theta}\big|\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta}. \tag{2.38}$$

For the third term in the expansion (2.34), we obtain

$$\begin{aligned}
&\int\limits_{\{|v|\geqslant 1\}} \partial_{\beta_1}\left[\frac{M-1}{M+1}\partial^{\alpha_1} g(v)\right]\partial_{kl}\psi^{ij}(\bar{v}) w^{2\theta}\partial_j\partial^\alpha_\beta h\, dv \cdot \int\limits_{2|u|\leqslant |v|} (u_l u_k)\partial_{\beta_2-\bar{\beta}_2}\left\{\frac{u_i}{2}\tilde{M}\right\}\partial^{\alpha_2}_{\bar{\beta}_2}\big(f^2\big)\, du \\
&\quad + \int\limits_{\{|v|\geqslant 1\}} \big[\partial_i \partial^{\alpha_1}_{\beta_1} g(v)\big]\partial_{kl}\psi^{ij}(\bar{v}) w^{2\theta}\partial_j\partial^\alpha_\beta h\, dv \cdot \int\limits_{2|u|\leqslant |v|} (u_l u_k)\partial_{\beta_2-\bar{\beta}_2}\tilde{M}\partial^{\alpha_2}_{\bar{\beta}_2}\big(f^2\big)\, du \\
&\quad \leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{\sigma,\theta}\big|\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta}.
\end{aligned} \tag{2.39}$$

This follows, since we can employ (2.33) to get that $\partial_{kl}\psi^{ij}(\bar{v}) \leqslant C(1+|v|)^{\gamma}$.

Now we consider I_6^3, since $|\partial_j[w^{2\theta}]| \leqslant C[w^{2\theta}](1+|v|)^{-1}$, by direct computation, we get that

$$\begin{aligned}
\big|I_6^3\big| &\leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\int\limits_{\mathbf{R}^3}\left|\left[\partial_{\beta_1}\left(\frac{v_i}{2}\frac{M-1}{M+1}\partial^{\alpha_1} g\right) + \partial_i\partial^{\alpha_1}_{\beta_1} g\right]\big(1+|v|\big)^{\gamma+1} w^{2\theta}\partial^\alpha_\beta h\right| dv \\
&\leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_2}_{\bar{\beta}'_2} f\big|_\infty \big|\partial^{\alpha_2-\bar{\alpha}_2}_{\bar{\beta}_2-\bar{\beta}'_2} f\big|_{2,\theta}\big|\partial^{\alpha_1}_{\bar{\beta}_1} g\big|_{\sigma,\theta}\big|\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta}.
\end{aligned} \tag{2.40}$$

Now we turn to compute I_8, because the estimate for I_7 and I_9 are similar (and easier). We note that

$$\left|\partial_\beta\left[\frac{v_j}{2}\frac{M-1}{M+1}\tilde{M}^{\frac{1}{2}}\right]\right| + \big|\partial_\beta \tilde{M}^{\frac{1}{2}}\big| \leqslant C\tilde{M}^{\frac{1}{4}},$$

for any $|\beta| \geqslant 0$, therefore applying the same type estimate as (2.28), we have

$$\begin{aligned}
|I_8| &\leqslant C\big|\tilde{M}^{\frac{1}{4}}\big(\big|\partial^{\alpha_2}_{\bar{\beta}_2} f\big| + \big|\partial_i\partial^{\alpha_2}_{\bar{\beta}_2} f\big|\big)\big|_2 \big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_1}_{\bar{\beta}'_1} g\big|_\infty \\
&\quad \times \int\limits_{\mathbf{R}^3}\big|\tilde{M}^{\frac{1}{4}}\partial^{\alpha_1-\bar{\alpha}_1}_{\bar{\beta}_1-\bar{\beta}'_1} g\big|\big(1+|v|\big)^{\gamma+2} w^{2\theta}\big\{\big|\partial^\alpha_\beta h\big| + \big|\partial_j\partial^\alpha_\beta h\big|\big\}\, dv \\
&\leqslant C\big|\tilde{M}^{\frac{1}{8}}\partial^{\bar{\alpha}_1}_{\bar{\beta}'_1} g\big|_\infty \big|\partial^{\alpha_1-\bar{\alpha}_1}_{\bar{\beta}_1-\bar{\beta}'_1} g\big|_{2,\theta}\big|\partial^{\alpha_2}_{\bar{\beta}_2} f\big|_{\sigma,\theta}\big|\partial^{\alpha}_{\beta} h\big|_{\sigma,\theta}.
\end{aligned} \tag{2.41}$$

Thus (2.25) is proved. □

There is a direct result from Lemma 2.5.

Lemma 2.6. *Assume $\gamma \geqslant -3$, $|\alpha| + |\beta| \leqslant N$, $\theta \geqslant 0$, there exists $C > 0$, such that if $\gamma \geqslant -2$, we have*

$$\begin{aligned}
\big|\big(\partial^\alpha_\beta \Gamma(f,g), w^{2\theta}\partial^\alpha_\beta h\big)\big| &\leqslant C\big\{\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f) + \mathcal{E}^{\frac{1}{2}}_{N,\theta}(g) + \mathcal{E}_{N,\theta}(f)\big\}\mathcal{D}^{\frac{1}{2}}_{N,\theta}(g)\mathcal{D}^{\frac{1}{2}}_{N,\theta}(h) \\
&\quad + C\big\{\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f) + \mathcal{E}^{\frac{1}{2}}_{N,\theta}(g) + \mathcal{E}_{N,\theta}(g)\big\}\mathcal{D}^{\frac{1}{2}}_{N,\theta}(f)\mathcal{D}^{\frac{1}{2}}_{N,\theta}(h),
\end{aligned} \tag{2.42}$$

if $-2 > \gamma \geqslant -3$, *we obtain*

$$\begin{aligned}\left|\left(\partial^\alpha_\beta \Gamma(f,g), w^{2\theta+2|\beta|}\partial^\alpha_\beta h\right)\right| \leqslant{}& C\left\{\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f)+\mathcal{E}^{\frac{1}{2}}_{N,\theta}(g)+\mathcal{E}_{N,\theta}(f)\right\}\mathcal{D}^{\frac{1}{2}}_{N,\theta}(g)\mathcal{D}^{\frac{1}{2}}_{N,\theta}(h)\\ &+C\left\{\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f)+\mathcal{E}^{\frac{1}{2}}_{N,\theta}(g)+\mathcal{E}_{N,\theta}(g)\right\}\mathcal{D}^{\frac{1}{2}}_{N,\theta}(f)\mathcal{D}^{\frac{1}{2}}_{N,\theta}(h). \qquad (2.43)\end{aligned}$$

Proof. Since (2.42) and (2.43) can be proved in the similar way, we only prove (2.43). Recalling Lemma 2.5, we compute only the following term for brevity.

$$T=\sum_{|\bar\alpha_2|+|\bar\beta_2'|\leqslant\frac{|\bar\beta_2|+|\alpha_2|}{2}}\int_{\mathbf{R}^3}\left|\tilde M^{\frac{1}{8}}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\right|_\infty\left|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\right|_{2,\theta+|\beta|}\left|\partial^{\alpha_1}_{\bar\beta_1}g\right|_{\sigma,\theta+|\beta|}\left|\partial^\alpha_\beta h\right|_{\sigma,\theta+|\beta|}dx. \qquad (2.44)$$

To compute (2.44), we divide our calculations into two parts. The first part is concerned with $\alpha>0$, and without lose of generality, we may suppose $|\alpha_2|\geqslant\frac{|\alpha|}{2}>0$. Therefore, if $|\alpha_2|+|\bar\beta_2|\leqslant\frac{N}{2}$, we have by splitting g into $g=\mathbf{P}g+(\mathbf{I}-\mathbf{P})g$ that

$$\begin{aligned}T\leqslant{}& C\int_{\mathbf{R}^3}\left\{\left|\partial^{\alpha_1}(a_g,b_g,c_g)\right|+\left|\partial^{\alpha_1}_{\bar\beta_1}(\mathbf{I}-\mathbf{P})g\right|_{\sigma,\theta+|\beta|}\right\}\left|\tilde M^{\frac{1}{8}}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\right|_\infty\left|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\right|_{2,\theta+|\beta|}\left|\partial^\alpha_\beta h\right|_{\sigma,\theta+|\beta|}dx\\ \leqslant{}& C\sup_x\left|\partial^{\alpha_1}(a_g,b_g,c_g)\right|\sup_{x,v}\left|\tilde M^{\frac{1}{8}}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\right|\int_{\mathbf{R}^3}\left|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\right|_{2,\theta+|\beta|}\left|\partial^\alpha_\beta h\right|_{\sigma,\theta+|\beta|}dx\\ &+C\sup_x\left|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2-\bar\beta_2'}f\right|_{2,\theta+|\beta|}\sup_{x,v}\left|\tilde M^{\frac{1}{8}}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\right|\int_{\mathbf{R}^3}\left|\partial^{\alpha_1}_{\bar\beta_1}(\mathbf{I}-\mathbf{P})g\right|_{\sigma,\theta+|\beta|}\left|\partial^\alpha_\beta h\right|_{\sigma,\theta+|\beta|}dx, \qquad (2.45)\end{aligned}$$

which can be further bounded by

$$C\mathcal{E}_{N,\theta}(f)\mathcal{D}^{\frac{1}{2}}_{N,\theta}(g)\mathcal{D}^{\frac{1}{2}}_{N,\theta}(h),$$

according to the following Sobolev's inequality:

$$\sup_x\left|\partial^{\alpha_1}(a_g,b_g,c_g)\right|\leqslant C\left\|\nabla_x\partial^{\alpha_1}(a_g,b_g,c_g)\right\|_{H^1_x}, \qquad (2.46)$$

$$\sup_{x,v}\left|\tilde M^{\frac{1}{8}}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\right|\leqslant C\left\|\nabla_x\partial^{\bar\alpha_2}(a_f,b_f,c_f)\right\|_{H^1_x}+C\left\|\tilde M^{\frac{1}{8}}\partial^{\bar\alpha_2}_{\bar\beta_2'}(\mathbf{I}-\mathbf{P})f\right\|_{H^4_{x,v}}, \qquad (2.47)$$

and

$$\begin{aligned}\sup_x\left|\partial^{\alpha_2}_{\bar\beta_2}f\right|^2_{2,\theta+|\beta|}\leqslant{}&\sum_{|\alpha'|\leqslant4}C\int_{\mathbf{R}^3\times\mathbf{R}^3}w^{2\theta+2|\beta|}\left|\partial^{\alpha_1'}\partial^{\alpha_2}_{\bar\beta_2}f\,\partial^{\alpha'-\alpha_1'}\partial^{\alpha_2}_{\bar\beta_2}f\right|dv\,dx\\ \leqslant{}& C\sum_{|\bar\alpha_2|+|\bar\beta_2|\leqslant N}\left\|\partial^{\bar\alpha_2}_{\bar\beta_2}f\right\|^2_{\theta+|\beta|}. \qquad (2.48)\end{aligned}$$

If $|\alpha_2|+|\bar\beta_2|\geqslant\frac{N}{2}$, by the same type estimates as (2.47) and (2.48) we obtain

$$T \leqslant C\sup_x\left|\partial^{\alpha_1}_{\bar\beta_1}g\right|_{\sigma,\theta+|\beta|}\sup_{x,v}\left|\tilde M^{\frac18}\partial^{\bar\alpha_2}_{\bar\beta_2'}f\right|\int_{\mathbf{R}^3}\left|\partial^{\alpha_2-\bar\alpha_2}_{\bar\beta_2}f\right|_{2,\theta+|\beta|}\left|\partial^{\alpha}_{\beta}h\right|_{\sigma,\theta+|\beta|}dx$$
$$\leqslant C\mathcal{E}_{N,\theta}(f)\mathcal{D}^{\frac12}_{N,\theta}(g)\mathcal{D}^{\frac12}_{N,\theta}(h). \tag{2.49}$$

As to the case $|\alpha|=0$, one can prove that (2.43) is true by using the same argument above. This completes the proof of Lemma 2.6. $\square$

For use later in the uniform spatial energy estimates, the following lemma is needed.

