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Weighted decay properties for the incompressible Stokes flow and Navier–Stokes equations in a half space

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ABSTRACT

Using the Stokes solution formula and L^q–L^r estimates of the Stokes operator semigroup, we establish the weighted decay properties for the Stokes flow and Navier–Stokes equations including their spatial derivatives in half spaces. In addition, the unboundedness of the projection operator $P: L^\infty(\mathbb{R}^n_+) \to L^\infty_\sigma(\mathbb{R}^n_+)$ is overcome by employing a decomposition for the nonlinear term, and L^∞-asymptotic behavior for the second derivatives of Navier–Stokes flows in half spaces is given.

1. Introduction and main results

In this paper, we are concerned with the weighted asymptotic behavior for the incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0,\infty), \\ u(x,0) = u_0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $n \geqslant 2$, and $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is the upper-half space of $\mathbb{R}^n$; $u = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ denote unknown velocity vector and the pressure respectively, while initial data $u_0(x)$ is assumed to satisfy a *compatibility condition*: $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$ and the normal component of u_0 equals to zero on $\partial\mathbb{R}^n_+$.

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Definition. u is called a weak solution of (1.1) if

$$u \in L^\infty\big(0,\infty; L^2_\sigma(\mathbb{R}^n_+)\big) \cap L^2_{loc}\big(0,\infty; H^1_0(\mathbb{R}^n_+)\big) \quad \text{satisfies}$$
$$-\int_0^\infty \int_{\mathbb{R}^n_+} u\partial_t\varphi \,dx\,dt + \int_0^\infty \int_{\mathbb{R}^n_+} \nabla u \cdot \nabla\varphi \,dx\,dt + \int_0^\infty \int_{\mathbb{R}^n_+} u\cdot\nabla u\cdot\varphi \,dx\,dt$$
$$= \int_{\mathbb{R}^n_+} u_0\varphi(0)\,dx \quad \text{for all } \varphi \in C_0^\infty\big([0,\infty); C^\infty_{0,\sigma}(\mathbb{R}^n_+)\big),$$

where $u_0 \in L^2_\sigma(\mathbb{R}^n_+) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_{L^2}}$, $C^\infty_{0,\sigma}(\mathbb{R}^n_+) = \{u \in C^\infty_0(\mathbb{R}^n_+);\ \nabla\cdot u = 0 \text{ in } \mathbb{R}^n_+\}$.

Let A denote the Stokes operator $-P\Delta$ in $\mathbb{R}^n_+$, where P is the projection: $L^r(\mathbb{R}^n_+) \to L^r_\sigma(\mathbb{R}^n_+)$, $1<r<\infty$. Then the solution u of Stokes problem

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \nabla\cdot u = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0,\infty), \\ u(x,0) = u_0 & \text{in } \mathbb{R}^n_+, \end{cases}$$

can be expressed by $u(t) = e^{-tA}u_0$. In the whole space $\mathbb{R}^n$, the Stokes flow $e^{-tA}u_0$ behaves just like that of the heat equation with initial data u_0. Moreover, for all $1 \leqslant q \leqslant \infty$, $\|\nabla e^{-tA}u_0\|_{L^q(\mathbb{R}^n)} \leqslant Ct^{-\frac{1}{2}}\|u_0\|_{L^q(\mathbb{R}^n)}$, which is valid for the half space $\mathbb{R}^n_+$ with $1<q<\infty$ (see [38]). The L^q–L^1 estimate of the Stokes flow in the half space has been derived by Desch, Hieber and Pruss [13], Borchers and Miyakawa [11] and Fujigaki and Miyakawa [15] via the semigroup method for $1<q\leqslant\infty$. Jin [24] obtained the weighted L^q–L^1 estimate of the Stokes flow in the half space with $1<q<\infty$. In addition, Jin [25] derived the spatial and temporal decay estimate of the Stokes solution in the half space when the prescribed initial data lies in a weighted L^1 space.

Fujigaki and Miyakawa [15] derived more rapid L^r-estimate of the Stokes flow with initial data in the weighted L^1 space:

$$\big\|e^{-tA}u_0\big\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\|x_n u_0\|_{L^1(\mathbb{R}^n_+)}$$

for any $u_0 \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$, whenever $1<r\leqslant\infty$.

Bae [3] considered more rapid L^1- and L^∞-estimates with an initial data with the special assumption $\int_{-\infty}^\infty u_0(y)\,dy_i = 0$ for some $i = 1,2,\dots,n-1$, and in this case

$$\big\|e^{-tA}u_0\big\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2}}\|x_n u_0\|_{L^r(\mathbb{R}^n_+)} \quad \text{for } r = 1,\infty.$$

In addition, if the initial data u_0 lies in an appropriate weighted space, Bae and Choe [4] showed the faster decay rate in $L^q(\mathbb{R}^n_+)$ $(1<q<\infty)$ of $u, \nabla u$. Furthermore, Bae [1,2] estimated the time decay rates of the gradient of Stokes solutions in $L^r(\mathbb{R}^n_+)$ with $r = 1,\infty$.

The decay rate for the first derivatives of the Stokes flow in $L^r_\sigma(\mathbb{R}^n_+)$ with $r = 1,\infty$ has been shown by Giga, Matsui and Shimizu [17] with $r=1$, Shimizu [33] with $r=\infty$ respectively. That is,

$$\big\|\nabla e^{-tA}u_0\big\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2}}\|u_0\|_{L^r(\mathbb{R}^n_+)} \quad \text{for } r = 1,\infty.$$

The present work is motivated by the results obtained by Bae [1–3], Bae and Jin [8], Fujigaki and Miyakawa [15] and Jin [24,25]. Now we state our main results as follows.

Theorem 1.1. *Let $u_0 = (u_{01}, u_{02}, \dots, u_{0n})$ satisfy $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$ $(n \geqslant 2)$, and $u_{0n}|_{\partial\mathbb{R}^n_+} = 0$. Then for any $t > 0$*

$$\left\|\nabla^k e^{-tA} u_0\right\|_{L^q(\mathbb{R}^n_+)} \leqslant C t^{-\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \left\|x_n^\alpha u_0\right\|_{L^r(\mathbb{R}^n_+)}, \quad \forall 0 \leqslant \alpha \leqslant 1,\ k = 0, 1, 2, \dots,$$

provided that $1 \leqslant r < q \leqslant \infty$.

There is a great literature on the decay rates for the Navier–Stokes flows of (1.1). Brandolese [10], Fujigaki and Miyakawa [15], Schonbek [31,32] considered the decay rates of solutions of (1.1) on the whole space $\mathbb{R}^n$. Bae and Choe [4], Borchers and Miyakawa [11], Kozono [26] studied asymptotic behavior for weak and strong solutions of (1.1) in $L^q(\mathbb{R}^n_+)$ with $1 < q < \infty$. For the exterior domains, see [5–7,9,20,21,27,28,30] for examples and the references therein.

Bae and Jin [8] showed the weighted energy inequality for weak solutions of the Cauchy problem under some conditions on the initial data. The situation changes in the case of the domain with boundary. The difficulty comes from the lack of the weighted estimate with respect to pressure because of the appearance of the boundary. In the case of exterior domain, Farwig [14] constructed a class of weak solutions such that the weighted energy inequality holds. He and Miyakawa [22] addressed the time-decay of weighted norms of weak and strong solutions to the Navier–Stokes equations in a 3D exterior domain under some constraint conditions on the initial data.

While in the half space case, these mentioned arguments cannot be applied because the boundary is non-compact. Recently, Choe and Jin [12] deduced the decay rates with respect to $\|x_3 u(t)\|_{L^2(\mathbb{R}^3_+)}$. Bae [3] and Fröhlich [16] showed the local existence of strong solution in weighted L^r-spaces, respectively. Under some constraint conditions on the initial data, He and Wang [23] considered the weighted energy inequality and the L^2-weighted decay properties for weak solutions of (1.1). To our knowledge, few weighted decay results are available on solutions of (1.1) and its first derivatives in $L^r(\mathbb{R}^n_+)$ with $1 < r \leqslant \infty$.

To solve problem (1.1), people usually invoke the projection P onto the solenoidal vector fields to eliminate the pressure gradient ∇p in (1.1) and then transform problem (1.1) into the integral equation: $u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} Pu(s) \cdot \nabla u(s)\,ds$. In the case of the Cauchy problem, the projection P commutes with the Laplacian Δ; so the semigroup $\{e^{-tA}\}_{t\geqslant 0}$ is essentially equal to the heat semigroup $\{e^{t\Delta}\}_{t\geqslant 0}$. Moreover, P can be written in terms of the Riesz transforms. However, all of these techniques are not applicable to problem (1.1) on the half spaces $\mathbb{R}^n_+$, because the projection operator $P : L^\infty(\mathbb{R}^n_+) \to L^\infty_\sigma(\mathbb{R}^n_+)$ is unbounded, which results in many difficulties in dealing with L^∞-asymptotic behavior of the second derivatives for the strong solution of (1.1).

It is known that if $u_0 \in L^2_\sigma(\mathbb{R}^n_+) \cap L^n(\mathbb{R}^n_+)$ $(n \geqslant 2)$. There exists a number $\eta_0 > 0$ such that if $\|u_0\|_{L^n(\mathbb{R}^n_+)} \leqslant \eta_0$ (small condition is unnecessary if $n = 2$), then problem (1.1) possesses a unique global strong solution. Whence, in the statement of our main results, we always assume that problem (1.1) admits a global strong solution.

Theorem 1.2. *Let $1 < r \leqslant \infty$ and $0 < \beta < \min\{1, n(1 - \frac{1}{r})\}$ $(n \geqslant 2)$. Assume $u_0 \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+) \cap W^{1,\frac{r}{r-1}}(\mathbb{R}^n_+)$ satisfies $\|x_n u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1 + x_n)\nabla u_0\|_{L^2(\mathbb{R}^n_+)} < \infty$. Let u be the strong solution of (1.1). Then there exists $t_0 > 0$ such that for any $t > t_0$*

$$\left\|x_n^\beta u(t)\right\|_{L^r(\mathbb{R}^n_+)} + \left\|x_n^\beta \nabla u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant C t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \tag{1.2}$$

with any $n \geqslant 3$ and $1 < r \leqslant \infty$.

Further if u_0 satisfies $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. Then the estimate (1.2) holds true for any $n \geqslant 2$ and $1 < r \leqslant \infty$.

Remark. When the half space (or some other fluid region with boundary) is concerned, pressure estimate is main obstacle since we do not have enough information of the pressure on the boundary. Let $1 < r \leqslant \infty$ and $0 < \beta < \min\{1, n(1 - \frac{1}{r})\}$, it is unknown whether the weighted energy inequality for the strong solution u of (1.1) holds:

$$\|x_n^\beta u(t)\|^2_{L^r(\mathbb{R}^n_+)} + \int_0^t \|x_n^\beta \nabla u(s)\|^2_{L^r(\mathbb{R}^n_+)}\,ds \leqslant C, \quad \forall t > 0.$$

Under some additional conditions on the initial data u_0, similar energy inequality is proved to be true for $r = 2$, $n = 3$ (see [23] for example).

Theorem 1.3. *Assume $u_0 \in L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$ $(n \geqslant 2)$ for all $1 < q < \infty$. Let u be a strong solution of (1.1) defined on $(0, \infty)$. Then for any $t > 0$*

$$\|\nabla^2 u(t)\|_{L^\infty(\mathbb{R}^n_+)} \leqslant \begin{cases} Ct^{-\frac{3}{2}}(1 + t^{-4}) & \text{if } n = 2; \\ Ct^{-\frac{n+2}{2}}(1 + t^{-\frac{5n-3}{2}}) & \text{if } n \geqslant 3. \end{cases}$$

Further if u_0 satisfies $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, it holds for any $t > 0$

$$\|\nabla^2 u(t)\|_{L^\infty(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{n+3}{2}}\big(1 + t^{-\frac{5n+2}{2}}\big).$$

Remark. Under the additional assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, Theorem 1.3 shows that the L^∞-decay rate of the second derivatives of the strong solution u of (1.1) becomes faster than observed. It should be pointed out that Theorem 1.3 has been verified by the author in [19]. However, the proof is incomplete in [19], because the following crucial estimate is employed in the proof, which in fact is unknown to us. Let $1 < q < \infty$. Assume that $a = (a_1, a_2, \ldots, a_n) \in W^{1,q}(\mathbb{R}^n_+)$ $(n \geqslant 2)$ satisfies $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ and $a_n|_{\partial\mathbb{R}^n_+} = 0$. Then for any $t > 0$

$$\|\nabla^2 e^{-tA} a\|_{L^\infty(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2} - \frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)}.$$

Up to now, we only can prove the weaker conclusion (see Lemma 3.4 below), which in fact is enough to the proof of Theorem 1.3.

To conclude this introduction, we explain some notations used in what follows: Let $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ denote the set of all C^∞ real vector-valued functions $\phi = (\phi_1, \ldots, \phi_n)$ with compact support in $\mathbb{R}^n_+$, such that $\nabla \cdot \phi = 0$ in $\mathbb{R}^n_+$. $L^q_\sigma(\mathbb{R}^n_+)$ $(1 < q < \infty)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ with respect to $\|\cdot\|_{L^q(\mathbb{R}^n_+)}$, where $L^q(\mathbb{R}^n_+)$ represents the usual Lebesgue space of vector-valued functions. The norm of $L^\infty(\mathbb{R}^n_+)$ is denoted by $\|u\|_{L^\infty(\mathbb{R}^n_+)} = \operatorname{ess\,sup}_{x \in \mathbb{R}^n_+} |u(x)|$. By symbol C, we denote a generic positive constant whose value may change from line to line.