Lemma 2.7. *Assume $\gamma\geqslant -3$, $|\alpha|\leqslant N$, let $\zeta(v)$ be a smooth function that decays exponentially, then there's a $C_\zeta>0$ such that*

$$\left\|\int\partial^\alpha\Gamma(f,f)\zeta\,dv\right\|\leqslant C_\zeta\left\{\mathcal{E}^{\frac12}_N(f)+\mathcal{E}_N(f)\right\}\mathcal{D}^{\frac12}_N(f). \tag{2.50}$$

Lemma 2.8. *It holds that*

$$\begin{aligned}\left\|\Gamma(f,g)\right\|_{Z_1}\leqslant{}& C\sum_{|\beta_1|\leqslant2}\|w\partial_{\beta_1}g\|\sum_{|\beta_2|\leqslant2}\|\partial_{\beta_2}f\|+C\sum_{|\beta_2|\leqslant2}\|\partial_{\beta_2}f\|^2\\&+C\sum_{\substack{|\alpha_2|+|\beta_2|\leqslant6\\|\alpha_2|>0}}\left\|\partial^{\alpha_2}_{\beta_2}f\right\|\sum_{|\beta_1|\leqslant2}\|w\partial_{\beta_1}g\|\sum_{|\beta_2|\leqslant1}\|\partial_{\beta_2}f\|+C\sum_{|\beta_1|\leqslant2}\|\partial_{\beta_1}g\|^2\\&+C\sum_{\substack{|\alpha_1|+|\beta_1|\leqslant6\\|\alpha_1|>0}}\left\|\partial^{\alpha_1}_{\beta_1}g\right\|\sum_{|\beta_1|\leqslant1}\|w\partial_{\beta_1}g\|\sum_{|\beta_2|\leqslant2}\|\partial_{\beta_2}f\|.\end{aligned}\tag{2.51}$$

Proof. Recalling from (1.9) and (1.10), we see that

$$\begin{aligned}\Gamma_{bi}(f,g)={}&\left[\frac{v_iv_j}{4}\frac{(M-1)^2}{(M+1)^2}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_ig(v)\right]\psi^{ij}*\left[\tilde M^{\frac12}f\right]\\&-\partial_j\left\{\left[\frac{v_i}{2}\frac{M-1}{M+1}g(v)+\partial_ig(v)\right]\psi^{ij}*\left[\tilde M^{\frac12}f\right]\right\}\\&-\left[\frac{v_iv_j}{4}\frac{(M-1)^2}{(M+1)^2}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_ig(v)\right]\psi^{ij}*\left[2M_q\tilde M^{\frac12}f\right]\\&+\partial_j\left\{\left[\frac{v_i}{2}\frac{M-1}{M+1}g(v)+\partial_ig(v)\right]\psi^{ij}*\left[2M_q\tilde M^{\frac12}f\right]\right\}\\&-\left[\frac{v_j}{2}\frac{M-1}{M+1}g(v)\right]\psi^{ij}*\left[\frac{u_i}{2}\frac{M-1}{M+1}\tilde M^{\frac12}f(u)+\tilde M^{\frac12}\partial_if(u)\right]\\&+\partial_j\left\{g(v)\psi^{ij}*\left[\frac{u_i}{2}\frac{M-1}{M+1}\tilde M^{\frac12}f(u)+\tilde M^{\frac12}\partial_i\partial^{\alpha_2}f(u)\right]\right\}\\&+\left[\frac{v_j}{2}M_q\frac{M-1}{M+1}g(v)\right]\psi^{ij}*\left[\frac{u_i}{2}\frac{M-1}{M+1}\tilde M^{\frac12}f(u)+\tilde M^{\frac12}\partial_if(u)\right]\\&-\partial_j\left\{\left[2M_qg(v)\right]\psi^{ij}*\left[\frac{u_i}{2}\frac{M-1}{M+1}\tilde M^{\frac12}\partial^{\alpha_2}f(u)+\tilde M^{\frac12}\partial_i\partial^{\alpha_2}f(u)\right]\right\}\\={}&\sum_{i=1}^{8}J_i,\end{aligned}\tag{2.52}$$

and

$$\begin{aligned}
\Gamma_{non}(f,g) &= \left[\frac{v_i v_j}{4}\frac{(M-1)^2}{(M+1)^2}g(v)+\frac{v_j}{2}\frac{M-1}{M+1}\partial_i g(v)\right]\psi^{ij}*\left[\tilde{M}f^2\right] \\
&\quad -\partial_j\left\{\left[\frac{v_i}{2}\frac{M-1}{M+1}g(v)+\partial_i g(v)\right]\psi^{ij}*\left[\tilde{M}f^2\right]\right\} \\
&\quad -\left[\frac{v_i v_j}{2}\frac{(M-1)^2}{(M+1)^2}M^{\frac{1}{2}}(v)\right]\psi^{ij}*\left[\tilde{M}f^2\right]+\partial_j\left\{\left[v_i\frac{M-1}{M+1}M^{\frac{1}{2}}(v)\right]\psi^{ij}*\left[\tilde{M}f^2\right]\right\} \\
&\quad -\left[\frac{v_j}{2}\frac{M-1}{M+1}\tilde{M}^{\frac{1}{2}}g^2(v)\right]\psi^{ij}*\left[\frac{u_i}{2}\frac{M-1}{M+1}\tilde{M}^{\frac{1}{2}}f(u)+\tilde{M}^{\frac{1}{2}}\partial_i f(u)\right] \\
&\quad +\partial_j\left\{\tilde{M}^{\frac{1}{2}}g^2(v)\psi^{ij}*\left[\frac{u_i}{2}\frac{M-1}{M+1}\tilde{M}^{\frac{1}{2}}f(u)+\tilde{M}^{\frac{1}{2}}\partial_i\partial^{\alpha_2}f(u)\right]\right\} \\
&\quad +\left[\frac{v_j}{2}\tilde{M}^{\frac{1}{2}}\frac{M-1}{M+1}g^2(v)\right]\psi^{ij}*[u_i\tilde{M}]-\partial_j\left\{\left[\tilde{M}^{\frac{1}{2}}g^2(v)\right]\psi^{ij}*[u_i\tilde{M}]\right\} \\
&=\sum_{i=9}^{16}J_i. \qquad (2.53)
\end{aligned}$$

Let $g=f$ in (2.52) and (2.53), now we turn to compute $\|J_i\|_{Z^1}$ ($1\leqslant i\leqslant 16$) term by term. For J_1, by the generalized Minkowski inequality, we have

$$\|J_1\|_{Z^1}\leqslant\int_{\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}\left|\left[\frac{v_i}{2}\frac{(M-1)^2}{(M+1)^2}f(v)+\frac{M-1}{M+1}\partial_i f(v)\right]\psi^{ij}*\left[\frac{u_j}{2}\tilde{M}^{\frac{1}{2}}f\right]\right|^2dv\right\}^{\frac{1}{2}}dx, \qquad (2.54)$$

recall (2.28), the second term in the right hand side of (2.54) is bounded by

$$C\int_{\mathbf{R}^3}\left|\tilde{M}^{\frac{1}{8}}f\right|_2\left\{\int_{\mathbf{R}^3}\left|\left(1+|v|\right)^{\gamma+2}\partial_i f(v)\right|^2dv\right\}^{\frac{1}{2}}dx\leqslant C\sum_{|\beta_1|=1}\|w\partial_{\beta_1}f\|\|wf\|.$$

For the first term, applying the same argument as computing I_6 in Lemma 2.5, we obtain its upper bound $C\|wf\|\|f\|$. Similarly, one can verify that $\|J_i\|_{Z^1}$ ($2\leqslant i\leqslant 8$) are all bounded by

$$C\sum_{|\beta_1|\leqslant 2}\|w\partial_{\beta_1}f\|\sum_{|\beta_2|\leqslant 2}\|w\partial_{\beta_2}f\|.$$

We now consider J_9, the generalized Minkowski inequality yields

$$\|J_9\|_{Z^1}\leqslant\int_{\mathbf{R}^3}\left\{\int_{\mathbf{R}^3}\left|\left[\frac{v_i}{2}\frac{(M-1)^2}{(M+1)^2}f(v)+\frac{M-1}{M+1}\partial_i f(v)\right]\psi^{ij}*\left[\frac{u_j}{2}\tilde{M}f^2\right]\right|^2dv\right\}^{\frac{1}{2}}dx, \qquad (2.55)$$

the second term in the right hand side of (2.55) is no more than

$$C\sup_{x,u}\left|\tilde{M}^{\frac{1}{8}}f\right|\int_{\mathbf{R}^3}\left|\tilde{M}^{\frac{1}{8}}f\right|_2\left\{\int_{\mathbf{R}^3}\left|\left(1+|v|\right)^{\gamma+2}\partial_i f(v)\right|^2dv\right\}^{\frac{1}{2}}dx$$
$$\leqslant C\sum_{|\beta_1|=1}\|w\partial_{\beta_1}f\|\sum_{\substack{|\alpha_2|+|\beta_2|\leqslant 6\\|\alpha_2|>0}}\left\|\partial^{\alpha_2}_{\beta_2}f\right\|\|f\|,$$

according to the same type estimates as (2.28) and the Sobolev inequality:

$$\|\cdot\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)}\leqslant C\|\nabla_{x,v}\cdot\|_{H^4(\mathbf{R}^3\times\mathbf{R}^3)}. \tag{2.56}$$

Applying the same method as computing I_6 again, we get that the first term in (2.55) is bounded by

$$C\sum_{\substack{|\alpha_2|+|\beta_2|\leqslant 6\\|\alpha_2|>0}}\left\|\partial^{\alpha_2}_{\beta_2}f\right\|\|wf\|\|f\|.$$

And similarly,

$$\|J_{10}\|_{Z^1}\leqslant C\sum_{\substack{|\alpha_2|+|\beta_2|\leqslant 6\\|\alpha_2|>0}}\left\|\partial^{\alpha_2}_{\beta_2}f\right\|\sum_{|\beta_1|\leqslant 2}\|w\partial_{\beta_1}f\|\sum_{|\beta_2|\leqslant 1}\|\partial_{\beta_2}f\|, \tag{2.57}$$

$$\|J_{13}\|_{Z^1}\leqslant C\sum_{\substack{|\alpha_1|+|\beta_1|\leqslant 6\\|\alpha_1|>0}}\left\|\partial^{\alpha_1}_{\beta_1}f\right\|\|wf\|\sum_{|\beta_2|\leqslant 1}\|\partial_{\beta_2}f\|, \tag{2.58}$$

$$\|J_{14}\|_{Z^1}\leqslant C\sum_{\substack{|\alpha_1|+|\beta_1|\leqslant 6\\|\alpha_1|>0}}\left\|\partial^{\alpha_1}_{\beta_1}f\right\|\sum_{|\beta_1|\leqslant 1}\|w\partial_{\beta_1}f\|\sum_{|\beta_2|\leqslant 2}\|\partial_{\beta_2}f\|. \tag{2.59}$$