2. Weighted L^q–L^r estimates for the Stokes flows in the half space $\mathbb{R}^n_+$

It is well known that the Hardy space $\mathcal{H}^q(\mathbb{R}^n)$ with $1 \leqslant q < \infty$ (see the definition in [29,36] for example) is a Banach space, and $\mathcal{H}^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ if $1 < q < \infty$; $\mathcal{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. The crucial fact for our purpose is the boundedness of the Riesz transforms R_j $(1 \leqslant j \leqslant n)$ on all of the spaces $\mathcal{H}^q(\mathbb{R}^n)$ with $1 \leqslant q < \infty$. Furthermore, a function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if $\sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n)$, where the symbol $*$ denotes the convolution with respect to the space variable x. The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \triangleq \Big\| \sup_{s>0} |G_s * f| \Big\|_{L^1(\mathbb{R}^n)}.$$

It is known (see [29,36]) that an L^1-function f is in $\mathcal{H}^1(\mathbb{R}^n)$ if and only if all its Riesz transforms $R_j f$ are in $L^1(\mathbb{R}^n)$ and that

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \approx \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)} \quad \text{(equivalent norm)}.$$

The Riesz operator norm of R_j on $\mathcal{H}^q(\mathbb{R}^n)$ $(1 \leqslant q < \infty)$ is denoted by $\|\!|R_j|\!\|$. The Hardy space on the half space is denoted by $\mathcal{H}^q(\mathbb{R}^n_+)$ $(1 \leqslant q < \infty)$, which norm is defined by

$$\|f\|_{\mathcal{H}^q(\mathbb{R}^n_+)} \triangleq \inf\big\{ \|\widetilde{f}\|_{\mathcal{H}^q(\mathbb{R}^n)} \;\big|\; \widetilde{f} \in \mathcal{H}^q\big(\mathbb{R}^n\big),\ \widetilde{f}|_{\mathbb{R}^n_+} = f \big\}.$$

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx.$$

The Riesz operators R_j $(j = 1, 2, \dots, n)$, S_j $(j = 1, 2, \dots, n-1)$, and the operator Λ are defined by

$$\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi),$$
$$\mathcal{F}(S_j f)(\xi) = \frac{i\xi_j}{|\xi'|}\mathcal{F}f(\xi),$$
$$\mathcal{F}(\Lambda f)(\xi) = \big|\xi'\big|\mathcal{F}f(\xi),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1}, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$.

Let r be the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$, and e is the extension operator from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, which is defined by

$$ef(x) = \begin{cases} f(x) & \text{for } x_n \geqslant 0, \\ 0 & \text{for } x_n < 0. \end{cases}$$

Define the operators $E(t)$ and $F(t)$ by

$$E(t)f(x) = \int_{\mathbb{R}^n_+} \big[G_t\big(x' - y', x_n - y_n\big) - G_t\big(x' - y', x_n + y_n\big)\big] f(y)\,dy,$$

and

$$F(t)f(x) = \int_{\mathbb{R}^n_+} \big[G_t\big(x' - y', x_n - y_n\big) + G_t\big(x' - y', x_n + y_n\big)\big] f(y)\,dy,$$

where G_t is the Gauss kernel $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.

By the solution formula in [38], the Stokes flow $u=(u',u_n)=e^{-tA}u_0$ can be represented as

$$\begin{cases} u_n = UE(t)V_1u_0, \\ u' = E(t)V_2u_0 - SUE(t)V_1u_0, \end{cases} \tag{2.1}$$

where the operators are defined by $Uf = rR'\cdot S(R'\cdot S+R_n)ef$, $V_1u_0 = -S\cdot u_0' + u_{0n}$, $V_2u_0 = u_0' + Su_{0n}$, $R' = (R_1, R_2, \dots, R_{n-1})$, $S = (S_1, S_2, \dots, S_{n-1})$, $u_0 = (u_{01}, u_{02}, \dots, u_{0n}) = (u_0', u_{0n})$.

Note that the Stokes flow $u=(u',u_n)=e^{-tA}u_0$ is given as a restriction $r\bar{u}$ of one vector field $\bar{u} = (\bar{u}', \bar{u}_n)$:

$$\begin{aligned} \bar{u}_n &= R'\cdot S\big(R'\cdot S + R_n\big)eE(t)V_1u_0 \\ &= R'\cdot S\big(R'\cdot S + R_n\big)\big(-S\cdot eE(t)u_0' + eE(t)u_{0n}\big); \end{aligned} \tag{2.2}$$

$$\begin{aligned} \bar{u}' &= E(t)V_2u_0 - SR'\cdot S\big(R'\cdot S + R_n\big)eE(t)V_1u_0 \\ &= E(t)u_0' + SE(t)u_{0n} - S\bar{u}_n. \end{aligned} \tag{2.3}$$

Lemma 2.1. *Assume that $u_0 = (u_{01}, u_{02}, \dots, u_{0n}) \in L^1(\mathbb{R}^n_+)$ satisfies $\nabla\cdot u_0 = 0$ in $\mathbb{R}^n_+$ $(n \geqslant 2)$, and $u_{0n}|_{\partial\mathbb{R}^n_+} = 0$. Let $\bar{u} = (\bar{u}', \bar{u}_n)$ be given in (2.2), (2.3). Then*

$$\begin{aligned} \bar{u}_n &= \sum_{k=1}^{n-1} R_nR_keE(t)u_{0k} - \sum_{k=1}^{n-1} R_k^2eE(t)u_{0n} \\ &\quad + \sum_{k,m=1}^{n-1} R_m^2\partial_k\Lambda^{-1}eE(t)u_{0k} + \sum_{k=1}^{n-1} R_nR_k\partial_k\Lambda^{-1}eE(t)u_{0n}; \end{aligned} \tag{2.4}$$

and for any $1 \leqslant j \leqslant n-1$

$$\begin{aligned} \bar{u}_j &= E(t)u_{0j} + R_nR_jeE(t)u_{0n} + \sum_{k=1}^{n-1} R_kR_jeE(t)u_{0k} \\ &\quad - R_nR_n\partial_j\Lambda^{-1}eE(t)u_{0n} - \sum_{k=1}^{n-1} R_nR_k\partial_j\Lambda^{-1}eE(t)u_{0k}. \end{aligned} \tag{2.5}$$

Proof. Note that $\mathcal{F}(\partial_j f)(\xi) = i\xi_j\mathcal{F}(f)(\xi)$ for $1 \leqslant j \leqslant n$. It follows from (2.2) that for any $t>0$

$$\begin{aligned} \mathcal{F}(\bar{u}_n)(\xi) &= \frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi'|}\left(\frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi'|} + \frac{i\xi_n}{|\xi|}\right)\left(-\frac{i\xi'}{|\xi'|}\cdot\mathcal{F}\big(eE(t)u_0'\big) + \mathcal{F}\big(eE(t)u_{0n}\big)\right) \\ &= \left(-\frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi|} + \frac{i\xi_n}{|\xi|}\frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi'|}\right)\left(-\frac{i\xi'}{|\xi'|}\cdot\mathcal{F}\big(eE(t)u_0'\big)\right) \\ &\quad + \left(-\frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi|} + \frac{i\xi_n}{|\xi|}\frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi'|}\right)\mathcal{F}\big(eE(t)u_{0n}\big) \\ &= \sum_{k=1}^{n-1}\frac{i\xi_n}{|\xi|}\frac{i\xi_k}{|\xi|}\mathcal{F}\big(eE(t)u_{0k}\big) + \sum_{k,m=1}^{n-1}\frac{i\xi_m}{|\xi|}\frac{i\xi_m}{|\xi|}\frac{i\xi_k}{|\xi'|}\mathcal{F}\big(eE(t)u_{0k}\big) \end{aligned}$$

$$+\sum_{k=1}^{n-1}\left(-\frac{i\xi_k}{|\xi|}\frac{i\xi_k}{|\xi|}+\frac{i\xi_n}{|\xi|}\frac{i\xi_k}{|\xi|}\frac{i\xi_k}{|\xi'|}\right)\mathcal{F}\big(eE(t)u_{0n}\big),$$

which implies that (2.4) holds.

From (2.3), (2.4), we get for any $1\leqslant j\leqslant n-1$ and $t>0$

$$\begin{aligned}\mathcal{F}(\bar{u}_j)(\xi)&=\mathcal{F}\big(E(t)u_{0j}\big)+\mathcal{F}\big(S_jE(t)u_{0n}\big)-\mathcal{F}(S_j\bar{u}_n)\\&=\mathcal{F}\big(E(t)u_{0j}\big)+\frac{i\xi_j}{|\xi'|}\mathcal{F}\big(E(t)u_{0n}\big)\\&\quad-\sum_{k=1}^{n-1}\frac{i\xi_j}{|\xi'|}\left(\frac{i\xi_n}{|\xi|}\frac{i\xi_k}{|\xi|}+\frac{i\xi'}{|\xi|}\cdot\frac{i\xi'}{|\xi|}\frac{i\xi_k}{|\xi'|}\right)\mathcal{F}\big(eE(t)u_{0k}\big)\\&\quad+\sum_{k=1}^{n-1}\frac{i\xi_j}{|\xi'|}\left(\frac{i\xi_k}{|\xi|}\frac{i\xi_k}{|\xi|}-\frac{i\xi_n}{|\xi|}\frac{i\xi_k}{|\xi|}\frac{i\xi_k}{|\xi'|}\right)\mathcal{F}\big(eE(t)u_{0n}\big)\\&=\mathcal{F}\big(E(t)u_{0j}\big)+\frac{i\xi_n}{|\xi|}\frac{i\xi_j}{|\xi|}\mathcal{F}\big(eE(t)u_{0n}\big)+\sum_{k=1}^{n-1}\frac{i\xi_k}{|\xi|}\frac{i\xi_j}{|\xi|}\mathcal{F}\big(eE(t)u_{0k}\big)\\&\quad-\frac{i\xi_n}{|\xi|}\frac{i\xi_n}{|\xi|}\frac{i\xi_j}{|\xi'|}\mathcal{F}\big(eE(t)u_{0n}\big)-\sum_{k=1}^{n-1}\frac{i\xi_n}{|\xi|}\frac{i\xi_k}{|\xi|}\frac{i\xi_j}{|\xi'|}\mathcal{F}\big(eE(t)u_{0k}\big),\end{aligned}$$

which implies that (2.5) is true. □

Lemma 2.2. *Let $G_t^{(n-1)}$ denote Gauss kernel in $\mathbb{R}^{n-1}$ $(n\geqslant 2)$, which is defined by $G_t^{(n-1)}(x')=(4\pi t)^{-\frac{n-1}{2}}\times e^{-\frac{|x'|^2}{4t}}$, $x'\in\mathbb{R}^{n-1}$. Then for any $1\leqslant j,k\leqslant n-1$, $0\leqslant\ell\leqslant n-1$, $x'\in\mathbb{R}^{n-1}$ and $t>0$*

$$\left|\partial_j\Lambda^{-1}G_t^{(n-1)}\big(x'\big)\right|\leqslant C_\ell t^{\frac{\ell+1-n}{2}}\left|x'\right|^{-\ell},\quad\forall 0\leqslant\ell\leqslant n-1;$$

and

$$\left|\Lambda G_t^{(n-1)}\big(x'\big)\right|+\left|\partial_j\partial_k\Lambda^{-1}G_t^{(n-1)}\big(x'\big)\right|\leqslant C_\ell t^{\frac{\ell-n}{2}}\left|x'\right|^{-\ell},\quad\forall 0\leqslant\ell\leqslant n.$$

Proof.

Case 1. $n\geqslant 3$. Note that Λ^{-1} is equal to $(-\Delta')^{-\frac{1}{2}}=(\sum_{k=1}^{n-1}\partial_k^2)^{-\frac{1}{2}}$, so the integral kernel of Λ^{-1} is $c_n|x'|^{-n+2}$ for $n\geqslant 3$, where c_n is a positive constant. Therefore we get for $n\geqslant 3$, $1\leqslant j\leqslant n-1$ and $t>0$

$$\partial_j\Lambda^{-1}G_t^{(n-1)}\big(x'\big)=c_n\partial_j\int_{\mathbb{R}^{n-1}}\left|x'-y'\right|^{-n+2}G_t^{(n-1)}\big(y'\big)\,dy'.$$

Set $x'=t^{\frac{1}{2}}z'$. Then

$$\begin{aligned}\partial_j\Lambda^{-1}G_t^{(n-1)}\big(x'\big)&=c_nt^{-\frac{n-1}{2}}\partial_{z_j}\int_{\mathbb{R}^{n-1}}\left|z'-y'\right|^{-n+2}G_1^{(n-1)}\big(y'\big)\,dy'\\&=t^{-\frac{n-1}{2}}\partial_{z_j}\Lambda^{-1}G_1^{(n-1)}\big(z'\big).\end{aligned}\tag{2.6}$$

So it is sufficient to prove that for $n \geqslant 3$, $1 \leqslant j \leqslant n-1$, $0 \leqslant \ell \leqslant n-1$ and $t > 0$

$$\left|\partial_{z_j} \Lambda^{-1} e G_1^{(n-1)}(z')\right| \leqslant C\left|z'\right|^{-\ell}. \tag{2.7}$$

In fact, if (2.7) is valid, then for $n \geqslant 3$, $1 \leqslant j \leqslant n-1$, $0 \leqslant \ell \leqslant n-1$ and $t > 0$

$$\left|\partial_j \Lambda^{-1} G_t^{(n-1)}(x')\right| = t^{-\frac{n-1}{2}}\left|\partial_{z_j} \Lambda^{-1} G_1^{(n-1)}(z')\right| \leqslant C t^{-\frac{n-1}{2}}\left|z'\right|^{-\ell} = C t^{\frac{\ell+1-n}{2}}\left|x'\right|^{-\ell}.$$

Let $\psi_1 \in C_0^\infty(\mathbb{R}^{n-1})$ be such that $0 \leqslant \psi_1 \leqslant 1$, $\operatorname{supp}\psi_1 \subset \{x' \in \mathbb{R}^{n-1} \mid |x'| < 1\}$, and $\psi_1 \equiv 1$ on $\{x' \in \mathbb{R}^{n-1} \mid |x'| < \frac{1}{2}\}$. Set $\psi_2 = 1 - \psi_1$. Then

$$\begin{aligned}
\partial_{z_j} \Lambda^{-1} e G_1^{(n-1)}(z') &= c_n(4\pi)^{-\frac{n-1}{2}} \partial_j \int_{\mathbb{R}^{n-1}} \left|z'-y'\right|^{-n+2} \psi_1(z'-y') e^{-\frac{|y'|^2}{4}}\, dy' \\
&\quad + c_n(4\pi)^{-\frac{n-1}{2}} \partial_j \int_{\mathbb{R}^{n-1}} \left|z'-y'\right|^{-n+2} \psi_2(z'-y') e^{-\frac{|y'|^2}{4}}\, dy' \\
&= I_1(z') + I_2(z'); \qquad (2.8)\\
\left|I_1(z')\right| &= c_n(4\pi)^{-\frac{n-1}{2}} \left|\partial_j \int_{|y'|\leqslant 1} \left|y'\right|^{-n+2} \psi_1(y') e^{-\frac{|z'-y'|^2}{4}}\, dy'\right| \\
&= c_n(4\pi)^{-\frac{n-1}{2}} \left|\int_{|y'|\leqslant 1} \left|y'\right|^{-n+2} \psi_1(y') \left(-\frac{z_j-y_j}{2}\right) e^{-\frac{|z'-y'|^2}{4}}\, dy'\right| \\
&\leqslant C \int_{|y'|\leqslant 1} \left|y'\right|^{-n+2} \left(\left|z'\right|+1\right) e^{-\frac{1}{4}(\frac{|z'|^2}{2}-1)}\, dy' \\
&\leqslant C_\ell \left|z'\right|^{-\ell} \quad \text{for any } \ell \geqslant 0; \qquad (2.9)
\end{aligned}$$

here we have used the fact: $|z'-y'|^2 \geqslant \frac{|z'|^2}{2} - 1$ for any $z', y' \in \mathbb{R}^{n-1}$ satisfying $|y'| \leqslant 1$.

$$\begin{aligned}
I_2(z') &= (4\pi)^{-\frac{n-1}{2}} c_n \int_{\mathbb{R}^{n-1}} \left|z'-y'\right|^{-n+2} \left(\partial_{z_j}\psi_2(z'-y')\right) e^{-\frac{|y'|^2}{4}}\, dy' \\
&\quad - (4\pi)^{-\frac{n-1}{2}} (n-2) c_n \int_{\mathbb{R}^{n-1}} \left|z'-y'\right|^{-n} (z_j-y_j) \psi_2(z'-y') e^{-\frac{|y'|^2}{4}}\, dy' \\
&= J_1(z') + J_2(z'); \qquad (2.10)\\
\left|J_1(z')\right| &\leqslant C\|\nabla\psi_2\|_{L^\infty(\mathbb{R}^{n-1})} \int_{\frac{1}{2}\leqslant|y'|\leqslant 1} \left|y'\right|^{-n+2} e^{-\frac{|z'-y'|^2}{4}}\, dy' \\
&\leqslant C \int_{\frac{1}{2}\leqslant|y'|\leqslant 1} \left|y'\right|^{-n+2} e^{-\frac{1}{4}(\frac{|z'|^2}{2}-1)}\, dy' \\
&\leqslant C_\ell \left|z'\right|^{-\ell} \quad \text{for any } \ell \geqslant 0; \qquad (2.11)
\end{aligned}$$

$$
\begin{aligned}
\left|J_2(z')\right| &\leqslant C\|\psi_2\|_{L^\infty(\mathbb{R}^{n-1})} \int\limits_{|z'-y'|\geqslant\frac12} \left|z'-y'\right|^{-n+1} e^{-\frac{|y'|^2}{4}}\,dy' \\
&\leqslant C\left|z'\right|^{-\ell} \int\limits_{|z'-y'|\geqslant\frac12} \left(\frac{|z'-y'|^\ell}{|z'-y'|^{n-1}} + \frac{|y'|^\ell}{|z'-y'|^{n-1}}\right) e^{-\frac{|y'|^2}{4}}\,dy' \\
&\leqslant C\left|z'\right|^{-\ell} \int\limits_{|z'-y'|\geqslant\frac12} \left(\left|z'-y'\right|^{\ell+1-n} + \left|y'\right|^\ell\left|z'-y'\right|^{-n+1}\right) e^{-\frac{|y'|^2}{4}}\,dy' \\
&\leqslant C\left|z'\right|^{-\ell} \int\limits_{|z'-y'|\geqslant\frac12} \left(2^{-\ell-1+n} + 2^{n-1}\left|y'\right|^\ell\right) e^{-\frac{|y'|^2}{4}}\,dy' \\
&\leqslant C_\ell\left|z'\right|^{-\ell} \quad \text{for any } 0\leqslant\ell\leqslant n-1. \qquad (2.12)
\end{aligned}
$$

From (2.8)–(2.12), we conclude that for $n\geqslant 3$, $1\leqslant j\leqslant n-1$, $0\leqslant\ell\leqslant n-1$ and $t>0$

$$
\left|\partial_{z_j}\Lambda^{-1}eG_1^{(n-1)}(z')\right| \leqslant C_\ell\left|z'\right|^{-\ell}, \quad \forall z'\in\mathbb{R}^{n-1},
$$

which is (2.7).

Case 2. $n=2$ and $j=1$. Note that $\partial_1\Lambda^{-1}=S_1$. So for any $t>0$ (see [37, p. 266])

$$
\begin{aligned}
\partial_{x_j}\Lambda^{-1}G_t^{(n-1)}(x') &= \partial_1\Lambda^{-1}G_t^{(1)}(x_1) \\
&= S_1G_t^{(1)}(x_1) \\
&= \frac{1}{\pi}\lim_{\epsilon\to0}\int\limits_{|y_1|>\epsilon} y_1^{-1}G_t^{(1)}(x_1-y_1)\,dy_1 \\
&= \frac{1}{\pi}\lim_{\epsilon\to0}\int\limits_{|y_1|>\epsilon} G_t^{(1)}(x_1-y_1)\,d\log|y_1| \\
&= \frac{1}{\pi}\lim_{\epsilon\to0}\Bigg\{\left[G_t^{(1)}(x_1-y_1)\log|y_1|\right]\Big|_\epsilon^\infty + \left[G_t^{(1)}(x_1-y_1)\log|y_1|\right]\Big|_{-\infty}^{-\epsilon} \\
&\quad - \int\limits_{|y_1|>\epsilon}\left(\log|y_1|\right)\partial_{y_1}G_t^{(1)}(x_1-y_1)\,dy_1\Bigg\} \\
&= \frac{1}{\pi}\lim_{\epsilon\to0}\Bigg\{\left[G_t^{(1)}(x_1+\epsilon)-G_t^{(1)}(x_1-\epsilon)\right]\log\epsilon \\
&\quad + \int\limits_{|y_1|>\epsilon}\left(\log|y_1|\right)\frac{x_1-y_1}{2t}G_t^{(1)}(x_1-y_1)\,dy_1\Bigg\} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\log|y_1|\right)\frac{x_1-y_1}{2t}G_t^{(1)}(x_1-y_1)\,dy_1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\pi} t^{-\frac{1}{2}} \int_{-\infty}^{\infty} \bigl(\log\sqrt{t} + \log|y_1|\bigr) \frac{z_1 - y_1}{2} G_1^{(1)}(z_1 - y_1)\, dy_1 \\
&= \frac{1}{\pi} t^{-\frac{1}{2}} \int_{-\infty}^{\infty} \bigl(\log|y_1|\bigr) \frac{z_1 - y_1}{2} G_1^{(1)}(z_1 - y_1)\, dy_1 \\
&= \frac{1}{\pi} t^{-\frac{1}{2}} \biggl(\int_{|y_1| \leqslant 1} + \int_{|y_1|>1} \biggr) \bigl(\log|y_1|\bigr) \frac{z_1 - y_1}{2} G_1^{(1)}(z_1 - y_1)\, dy_1 \\
&= \frac{1}{\pi} t^{-\frac{1}{2}} \bigl(L_1(z_1) + L_2(z_1)\bigr); \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
\bigl|L_1(z_1)\bigr| &\leqslant \frac{1}{2} \biggl| \int_{|y_1| \leqslant 1} \bigl(\log|y_1|\bigr) |z_1 - y_1| G_1^{(1)}(z_1 - y_1)\, dy_1 \biggr| \\
&\leqslant C\bigl(1 + |z_1|\bigr) e^{-\frac{|z_1|^2}{8} + \frac{1}{4}} \int_{|y_1| \leqslant 1} \bigl|\log|y_1|\bigr|\, dy_1 \\
&\leqslant C(\ell)|z_1|^{-\ell} \quad \text{for any } \ell \geqslant 0. \qquad (2.14)
\end{aligned}
$$

Note that

$$
\begin{aligned}
L_2(z_1) &= \int_{|y_1|>1} \bigl(\log|y_1|\bigr) \frac{z_1 - y_1}{2} G_1^{(1)}(z_1 - y_1)\, dy_1 \\
&= - \int_{|y_1|>1} \bigl(\log|y_1|\bigr) \partial_{y_1} G_1^{(1)}(z_1 - y_1)\, dy_1 \\
&= - \bigl[\bigl(\log|y_1|\bigr) G_1^{(1)}(z_1 - y_1)\bigr]\big|_1^{\infty} - \bigl[\bigl(\log|y_1|\bigr) G_1^{(1)}(z_1 - y_1)\bigr]\big|_{-\infty}^{-1} \\
&\quad + \int_{|y_1|>1} y_1^{-1} G_1^{(1)}(z_1 - y_1)\, dy_1 \\
&= \int_{|y_1|>1} y_1^{-1} G_1^{(1)}(z_1 - y_1)\, dy_1.
\end{aligned}
$$

Thus for any $0 \leqslant \ell \leqslant 1$ and $t > 0$

$$
\bigl|L_2(z_1)\bigr| \leqslant C(\ell)|z_1|^{-\ell} \int_{|y_1|>1} \bigl(|y_1|^{\ell-1} + |z_1 - y_1|^{\ell} |y_1|^{-1}\bigr) e^{-\frac{|z_1 - y_1|^2}{4}}\, dy_1 \leqslant C(\ell)|z_1|^{-\ell}. \qquad (2.15)
$$

Whence from (2.13)–(2.15), we obtain for $n = 2$, $j = 1$ and $t > 0$

$$
\bigl|\partial_j \Lambda^{-1} G_t^{(n-1)}\bigl(x'\bigr)\bigr| \leqslant C_\ell t^{\frac{\ell-1}{2}} \bigl|x'\bigr|^{-\ell},
$$

for any $0 \leqslant \ell \leqslant 1$.

From the above arguments on the two cases: $n \geqslant 3$; $n = 2$, we complete the proof of the first estimate in Lemma 2.2. Note that $\partial_j \partial_k \Lambda^{-1} = -\Lambda$ for $n = 2$ and $j = k = 1$. Similar to the proof of the above arguments, we can verify the validity of the second estimate in Lemma 2.2, and here we omit the details. $\square$

Lemma 2.3. *Assume that $u_0 = (u_{01}, u_{02}, \dots, u_{0n})$ satisfies $\nabla \cdot u_0 = 0$ in $\mathbb{R}^n_+$ $(n \geqslant 2)$, $u_{0n}|_{\partial\mathbb{R}^n_+} = 0$. Then for any $0 \leqslant \alpha \leqslant 1$, $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n-1$ and $t > 0$*

$$\big\| E(t)u_{0k} \big\|_{L^q(\mathbb{R}^n)} + \big\| \partial_j \Lambda^{-1} e E(t) u_{0k} \big\|_{L^q(\mathbb{R}^n)} \leqslant C t^{-\frac{\alpha}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \big\| x_n^\alpha u_0 \big\|_{L^r(\mathbb{R}^n_+)}$$

provided that $1 \leqslant r < q < \infty$.

Proof. Let $G_t^{(n-1)}$, $G_t^{(1)}$ denote Gauss kernels in $\mathbb{R}^{n-1}$ and $\mathbb{R}^1$, respectively:

$$G_t^{(n-1)}(x') = (4\pi t)^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{4t}}, \qquad G_t^{(1)}(x_n) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x_n|^2}{4t}}, \quad \forall x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}^1.$$

Then the Gauss kernel $G_t(x)$ in $\mathbb{R}^n$ can be written as:

$$G_t(x) = G_t^{(n-1)}(x') G_t^{(1)}(x_n), \quad \forall x = (x', x_n) \in \mathbb{R}^n.$$

Let $1 \leqslant k \leqslant n$, $0 \leqslant \alpha \leqslant 1$. Then for any $x = (x', x_n) \in \mathbb{R}^n$ and $t > 0$

$$\begin{aligned}
\big|\big[E(t)u_{0k}\big](x)\big| &= \left| \int_{\mathbb{R}^n} G_t^{(n-1)}(x'-y') \big(G_t^{(1)}(x_n - y_n) - G_t^{(1)}(x_n + y_n)\big) \theta(y_n) u_{0k}(y', y_n)\, dy'\, dy_n \right| \\
&\leqslant \int_{\mathbb{R}^n} G_t^{(n-1)}(x'-y') \big(G_t^{(1)}(x_n - y_n) + G_t^{(1)}(x_n + y_n)\big)^{1-\alpha} \\
&\quad \times \left| \int_{-1}^{1} \partial_{x_n} G_t^{(1)}(x_n - \tau y_n)\, d\tau \right|^\alpha \theta(y_n) y_n^\alpha \big|u_{0k}(y', y_n)\big|\, dy'\, dy_n \\
&\triangleq \int_{\mathbb{R}^n} k_1(x, y) \theta(y_n) y_n^\alpha \big|u_{0k}(y', y_n)\big|\, dy'\, dy_n,
\end{aligned} \tag{2.16}$$

where

$$\theta(x_n) = \begin{cases} 1 & \text{if } x_n \geqslant 0, \\ 0 & \text{if } x_n < 0. \end{cases}$$

Let $1 < s < \infty$. Then for any $0 \leqslant \alpha \leqslant 1$ and $t > 0$

$$\begin{aligned}
\big\| k_1(x, y) \big\|_{L^s(\mathbb{R}^n_x)} &\leqslant \big\| G_t^{(n-1)}(x'-y') \big(G_t^{(1)}(x_n - y_n) + G_t^{(1)}(x_n + y_n)\big) \big\|_{L^s(\mathbb{R}^n_x)}^{1-\alpha} \\
&\quad \times \left\| G_t^{(n-1)}(x'-y') \int_{-1}^{1} \partial_{x_n} G_t^{(1)}(x_n - \tau y_n)\, d\tau \right\|_{L^s(\mathbb{R}^n_x)}^\alpha \\
&\leqslant C t^{-\frac{\alpha}{2} - \frac{n}{2}(1 - \frac{1}{s})}.
\end{aligned} \tag{2.17}$$

Similarly,

$$\|k_1(x,y)\|_{L^s(\mathbb{R}^n_y)} \leqslant Ct^{-\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{s})}. \tag{2.18}$$

From (2.16)–(2.18), using Young's inequality, we conclude for any $1 \leqslant k \leqslant n$, $0 \leqslant \alpha \leqslant 1$ and $t > 0$

$$\|E(t)u_{0k}\|_{L^q(\mathbb{R}^n)} \leqslant Ct^{-\frac{\alpha}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|y_n^\alpha u_{0k}\|_{L^r(\mathbb{R}^n_+)}, \tag{2.19}$$

for any $1 \leqslant r < q < \infty$.