As to J_{11}, J_{12}, J_{16}, utilizing the Sobolev embedding $H^2(\mathbf{R}^3)\subset L^\infty(\mathbf{R}^3)$, we have

$$\|J_{11}\|_{Z^1}\leqslant C\int_{\mathbf{R}^3}\sup_u|f||f|_2\,dx\leqslant C\sum_{|\beta_2|\leqslant 2}\|\partial_{\beta_2}f\|\|f\|,$$
$$\|J_{12}\|_{Z^1}\leqslant C\sum_{|\bar\beta_2|\leqslant 1}\int_{\mathbf{R}^3}\sup_u|f||\partial_{\bar\beta_2}f|_2\,dx\leqslant C\sum_{|\beta_2|\leqslant 2}\|\partial_{\beta_2}f\|\sum_{|\bar\beta_2|\leqslant 1}\|\partial_{\bar\beta_2}f\|,$$
$$\|J_{15}\|_{Z^1}\leqslant C\int_{\mathbf{R}^3}\sup_v|f||f|_2\,dx\leqslant C\sum_{|\beta_1|\leqslant 2}\|\partial_{\beta_1}f\|\|f\|,$$
$$\|J_{16}\|_{Z^1}\leqslant C\sum_{|\bar\beta_1|\leqslant 1}\int_{\mathbf{R}^3}\sup_v|f||\partial_{\bar\beta_1}f|_2\,dx\leqslant C\sum_{|\beta_1|\leqslant 2}\|\partial_{\beta_1}f\|\sum_{|\bar\beta_1|\leqslant 1}\|\partial_{\bar\beta_1}f\|.$$

Thus the proof of Lemma 2.8 is completed. □

Lemma 2.9. *For any $\theta\geqslant 1$, it holds that*

$$\sum_{|\alpha|\leqslant 1}\left\|\partial^\alpha\Gamma(f,f)\right\|^2\leqslant C\left\{\mathcal{E}^h_{N,0}(f)+\left[\mathcal{E}^h_{N,0}(f)\right]^2\right\}\mathcal{E}_{N,\theta}(f). \tag{2.60}$$

Proof. Recall (2.52) and (2.53), let $f=g$, we have to estimate 16 terms. For brevity, we only compute $\|J_1\|^2$ and $\|J_9\|^2$ in the following. If $|\alpha|=0$, for J_1, we have

$$\|J_1\|^2=\int\limits_{\mathbf{R}^3\times\mathbf{R}^3}\left|\left[\frac{v_i}{2}\frac{(M-1)^2}{(M+1)^2}f(v)+\frac{M-1}{M+1}\partial_i f(v)\right]\psi^{ij}*\left[\frac{u_j}{2}\tilde{M}^{\frac{1}{2}}f\right]\right|^2 dv\,dx. \tag{2.61}$$

Set $\theta\geqslant 1$, in light of (2.56) and (2.28), the second term in the right hand side of (2.61) is bounded by

$$C\sup_{x,u}\big|\tilde{M}^{\frac{1}{8}}f\big|^2\int\limits_{\mathbf{R}^3\times\mathbf{R}^3}\big|\big(1+|v|\big)^{\gamma+2}\partial_i f(v)\big|^2 dv\,dx\leqslant C\mathcal{E}^h_{N,0}(f)\mathcal{E}_{N,\theta}(f). \tag{2.62}$$

For the first term, borrow the same procedure as estimating I_6, we can get the same upper bound as above.

As to J_9, we obtain

$$\|J_9\|^2=\int\limits_{\mathbf{R}^3\mathbf{R}^3}\left|\left[\frac{v_i}{2}\frac{(M-1)^2}{(M+1)^2}f(v)+\frac{M-1}{M+1}\partial_i f(v)\right]\psi^{ij}*\left[\frac{u_j}{2}\tilde{M}f^2\right]\right|^2 dv\,dx. \tag{2.63}$$

The second term in the right hand side of (2.63) is no more than

$$C\sup_{x,u}\big|\tilde{M}^{\frac{1}{8}}f\big|^4\int\limits_{\mathbf{R}^3\times\mathbf{R}^3}\big|\big(1+|v|\big)^{\gamma+2}\partial_i f(v)\big|^2 dv\,dx\leqslant C\big\{\mathcal{E}^h_{N,0}(f)\big\}^2\mathcal{E}_{N,\theta}(f), \tag{2.64}$$

according to (2.56) and (2.28). For the first term, one can get the upper bound by applying the same method as computing I_6 again. As to the case $|\alpha|=1$, one can prove that (2.60) is true by using the same argument above. Thus Lemma 2.9 is valid. □

3. The linearized equation

In this section, our goal is to estimate $\mathbf{P}f$ in terms of $(\mathbf{I}-\mathbf{P})f$. We first recall Kawashima's compensating function method. Let $\widetilde{W}$ denote the subspace of thirteen moments, precisely, $\widetilde{W}$ is defined as the space generated by $\mathcal{N}$ and the image of $\mathcal{N}$ under the mapping $f(v)\to v_j f(v)$, thus

$$\widetilde{W}=span\big\{\tilde{M}^{\frac{1}{2}}\phi_k\big\}_{k=1}^{13},$$

where $\phi_1=1$, $\phi_j=v_j$, $\phi_{j+4}=v_j^2$, $\phi_8=v_1v_2$, $\phi_9=v_2v_3$, $\phi_{10}=v_1v_3$, $\phi_{j+10}=|v|^2v_j$, $j=1,2,3$.

Note that $v\cdot\xi:\mathcal{N}(L)\to\widetilde{W}$ for all $\xi\in\mathbf{R}^3$, and that $\mathcal{N}(L)\subset\widetilde{W}$. We denote an orthogonal basis for this thirteen dimensional space by e_j $(1\leqslant j\leqslant 13)$. Let

$$\big\{\tilde{M}^{\frac{1}{2}}\phi_k\big\}_{k=1}^{13}O_{13\times 13}=[e_j]_{j=1}^{13},$$

where $\det O\neq 0$, and $[e_j]_{j=1}^{5}$ is the orthogonal basis of the null space of L. Let $\mathbf{P}_0$ be the orthogonal projection from $L^2(\mathbf{R}^3_v)$ into $\widetilde{W}$

$$\mathbf{P}_0 f=\sum_{k=1}^{13}\langle f,e_k\rangle e_k.$$

Now we consider the following linear quantum Landau equation

$$\partial_t f + v \cdot \nabla_x f + Lf = g. \tag{3.1}$$

From which and by putting $W_k = \langle f, e_k \rangle$ we have

$$\partial_t W + \sum_{j=1}^{3} V^j \partial_{x_j} W + \bar{L} W = \bar{g} + \bar{R},$$

where V^j $(j = 1, 2, 3)$ and $\bar{L}$ are symmetric matrices

$$V(\xi) = \sum_{j=1}^{3} V^j \xi_j = \left\{ \langle (v \cdot \xi) e_k, e_l \rangle \right\}_{k,l=1}^{13}, \qquad \bar{L} = \left\{ \langle L[e_l], e_k \rangle \right\}_{k,l=1}^{13},$$

$\bar{g}$ is the vector with components $\langle g, e_j \rangle$ and $\bar{R}$ is a sum of terms involving $(\mathbf{I} - \mathbf{P}_0) f$.

Next we introduce the notation

$$W = [W_I, W_{II}]^T, \qquad W_I = [W_1, W_2, \ldots, W_5]^T, \qquad W_{II} = [W_6, W_7, \ldots, W_{13}]^T$$

and write $\mathcal{R}z$ for the real part of $z \in \mathbb{C}$.

With the above preparations in hand, now we can define the compensating function of (3.1) as follows [11,16].

Definition 3.1. Let $S(\omega)$, $\omega \in \mathbf{S}^2$ be a bounded linear operator on $L^2(\mathbf{R}^3)$ is called a compensating function for (3.1) if:

(1) $S(\cdot)$ is C^∞ on $\mathbf{S}^2$ with values in the space of bounded linear operators on $L^2(\mathbf{R}^3)$, and $S(-\omega) = -S(\omega)$ for all $\omega \in \mathbf{S}^2$,
(2) $iS(\omega)$ is self-adjoint on $L^2(\mathbf{R}^3)$ for all $\omega \in \mathbf{S}^2$,
(3) there exists $c_0 > 0$ such that for all $\partial^\alpha f \in L^2(\mathbf{R}^3)$ and $\omega \in \mathbf{S}^2$,

$$\mathcal{R}\left\langle S(\omega)(v \cdot \omega) \partial^\alpha f, \partial^\alpha f \right\rangle + \left\langle L \partial^\alpha f, \partial^\alpha f \right\rangle \geqslant c_0 \left(\left| \partial^\alpha \mathbf{P} f \right|^2 + \left| \partial^\alpha (\mathbf{I} - \mathbf{P}) f \right|_\sigma^2 \right).$$

In order to find the compensating function $S(\omega)$ of (3.1), we need the following lemma which has been proved in [16].

Lemma 3.1. *There exist* 13×13 *real constant skew-symmetric matrices* R^j $(j = 1, 2, 3)$ *and positive constants* c_1, c_2 *such that for*

$$R(\omega) = \sum_{j=1}^{3} R^j \omega_j,$$

we have

$$\mathcal{R}\left\langle R(\omega) V(\omega) W \middle| W \right\rangle \geqslant c_1 |W_I|^2 - c_2 |W_{II}|^2,$$

for all $W \in \mathbb{C}^{13}$. *Here* $\langle \cdot | \cdot \rangle$ *denotes the complex inner product on* $\mathbb{C}^{13}$.

Next we exhibit a compensating function for (3.1). Given $\omega \in \mathbf{S}^2$, $\partial^\alpha f \in L^2(\mathbf{R}^3)$, we write $R(\omega) = \{r_{ij}\}_{i,j=1}^{13}$. Let

$$S(\omega)\partial^\alpha f = \sum_{k,l=1}^{13} \lambda r_{kl}(\omega)\langle \partial^\alpha f, e_l\rangle e_k,$$

for some $\lambda > 0$.

Lemma 3.2. *There exists $\lambda > 0$ such that $S(\omega)$ is a compensating function for* (3.1). *Moreover $S(\omega) : L^2(\mathbf{R}^3) \to \widetilde{W}$.*

Since (2.1) is obtained, the proof of Lemma 3.2 is the same as that of [16], we omit the details here.

Now, we are in a position to estimate the macroscopic component of the solution to (1.5) by virtue of the above compensating function.