Let $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n-1$, $0 \leqslant \alpha \leqslant 1$. Set

$$\begin{aligned} k_2(x,y) = {} & \partial_{x_j}\Lambda^{-1}G_t^{(n-1)}(x'-y')\big(G_t^{(1)}(x_n-y_n)+G_t^{(1)}(x_n+y_n)\big)^{1-\alpha} \\ & \times \left|\int_{-1}^{1}\partial_{x_n}G_t^{(1)}(x_n-\tau y_n)\,d\tau\right|^{\alpha}. \end{aligned} \tag{2.20}$$

From (2.16) and (2.20), we infer that for any $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n-1$, $0 \leqslant \alpha \leqslant 1$, $x = (x', x_n) \in \mathbb{R}^n$ and $t > 0$

$$\big|\partial_{x_j}\Lambda^{-1}\big[eE(t)u_{0k}\big](x)\big| \leqslant \int_{\mathbb{R}^n}\big|k_2(x,y)\big|\theta(y_n)y_n^\alpha\big|u_{0k}(y',y_n)\big|\,dy'\,dy_n. \tag{2.21}$$

Let $1 < s < \infty$. Then for any $1 \leqslant j \leqslant n-1$, $0 \leqslant \alpha \leqslant 1$, $y = (y', y_n) \in \mathbb{R}^n$ and $t > 0$

$$\begin{aligned} \|k_2(x,y)\|_{L^s(\mathbb{R}^n_x)} \leqslant {} & \big\|\partial_{x_j}\Lambda^{-1}G_t^{(n-1)}(x'-y')\big(G_t^{(1)}(x_n-y_n)+G_t^{(1)}(x_n+y_n)\big)\big\|_{L^s(\mathbb{R}^n_x)}^{1-\alpha} \\ & \times \left\|\partial_{x_j}\Lambda^{-1}G_t^{(n-1)}(x'-y')\int_{-1}^{1}\partial_{x_n}G_t^{(1)}(x_n-\tau y_n)\,d\tau\right\|_{L^s(\mathbb{R}^n_x)}^{\alpha} \\ \leqslant {} & \big\|\partial_{x_j}\Lambda^{-1}G_t^{(n-1)}(x'-y')\big\|_{L^s(\mathbb{R}^{n-1}_{x'})}^{1-\alpha}\big\|G_t^{(1)}(x_n-y_n)+G_t^{(1)}(x_n+y_n)\big\|_{L^s(\mathbb{R}^1_{x_n})}^{1-\alpha} \\ & \times \big\|\partial_{x_j}\Lambda^{-1}G_t^{(n-1)}(x'-y')\big\|_{L^s(\mathbb{R}^{n-1}_{x'})}^{\alpha}\left\|\int_{-1}^{1}\partial_{x_n}G_t^{(1)}(x_n-\tau y_n)\,d\tau\right\|_{L^s(\mathbb{R}^1_{x_n})}^{\alpha} \\ \leqslant {} & Ct^{-\frac{\alpha}{2}-\frac{1}{2}(1-\frac{1}{s})}\big\|\partial_j\Lambda^{-1}G_t^{(n-1)}\big\|_{L^s(\mathbb{R}^{n-1})}. \end{aligned} \tag{2.22}$$

Let $1 < s < \infty$. Using Lemma 2.2, we have for any $1 \leqslant j \leqslant n-1$ and $t > 0$

$$\begin{aligned} \big\|\partial_j\Lambda^{-1}G_t^{(n-1)}\big\|_{L^s(\mathbb{R}^{n-1})}^s = {} & \left(\int_{|x'|\leqslant t^{\frac{1}{2}}} + \int_{|x'|\geqslant t^{\frac{1}{2}}}\right)\big|\partial_j\Lambda^{-1}G_t^{(n-1)}(x')\big|^s\,dx' \\ \leqslant {} & C_0t^{-\frac{(n-1)s}{2}}\int_{|x'|\leqslant t^{\frac{1}{2}}}dx' + C_{n-1}\int_{|x'|\geqslant t^{\frac{1}{2}}}|x'|^{-(n-1)s}\,dx' \\ \leqslant {} & Ct^{-\frac{(n-1)(s-1)}{2}}. \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), we derive for any $1 < s < \infty$, $y = (y', y_n) \in \mathbb{R}^n$ and $t > 0$

$$\|k_2(x, y)\|_{L^s(\mathbb{R}^n_x)} \leqslant Ct^{-\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{s})}. \tag{2.24}$$

Similarly,

$$\|k_2(x, y)\|_{L^s(\mathbb{R}^n_y)} \leqslant Ct^{-\frac{\alpha}{2}-\frac{n}{2}(1-\frac{1}{s})}. \tag{2.25}$$

From (2.21), (2.24), (2.25) and using Young's inequality, we deduce for any $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n-1$, $0 \leqslant \alpha \leqslant 1$ and $t > 0$

$$\|\partial_j \Lambda^{-1} E(t) u_{0k}\|_{L^q(\mathbb{R}^n)} \leqslant Ct^{-\frac{\alpha}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|y_n^\alpha u_{0k}\|_{L^r(\mathbb{R}^n_+)}, \tag{2.26}$$

for any $1 \leqslant r < q < \infty$.

Combining (2.19) and (2.26), we complete the proof of Lemma 2.3. □

Proof of Theorem 1.1. Note that the Riesz operator R_j $(1 \leqslant j \leqslant n)$ is bounded in $\mathcal{H}^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ for any $1 < q < \infty$. $e^{-tA}u_0 = \bar{u}|_{\mathbb{R}^n_+}$, where $\bar{u}$ (given in (2.2), (2.3)) is the extension of the Stokes flow $e^{-tA}u_0$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. From Lemmata 2.1, 2.3, we derive for any $0 \leqslant \alpha \leqslant 1$ and $t > 0$

$$\|e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)} \leqslant \|\bar{u}(t)\|_{L^q(\mathbb{R}^n)} \leqslant Ct^{-\frac{\alpha}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|x_n^\alpha u_0\|_{L^r(\mathbb{R}^n_+)}, \tag{2.27}$$

for any $1 \leqslant r < q < \infty$.

Now we recall the estimates, which can be found in [15]:

$$\|\nabla^k e^{-tA} a\|_{L^q(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|a\|_{L^r(\mathbb{R}^n_+)}, \quad \forall a \in L^r_\sigma(\mathbb{R}^n_+), \tag{2.28}$$

with $k = 0, 1, \ldots,$ provided that $1 \leqslant r < q \leqslant \infty$ or $1 < r \leqslant q < \infty$.

Let $1 \leqslant r < q \leqslant \infty$ and take $r < r_1 < q$. Then from (2.27) and (2.28), we conclude for any $0 \leqslant \alpha \leqslant 1$ and $t > 0$

$$\begin{aligned}\|\nabla^k e^{-tA} u_0\|_{L^q(\mathbb{R}^n_+)} &\leqslant Ct^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r_1}-\frac{1}{q})} \|e^{-\frac{t}{2}A}u_0\|_{L^{r_1}(\mathbb{R}^n_+)} \\ &\leqslant Ct^{-\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|x_n^\alpha u_0\|_{L^r(\mathbb{R}^n_+)}. \qquad \square\end{aligned}$$

3. Decay properties for the Navier–Stokes flows in half spaces

Let $g = \mathcal{N} f$ denote the solution of the Neumann problem

$$\begin{cases} -\Delta g = f & \text{in } \mathbb{R}^n_+, \\ \partial_\nu g|_{\partial\mathbb{R}^n_+} = 0. \end{cases}$$

Then (see [18])

$$\mathcal{N} = \int_0^\infty F(\tau)\, d\tau. \tag{3.1}$$

Moreover for any $u \in L^2_\sigma(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$

$$P(u \cdot \nabla u) = u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j). \tag{3.2}$$

Lemma 3.1. *Let $0 < \theta < 1$, $0 \leqslant \alpha < 1$, $1 \leqslant k \leqslant n$ and $1 \leqslant q \leqslant \infty$. Then for any $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$*

$$\left\| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \leqslant C \bigl(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} \bigr); \tag{3.3}$$

$$\left\| \sum_{i,j=1}^n x_n^{\alpha} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)}$$
$$\leqslant C \Bigl(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \bigl\| y_n^{\frac{\alpha}{2}} u \bigr\|^2_{L^{2q}(\mathbb{R}^n_+)} + \bigl\| y_n^{\frac{\alpha}{2}} \nabla u \bigr\|^2_{L^{2q}(\mathbb{R}^n_+)} \Bigr). \tag{3.4}$$

Especially,

$$\left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \leqslant C \bigl(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} \bigr).$$

Proof. Denote the odd and even extensions of a function f from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, respectively by

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geqslant 0, \\ -f(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

and

$$f_*(x', x_n) = \begin{cases} f(x', x_n) & \text{if } x_n \geqslant 0, \\ f(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Recall that the following result holds (see [18]): Let $1 \leqslant k \leqslant n$ and $1 \leqslant q \leqslant \infty$, then for any $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$

$$\left\| \sum_{i,j=1}^n \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \leqslant C \bigl(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} \bigr). \tag{3.5}$$

Let $1 \leqslant q \leqslant \infty$. From (3.1), (3.5), one has for any $1 \leqslant k \leqslant n$

$$\left\| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)}$$
$$\leqslant \left\| \sum_{i,j=1}^n x_n^{-\theta} \partial_k \int_0^1 G_\tau * \bigl[\partial_i \partial_j (u_i u_j) \bigr]_* \, d\tau \right\|_{L^q(\mathbb{R}^{n-1} \times (0,1))}$$

$$+\left\| \sum_{i,j=1}^{n} x_n^{-\theta} \partial_k \int_1^\infty G_\tau * \left[\partial_i \partial_j (u_i u_j)\right]_* d\tau \right\|_{L^q(\mathbb{R}^{n-1}\times(0,1))}$$

$$+\left\| \sum_{i,j=1}^{n} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)}$$

$$\leqslant C \sup_{y\in\mathbb{R}^n} \left\| \int_0^1 x_n^{-\theta} \partial_k G_\tau(x-y)\, d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \left\| \sum_{i,j=1}^{n} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)}$$

$$+ C \sup_{y\in\mathbb{R}^n} \sum_{i,j=1}^{n} \left\| \int_1^\infty x_n^{-\theta} \partial_k \partial_i \partial_j G_\tau(x-y)\, d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \|w_{ij}\|_{L^q(\mathbb{R}^n_+)}$$

$$+\left\| \sum_{i,j=1}^{n} \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)}, \tag{3.6}$$

where $w_{ij} = (u_i u_j)_*$ if $1 \leqslant i, j \leqslant n-1$ or $i = j = n$; $w_{in} = (u_i u_n)^*$ if $1 \leqslant i \leqslant n-1$; $w_{nj} = (u_n u_j)^*$ if $1 \leqslant j \leqslant n-1$.

Let $1 \leqslant i, j, k \leqslant n$. $0 < \theta < 1$, $q_1, q_2 \in (1, \infty)$ such that $\frac{1}{q_1} + \frac{1}{q_2} = 1$ and $\theta q_1 < 1$. Then for any $y = (y', y_n) \in \mathbb{R}^n$

$$\begin{aligned}
\left\| \int_0^1 x_n^{-\theta} \partial_k G_\tau(x-y)\, d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} &\leqslant \int_0^1 \int_{\mathbb{R}^{n-1}} \int_0^1 \tau^{-\frac{1}{2}} x_n^{-\theta} \frac{|x_k - y_k|}{2\sqrt{\tau}} G_\tau(x-y)\, dx\, d\tau \\
&\leqslant C \int_0^1 \int_0^1 \tau^{-1} x_n^{-\theta} e^{-\frac{(x_n-y_n)^2}{8\tau}}\, dx_n\, d\tau \\
&\leqslant C \int_0^1 \tau^{-1} \left(\int_0^1 x_n^{-\theta q_1}\, dx_n \right)^{\frac{1}{q_1}} \left(\int_0^1 e^{-\frac{(x_n-y_n)^2 q_2}{8\tau}}\, dx_n \right)^{\frac{1}{q_2}} d\tau \\
&\leqslant C \int_0^1 \tau^{-1+\frac{1}{2q_2}}\, d\tau \\
&\leqslant C;
\end{aligned} \tag{3.7}$$

and

$$\left\| \int_1^\infty x_n^{-\theta} \partial_k \partial_i \partial_j G_\tau(x-y)\, d\tau \right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}$$

$$\leqslant \int_1^\infty \tau^{-\frac{3}{2}} \int_{\mathbb{R}^{n-1}} \int_0^1 x_n^{-\theta} \left(\frac{|x_k - y_k|\delta_{i,j} + |x_i - y_i|\delta_{k,j} + |x_j - y_j|\delta_{k,i}}{4\sqrt{\tau}} \right.$$

$$+\frac{|x_i-y_i|}{2\sqrt{\tau}}\frac{|x_j-y_j|}{2\sqrt{\tau}}\frac{|x_k-y_k|}{2\sqrt{\tau}}\bigg)(4\pi\tau)^{-\frac{n}{2}}e^{-\frac{|x'-y'|^2}{4\tau}}e^{-\frac{|x_n-y_n|^2}{4\tau}}\,dx'\,dx_n\,d\tau$$

$$\begin{aligned}
&\leqslant C\int_1^\infty\int_0^1 \tau^{-2}x_n^{-\theta}e^{-\frac{(x_n-y_n)^2}{8\tau}}\,dx_n\,d\tau\\
&\leqslant C\int_1^\infty \tau^{-2}\bigg(\int_0^1 x_n^{-\theta q_1}\,dx_n\bigg)^{\frac{1}{q_1}}\bigg(\int_0^1 e^{-\frac{(x_n-y_n)^2q_2}{8\tau}}\,dx_n\bigg)^{\frac{1}{q_2}}\,d\tau\\
&\leqslant C\int_1^\infty \tau^{-2+\frac{1}{2q_2}}\,d\tau\\
&\leqslant C.
\end{aligned}\tag{3.8}$$

Note that for any $1\leqslant q\leqslant\infty$

$$\bigg\|\sum_{i,j=1}^n\partial_i\partial_j(u_iu_j)\bigg\|_{L^q(\mathbb{R}^n_+)}+\sum_{i,j=1}^n\|w_{ij}\|_{L^q(\mathbb{R}^n_+)}\leqslant C\big(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)}+\|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}\big).\tag{3.9}$$

From (3.5)–(3.9), we conclude that for $0<\theta<1$, $1\leqslant k\leqslant n$ and $1\leqslant q\leqslant\infty$

$$\bigg\|\sum_{i,j=1}^n x_n^{-\theta}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\bigg\|_{L^q(\mathbb{R}^n_+)}\leqslant C\big(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)}+\|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}\big),$$

which is (3.3).

Now we show the validity of (3.4). Let $1\leqslant k\leqslant n$ and $0\leqslant\alpha<1$. From (3.1), we get for any $1\leqslant q\leqslant\infty$

$$\begin{aligned}
&\bigg\|\sum_{i,j=1}^n x_n^{\alpha}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\bigg\|_{L^q(\mathbb{R}^n_+)}\\
&\quad=\bigg\|\sum_{i,j=1}^n x_n^{\alpha}\partial_k\int_0^\infty F(\tau)\partial_i\partial_j(u_iu_j)\,d\tau\bigg\|_{L^q(\mathbb{R}^n_+)}\\
&\quad=C\bigg\|\sum_{i,j=1}^n \theta(x_n)x_n^{\alpha}\partial_k\bigg(\int_0^1+\int_1^\infty\bigg)G_\tau*\big[\partial_i\partial_j(u_iu_j)\big]_*\,d\tau\bigg\|_{L^q(\mathbb{R}^n)}\\
&\quad\leqslant C\bigg\|\int_0^1\int_{\mathbb{R}^n}|x_n-y_n|^\alpha\big|\partial_kG_\tau(x-y)\big|\bigg|\bigg[\sum_{i,j=1}^n\partial_i\partial_j(u_iu_j)\bigg]_*(y)\bigg|\,dy\,d\tau\bigg\|_{L^q(\mathbb{R}^n)}\\
&\qquad+C\bigg\|\int_0^1\int_{\mathbb{R}^n}\big|\partial_kG_\tau(x-y)\big||y_n|^\alpha\bigg|\bigg[\sum_{i,j=1}^n\partial_i\partial_j(u_iu_j)\bigg]_*(y)\bigg|\,dy\,d\tau\bigg\|_{L^q(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{i,j=1}^{n}\left\| \int_1^\infty \int_{\mathbb{R}^n} |x_n - y_n|^\alpha \left|\partial_k\partial_i\partial_j G_\tau(x-y)\right|\left|w_{ij}(y)\right| dy\, d\tau \right\|_{L^q(\mathbb{R}^n)} \\
&+ C\sum_{i,j=1}^{n}\left\| \int_1^\infty \int_{\mathbb{R}^n} \left|\partial_k\partial_i\partial_j G_\tau(x-y)\right| |y_n|^\alpha \left|w_{ij}(y)\right| dy\, d\tau \right\|_{L^q(\mathbb{R}^n)} \\
&= I_1 + I_2 + I_3 + I_4; \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
I_1 + I_2 &\leqslant C\int_0^1 \left\| |x_n|^\alpha \partial_k G_\tau\left(x', x_n\right)\right\|_{L^1(\mathbb{R}^n)} d\tau \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + C\int_0^1 \|\partial_k G_\tau\|_{L^1(\mathbb{R}^n)} d\tau \left\| y_n^{\frac{\alpha}{2}} \nabla u\right\|^2_{L^{2q}(\mathbb{R}^n_+)} \\
&\leqslant C\left\| |x_n|^\alpha \partial_k G_1\right\|_{L^1(\mathbb{R}^n)} \int_0^1 \tau^{\frac{\alpha}{2}-\frac{1}{2}} d\tau \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + C\|\partial_k G_1\|_{L^1(\mathbb{R}^n)} \int_0^1 \tau^{-\frac{1}{2}} d\tau \left\| y_n^{\frac{\alpha}{2}} \nabla u\right\|^2_{L^{2q}(\mathbb{R}^n_+)} \\
&\leqslant C\left(\|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \left\| y_n^{\frac{\alpha}{2}} \nabla u\right\|^2_{L^{2q}(\mathbb{R}^n_+)}\right); \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\leqslant C\sum_{i,j=1}^{n} \int_1^\infty \int_{\mathbb{R}^n} |x_n|^\alpha \left|\partial_k\partial_i\partial_j G_\tau\left(x', x_n\right)\right| dx\, d\tau \|w_{ij}\|_{L^q(\mathbb{R}^n_+)} \\
&\leqslant C\int_1^\infty \tau^{\frac{\alpha}{2}-\frac{3}{2}} d\tau \|u\|^2_{L^{2q}(\mathbb{R}^n_+)} \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} |x_n|^\alpha \left|\partial_k\partial_i\partial_j G_1\left(x', x_n\right)\right| dx \\
&\leqslant C\|u\|^2_{L^{2q}(\mathbb{R}^n_+)}; \qquad (3.12)
\end{aligned}
$$

$$
I_4 \leqslant C\sum_{i,j=1}^{n} \int_1^\infty \tau^{-\frac{3}{2}} d\tau \|\partial_i\partial_j\partial_k G_1\|_{L^1(\mathbb{R}^n)} \left\| y_n^\alpha w_{ij}\right\|_{L^q(\mathbb{R}^n_+)} \leqslant C\left\| y_n^{\frac{\alpha}{2}} u\right\|^2_{L^{2q}(\mathbb{R}^n_+)}. \qquad (3.13)
$$

From (3.10)–(3.13), we infer that (3.4) holds. □

Lemma 3.2. *(See [15,19].) Suppose that $u_0 \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+)$ $(n \geqslant 2)$ for $1 < r < \infty$. Let u be the strong solution of (1.1). Then for all $t > 0$*

$$
\begin{aligned}
&\left\|u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{r})}; \\
&\left\|u(t)\right\|_{L^\infty(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{n}{2}}\left(1+t^{-\frac{n-1}{2}}\right); \\
&\left\|\nabla u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}; \\
&\left\|\nabla^2 u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\left(1+t^{1-n}\right).
\end{aligned}
$$

Further if u_0 satisfies $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. Then for any $t > 0$,

$$
\begin{aligned}
&\left\|u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}; \\
&\left\|u(t)\right\|_{L^\infty(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{n+1}{2}}\left(1+t^{-\frac{n}{2}}\right);
\end{aligned}
$$

$$\left\|\nabla u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-1-\frac{n}{2}(1-\frac{1}{r})};$$

and

$$\left\|\nabla^2 u(t)\right\|_{L^r(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{3}{2}-\frac{n}{2}(1-\frac{1}{r})}\left(1+t^{-n}\right).$$

Proof of Theorem 1.2. Let u be the strong solution of (1.1). Then it can be expressed as (see [34,35])

$$u(x,t) = \int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t)u_0(y)\,dy - \int_0^t \int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds,$$

where $\mathcal{M} = (M_{ij})_{i,j=1,2,\ldots,n}$ is defined as follows for $1 \leqslant i, j \leqslant n$

$$M_{ij}(x,y,t) = \delta_{ij}\left(G_t(x-y) - G_t\left(x-y^*\right)\right) + M^*_{ij}(x,y,t) = \delta_{ij}G_t(x-y) + N^*_{ij}(x,y,t).$$

Here

$$M^*_{ij}(x,y,t) = 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_i} G_t\left(z-y^*\right)dz,$$

and

$$N^*_{ij}(x,y,t) = -\delta_{ij}G_t\left(x-y^*\right) + M^*_{ij}(x,y,t),$$

$y^* = (y_1, y_2, \ldots, -y_n)$, $G_t(x) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$ is the Gauss kernel, and

$$E(z) = \begin{cases} -\frac{\Gamma(\frac{n}{2})}{2(n-2)\pi^{\frac{n}{2}}}\frac{1}{|z|^{n-2}} & \text{if } n > 2, \\ \frac{1}{2\pi}\log|z| & \text{if } n = 2, \end{cases}$$

is the fundamental solution of the Laplace equation. In addition, it holds for M^*_{ij}, N^*_{ij}:

$$\begin{aligned} &\left|\partial_t^s\partial_x^\ell\partial_y^m M^*_{ij}(x,y,t)\right| + \left|\partial_t^s\partial_x^\ell\partial_y^m N^*_{ij}(x,y,t)\right| \\ &\quad \leqslant Ct^{-s-\frac{m_n}{2}}\left(t+x_n^2\right)^{-\frac{\ell_n}{2}}\left(\left|x-y^*\right|^2+t\right)^{-\frac{n+|\ell'|+|m'|}{2}}e^{-\frac{cy_n^2}{t}}, \end{aligned} \tag{3.14}$$

where $m = (m_1, m_2, \ldots, m_{n-1}, m_n) = (m', m_n)$, $\ell = (\ell_1, \ell_2, \ldots, \ell_{n-1}, \ell_n) = (\ell', \ell_n)$.

We firstly verify that (1.2) holds for $n \geqslant 2$ under the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. Let $1 < r \leqslant \infty$ and $0 < \beta < \min\{1, n(1-\frac{1}{r})\}$, $n \geqslant 2$. Then for any $t > 0$

$$\begin{aligned} &\left\|\int_{\mathbb{R}^n_+} x_n^\beta \mathcal{M}(\cdot,y,t)u_0(y)\,dy\right\|_{L^r(\mathbb{R}^n_+)} \\ &\quad \leqslant C\int_{\mathbb{R}^n_+}\left(\left\||x_n-y_n|^\beta G_t(\cdot-y)\right\|_{L^r(\mathbb{R}^n_+)} + \|G_t\|_{L^r(\mathbb{R}^n_+)}y_n^\beta\right)\left|u_0(y)\right|dy \end{aligned}$$

$$
\begin{aligned}
&+C\|u_0\|_{L^1(\mathbb{R}^n_+)} \sup_{y=(y',y_n)\in\mathbb{R}^n_+} \big\| x_n^\beta \big(|x'-y'| + (x_n+y_n) + \sqrt{t} \big)^{-n} \big\|_{L^r(\mathbb{R}^n_+)} \\
&\leqslant C \big(t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + t^{-\frac{n}{2}(1-\frac{1}{r})} \big);
\end{aligned}
\tag{3.15}
$$

and for any $1 \leqslant k \leqslant n$

$$
\begin{aligned}
&\Big\| \int_{\mathbb{R}^n_+} x_n^\beta \partial_{x_k} \mathcal{M}(\cdot, y, t) u_0(y)\, dy \Big\|_{L^r(\mathbb{R}^n_+)} \\
&\leqslant C \int_{\mathbb{R}^n_+} \big(\big\| |x_n - y_n|^\beta \partial_k G_t(\cdot - y) \big\|_{L^r(\mathbb{R}^n_+)} + \|\partial_k G_t\|_{L^r(\mathbb{R}^n_+)} y_n^\beta \big) \big| u_0(y) \big|\, dy \\
&\quad + C\|u_0\|_{L^1(\mathbb{R}^n_+)} \sup_{y=(y',y_n)\in\mathbb{R}^n_+} \big\| x_n^\beta (x_n + \sqrt{t})^{-1_n} \big(|x'-y'| + (x_n+y_n) + \sqrt{t} \big)^{-n-|1'|} dx \big\|_{L^r(\mathbb{R}^n_+)} \\
&\leqslant C \big(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} \big) \quad \text{where } 1 = \big| (1', 1_n) \big|.
\end{aligned}
\tag{3.16}
$$

By (3.2) and Lemmata 3.1, 3.2, one has for any $0 < \beta < \min\{1, n(1-\frac{1}{r})\}$, $1 < r < \infty$ and $t > 0$

$$
\begin{aligned}
&\Big\| x_n^\beta \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot, y, t-s) Pu(y,s) \cdot \nabla u(y,s)\, dy\, ds \Big\|_{L^r(\mathbb{R}^n_+)} \\
&\leqslant C \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n_+} \big\| |x_n - y_n|^\beta G_{t-s}(\cdot - y) \big\|_{L^r(\mathbb{R}^n_+)} \big| Pu(y,s) \cdot \nabla u(y,s) \big|\, dy\, ds \\
&\quad + C \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n_+} \|G_{t-s}\|_{L^r(\mathbb{R}^n_+)} y_n^\beta \big| Pu(y,s) \cdot \nabla u(y,s) \big|\, dy\, ds \\
&\quad + C \int_0^{\frac{t}{2}} \sup_{y\in\mathbb{R}^n_+} \big\| x_n^\beta \mathcal{N}^*(\cdot, y, t-s) \big\|_{L^r(\mathbb{R}^n_+)} \big\| Pu(s) \cdot \nabla u(s) \big\|_{L^1(\mathbb{R}^n_+)}\, ds \\
&\leqslant C t^{-\frac{n}{2}(1-\frac{1}{r})} \int_0^{\frac{t}{2}} \big\| y_n^\beta Pu(s) \cdot \nabla u(s) \big\|_{L^1(\mathbb{R}^n_+)}\, ds \\
&\quad + C t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \int_0^{\frac{t}{2}} \big(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \big)\, ds \\
&\quad + C \int_0^{\frac{t}{2}} \sup_{y=(y',y_n)\in\mathbb{R}^n_+} \big\| x_n^\beta \big(|x'-y'|^2 + (x_n+y_n)^2 + t - s \big)^{-\frac{n}{2}} \big\|_{L^r(\mathbb{R}^n_+)} \big\| Pu(s) \cdot \nabla u(s) \big\|_{L^1(\mathbb{R}^n_+)}\, ds
\end{aligned}
$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}\big(\big\|x_n^{\frac{\beta}{2}}u(s)\big\|^2_{L^2(\mathbb{R}^n_+)}+\big\|x_n^{\frac{\beta}{2}}\nabla u(s)\big\|^2_{L^2(\mathbb{R}^n_+)}\big)\,ds$$

$$+C\big(t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}+t^{-\frac{n}{2}(1-\frac{1}{r})}\big)\left(\int_0^{\frac{t}{2}}(1+s)^{-\frac{n+2}{2}}\,ds+\int_0^{\infty}\big\|\nabla u(s)\big\|^2_{L^2(\mathbb{R}^n_+)}\,ds\right)$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}\big(\big\|x_n^{\beta}u(s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|u(s)\big\|_{L^{\frac{r}{r-1}}(\mathbb{R}^n_+)}$$

$$+\big\|x_n^{\beta}\nabla u(s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|\nabla u(s)\big\|_{L^{\frac{r}{r-1}}(\mathbb{R}^n_+)}\big)\,ds$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{n}{2}(1-\frac{1}{r})}f_1(t)\int_0^{\frac{t}{2}}(1+s)^{-\frac{1}{2}-\frac{n}{2r}}\,ds$$

$$+Ct^{-\frac{n}{2}(1-\frac{1}{r})}f_2(t)\int_0^{\frac{t}{2}}(1+s)^{-1-\frac{n}{2r}}\,ds$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{n}{2}(1-\frac{1}{r})}\big(L(t)f_1(t)+f_2(t)\big), \tag{3.17}$$

where $f_1(t)=\sup_{0<s\leqslant t}\|x_n^{\beta}u(s)\|_{L^r(\mathbb{R}^n_+)}$, $f_2(t)=\sup_{0<s\leqslant t}\|x_n^{\beta}\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}$ and

$$L(t)=\begin{cases}(1+t)^{\frac{1}{2}(1-\frac{n}{r})} & \text{if } r>n,\\ \log(1+t) & \text{if } r=n,\\ 1 & \text{if } r<n.\end{cases}$$