Lemma 3.3. *For Eq.* (1.5), *we have the following estimate*:

$$\begin{aligned}
&\frac{d}{dt}\left\{ \sum_{|\alpha|\leqslant N} \left\|\partial^\alpha f\right\|^2 - \kappa \sum_{|\alpha|\leqslant N-1} \int_{\mathbf{R}^3} |\xi| \langle \mathrm{i} S(\omega)\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\rangle \, d\xi \right\} \\
&\quad + \delta_1 \sum_{1\leqslant|\alpha|\leqslant N} \left\|\partial^\alpha \mathbf{P} f\right\|^2 + \delta_2 \sum_{|\alpha|\leqslant N} \left\|\partial^\alpha (\mathbf{I}-\mathbf{P}) f\right\|_\sigma^2 \\
&\leqslant C\left\{\mathcal{E}_N^{\frac{1}{2}}(f) + \mathcal{E}_N(f) + \mathcal{E}_N^2(f)\right\}\mathcal{D}_N(f),
\end{aligned} \tag{3.2}$$

and especially,

$$\begin{aligned}
&\frac{d}{dt}\left\{ \sum_{1\leqslant|\alpha|\leqslant N} \left\|\partial^\alpha f\right\|^2 - \kappa \sum_{1\leqslant|\alpha|\leqslant N-1} \int_{\mathbf{R}^3} |\xi| \langle \mathrm{i} S(\omega)\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\rangle \, d\xi \right\} \\
&\quad + \delta_1 \sum_{2\leqslant|\alpha|\leqslant N} \left\|\partial^\alpha \mathbf{P} f\right\|^2 + \delta_2 \sum_{1\leqslant|\alpha|\leqslant N} \left\|\partial^\alpha (\mathbf{I}-\mathbf{P}) f\right\|_\sigma^2 \\
&\leqslant C\left\{\mathcal{E}_N^{\frac{1}{2}}(f) + \mathcal{E}_N(f) + \mathcal{E}_N^2(f)\right\}\mathcal{D}_N(f),
\end{aligned} \tag{3.3}$$

where $\delta_1 > 0$, $\delta_2 > 0$.

Proof. Let $\omega = \xi/|\xi|$, by acting ∂^α to (3.1) and taking the Fourier transform in x of the resulting equation, we have

$$\partial_t \widehat{\partial^\alpha f} + \mathrm{i}|\xi|(v\cdot\xi)\widehat{\partial^\alpha f} + L\widehat{\partial^\alpha f} = \widehat{\partial^\alpha g}. \tag{3.4}$$

Further taking complex inner product with $\widehat{\partial^\alpha f}$ and taking the real part yield

$$\frac{1}{2}\partial_t \left|\widehat{\partial^\alpha f}\right|_2^2 + \langle L\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\rangle = \mathcal{R}\langle \widehat{\partial^\alpha g}, \widehat{\partial^\alpha f}\rangle. \tag{3.5}$$

On the other hand, applying $-\mathrm{i}|\xi|S(\omega)$ to (3.4) gives

$$-\mathrm{i}|\xi|S(\omega)\partial_t\widehat{\partial^\alpha f} + |\xi|^2 S(\omega)\big((v\cdot\xi)\widehat{\partial^\alpha f}\big) - \mathrm{i}|\xi|S(\omega)L\widehat{\partial^\alpha f} = -\mathrm{i}|\xi|S(\omega)\widehat{\partial^\alpha g}. \tag{3.6}$$

By taking the inner product of the above equation with $\widehat{\partial^\alpha f}$ and taking the real part, we obtain

$$\begin{aligned}&\mathcal{R}\big\langle -\mathring{\imath}|\xi|S(\omega)\partial_t\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\big\rangle + |\xi|^2\mathcal{R}\big\langle S(\omega)\big((v\cdot\xi)\widehat{\partial^\alpha f}\big), \widehat{\partial^\alpha f}\big\rangle \\ &\quad = |\xi|\mathcal{R}\big\{\big\langle \mathring{\imath}S(\omega)L\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\big\rangle - \big\langle \mathring{\imath}S(\omega)\widehat{\partial^\alpha g}, \widehat{\partial^\alpha f}\big\rangle\big\}. \end{aligned} \tag{3.7}$$

Next, choosing $\kappa > 0$ small enough, Definition 3.1, Lemma 3.1, as well as (3.5) and (3.7) imply that there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\begin{aligned}&\frac{d}{dt}\big\{\big(1+|\xi|^2\big)\big|\widehat{\partial^\alpha f}\big|_2^2 - \kappa|\xi|\big\langle \mathring{\imath}S(\omega)\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\big\rangle\big\} \\ &\qquad + \delta_1|\xi|^2\big|\mathbf{P}\widehat{\partial^\alpha f}\big|^2 + \delta_2\big(1+|\xi|^2\big)\big|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}\big|_\sigma^2 \\ &\quad \leqslant \big(1+|\xi|^2\big)\mathcal{R}\big\langle \widehat{\partial^\alpha g}, \widehat{\partial^\alpha f}\big\rangle - \sum_{k=1}^{13}\mathcal{R}\big\langle \widehat{\partial^\alpha g}, e_k\big\rangle^2, \end{aligned} \tag{3.8}$$

where we also used the fact that $\langle Lf, e_k\rangle \leqslant C|(\mathbf{I}-\mathbf{P})f|_\sigma$. Integrating (3.8) over $\mathbf{R}^3$ with respect to ξ and summing over $0 \leqslant |\alpha| \leqslant N-1$ give

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2 - \sum_{|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\kappa|\xi|\big\langle \mathring{\imath}S(\omega)\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\big\rangle d\xi\Big\} \\ &\qquad + \delta_1\sum_{1\leqslant|\alpha|\leqslant N}\big\|\mathbf{P}\partial^\alpha f\big\|^2 + \delta_2\sum_{|\alpha|\leqslant N}\big\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\big\|_\sigma^2 \\ &\quad \leqslant \underbrace{\sum_{|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big(1+|\xi|^2\big)\mathcal{R}\big\langle \widehat{\partial^\alpha g}, \widehat{\partial^\alpha f}\big\rangle d\xi}_{I_{10}} - \underbrace{\sum_{k=1}^{13}\sum_{|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\mathcal{R}\big\langle \widehat{\partial^\alpha g}, e_k\big\rangle^2 d\xi}_{I_{11}}. \end{aligned} \tag{3.9}$$

Now, we turn to estimate I_{10} and I_{11} with $g = \Gamma(f,f)$. For I_{10}, we get from Parseval's identity that

$$\begin{aligned}|I_{10}| &\leqslant \sum_{|\alpha|\leqslant N-1}\big|\big(\partial^\alpha\Gamma(f,f), \partial^\alpha f\big)\big| \\ &\quad + \sum_{|\alpha|\leqslant N-1}\big|\big(\nabla_x\partial^\alpha\Gamma(f,f), \partial^\alpha f\big) + \big(\partial^\alpha\Gamma(f,f), \nabla_x\partial^\alpha f\big)\big|, \end{aligned} \tag{3.10}$$

which can be bounded by

$$C\big\{\mathcal{E}_N^{\frac{1}{2}}(f) + \mathcal{E}_N(f)\big\}\mathcal{D}_N(f),$$

according to Lemma 2.6. As to I_{11}, Lemma 2.7 and Plancherel's identity imply

$$|I_{11}| \leqslant C\big\{\mathcal{E}_N(f) + \mathcal{E}_N^2(f)\big\}\mathcal{D}_N(f).$$

We conclude from all the estimates above that (3.2) is valid, and the same argument can be used to prove that (3.3) is true. This completes the proof of the lemma. $\square$

For the study on the optimal time decay, we need the following estimate on the solution operator of the homogeneous linearized quantum Landau equation

$$\partial_t f + v \cdot \nabla_x f + Lf = 0, \qquad f(0,x,v) = f_0(x,v). \tag{3.11}$$

Formally, the solution to the Cauchy problem (3.11) can be written as the mild form

$$f(t) = S(t) f_0, \tag{3.12}$$

where $S(t)$ denotes the solution operator to the Cauchy problem of the linearized equation (3.11). The solution operator has the following decay estimates.

Lemma 3.4. *(See [11,16].) Suppose that $f_0 \in H^N \cap Z_q$, and assume that $f(t,x,v)$ defined in (3.12) is a solution of (3.11), we have*

$$\left\| \nabla_x^k S(t) f_0 \right\|^2 \leqslant C(1+t)^{-2\sigma_{q,k}} \left(\| f_0 \|_{Z_q}^2 + \left\| \nabla_x^k f_0 \right\|^2 \right), \tag{3.13}$$

where $q \in [1,2]$, $0 \leqslant k \leqslant N$ and the decay rate is measured by

$$\sigma_{q,k} = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{k}{2}.$$

4. Local existence

In this section, we construct local-in-time solution to the quantum Landau equation (1.2). Recall (1.3), the construction is based on a uniform energy estimate for the following sequence of iterating approximate solutions:

$$\{\partial_t + v \cdot \nabla_x\} F^{n+1} = Q\left(F^n, F^{n+1}\right), \qquad F^{n+1}(0,x,v) = F_0(x,v), \tag{4.1}$$

starting with $F^0(t,x,v) = F_0(x,v)$. Since $F^{n+1} = M_q + \sqrt{\tilde{M}} f^{n+1}$, equivalently, we need to solve f^{n+1} such that

$$\{\partial_t + v \cdot \nabla_x + A\} f^{n+1} + K f^n = \Gamma\left(f^n, f^{n+1}\right), \qquad f^{n+1}(0,x,v) = f_0(x,v). \tag{4.2}$$

Our discussion is based on the uniform bound in n for $\mathcal{E}_{N,\theta}(f^{n+1})$ for a small time interval. The main results is as follows.

Lemma 4.1. *Assume $\gamma \geqslant -3$, for any sufficiently small $W_0 > 0$, there exists $T^*(W_0) > 0$ such that if*

$$\mathcal{E}_{N,\theta}(f_0) \leqslant W_0,$$

then there is a unique classical solution $f(t,x,v)$ to the quantum Landau equation (1.5) in $[0, T^*(W_0)) \times \mathbf{R}^3 \times \mathbf{R}^3$ such that

$$\sup_{0 \leqslant t \leqslant T^*(W_0)} \mathcal{E}_{N,\theta}(f)(t) + c \int_0^t \overline{\mathcal{D}}_{N,\theta}(f)(s)\, ds \leqslant C W_0, \tag{4.3}$$

where $c, C > 0$ and $\overline{\mathcal{D}}_{N,\theta}(f)(t)$ is defined as

$$\overline{\mathcal{D}}_{N,\theta}(g)(t) \sim \begin{cases} \sum_{|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_{\sigma,\theta}, & \gamma \geqslant -2, \\ \sum_{|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta (\mathbf{I}-\mathbf{P}) f\|^2_{\sigma,\theta+|\beta|}, & -3 \leqslant \gamma < -2. \end{cases}$$

Moreover, $\mathcal{E}_{N,\theta}(f)(t)$ is continuous over $[0, T^*(W_0))$. If $0 \leqslant F_0(x, v) = M_q + \sqrt{\tilde{M}} f_0 \leqslant 1$, then

$$0 \leqslant F(t, x, v) = M_q + \sqrt{\tilde{M}} f(t, x, v) \leqslant 1.$$

Proof. Recall $F^n = M_q + \sqrt{\tilde{M}} f^n$, for each $n \geqslant 0$, we first claim that

$$y^n(t) \leqslant C W_0, \tag{4.4}$$

where $y^n(t)$ is defined as

$$y^n(t) \sim \sup_{0\leqslant s\leqslant t} \mathcal{E}_{N,\theta}\big(f^n\big)(s) + \int_0^t \overline{\mathcal{D}}_{N,\theta}\big(f^n\big)\, ds. \tag{4.5}$$

To verify (4.4), we use an induction over k. Clearly $k = 0$ is valid and we assume (4.4) is true for $k = n$ so that $0 \leqslant F^n(t, x, v) \leqslant 1$. We notice that the quantum Landau collision operator in (4.1) has the non-divergence form of

$$\begin{aligned} Q\left(F^n, F^{n+1}\right) &= \left\{\psi^{ij} * \left[F^n\left(1 - F^n\right)\right]\right\} \partial_{ij} F^{n+1} + \left\{\psi^{ij} * \partial_i\left[F^n\left(1 - F^n\right)\right]\right\} \partial_j F^{n+1} \\ &\quad - \left\{\partial_{ij}\psi^{ij} * F^n\right\}\left[F^{n+1}\left(1 - F^{n+1}\right)\right], \end{aligned} \tag{4.6}$$

provided $\gamma > -3$. On the other hand, one can get that

$$\begin{aligned} -\partial_{ij}\psi^{ij} &= 2(\gamma+3)|v|^\gamma \geqslant 0, \quad \gamma > -3, \\ -\left\{\partial_{ij}\psi^{ij} * F^n\right\} &= 8\pi F^n \geqslant 0, \quad \gamma = -3. \end{aligned}$$

Therefore, by following the same procedure as Appendix in [2] or by the generalized maximum principle in [22], we can see that (4.1) has a smooth solution F^{n+1} satisfying $0 \leqslant F^{n+1} \leqslant 1$ for some time $T_1^* > 0$. We now turn to prove (4.4). In order to make our presentation easy to read, we divide our computations into following three steps.