Similarly, let $1\leqslant k\leqslant n$, then for $1<r<\infty$ and $0<\beta<\min\{1,n(1-\frac{1}{r})\}$

$$\left\|x_n^{\beta}\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\partial_{x_k}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^r(\mathbb{R}^n_+)}$$

$$\leqslant C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\big\||x_n-y_n|^{\beta}\partial_{x_k}G_{t-s}(\cdot-y)\big\|_{L^r(\mathbb{R}^n_+)}\big|Pu(y,s)\cdot\nabla u(y,s)\big|\,dy\,ds$$

$$+C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\|\partial_k G_{t-s}\|_{L^r(\mathbb{R}^n_+)}y_n^{\beta}\big|Pu(y,s)\cdot\nabla u(y,s)\big|\,dy\,ds$$

$$+C\int_0^{\frac{t}{2}}\sup_{y\in\mathbb{R}^n_+}\big\|x_n^{\beta}\partial_{x_k}\mathcal{N}^*(\cdot,y,t-s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|Pu(s)\cdot\nabla u(s)\big\|_{L^1(\mathbb{R}^n_+)}\,ds$$

$$
\begin{aligned}
&\leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} \int_0^{\frac{t}{2}} \| y_n^{\beta} Pu(s) \cdot \nabla u(s) \|_{L^1(\mathbb{R}^n_+)} \, ds \\
&\quad + Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \int_0^{\frac{t}{2}} \left(\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \right) ds \\
&\quad + C \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \| Pu(s) \cdot \nabla u(s) \|_{L^1(\mathbb{R}^n_+)} \, ds \\
&\leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} \int_0^{\frac{t}{2}} \left(\|x_n^{\frac{\beta}{2}} u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|x_n^{\frac{\beta}{2}} \nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \right) ds \\
&\quad + C\left(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\right) \left(\int_0^{\frac{t}{2}} (1+s)^{-\frac{n+2}{2}} \, ds + \int_0^{\infty} \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \, ds \right) \\
&\leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \left(1 + t^{-\frac{\beta}{2}}\right) \\
&\quad + Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} \int_0^{\frac{t}{2}} \left(\|x_n^{\beta} u(s)\|_{L^r(\mathbb{R}^n_+)} \|u(s)\|_{L^{\frac{r}{r-1}}(\mathbb{R}^n_+)} + \|x_n^{\beta} \nabla u(s)\|_{L^r(\mathbb{R}^n_+)} \|\nabla u(s)\|_{L^{\frac{r}{r-1}}(\mathbb{R}^n_+)} \right) ds \\
&\leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \left(1 + t^{-\frac{\beta}{2}}\right) + Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})} \left(L(t) f_1(t) + f_2(t) \right). \qquad (3.18)
\end{aligned}
$$

Let $1 < r \leqslant \infty$ and $0 < \beta < \min\{1, n(1-\frac{1}{r})\}$, and take $r_1, r_2 > 1$ such that $1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Moreover, $1 - \frac{2+\beta}{n} < \frac{1}{r_1} < 1 - \frac{\beta}{n}$. By (3.2), Young's inequality and Lemmata 3.1, 3.2, we obtain for any $t > 0$

$$
\begin{aligned}
&\left\| x_n^{\beta} \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot, y, t-s) Pu(y,s) \cdot \nabla u(y,s) \, dy \, ds \right\|_{L^r(\mathbb{R}^n_+)} \\
&\quad \leqslant C \int_{\frac{t}{2}}^{t} \sup_{y \in \mathbb{R}^n_+} \left\| |x_n - y_n|^{\beta} G_{t-s}(\cdot - y) \right\|_{L^1(\mathbb{R}^n_+)} \| Pu(s) \cdot \nabla u(s) \|_{L^r(\mathbb{R}^n_+)} \, ds \\
&\qquad + C \int_{\frac{t}{2}}^{t} \|G_{t-s}\|_{L^1(\mathbb{R}^n_+)} \| y_n^{\beta} Pu(s) \cdot \nabla u(s) \|_{L^r(\mathbb{R}^n_+)} \, ds \\
&\qquad + C \int_{\frac{t}{2}}^{t} \sup_{y \in \mathbb{R}^n_+} \| x_n^{\beta} \mathcal{N}^*(\cdot, y, t-s) \|_{L^{r_1}(\mathbb{R}^n_+)} \| Pu(s) \cdot \nabla u(s) \|_{L^{r_2}(\mathbb{R}^n_+)} \, ds
\end{aligned}
$$

$$\leqslant C\int_{\frac{t}{2}}^{t}\left(1+(t-s)^{\frac{\beta}{2}}\right)\left(\left\|u(s)\right\|_{L^{2r}(\mathbb{R}^n_+)}^2+\left\|\nabla u(s)\right\|_{L^{2r}(\mathbb{R}^n_+)}^2\right.$$

$$\left.+\left\|y_n^{\frac{\beta}{2}}u(s)\right\|_{L^{2r}(\mathbb{R}^n_+)}^2+\left\|y_n^{\frac{\beta}{2}}\nabla u(s)\right\|_{L^{2r}(\mathbb{R}^n_+)}^2\right)ds$$

$$+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{2}(1-\frac{1}{r_1})+\frac{\beta}{2}}\left(\left\|u(s)\right\|_{L^{2r_2}(\mathbb{R}^n_+)}^2+\left\|\nabla u(s)\right\|_{L^{2r_2}(\mathbb{R}^n_+)}^2\right)ds$$

$$\leqslant C\int_{\frac{t}{2}}^{t}\left(1+(t-s)^{\frac{\beta}{2}}\right)\left(s^{-1-n(1-\frac{1}{2r})}+s^{-2-n(1-\frac{1}{2r})}\right)ds$$

$$+C\int_{\frac{t}{2}}^{t}\left(1+(t-s)^{\frac{\beta}{2}}\right)\left(\left\|x_n^{\beta}u(s)\right\|_{L^{r}(\mathbb{R}^n_+)}\left\|u(s)\right\|_{L^{\infty}(\mathbb{R}^n_+)}\right.$$

$$\left.+\left\|x_n^{\beta}\nabla u(s)\right\|_{L^{r}(\mathbb{R}^n_+)}\left\|\nabla u(s)\right\|_{L^{2n}(\mathbb{R}^n_+)}^{\frac{1}{2}}\left\|\nabla^2 u(s)\right\|_{L^{2n}(\mathbb{R}^n_+)}^{\frac{1}{2}}\right)ds$$

$$+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{n}{2}(1-\frac{1}{r_1})+\frac{\beta}{2}}\left(s^{-1-n(1-\frac{1}{2r_2})}+s^{-2-n(1-\frac{1}{2r_2})}\right)ds$$

$$\leqslant Ct^{-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right)\left(1+t^{-1}\right)$$

$$+Ct^{-\frac{n-1}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right)\left(1+t^{-\frac{n}{2}}\right)\left(f_1(t)+t^{-\frac{1}{2}}f_2(t)\right), \tag{3.19}$$

where $f_1(t)$, $f_2(t)$ are given in the proof of (3.17), and we have used the Gagliardo–Nirenberg inequality on the half space (see (4.1) in [11] for example): Let $f\in W^{1,s}(\mathbb{R}^n_+)$, $n<s<\infty$. Then

$$\|h\|_{L^\infty(\mathbb{R}^n_+)}\leqslant C\|h\|_{L^s(\mathbb{R}^n_+)}^{1-\frac{n}{s}}\|\nabla h\|_{L^s(\mathbb{R}^n_+)}^{\frac{n}{s}}.$$

Similarly, let $1\leqslant k\leqslant n$, then for $1<r\leqslant\infty$ and $0<\beta<\min\{1,n(1-\frac{1}{r})\}$

$$\left\|x_n^{\beta}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_k}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^r(\mathbb{R}^n_+)}$$

$$\leqslant C\int_{\frac{t}{2}}^{t}\sup_{y\in\mathbb{R}^n_+}\left\||x_n-y_n|^{\beta}\partial_{x_k}G_{t-s}(x-y)\right\|_{L^1(\mathbb{R}^n_+)}\left\|Pu(s)\cdot\nabla u(s)\right\|_{L^r(\mathbb{R}^n_+)}ds$$

$$+C\int_{\frac{t}{2}}^{t}\|\partial_k G_{t-s}\|_{L^1(\mathbb{R}^n_+)}\left\|y_n^{\beta}Pu(s)\cdot\nabla u(s)\right\|_{L^r(\mathbb{R}^n_+)}ds$$

$$
\begin{aligned}
&+C\int_{\frac{t}{2}}^{t}\sup_{y\in\mathbb{R}^n_+}\big\|x_n^\beta\partial_{x_k}\mathcal{N}^*(x,y,t-s)\big\|_{L^1(\mathbb{R}^n_+)}\big\|Pu(s)\cdot\nabla u(s)\big\|_{L^r(\mathbb{R}^n_+)}\,ds\\
&\leqslant C\int_{\frac{t}{2}}^{t}\big((t-s)^{-\frac12}+(t-s)^{-\frac12+\frac{\beta}{2}}\big)\big(\big\|u(s)\big\|^2_{L^{2r}(\mathbb{R}^n_+)}+\big\|\nabla u(s)\big\|^2_{L^{2r}(\mathbb{R}^n_+)}\\
&\quad+\big\|x_n^\beta u(s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|u(s)\big\|_{L^\infty(\mathbb{R}^n_+)}+\big\|x_n^\beta\nabla u(s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|\nabla u(s)\big\|^{\frac12}_{L^{2n}(\mathbb{R}^n_+)}\big\|\nabla^2u(s)\big\|^{\frac12}_{L^{2n}(\mathbb{R}^n_+)}\big)\,ds\\
&\leqslant Ct^{-\frac12-\frac{n}{2}-\frac{n}{2}(1-\frac1r)+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-1}\big)\\
&\quad+Ct^{-\frac{n}{2}+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac{n}{2}}\big)\big(f_1(t)+t^{-\frac12}f_2(t)\big),
\end{aligned}
\tag{3.20}
$$

where $f_1(t)$, $f_2(t)$ are given in (3.17).

From (3.15)–(3.20), we obtain for $1<r<\infty$ and $0<\beta<\min\{1,n(1-\frac1r)\}$

$$
\begin{aligned}
f_1(t)+f_2(t)\leqslant{}&Ct^{-\frac{n}{2}(1-\frac1r)+\frac{\beta}{2}}\big(1+t^{-\frac12}\big)\big(1+t^{-\frac{\beta}{2}}\big)\\
&+Ct^{-\frac{n}{2}-\frac{n}{2}(1-\frac1r)+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac12}\big)\big(1+t^{-1}\big)\\
&+Ct^{-\frac{n}{2}(1-\frac1r)}\big(1+t^{-\frac12}\big)\big(L(t)f_1(t)+f_2(t)\big)\\
&+Ct^{-\frac{n-1}{2}+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac12}\big)\big(1+t^{-\frac{n}{2}}\big)\big(f_1(t)+t^{-\frac12}f_2(t)\big).
\end{aligned}
$$

So there exists $t_0>0$, which is independent of t, such that for any $t\geqslant t_0$

$$
f_1(t)+f_2(t)\leqslant Ct^{-\frac{n}{2}(1-\frac1r)+\frac{\beta}{2}},
$$

which implies that for any $t\geqslant t_0$, (1.2) holds with $n\geqslant 2$ and $1<r<\infty$.