Step 1. *Mixed derivatives*: Taking ∂^α_β ($|\alpha| + |\beta| \leqslant N$, $|\beta| \neq 0$) of (4.2), we have

$$\begin{aligned} &[\partial_t + v\cdot\nabla_x]\partial^\alpha_\beta f^{n+1} + \partial_\beta A\left[\partial^\alpha f^{n+1}\right] + \partial_\beta K\left[\partial^\alpha f^n\right] + C^{\beta_1}_\beta \partial_{\beta_1} v \cdot \partial_{\beta-\beta_1}\nabla_x f^{n+1} \\ &\quad = \partial^\alpha_\beta \Gamma\left(f^n, f^{n+1}\right), \end{aligned} \tag{4.7}$$

where $|\beta_1| \geqslant 1$. If $\gamma \geqslant -2$, taking the inner product of (4.7) over $\mathbf{R}^3 \times \mathbf{R}^3$ with $w^{2\theta}\partial^\alpha_\beta f^{n+1}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|w^\theta \partial^\alpha_\beta f^{n+1}\right\|^2 + \left(\partial_\beta A\left[\partial^\alpha f^{n+1}\right], w^{2\theta}\partial^\alpha_\beta f^{n+1}\right) + \left(\partial_\beta K\left[\partial^\alpha f^n\right], w^{2\theta}\partial^\alpha_\beta f^{n+1}\right) \\ &\quad = -\left(C^{\beta_1}_\beta \partial_{\beta_1} v \cdot \partial_{\beta-\beta_1}\nabla_x f^{n+1}, w^{2\theta}\partial^\alpha_\beta f^{n+1}\right) + \left(\partial^\alpha_\beta \Gamma\left(f^n, f^{n+1}\right), w^{2\theta}\partial^\alpha_\beta f^{n+1}\right). \end{aligned} \tag{4.8}$$

We now estimate (4.8) term by term. Applying Lemma 2.3, we deduce, for any $\eta > 0$,

$$\big(\partial_\beta A\big[\partial^\alpha f^{n+1}\big], w^{2\theta}\partial^\alpha_\beta f^{n+1}\big) \geqslant \big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta} - \eta\sum_{\beta_1\leqslant\beta}\big\|\partial^\alpha_{\beta_1} f^{n+1}\big\|^2_{\sigma,\theta} - C_{\eta,|\beta|}\big\|\partial^\alpha f^{n+1}\big\|^2_\sigma,$$

$$\begin{aligned}\big(\partial_\beta K\big[\partial^\alpha f^{n}\big], w^{2\theta}\partial^\alpha_\beta f^{n+1}\big) &\leqslant \Big\{\eta\sum_{\beta_1\leqslant\beta}\big\|\partial^\alpha_{\beta_1} f^{n}\big\|_{\sigma,\theta} + C_{\eta,|\beta|}\big\|\partial^\alpha f^{n}\big\|_\sigma\Big\}\big\|\partial^\alpha_\beta f^{n+1}\big\|_{\sigma,\theta}\\ &\leqslant \eta\sum_{\beta_1\leqslant\beta}\big\|\partial^\alpha_{\beta_1} f^{n}\big\|^2_{\sigma,\theta} + \eta\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta} + C_{\eta,|\beta|}\big\|\partial^\alpha f^{n}\big\|^2_\sigma.\end{aligned}$$

And if $-3 \leqslant \gamma < -2$, taking the inner product of (4.7) over $\mathbf{R}^3 \times \mathbf{R}^3$ with $w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|w^{\theta+|\beta|}\partial^\alpha_\beta f^{n+1}\big\|^2 + \big(\partial_\beta A\big[\partial^\alpha f^{n+1}\big], w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big) + \big(\partial_\beta K\big[\partial^\alpha f^{n}\big], w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big)\\ &\quad= -\big(C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\partial_{\beta-\beta_1}\nabla_x f^{n+1}, w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big) + \big(\partial^\alpha_\beta\Gamma\big(f^n, f^{n+1}\big), w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big).\end{aligned}\tag{4.9}$$

Since $w^{|\beta_1|} \geqslant w^{|\beta|}$ for $\beta_1 \leqslant \beta$, one can get from Lemma 2.3 that

$$\begin{aligned}&\big(\partial_\beta A\big[\partial^\alpha f^{n+1}\big], w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big)\\ &\quad\geqslant \big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta+|\beta|} - \eta\sum_{\beta_1\leqslant\beta}\big\|\partial^\alpha_{\beta_1} f^{n+1}\big\|^2_{\sigma,\theta+|\beta_1|} - C_{\eta,|\beta|}\big\|\partial^\alpha f^{n+1}\big\|^2_\sigma,\\ &\big(\partial_\beta K\big[\partial^\alpha f^{n}\big], w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big) \leqslant \eta\sum_{\beta_1\leqslant\beta}\big\|\partial^\alpha_{\beta_1} f^{n}\big\|^2_{\sigma,\theta+|\beta_1|} + C_{\eta,|\beta|}\big\|\partial^\alpha f^{n}\big\|^2_\sigma.\end{aligned}$$

For the free streaming term, if $\gamma \geqslant -2$, we get from Cauchy–Schwarz's inequality with η that

$$\big(C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\partial_{\beta-\beta_1}\nabla_x f^{n+1}, w^{2\theta}\partial^\alpha_\beta f^{n+1}\big) \leqslant C_\eta\sum_{|\bar\beta|=|\beta|-1}\big\|\partial^\alpha_{\bar\beta} f^{n+1}\big\|^2_{\sigma,\theta} + \eta\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta}.$$

If $-3 \leqslant \gamma < -2$, we have

$$\big(C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\partial_{\beta-\beta_1}\nabla_x f^{n+1}, w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big) \leqslant C_\eta\sum_{|\bar\beta|=|\beta|-1}\big\|\partial^\alpha_{\bar\beta} f^{n+1}\big\|^2_{\sigma,\theta+|\bar\beta|} + \eta\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta+|\beta|}.$$

For the nonlinear term, by Lemma 2.5, we see that

$$\big(\partial^\alpha_\beta\Gamma\big(f^n, f^{n+1}\big), w^{2\theta}\partial^\alpha_\beta f^{n+1}\big)$$

and

$$\big(\partial^\alpha_\beta\Gamma\big(f^n, f^{n+1}\big), w^{2(\theta+|\beta|)}\partial^\alpha_\beta f^{n+1}\big)$$

are bounded by

$$
\begin{aligned}
& C\mathcal{E}_{N,\theta}\big(f^n\big)\big\{\overline{\mathcal{D}}^{\frac{1}{2}}_{N,\theta}\big(f^{n+1}\big)+\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\big\}+C\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}^{\frac{1}{2}}_{N,\theta}\big(f^{n}\big)\overline{\mathcal{D}}^{\frac{1}{2}}_{N,\theta}\big(f^{n+1}\big) \\
&\quad + C\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}^{\frac{1}{2}}_{N,\theta}\big(f^{n+1}\big) \\
&\leqslant C\mathcal{E}_{N,\theta}\big(f^n\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)+C\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)+C\mathcal{E}^2_{N,\theta}\big(f^n\big) \\
&\quad + C\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n}\big)+C\mathcal{E}^2_{N,\theta}\big(f^{n+1}\big)+\eta\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big),
\end{aligned}
$$

where we have used the Cauchy–Schwarz inequality with η.

Next, integrating (4.8) over $[0,t]$, collecting the above estimates and summing up over $|\alpha|+|\beta|\leqslant N$, $|\beta|=1,2,\ldots,$ we obtain directly

$$
\begin{aligned}
& \sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_\theta + c_1\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\int_0^t\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta}\,ds \\
&\leqslant \sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}} C\big\|\partial^\alpha_\beta f^{n+1}(0)\big\|^2_\theta + C\int_0^t\mathcal{D}_{N,\theta}\big(f^n\big)\,ds + C\sum_{|\alpha|\leqslant N}\int_0^t\big\|\partial^\alpha f^{n+1}\big\|^2_{\sigma,\theta}\,ds \\
&\quad + C\int_0^t\mathcal{E}_{N,\theta}\big(f^n\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds + C\int_0^t\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds \\
&\quad + C\int_0^t\mathcal{E}^2_{N,\theta}\big(f^n\big)\,ds + C\int_0^t\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n}\big)\,ds \\
&\quad + C\int_0^t\mathcal{E}^2_{N,\theta}\big(f^{n+1}\big)\,ds, \quad \text{for } \gamma\geqslant -2.
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
& \sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\theta+|\beta|} + c_1\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\int_0^t\big\|\partial^\alpha_\beta f^{n+1}\big\|^2_{\sigma,\theta+|\beta|}\,ds \\
&\leqslant \sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}} C\big\|\partial^\alpha_\beta f^{n+1}(0)\big\|^2_{\theta+|\beta|} + C\int_0^t\mathcal{D}_{N,\theta}\big(f^n\big)\,ds + C\sum_{|\alpha|\leqslant N}\int_0^t\big\|\partial^\alpha f^{n+1}\big\|^2_{\sigma,\theta}\,ds \\
&\quad + C\int_0^t\mathcal{E}_{N,\theta}\big(f^n\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds + C\int_0^t\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds \\
&\quad + C\int_0^t\mathcal{E}^2_{N,\theta}\big(f^n\big)\,ds + C\int_0^t\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n}\big)\,ds \\
&\quad + C\int_0^t\mathcal{E}^2_{N,\theta}\big(f^{n+1}\big)\,ds, \quad \text{for } -3\leqslant\gamma<-2.
\end{aligned}
\tag{4.11}
$$

Step 2. *x-Derivatives with weight*: We now consider the pure x-derivatives with velocity weight. Taking ∂^α ($|\alpha| \leqslant N$) of (4.2), we have

$$[\partial_t + v \cdot \nabla_x]\partial^\alpha f^{n+1} + A\big[\partial^\alpha f^{n+1}\big] + K\big[\partial^\alpha f^n\big] = \partial^\alpha \Gamma\big(f^n, f^{n+1}\big). \tag{4.12}$$

Therefore, applying Lemma 2.5 yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|w^\theta \partial^\alpha f^{n+1}\big\|^2 + \big(A\big[\partial^\alpha f^{n+1}\big], w^{2\theta}\partial^\alpha f^{n+1}\big) + \big(K\big[\partial^\alpha f^n\big], w^{2\theta}\partial^\alpha f^{n+1}\big) \\
&\quad \leqslant C\mathcal{E}_{N,\theta}\big(f^n\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big) + C\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big) + C\mathcal{E}^2_{N,\theta}\big(f^n\big) \\
&\qquad + C\mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^n\big) + C\mathcal{E}^2_{N,\theta}\big(f^{n+1}\big) + \eta\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big).
\end{aligned} \tag{4.13}$$

Furthermore, integrating the above over $[0, t]$ yields

$$\begin{aligned}
&\frac{1}{2}\big\|w^\theta \partial^\alpha f^{n+1}\big\|^2 + \int_0^t \big(A\big[\partial^\alpha f^{n+1}\big], w^{2\theta}\partial^\alpha f^{n+1}\big)\,ds + \int_0^t \big(K\big[\partial^\alpha f^n\big], w^{2\theta}\partial^\alpha f^{n+1}\big)\,ds \\
&\quad \leqslant C\int_0^t \mathcal{E}_{N,\theta}\big(f^n\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds + C\int_0^t \mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds \\
&\qquad + C\int_0^t \mathcal{E}^2_{N,\theta}\big(f^n\big)\,ds + C\int_0^t \mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^n\big)\,ds + C\int_0^t \mathcal{E}^2_{N,\theta}\big(f^{n+1}\big)\,ds.
\end{aligned} \tag{4.14}$$

We notice that from Remark 2.1,

$$\begin{aligned}
&\int_0^t \big(A\big[\partial^\alpha f^{n+1}\big], w^{2\theta}\partial^\alpha f^{n+1}\big)\,ds + \int_0^t \big(K\big[\partial^\alpha f^n\big], w^{2\theta}\partial^\alpha f^{n+1}\big)\,ds \\
&\quad \geqslant \frac{1}{2}\int_0^t \big\|\partial^\alpha f^{n+1}\big\|^2_{\sigma,\theta}\,ds - C\int_0^t \big\|\partial^\alpha f^{n+1}\big\|^2_\sigma\,ds - \eta\int_0^t \big\|\partial^\alpha f^n\big\|^2_{\sigma,\theta}\,ds - C_\eta\int_0^t \big\|\partial^\alpha f^n\big\|^2_\sigma\,ds.
\end{aligned} \tag{4.15}$$

Step 3. *x-Derivatives without weight*: We now turn to estimate the pure x-derivatives without velocity weight. Taking the inner product of (4.12) with $\partial^\alpha f^{n+1}$ and utilizing Lemma 2.5, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\partial^\alpha f^{n+1}\big\|^2 + \big(A\big[\partial^\alpha f^{n+1}\big], \partial^\alpha f^{n+1}\big) + \big(K\big[\partial^\alpha f^n\big], \partial^\alpha f^{n+1}\big) \\
&\quad \leqslant C\mathcal{E}_N\big(f^n\big)\overline{\mathcal{D}}_N\big(f^{n+1}\big) + C\mathcal{E}_N\big(f^{n+1}\big)\overline{\mathcal{D}}_N\big(f^{n+1}\big) + C\mathcal{E}^2_N\big(f^n\big) \\
&\qquad + C\mathcal{E}_N\big(f^{n+1}\big)\overline{\mathcal{D}}_N\big(f^n\big) + C\mathcal{E}^2_N\big(f^{n+1}\big) + \eta\overline{\mathcal{D}}_N\big(f^{n+1}\big).
\end{aligned} \tag{4.16}$$

On the other hand, one get from Remark 2.1 that

$$\int_0^t \big(A\big[\partial^\alpha f^{n+1}\big], \partial^\alpha f^{n+1}\big)\,ds + \int_0^t \big(K\big[\partial^\alpha f^{n}\big], \partial^\alpha f^{n+1}\big)\,ds$$
$$\geqslant \frac{1}{2}\int_0^t \big\|\partial^\alpha f^{n+1}\big\|_\sigma^2\,ds - C\int_0^t \big\|\tilde{M}^{\frac{1}{16}}\partial^\alpha f^{n+1}\big\|^2\,ds - \eta\int_0^t \big\|\partial^\alpha f^{n}\big\|_\sigma^2\,ds - C_\eta\int_0^t \big\|\tilde{M}^{\frac{1}{16}}\partial^\alpha f^{n}\big\|^2\,ds. \tag{4.17}$$

Then a suitable linear combination of (4.10)–(4.17) yields

$$\begin{aligned}
&\mathcal{E}_{N,\theta}\big(f^{n+1}\big)(t) + c_2\int_0^t \overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds \\
&\quad\leqslant \mathcal{E}_{N,\theta}\big(f^{n+1}\big)(0) + C\int_0^t \overline{\mathcal{D}}_{N,\theta}\big(f^{n}\big)\,ds + C\sum_{|\alpha|\leqslant N}\int_0^t \big\|\partial^\alpha f^{n+1}\big\|^2\,ds \\
&\qquad + C\int_0^t \mathcal{E}_{N,\theta}\big(f^{n}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds + C\int_0^t \mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n+1}\big)\,ds \\
&\qquad + C\int_0^t \mathcal{E}^2_{N,\theta}\big(f^{n}\big)\,ds + C\int_0^t \mathcal{E}_{N,\theta}\big(f^{n+1}\big)\overline{\mathcal{D}}_{N,\theta}\big(f^{n}\big)\,ds + C\int_0^t \mathcal{E}^2_{N,\theta}\big(f^{n+1}\big)\,ds,
\end{aligned} \tag{4.18}$$

where $0 < c_2 < 1$.

Recall (4.5), we deduce from (4.18) that

$$y^{n+1}(t) \leqslant y^{n+1}(0) + Cy^n(t) + Cty^{n+1}(t) + Cy^n(t)y^{n+1}(t) + C\big(y^{n+1}(t)\big)^2 + C\big(y^n(t)\big)^2. \tag{4.19}$$

Choosing $T^*(W_0) \leqslant T_1^*$ small enough, letting $0 \leqslant t \leqslant T^*(W_0)$, and noticing $y^n(t) \leqslant CW_0$, one get from (4.19) that

$$y^{n+1}(t) \leqslant CW_0 + C\big(y^{n+1}(t)\big)^2,$$

which implies

$$y^{n+1}(t) \leqslant CW_0,$$

for $0 \leqslant t \leqslant T^*(W_0)$. Thus (4.4) is true. Finally, by taking $n \to +\infty$ in (4.4), we obtain a classical solution $f(t,x,v)$ which satisfies (4.3). The proof for the uniqueness of $f(t,x,v)$ is the same as that of Theorem 4 in [12], we omit the details. This completes the proof of Lemma 4.1. □

Remark 4.1. Notice that in this local existence result, $\overline{\mathcal{D}}_{N,\theta}(f)$ includes the macroscopic part $\|\mathbf{P}f\|^2$, therefore it is stronger than the dissipation rate $\mathcal{D}_{N,\theta}(f)$.

5. Energy estimates on the nonlinear equation

In this section, we establish our Theorem 1.1, which follows from the local existence together with the uniform a priori estimates as well as the standard continuum argument. Now we derive the uniform a priori estimates.

Lemma 5.1. *Under the conditions listed in Theorem* 1.1, *there exist* $\lambda_N > 0$ *and* $C > 0$ *such that*

$$\frac{d}{dt}\mathcal{E}_N(f) + \lambda_N \mathcal{D}_N(f) \leqslant C\big\{\mathcal{E}_N^{\frac{1}{2}}(f) + \mathcal{E}_N(f) + \mathcal{E}_N^2(f)\big\}\mathcal{D}_N(f). \tag{5.1}$$

Proof. Let $-3 \leqslant \gamma < -2$, in order to make our presentation clear, we divide our computation into following two steps. The first step is concerned with the case that $\beta = 0$. Recalling Lemma 3.3, and noticing that

$$\sum_{|\alpha| \leqslant N-1} \int_{\mathbf{R}^3} \kappa|\xi|\big\langle \mathrm{i}S(\omega)\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\big\rangle d\xi \leqslant \sum_{|\alpha| \leqslant N} \|\partial^\alpha f\|^2, \tag{5.2}$$

one can see that (5.1) is true in the case of $\beta = 0$. The second step is devoted to the estimates for x, v derivatives. Letting $|\alpha| + |\beta| \leqslant N$ and $|\beta| \geqslant 1$, taking ∂^α_β of (1.5) to get

$$\begin{aligned}
&\{\partial_t + v \cdot \nabla_x\}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f + \partial^\alpha_\beta L(\mathbf{I}-\mathbf{P})f \\
&\quad = -\{\partial_t + v \cdot \nabla_x\}\partial^\alpha_\beta \mathbf{P}f - \sum_{|\beta_1'|>0} C_\beta^{\beta_1'} \partial_{\beta_1'} v \cdot \nabla_x \partial^\alpha_{\beta-\beta_1'} f + \partial^\alpha_\beta \Gamma(f,f).
\end{aligned} \tag{5.3}$$

Taking the inner product of (5.3) over $\mathbf{R}^3 \times \mathbf{R}^3$ with $w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|^2_{|\beta|} + \big(\partial^\alpha_\beta L(\mathbf{I}-\mathbf{P})f, w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big) \\
&\quad = \underbrace{-\big(\{\partial_t + v\cdot\nabla_x\}\partial^\alpha_\beta \mathbf{P}f, w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)}_{I_{12}} \underbrace{- \sum_{|\beta_1'|>0} C_\beta^{\beta_1'}\big(\partial_{\beta_1'} v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1'} f, w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)}_{I_{13}} \\
&\qquad + \underbrace{\big(\partial^\alpha_\beta \Gamma(f,f), w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)}_{I_{14}}.
\end{aligned} \tag{5.4}$$

We apply Lemma 2.4 to get that

$$\big(\partial^\alpha_\beta L(\mathbf{I}-\mathbf{P})f, w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big) \geqslant \frac{1}{2}\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|^2_{\sigma,|\beta|} - C_{|\beta|}\big\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2_\sigma. \tag{5.5}$$

We are now in a position to compute I_j $(12 \leqslant j \leqslant 14)$ term by term. There are two parts in I_{12}, we estimate each part as follows. For the first part, we obtain

$$\begin{aligned}
\big|\big(\partial^\alpha_\beta \partial_t \mathbf{P}f, w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)\big| &\leqslant C\big|\big(\partial^\alpha \partial_t(a,b,c), w^{2|\beta|}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)\big| \\
&\leqslant \epsilon\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|^2_{\sigma,|\beta|} + C(\epsilon)\sum_{|\alpha|\leqslant N-1}\big\|\partial^\alpha \partial_t(a,b,c)\big\|^2,
\end{aligned} \tag{5.6}$$

according to the Cauchy–Schwarz inequality with ϵ. Next, we have by taking the inner product of (1.5) over $\mathbf{R}^3\times\mathbf{R}^3$ with $\partial_t\partial^\alpha\mathbf{P}f$ that

$$C(\epsilon)\sum_{|\alpha|\leqslant N-1}\left\|\partial^\alpha\partial_t(a,b,c)\right\|^2\leqslant C\sum_{|\alpha|\leqslant N}\left\|\nabla\partial^\alpha f\right\|^2+C\left\{\mathcal{E}_N^2(f)+\mathcal{E}_N(f)\right\}\mathcal{D}_N(f), \tag{5.7}$$

where we have used Lemma 2.7 to estimate $\sum_{|\alpha|\leqslant N-1}\|\langle\partial^\alpha\Gamma(f,f),\zeta\rangle\|^2$.