Now we verify that (1.2) holds with $n\geqslant 2$ and $r=\infty$ under the assumption: $\|x_nu_0\|_{L^1(\mathbb{R}^n_+)}<\infty$. Following the proof of (3.17) and using (1.2) with $n\geqslant 2$ and $r=2$, one has for $0<\beta<1$ and $t>0$

$$
\begin{aligned}
&\bigg\|x_n^\beta\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\bigg\|_{L^\infty(\mathbb{R}^n_+)}\\
&\quad\leqslant C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\big\||x_n-y_n|^\beta G_{t-s}(\cdot-y)\big\|_{L^\infty(\mathbb{R}^n_+)}\big|Pu(y,s)\cdot\nabla u(y,s)\big|\,dy\,ds\\
&\qquad+C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\|G_{t-s}\|_{L^\infty(\mathbb{R}^n_+)}y_n^\beta\big|Pu(y,s)\cdot\nabla u(y,s)\big|\,dy\,ds\\
&\qquad+C\int_0^{\frac{t}{2}}\sup_{y\in\mathbb{R}^n_+}\big\|x_n^\beta\mathcal{N}^*(\cdot,y,t-s)\big\|_{L^\infty(\mathbb{R}^n_+)}\big\|Pu(s)\cdot\nabla u(s)\big\|_{L^1(\mathbb{R}^n_+)}\,ds
\end{aligned}
$$

$$\leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right)+Ct^{-\frac{n}{2}}\int_0^{\frac{t}{2}}\left(\left\|x_n^{\beta}u(s)\right\|_{L^2(\mathbb{R}^n_+)}\left\|u(s)\right\|_{L^2(\mathbb{R}^n_+)}\right.$$
$$\left.+\left\|x_n^{\beta}\nabla u(s)\right\|_{L^2(\mathbb{R}^n_+)}\left\|\nabla u(s)\right\|_{L^2(\mathbb{R}^n_+)}\right)ds$$
$$\leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right)+Ct^{-\frac{n}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{n+1}{2}+\frac{\beta}{2}}\,ds$$
$$\leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right). \tag{3.21}$$

Similarly, following the proof of (3.18) and using Lemma 3.2, (1.2) with $n \geqslant 2$ and $r = 2$, we get for any $1 \leqslant k \leqslant n$, $0 < \beta < 1$ and $t > 0$

$$\left\|x_n^{\beta}\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\partial_{x_k}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^\infty(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2}-\frac{n}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right). \tag{3.22}$$

Note that (3.15), (3.16), (3.19) and (3.20) hold for $r=\infty$. From (3.21) and (3.22), we conclude for any $t > 0$ and $0 < \beta < 1$

$$\begin{aligned} g_1(t)+g_2(t) \leqslant\ & Ct^{-\frac{n}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{1}{2}}\right)\left(1+t^{-\frac{\beta}{2}}\right) \\ & + Ct^{-n+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right)\left(1+t^{-\frac{1}{2}}\right)\left(1+t^{-1}\right) \\ & + Ct^{-\frac{n-1}{2}+\frac{\beta}{2}}\left(1+t^{-\frac{\beta}{2}}\right)\left(1+t^{-\frac{1}{2}}\right)\left(1+t^{-\frac{n}{2}}\right)\left(g_1(t)+t^{-\frac{1}{2}}g_2(t)\right), \end{aligned}$$

where $g_1(t) = \sup_{0<s\leqslant t}\|x_n^{\beta}u(s)\|_{L^\infty(\mathbb{R}^n_+)}$, $g_2(t) = \sup_{0<s\leqslant t}\|x_n^{\beta}\nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)}$.

So there exists $t_1 > 0$, which is independent of t, such that for any $t \geqslant t_1$ and $0 < \beta < 1$

$$g_1(t)+g_2(t) \leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}},$$

which implies that for any $t \geqslant t_1$, (1.2) holds with $n \geqslant 2$ and $r = \infty$.

Now we show that (1.2) is true for $n \geqslant 3$ without the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. From Lemma 3.2, following the proof of (3.17), one has for any $0 < \beta < \min\{1, n(1-\frac{1}{r})\}$, $n \geqslant 3$, $1 < r < \infty$ and $t > 0$

$$\left\|x_n^{\beta}\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^r(\mathbb{R}^n_+)}$$
$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}\left(\left\|x_n^{\frac{\beta}{2}}u(s)\right\|^2_{L^2(\mathbb{R}^n_+)}+\left\|x_n^{\frac{\beta}{2}}\nabla u(s)\right\|^2_{L^2(\mathbb{R}^n_+)}\right)ds$$
$$+C\left(t^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}+t^{-\frac{n}{2}(1-\frac{1}{r})}\right)\left(\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2}}\,ds+\int_0^{\infty}\left\|\nabla u(s)\right\|^2_{L^2(\mathbb{R}^n_+)}\,ds\right)$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}\big(\big\|x_n^\beta u(s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|u(s)\big\|_{L^{\frac{r}{r-1}}(\mathbb{R}^n_+)}$$

$$+\big\|x_n^\beta\nabla u(s)\big\|_{L^r(\mathbb{R}^n_+)}\big\|\nabla u(s)\big\|_{L^{\frac{r}{r-1}}(\mathbb{R}^n_+)}\big)\,ds$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{n}{2}(1-\frac{1}{r})}f_1(t)\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2r}}\,ds$$

$$+Ct^{-\frac{n}{2}(1-\frac{1}{r})}f_2(t)\int_0^{\frac{t}{2}}(1+s)^{-\frac{1}{2}-\frac{n}{2r}}\,ds$$

$$\leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{n}{2}(1-\frac{1}{r})}\big(X(t)f_1(t)+L(t)f_2(t)\big), \tag{3.23}$$

where $f_1(t)$, $f_2(t)$, $L(t)$ are given in (3.17); and

$$X(t)=\begin{cases}(1+t)^{1-\frac{n}{2r}} & \text{if } r>\frac{n}{2},\\ \log(1+t) & \text{if } r=\frac{n}{2},\\ 1 & \text{if } r<\frac{n}{2}.\end{cases}$$

Similar to the proof of (3.18), we have for $n\geqslant 3$, $0<\beta<\min\{1,n(1-\frac{1}{r})\}$, $1<r<\infty$ and $t>0$

$$\left\|x_n^\beta\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\partial_{x_k}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^r(\mathbb{R}^n_+)}$$

$$\leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)+Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\big(X(t)f_1(t)+L(t)f_2(t)\big). \tag{3.24}$$

Following the proofs of (3.19) and (3.20), we infer that for $n\geqslant 3$, $0<\beta<\min\{1,n(1-\frac{1}{r})\}$, $1<r\leqslant\infty$ and $t>0$

$$\left\|x_n^\beta\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^r(\mathbb{R}^n_+)}$$

$$\leqslant Ct^{-\frac{n-2}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-1}\big)$$

$$+Ct^{-\frac{n-2}{2}+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac{n-1}{2}}\big)\big(f_1(t)+t^{-\frac{1}{2}}f_2(t)\big), \tag{3.25}$$

and

$$\left\|x_n^\beta\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_k}\mathcal{M}(\cdot,y,t-s)Pu(y,s)\cdot\nabla u(y,s)\,dy\,ds\right\|_{L^r(\mathbb{R}^n_+)}$$

$$\leqslant Ct^{-\frac{n-1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-1}\big)$$

$$+Ct^{-\frac{n-1}{2}+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac{n-1}{2}}\big)\big(f_1(t)+t^{-\frac{1}{2}}f_2(t)\big). \tag{3.26}$$

From (3.15), (3.16) and (3.23)–(3.26), we conclude for $n \geqslant 3$, $1 < r < \infty$ and $0 < \beta < \min\{1, n(1 - \frac{1}{r})\}$

$$\begin{aligned} f_1(t) + f_2(t) \leqslant{} & Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr)\bigl(1 + t^{-\frac{1}{2}}\bigr) \\ & + Ct^{-\frac{n-2}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr)\bigl(1 + t^{-\frac{1}{2}}\bigr)\bigl(1 + t^{-1}\bigr) \\ & + Ct^{-\frac{n}{2}(1-\frac{1}{r})}\bigl(1 + t^{-\frac{1}{2}}\bigr)\bigl(X(t) f_1(t) + L(t) f_2(t)\bigr) \\ & + Ct^{-\frac{n-2}{2}+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr)\bigl(1 + t^{-\frac{1}{2}}\bigr)\bigl(1 + t^{-\frac{n-1}{2}}\bigr)\bigl(f_1(t) + t^{-\frac{1}{2}} f_2(t)\bigr). \end{aligned}$$

So there exists $t_2 > 0$, which is independent of t, such that for $n \geqslant 3$, $1 < r < \infty$, $0 < \beta < \min\{1, n(1 - \frac{1}{r})\}$ and $t \geqslant t_2$

$$f_1(t) + f_2(t) \leqslant Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \tag{3.27}$$

which implies that for any $t \geqslant t_2$, (1.2) holds with $n \geqslant 3$ and $1 < r < \infty$.

Now we show the validity of (1.2) for $n \geqslant 3$ and $r = \infty$ without the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. Following the proof of (3.17) and using Lemma 3.2, (1.2) with $n \geqslant 3$ and $r = 2$, one has for $0 < \beta < 1$ and $t > 0$

$$\begin{aligned} & \Biggl\| x_n^\beta \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n_+} \mathcal{M}(\cdot, y, t - s) P u(y, s) \cdot \nabla u(y, s)\, dy\, ds \Biggr\|_{L^\infty(\mathbb{R}^n_+)} \\ & \quad \leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr) \int_0^{\frac{t}{2}} (1 + s)^{-\frac{n}{2}}\, ds \\ & \qquad + Ct^{-\frac{n}{2}} \int_0^{\frac{t}{2}} \bigl(\bigl\|x_n^\beta u(s)\bigr\|_{L^2(\mathbb{R}^n_+)} \bigl\|u(s)\bigr\|_{L^2(\mathbb{R}^n_+)} + \bigl\|x_n^\beta \nabla u(s)\bigr\|_{L^2(\mathbb{R}^n_+)} \bigl\|\nabla u(s)\bigr\|_{L^2(\mathbb{R}^n_+)}\bigr)\, ds \\ & \quad \leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr) + Ct^{-\frac{n}{2}} \int_0^{\frac{t}{2}} (1 + s)^{-\frac{n}{2}+\frac{\beta}{2}}\, ds \\ & \quad \leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr). \end{aligned} \tag{3.28}$$

Let $1 \leqslant k \leqslant n$, following the proof of (3.18) and using Lemma 3.2, (1.2) with $n \geqslant 3$ and $r = 2$, we get for $0 < \beta < 1$ and $t > 0$

$$\Biggl\| x_n^\beta \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n_+} \partial_{x_k} \mathcal{M}(\cdot, y, t - s) P u(y, s) \cdot \nabla u(y, s)\, dy\, ds \Biggr\|_{L^\infty(\mathbb{R}^n_+)} \leqslant Ct^{-\frac{1}{2}-\frac{n}{2}+\frac{\beta}{2}}\bigl(1 + t^{-\frac{\beta}{2}}\bigr). \tag{3.29}$$

Note that (3.15), (3.16), (3.25) and (3.26) hold for $r = \infty$. Using (3.28) and (3.29), we conclude for any $t > 0$ and $0 < \beta < 1$

$$
\begin{aligned}
g_1(t)+g_2(t) \leqslant{} & Ct^{-\frac{n}{2}+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac{1}{2}}\big) \\
& + Ct^{-n+1+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac{1}{2}}\big)\big(1+t^{-1}\big) \\
& + Ct^{-\frac{n-2}{2}+\frac{\beta}{2}}\big(1+t^{-\frac{\beta}{2}}\big)\big(1+t^{-\frac{1}{2}}\big)\big(1+t^{-\frac{n-1}{2}}\big)\big(g_1(t)+t^{-\frac{1}{2}}g_2(t)\big),
\end{aligned}
$$

where $g_1(t)$, $g_2(t)$ are given in the above proof of (1.2) with $n \geqslant 2$ and $r=\infty$.

So there exists $t_3>0$, which is independent of t, such that for any $t \geqslant t_3$ and $0<\beta<1$

$$
g_1(t)+g_2(t) \leqslant Ct^{-\frac{n}{2}+\frac{\beta}{2}}
$$

which implies that for any $t \geqslant t_3$, (1.2) holds with $n \geqslant 3$ and $r=\infty$. $\square$

Lemma 3.3. *Let $1<r<\infty$ and $1 \leqslant i,j,k \leqslant n-1$, $n \geqslant 2$. Then the estimates hold for any $t>0$*

$$
\|\Lambda eG_t\|_{L^r(\mathbb{R}^n)}+\|\partial_i eG_t\|_{L^r(\mathbb{R}^n)}+\big\|\partial_j\partial_k\Lambda^{-1}eG_t\big\|_{L^r(\mathbb{R}^n)} \leqslant Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{3.30}
$$

Proof. The following estimates for the Gauss kernel G_t are from [17]: Assume that real parameters ℓ and m satisfy $0 \leqslant \ell \leqslant n$ and $m \geqslant 0$. Then for any $t>0$

$$
\begin{aligned}
\big|\partial_j G_t(x)\big| &\leqslant C_{\ell,m}t^{\frac{\ell+m-n-1}{2}}\big|x'\big|^{-\ell}|x_n|^{-m} \quad \text{for } 1 \leqslant j \leqslant n \text{ with } n \geqslant 2; \\
\big|\partial_j\partial_k\Lambda^{-1}G_t(x)\big| &\leqslant C_{\ell,m}t^{\frac{\ell+m-n-1}{2}}\big|x'\big|^{-\ell}|x_n|^{-m} \quad \text{for } 1 \leqslant j,k \leqslant n-1 \text{ with } n \geqslant 3; \\
\big|\Lambda G_t(x)\big| &\leqslant C_{\ell,m}t^{\frac{\ell+m-n-1}{2}}\big|x'\big|^{-\ell}|x_n|^{-m} \quad \text{with } n \geqslant 2.
\end{aligned}
$$

Note that $\partial_j\partial_k\Lambda^{-1}=-\Lambda$ if $n=2$, $j=k=1$. Let $1<r<\infty$ and $1 \leqslant i,j,k \leqslant n-1$. Then for any $0 \leqslant \ell \leqslant n$ and $m \geqslant 0$,

$$
\begin{aligned}
& \|\Lambda eG_t\|_{L^r(\mathbb{R}^n)}+\|\partial_i eG_t\|_{L^r(\mathbb{R}^n)}+\big\|\partial_j\partial_k\Lambda^{-1}eG_t\big\|_{L^r(\mathbb{R}^n)} \\
&\quad \leqslant \|\Lambda G_t\|_{L^r(\mathbb{R}^n)}+\|\partial_i G_t\|_{L^r(\mathbb{R}^n)}+\|\partial_j\partial_k\Lambda^{-1}G_t\|_{L^r(\mathbb{R}^n)} \\
&\quad \leqslant \sum_{q=1}^{4} C_{\ell,m}t^{\frac{\ell+m-n-1}{2}}\Bigg(\int\limits_{\Omega_q}\big|x'\big|^{-\ell r}|x_n|^{-mr}\,dx'\,dx_n\Bigg)^{\frac{1}{r}},
\end{aligned} \tag{3.31}
$$

where Ω_q $(q=1,2,3,4)$ are defined as follows:

$$
\begin{aligned}
\Omega_1 &= \big\{\big(x',x_n\big)\in\mathbb{R}^n;\ \big|x'\big| \leqslant t^{\frac{1}{2}} \text{ and } |x_n| \leqslant t^{\frac{1}{2}}\big\}; \\
\Omega_2 &= \big\{\big(x',x_n\big)\in\mathbb{R}^n;\ \big|x'\big| > t^{\frac{1}{2}} \text{ and } |x_n| \leqslant t^{\frac{1}{2}}\big\}; \\
\Omega_3 &= \big\{\big(x',x_n\big)\in\mathbb{R}^n;\ \big|x'\big| \leqslant t^{\frac{1}{2}} \text{ and } |x_n| > t^{\frac{1}{2}}\big\}; \\
\Omega_4 &= \big\{\big(x',x_n\big)\in\mathbb{R}^n;\ \big|x'\big| > t^{\frac{1}{2}} \text{ and } |x_n| > t^{\frac{1}{2}}\big\}.
\end{aligned}
$$

For each integration in (3.31), we take suitable ℓ and m such that $\ell=m=0$ in Ω_1; $\ell=n$, $m=0$ in Ω_2; $\ell=0$, $m=2$ in Ω_3; $\frac{n-1}{r}<\ell \leqslant n$, $m>\frac{1}{r}$ with $1<r<\infty$ in Ω_4. Then for any $t>0$

$$\sum_{q=1}^{4} C_{\ell,m} t^{\frac{\ell+m-n-1}{2}} \left(\int_{\Omega_q} |x'|^{-\ell r} |x_n|^{-mr} \, dx' \, dx_n \right)^{\frac{1}{r}} \leqslant C t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}. \tag{3.32}$$

Combining (3.31) and (3.32), we conclude that (3.30) is true. □

The following lemma plays a crucial role in the proof of the main result, which can make us to avoid the singularity in considering the decay of the second derivatives of solutions of (1.1).