For the second part in I_{12}, it is bounded by

$$C(\epsilon)\sum_{0<|\alpha|\leqslant N}\left\|\partial^\alpha(a,b,c)\right\|^2+\epsilon\sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_{\sigma,|\beta|}^2, \tag{5.8}$$

according to the Cauchy–Schwarz inequality with ϵ.

Utilizing the Cauchy–Schwarz inequality with ϵ again, I_{13} can be estimated as follows

$$\epsilon\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_{\sigma,|\beta|}^2+C(\epsilon)\sum_{\substack{|\alpha|+|\beta'|\leqslant N-1\\|\beta'|=|\beta|-1}}\left\|\partial_{\beta'}^\alpha\nabla_x f\right\|_{\sigma,|\beta'|}^2. \tag{5.9}$$

Applying the nonlinear estimates in Lemma 2.6, we get

$$|I_{14}|\leqslant C\left\{\mathcal{E}_N^{\frac{1}{2}}(f)+\mathcal{E}_N(f)\right\}\mathcal{D}_N(f). \tag{5.10}$$

Finally, putting all the estimates above together, summing over $0\leqslant|\beta|\leqslant N$ and adjusting constants, we can see that (5.1) with $-3\leqslant\gamma<-2$ is true. The case of $\gamma\geqslant-2$ follows the same argument without using the weight function $w^{|\beta|}$. Thus the proof of Lemma 5.1 is completed. □

Moreover, we have the following weighted energy estimates.

Lemma 5.2. *Under the conditions listed in Theorem* 1.1, *for any $\theta>0$, there exist $\bar\lambda_N>0$ and $C>0$ such that*

$$\frac{d}{dt}\mathcal{E}_{N,\theta}(f)+\bar\lambda_N\mathcal{D}_{N,\theta}(f)\leqslant C\left\{\mathcal{E}_{N,\theta}^{\frac{1}{2}}(f)+\mathcal{E}_{N,\theta}(f)+\mathcal{E}_N^2(f)\right\}\mathcal{D}_{N,\theta}(f). \tag{5.11}$$

Proof. As in Lemma 5.1, we only discuss the case $-3\leqslant\gamma<-2$, for this, we divide our proof into following three parts. The first part is devoted to the case that $|\alpha|=0$, $|\beta|=0$ and $\theta>0$. Letting $|\alpha|=0$, $|\beta|=0$ in (5.3), taking the inner product of the resulting equation with $w^{2\theta}(\mathbf{I}-\mathbf{P})f$ over $\mathbf{R}^3\times\mathbf{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|(\mathbf{I}-\mathbf{P})f\right\|_\theta^2+\underbrace{\left(L(\mathbf{I}-\mathbf{P})f,w^{2\theta}(\mathbf{I}-\mathbf{P})f\right)}_{I_{15}}\\&\quad=\underbrace{-\left(\{\partial_t+v\cdot\nabla_x\}\mathbf{P}f,w^{2\theta}(\mathbf{I}-\mathbf{P})f\right)}_{I_{16}}\underbrace{+\left(\Gamma(f,f),w^{2\theta}(\mathbf{I}-\mathbf{P})f\right)}_{I_{17}}.\end{aligned} \tag{5.12}$$

Now we turn to estimate I_j $(15\leqslant j\leqslant17)$ term by term. Firstly, we have get from Remark 2.1 that

$$I_{15}\geqslant\frac{1}{2}\left\|(\mathbf{I}-\mathbf{P})f\right\|_{\sigma,\theta}^2-C_\theta\left\|(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2. \tag{5.13}$$

By Cauchy–Schwarz's inequality with ϵ, we can see that

$$|I_{16}| \leqslant C(\epsilon)\big\|\partial_t(a,b,c)\big\|^2 + C(\epsilon)\big\|\nabla(a,b,c)\big\|^2 + \epsilon\big\|(\mathbf{I}-\mathbf{P})f\big\|^2_{\sigma,\theta}. \tag{5.14}$$

For the first term in (5.14), just applying the same argument as (5.7) we get

$$\big\|\partial_t(a,b,c)\big\|^2 \leqslant C\big\|\nabla(a,b,c)\big\|^2 + C\big\|\nabla(\mathbf{I}-\mathbf{P})f\big\|^2_{\sigma} + \big\{\mathcal{E}_N^2(f)+\mathcal{E}_N(f)\big\}\mathcal{D}_N(f). \tag{5.15}$$

Applying the nonlinear estimates in Lemma 2.6, we get

$$|I_{17}| \leqslant C\big\{\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f)+\mathcal{E}_{N,\theta}(f)\big\}\mathcal{D}_N(f). \tag{5.16}$$

Combing the estimates (5.12)–(5.16) together, we deduce that

$$\frac{d}{dt}\big\|(\mathbf{I}-\mathbf{P})f\big\|^2_{\theta} + \lambda_1\big\|(\mathbf{I}-\mathbf{P})f\big\|^2_{\sigma,\theta} \leqslant C\mathcal{D}_N(f) + C\big\{\mathcal{E}^{1/2}_{N,\theta}(f)+\mathcal{E}_{N,\theta}(f)\big\}\mathcal{D}_{N,\theta}(f), \tag{5.17}$$

where $\lambda_1 > 0$.

The second part is concerned with that case that $|\beta| = 0$, $1 \leqslant |\alpha| \leqslant N$. Letting $|\alpha|+|\beta| \leqslant N$, taking ∂^α of (1.5) to get

$$\big\{\partial_t\partial^\alpha + v\cdot\nabla_x\partial^\alpha\big\}f + \partial^\alpha Lf = \partial^\alpha \Gamma(f,f). \tag{5.18}$$

Multiplying the above equation by $w^{2\theta}\partial^\alpha f$, taking the integrations in x, v and applying Remark 2.1, one has

$$\frac{d}{dt}\big\|\partial^\alpha f\big\|^2_{\theta} + \lambda_2\big\|\partial^\alpha f\big\|^2_{\sigma,\theta} \leqslant C_\theta \sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2_{\sigma} + \big|\big(\partial^\alpha\Gamma(f,f), w^{2\theta}\partial^\alpha f\big)\big|, \tag{5.19}$$

where $\lambda_2 > 0$. For the second term on the right hand side of (5.19), by Lemma 2.6 we have

$$I_{16} \leqslant C\big\{\mathcal{E}^{1/2}_{N,\theta}(f)+\mathcal{E}_{N,\theta}(f)\big\}\mathcal{D}_{N,\theta}(f). \tag{5.20}$$

Therefore, putting the estimates (5.19) and (5.20) together, we have

$$\frac{d}{dt}\big\|\partial^\alpha f\big\|^2_{\theta} + \lambda_2\big\|\partial^\alpha f\big\|^2_{\sigma,\theta} \leqslant C\mathcal{D}_N(f) + C\big\{\mathcal{E}^{1/2}_{N,\theta}(f)+\mathcal{E}_{N,\theta}(f)\big\}\mathcal{D}_{N,\theta}(f). \tag{5.21}$$

Now, we turn to the remaining third part $|\beta| \geqslant 1$, $|\alpha|+|\beta| \leqslant N$. Applying $\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f$ to (5.3), taking the inner product of the resulting equation over $\mathbf{R}^3\times\mathbf{R}^3$ with $w^{2(\theta+|\beta|)}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|^2_{\theta+|\beta|} + \underbrace{\big(\partial^\alpha_\beta L(\mathbf{I}-\mathbf{P})f, w^{2(\theta+|\beta|)}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)}_{I_{18}} \\
&\quad = \underbrace{-\big(\{\partial_t + v\cdot\nabla_x\}\partial^\alpha_\beta \mathbf{P}f, w^{2(\theta+|\beta|)}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)}_{I_{19}}
\end{aligned}$$

$$\underbrace{-\sum_{|\beta_1'|>0} C_\beta^{\beta_1'}\left(\partial_{\beta_1'} v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1'} f, w^{2(\theta+|\beta|)}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right)}_{I_{20}}$$
$$+\underbrace{\left(\partial^\alpha_\beta\Gamma(f,f), w^{2(\theta+|\beta|)}\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right)}_{I_{21}}. \tag{5.22}$$

Now we turn to estimate I_j $(18\leqslant j\leqslant 21)$ term by term.

For I_{18}, by Lemma 2.4, we obtain

$$\begin{aligned} I_{18} &\geqslant \frac{1}{2}\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma,\theta+|\beta|} - C_{|\beta|,\theta}\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma,\theta+|\beta|} \\ &\geqslant \frac{1}{2}\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma,\theta+|\beta|} - C_{|\beta|,\theta}\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma}, \end{aligned} \tag{5.23}$$

since $w\leqslant 1$ in this case.

For I_{19}, by performing the similar calculations as I_{12} one can get that

$$\begin{aligned} |I_{19}| \leqslant{}& C(\epsilon)\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial^\alpha f\right\|^2 + \epsilon\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma,\theta+|\beta|} \\ &+ C\left\{\mathcal{E}_N(f)+\mathcal{E}_N^2(f)\right\}\mathcal{D}_N(f). \end{aligned} \tag{5.24}$$

According to Cauchy–Schwarz's inequality, I_{20} is no more than

$$C(\epsilon)\sum_{|\beta_1|=|\beta|-1}\left\|\nabla_x\partial^\alpha_{\beta_1} f\right\|^2_{\sigma,\theta+|\beta_1|} + \epsilon\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma,\theta+|\beta|}. \tag{5.25}$$

By Lemma 2.6, we can see that

$$|I_{21}| \leqslant C\left\{\mathcal{E}^{1/2}_{N,\theta}(f)+\mathcal{E}_{N,\theta}(f)\right\}\mathcal{D}_{N,\theta}(f). \tag{5.26}$$

Therefore, we get from (5.22)–(5.26) that there exists $\lambda_3>0$ such that

$$\begin{aligned} &\frac{d}{dt}\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{\theta+|\beta|} + \lambda_3\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{\sigma,\theta+|\beta|} \\ &\quad\leqslant C\mathcal{D}_N(f) + C\left\{\mathcal{E}^{1/2}_{N,\theta}(f)+\mathcal{E}_{N,\theta}(f)+\mathcal{E}_N^2(f)\right\}\mathcal{D}_{N,\theta}(f). \end{aligned} \tag{5.27}$$

A linear combination of (5.1), (5.17), (5.21) and (5.27) yields that (5.11) is true. This completes the proof of Lemma 5.2 with $-3\leqslant\gamma<-2$. The proof for the case of $\gamma\geqslant-2$ is similar and much easier. Therefore Lemma 5.2 holds for all potentials. Recall $\mathcal{E}^{\frac{1}{2}}_{N,\theta}(f)$ is sufficiently small uniformly in time, then (1.17) is true for all $\theta\geqslant 0$, this completes the first part of Theorem 1.1. Now we turn to prove the second part of Theorem 1.1. Firstly, we denote

$$\begin{aligned} \widetilde{\mathcal{E}}_{N,\theta}(f) ={}& \sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha_\beta f\right\|^2_\theta + \sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_\theta \\ &- \kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\left\langle \mathrm{i}S(\omega)\widehat{\partial^\alpha f}, \widehat{\partial^\alpha f}\right\rangle d\xi, \quad \text{for } \gamma\geqslant-2, \end{aligned} \tag{5.28}$$

and

$$\widetilde{\mathcal{E}}_{N,\theta}(f)=\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial_\beta^\alpha f\big\|_{\theta+|\beta|}^2+\sum_{|\alpha|+|\beta|\leqslant N}\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_{\theta+|\beta|}^2 \\ -\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle \mathrm{i}S(\omega)\widehat{\partial^\alpha f},\widehat{\partial^\alpha f}\big\rangle d\xi,\quad \text{for } -3\leqslant\gamma<-2, \tag{5.29}$$

then from Lemma 3.3, (5.17), (5.21) and (5.27), we deduce that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_{N,\theta}(f)+c\mathcal{D}_{N,\theta}(f)\leqslant C\|\nabla_x\mathbf{P}f\|^2+C\big\{\mathcal{E}_{N,\theta}^{1/2}(f)+\mathcal{E}_{N,\theta}(f)+\mathcal{E}_N^2(f)\big\}\mathcal{D}_{N,\theta}(f). \tag{5.30}$$

Noticing (5.2) and the definition (1.15) for $\mathcal{E}_N^h$, we have

$$\frac{d}{dt}\mathcal{E}_{N,\theta}^h(f)+c\mathcal{D}_{N,\theta}(f)\leqslant C\|\nabla_x\mathbf{P}f\|^2+C\big\{\mathcal{E}_{N,\theta}^{1/2}(f)+\mathcal{E}_{N,\theta}(f)++\mathcal{E}_N^2(f)\big\}\mathcal{D}_{N,\theta}(f), \tag{5.31}$$

which implies the desired estimate (1.18) since $\mathcal{E}_{N,\theta}^{\frac{1}{2}}(f)$ is small enough uniformly in all $t\geqslant 0$. This completes the second part of Theorem 1.1. □

6. Time decay for the nonlinear equation

In this section, we turn to prove Theorem 1.2. And the following proofs on the optimal decay estimates are motivated by the work of Duan, Ukai, Yang, Zhao [10] and Yang, Yu [27–29] on the Boltzmann equation and related models. In order to make our presentation easy to read, we divide our proofs into following two parts, the first part is devoted to getting the decay estimate for higher instant energy. From the definition (1.16) and (1.14) it is easy to see that

$$\mathcal{D}_{N,\theta}(t)\geqslant\mathcal{E}_{N,\theta}^h(t)-\|\nabla_x\mathbf{P}f\|^2. \tag{6.1}$$

Therefore we get from Lemma 5.2 that

$$\frac{d}{dt}\mathcal{E}_{N,\theta}^h(t)+c\mathcal{E}_{N,\theta}^h(t)\leqslant C\|\nabla_x\mathbf{P}f\|^2. \tag{6.2}$$

On the other hand, by using Lemma 3.1 we obtain

$$\|\nabla_x\mathbf{P}f\|^2\leqslant\|\nabla_x f\|^2\leqslant C\lambda_0(1+t)^{-\frac{5}{2}}+C\left[\int_0^t(1+t-s)^{-\frac{5}{4}}\big(\big\|\Gamma(f,f)\big\|_{Z_1}+\big\|\nabla\Gamma(f,f)\big\|\big)\,ds\right]^2,$$

where $\lambda_0=(\|f_0\|_{Z_1}^2+\|\nabla_x f_0\|^2)$. Moreover by Lemmas 2.9, 2.8 and the micro–macro decomposition (1.12), we have

$$\big\|\nabla_x\Gamma(f,f)\big\|\leqslant C\Big\{\sqrt{\mathcal{E}_N^h(t)}+\mathcal{E}_N^h(f)\Big\}\mathcal{E}_{N,1}^{\frac{1}{2}}(f), \tag{6.3}$$

and

$$
\begin{aligned}
\big\|\Gamma(f,f)\big\|_{Z_1} \leqslant{}& C\sum_{|\beta_1|\leqslant 2}\|w\partial_{\beta_1}f\|\|\partial_{\beta_1}f\| + C\sum_{|\beta_2|\leqslant 2}\|\partial_{\beta_2}f\|^2\\
&+C\sum_{|\beta_1|\leqslant 2}\|w\partial_{\beta_1}f\|^2\sum_{\substack{|\alpha_2|+|\beta_2|\leqslant 6\\ |\alpha_2|\geqslant 1}}\big\|\partial^{\alpha_2}_{\beta_2}f\big\|\\
\leqslant{}& C\sum_{|\beta_1|\leqslant 2}\big\|w\partial_{\beta_1}(\mathbf{I}-\mathbf{P})f\big\|^2 + C\|\mathbf{P}f\|^2 + C\mathcal{E}_{N,1}(t)\sqrt{\mathcal{E}^h_N(t)}.
\end{aligned}
\tag{6.4}
$$

Define

$$
M(t)=\sup_{0\leqslant s\leqslant t}\big\{(1+s)^{\frac{5}{2}}\mathcal{E}^h_{N,\theta}(t)\big\},\qquad M_0(t)=\sup_{0\leqslant s\leqslant t}\big\{(1+s)^{\frac{3}{2}}\|f\|^2\big\}.
\tag{6.5}
$$

Then we easily have

$$
\begin{aligned}
\|\nabla_x\mathbf{P}f\|^2 \leqslant{}& C\lambda_0(1+t)^{-\frac{5}{2}}\\
&+C\big\{\big[\mathcal{E}_{N,1}(0)+\mathcal{E}^2_{N,1}(0)\big]M(t)+M_0^2(t)\big\}\left[\int_0^t(1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}}\,ds\right]^2\\
\leqslant{}& C(1+t)^{-\frac{5}{2}}\big\{\lambda_0+\big[\mathcal{E}_{N,1}(0)+\mathcal{E}^2_{N,1}(0)\big]M(t)+M_0^2(t)\big\}.
\end{aligned}
\tag{6.6}
$$

On the other hand, we have from (6.2) that

$$
\mathcal{E}^h_{N,\theta}(t)\leqslant e^{-ct}\mathcal{E}^h_{N,\theta}(0)+\int_0^t e^{-c(t-s)}\big\|\nabla\mathbf{P}f(s)\big\|^2\,ds.
\tag{6.7}
$$

Substituting (6.6) into (6.7), we can deduce that

$$
\begin{aligned}
\mathcal{E}^h_{N,\theta}(t)\leqslant{}& e^{-ct}\mathcal{E}^h_{N,\theta}(0)\\
&+\big\{\lambda_0+\big[\mathcal{E}_{N,1}(0)+\mathcal{E}^2_{N,1}(0)\big]M(t)+M_0^2(t)\big\}\int_0^t e^{-c(t-s)}(1+s)^{-\frac{5}{2}}\,ds.
\end{aligned}
\tag{6.8}
$$

Then letting $\mathcal{E}_{N,1}(0)$ small enough, we can get

$$
\mathcal{E}^h_{N,\theta}(t)\leqslant C\big(\epsilon_{N,\theta\vee 1}+M_0^2(t)\big)(1+t)^{-\frac{5}{2}}.
\tag{6.9}
$$

The second part is concerned with the decay estimate for $\|f(t)\|$. By the definition (1.16) and (1.15) it is easy to see that

$$
\mathcal{D}_N(t)\geqslant\mathcal{E}_N(t)-C\|\mathbf{P}f\|^2,
\tag{6.10}
$$

from which and (5.31), we deduce that

$$
\frac{d}{dt}\mathcal{E}_N(f)+c\mathcal{E}_N(f)\leqslant C\|\mathbf{P}f\|^2,
\tag{6.11}
$$

for some suitable $c>0$.

Next, employing Lemma 3.1 again, we obtain

$$\|\mathbf{P}f\|^2 \leqslant \|f\|^2 \leqslant C\lambda_1(1+t)^{-\frac{3}{2}} + C\left[\int_0^t (1+t-s)^{-\frac{3}{4}}\left(\left\|\Gamma(f,f)\right\|_{Z_1} + \left\|\Gamma(f,f)\right\|\right)ds\right]^2, \tag{6.12}$$

where $\lambda_1 = \|f_0\|_{Z_1}^2 + \|f_0\|^2$. Performing the similar calculations as (6.3), (6.4), we have

$$\begin{aligned}\left\|\Gamma(f,f)\right\|_{Z_1} + \left\|\Gamma(f,f)\right\| \leqslant C\left\{\sqrt{\mathcal{E}_N^h(t)} + \mathcal{E}_N^h(t)\right\}\mathcal{E}_{N,1}^{\frac{1}{2}}(t) \\ + \mathcal{E}_{N,1}^{\frac{1}{2}}(t)\sqrt{\mathcal{E}_{N,1}^h(t)} + \|\mathbf{P}f\|^2 + \mathcal{E}_{N,1}(t)\sqrt{\mathcal{E}_N^h(t)}.\end{aligned}$$

From which and (6.12), (6.9), we get

$$\begin{aligned}\|\mathbf{P}f\|^2 &\leqslant C\lambda_1(1+t)^{-\frac{3}{2}} + C\left\{\epsilon_{N,\theta\vee1}\left[\mathcal{E}_{N,1}^{\frac{1}{2}}(0) + \mathcal{E}_{N,1}(0)\right]^2 \right. \\ &\quad \left. + M_0^2(t)\right\}\left[\int_0^t (1+t-s)^{-\frac{5}{4}}(1+t)^{-\frac{3}{4}}\,ds\right]^2 \\ &\leqslant C\left\{\epsilon_{N,\theta\vee1} + M_0^2(t)\right\}(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{6.13}$$

Recalling (6.11), we deduce

$$\begin{aligned}\left\|f(t)\right\|^2 &\leqslant e^{-\lambda t}\mathcal{E}_N^h(0) + C\left\{\epsilon_{N,\theta\vee1} + M_0^2(t)\right\}\int_0^t e^{-c(t-s)}(1+s)^{-\frac{3}{2}}\,ds \\ &\leqslant (1+t)^{-\frac{3}{2}}\mathcal{E}_N(0) + C(1+t)^{-\frac{3}{2}}\left\{\epsilon_{N,\theta\vee1} + M_0^2(t)\right\} \\ &\leqslant C\epsilon_{N,\theta\vee1}(1+t)^{-\frac{3}{2}} + CM_0^2(t)(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{6.14}$$

Since $\epsilon_{N,\theta\vee1}$ can be small enough, we have from (6.14) that

$$\left\|f(t)\right\|^2 \leqslant C\epsilon_{N,\theta\vee1}(1+t)^{-\frac{3}{2}},$$

from which and (6.9), we obtain

$$\mathcal{E}_{N,\theta}^h(t) \leqslant C\epsilon_{N,\theta\vee1}(1+t)^{-\frac{5}{2}}.$$

Hence (1.21) holds. This completes the proof of Theorem 1.2.

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