Lemma 3.4. *Let $1 < q < \infty$. Assume that $a = (a_1, a_2, \dots, a_n) \in W^{1,q}(\mathbb{R}^n_+)$ $(n \geqslant 2)$ satisfies $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ and $a_n|_{\partial \mathbb{R}^n_+} = 0$. Then for any $0 < \theta < 1$ and $t > 0$*

$$\left\| \nabla^2 e^{-tA} a \right\|_{L^\infty(\mathbb{R}^n_+)} \leqslant C(\theta, q) \left(t^{-\frac{1}{2}-\frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)} + t^{-1+\frac{\theta}{2}-\frac{n}{2q}} \left\| y_n^{-\theta} a \right\|_{L^q(\mathbb{R}^n_+)} \right). \tag{3.33}$$

Proof. Note that the Stokes flow $e^{-tA} a$ is given as a restriction $r\bar{u}$ of one vector field $\bar{u} = (\bar{u}', \bar{u}_n)$, where $\bar{u}_n, \bar{u}'$ are given in (2.2), (2.3) respectively. Moreover, from Lemma 1.2 in [17], we conclude for any $1 \leqslant k, j \leqslant n$ and $t > 0$

$$\begin{aligned}
\partial_k \partial_j \bar{u}_n &= -R_j \left\{ R' \cdot \partial_k \Lambda e E(t) a' - R_n \nabla' \cdot \partial_k e E(t) a' + R' \cdot \nabla' \partial_k e E(t) a_n + R_n \partial_k \Lambda e E(t) a_n \right\} \\
&= -R_j \left\{ R' \cdot \Lambda e E(t)(1-\delta_{kn}) \partial_k a' + \delta_{kn} R_n \Lambda e E(t) \nabla' \cdot a' \right. \\
&\quad - R_n \nabla' \cdot e E(t)(1-\delta_{kn}) \partial_k a' - \delta_{kn} R_n \partial_n e E(t) \nabla' \cdot a' \\
&\quad + R' \cdot \nabla' e E(t)(1-\delta_{kn}) \partial_k a_n + \delta_{kn} R' \cdot \partial_n e E(t) \nabla' a_n \\
&\quad \left. + R_n \Lambda e E(t)(1-\delta_{kn}) \partial_k a_n + \delta_{kn} R_n \Lambda e F(t) \partial_n a_n \right\},
\end{aligned} \tag{3.34}$$

where $\delta_{kn} = 0$ if $1 \leqslant k \leqslant n-1$; $\delta_{kn} = 1$ if $k = n$; $\nabla' = (\partial_1, \partial_2, \dots, \partial_{n-1})$.

Let $1 \leqslant k \leqslant n$. Then for any $t > 0$

$$\begin{aligned}
\partial_k \partial_n \bar{u}' &= \partial_k \partial_n E(t) a' - \partial_k \nabla' \left(\nabla' \Lambda^{-1} \cdot F(t) a' \right) \\
&\quad + R_n \left\{ R' \nabla' \cdot \partial_k e E(t) a' - R_n \partial_k \nabla' \left(\nabla' \Lambda^{-1} \cdot e E(t) a' \right) - R' \Lambda \partial_k e E(t) a_n + R_n \nabla' \partial_k e E(t) a_n \right\} \\
&= \partial_n E(t)(1-\delta_{kn}) \partial_k a' + \delta_{kn} \partial_n \partial_n E(t) a' \\
&\quad - \nabla' \left(\nabla' \Lambda^{-1} \cdot F(t)(1-\delta_{kn}) \partial_k a' \right) - \nabla' \left(\nabla' \Lambda^{-1} \cdot E(t) \delta_{kn} \partial_n a' \right) \\
&\quad + R_n \left\{ R' \nabla' \cdot e E(t)(1-\delta_{kn}) \partial_k a' + R' \delta_{kn} \partial_n e E(t) \nabla' \cdot a' \right. \\
&\quad - R_n \nabla' \left(\nabla' \Lambda^{-1} \cdot e E(t)(1-\delta_{kn}) \partial_k a' \right) - R_n \nabla' \left(\nabla' \Lambda^{-1} \cdot \delta_{kn} \partial_n e E(t) a' \right) \\
&\quad - R' \Lambda e E(t)(1-\delta_{kn}) \partial_k a_n - R' \Lambda \delta_{kn} e F(t) \partial_n a_n \\
&\quad \left. + R_n \nabla' e E(t)(1-\delta_{kn}) \partial_k a_n + R_n \delta_{kn} \nabla' e F(t) \partial_n a_n \right\}.
\end{aligned} \tag{3.35}$$

Let $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n-1$. Then for any $t > 0$

$$\begin{aligned}
\partial_k \partial_j \bar{u}' &= \partial_k \partial_j E(t) a' + \partial_j \partial_k \nabla' \Lambda^{-1} E(t) a_n \\
&\quad + R_k \left\{ R' \nabla' \cdot \partial_j e E(t) a' - R_n \nabla' \left(\partial_j \nabla' \Lambda^{-1} \cdot e E(t) a' \right) \right. \\
&\quad \left. - R' \partial_j \Lambda e E(t) a_n + R_n \nabla' \partial_j e E(t) a_n \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \partial_k E(t)\partial_j a' + \partial_j \nabla' \Lambda^{-1} E(t)(1-\delta_{kn})\partial_k a_n + \partial_j \nabla' \Lambda^{-1} F(t)\delta_{kn}\partial_n a_n \\
&\quad + R_k \big\{ R'\nabla' \cdot eE(t)\partial_j a' - R_n \nabla' \big(\nabla' \Lambda^{-1} \cdot eE(t)\partial_j a'\big) \\
&\quad - R' \Lambda eE(t)\partial_j a_n + R_n \nabla' eE(t)\partial_j a_n \big\}.
\end{aligned} \tag{3.36}
$$

To proceed, we recall a known result, which will be used frequently. Let K be an integral operator defined by

$$
Kf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\,dy, \quad \forall x \in \mathbb{R}^n.
$$

Note that the Riesz operators R_j $(j = 1, 2, \ldots, n)$ are bounded on $\mathcal{H}^r(\mathbb{R}^n) = L^r(\mathbb{R}^n)$ with $1 < r < \infty$ (see [36] for example). Using Young's inequality, we have for all $1 \leqslant j, m \leqslant n$, $1 < q < \infty$ and $t > 0$

$$
\begin{aligned}
\|R_j R_m K f\|_{L^\infty(\mathbb{R}^n)} &= \bigg\| \int_{\mathbb{R}^n} R_{x_j} R_{x_m} k(x,y) f(y)\,dy \bigg\|_{L^\infty(\mathbb{R}^n_x)} \\
&\leqslant \sup_{x \in \mathbb{R}^n} \big\| R_{x_j} R_{x_m} k(x,y) \big\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n_y)} \|f\|_{L^q(\mathbb{R}^n)} \\
&\leqslant |||R_j||| \, |||R_m||| \sup_{x \in \mathbb{R}^n} \big\| k(x,y) \big\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n_y)} \|f\|_{L^q(\mathbb{R}^n)}.
\end{aligned}
$$

Let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. From (3.34)–(3.36) and Lemma 3.3, we conclude for any $1 \leqslant k, j \leqslant n$ and $t > 0$

$$
\begin{aligned}
\|\partial_k \partial_j \bar{u}_n\|_{L^\infty(\mathbb{R}^n)} &\leqslant C |||R_j||| \Big\{ \big|\big|\big| R' \big|\big|\big| \, \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \big\|\partial_k a'\big\|_{L^q(\mathbb{R}^n)} \\
&\quad + |||R_n||| \, \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \big\|\nabla' \cdot a'\big\|_{L^q(\mathbb{R}^n)} \\
&\quad + |||R_n||| \, \big\|\nabla' e G_t\big\|_{L^{q'}(\mathbb{R}^n)} \big\|\partial_k a'\big\|_{L^q(\mathbb{R}^n)} \\
&\quad + |||R_n||| \, \|\partial_n e G_t\|_{L^{q'}(\mathbb{R}^n)} \big\|\nabla' \cdot a'\big\|_{L^q(\mathbb{R}^n)} \\
&\quad + \big|\big|\big| R' \big|\big|\big| \, \big\|\nabla' e G_t\big\|_{L^{q'}(\mathbb{R}^n)} \|\partial_k a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + \big|\big|\big| R' \big|\big|\big| \, \|\partial_n e G_t\|_{L^{q'}(\mathbb{R}^n)} \big\|\nabla' a_n\big\|_{L^q(\mathbb{R}^n)} \\
&\quad + |||R_n||| \, \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \|\partial_k a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + |||R_n||| \, \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \|\partial_n a_n\|_{L^q(\mathbb{R}^n)} \Big\} \\
&\leqslant C t^{-\frac{1}{2} - \frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)}.
\end{aligned} \tag{3.37}
$$

Let $1 \leqslant k \leqslant n$, $1 \leqslant j \leqslant n-1$. Then for any $t > 0$

$$
\begin{aligned}
\big\|\partial_k \partial_j \bar{u}'\big\|_{L^\infty(\mathbb{R}^n)} &\leqslant \|\partial_k G_t\|_{L^{q'}(\mathbb{R}^n)} \big\|\partial_j a'\big\|_{L^q(\mathbb{R}^n)} + \big\|\partial_j \nabla' \Lambda^{-1} G_t\big\|_{L^{q'}(\mathbb{R}^n)} \|\partial_k a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + \big\|\partial_j \nabla' \Lambda^{-1} G_t\big\|_{L^{q'}(\mathbb{R}^n)} \|\partial_n a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + C |||R_k||| \Big\{ \big|\big|\big| R' \big|\big|\big| \, \big\|\nabla' e G_t\big\|_{L^{q'}(\mathbb{R}^n)} \big\|\partial_j a'\big\|_{L^q(\mathbb{R}^n)} \\
&\quad + |||R_n||| \, \big\|\nabla' \nabla' \Lambda^{-1} e G_t\big\|_{L^{q'}(\mathbb{R}^n)} \big\|\partial_j a'\big\|_{L^q(\mathbb{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|\!\left|\!\left| R' \right|\!\right|\!\right| \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \|\partial_j a_n\|_{L^q(\mathbb{R}^n)} \\
&+ |\!|\!| R_n |\!|\!| \left\|\nabla' e G_t\right\|_{L^{q'}(\mathbb{R}^n)} \|\partial_j a_n\|_{L^q(\mathbb{R}^n)} \Big\} \\
\leqslant{}& C t^{-\frac{1}{2}-\frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)};
\end{aligned} \tag{3.38}
$$

and

$$
\begin{aligned}
&\left\|\partial_k \partial_n \bar{u}' - \delta_{kn}\partial_n\partial_n E(t) a'\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leqslant C\|\partial_n G_t\|_{L^{q'}(\mathbb{R}^n)} \left\|\partial_k a'\right\|_{L^q(\mathbb{R}^n)} + C\left\|\nabla'\nabla'\Lambda^{-1} G_t\right\|_{L^{q'}(\mathbb{R}^n)} \left\|\partial_k a'\right\|_{L^q(\mathbb{R}^n)} \\
&\quad + C\left\|\nabla'\nabla'\Lambda^{-1} G_t\right\|_{L^{q'}(\mathbb{R}^n)} \left\|\partial_n a'\right\|_{L^q(\mathbb{R}^n)} \\
&\quad + C|\!|\!| R_n |\!|\!| \Big\{ \left|\!\left|\!\left| R' \right|\!\right|\!\right| \left\|\nabla' e G_t\right\|_{L^{q'}(\mathbb{R}^n)} \left\|\partial_k a'\right\|_{L^q(\mathbb{R}^n)} \\
&\quad + \left|\!\left|\!\left| R' \right|\!\right|\!\right| \|\partial_n e G_t\|_{L^{q'}(\mathbb{R}^n)} \left\|\nabla' \cdot a'\right\|_{L^q(\mathbb{R}^n)} \\
&\quad + |\!|\!| R_n |\!|\!| \left\|\nabla'\nabla'\Lambda^{-1} e G_t\right\|_{L^{q'}(\mathbb{R}^n)} \left\|\partial_k a'\right\|_{L^q(\mathbb{R}^n)} \\
&\quad + \left|\!\left|\!\left| R' \right|\!\right|\!\right| \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \|\partial_k a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + \left|\!\left|\!\left| R' \right|\!\right|\!\right| \|\Lambda e G_t\|_{L^{q'}(\mathbb{R}^n)} \|\partial_n a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + |\!|\!| R_n |\!|\!| \left\|\nabla' e G_t\right\|_{L^{q'}(\mathbb{R}^n)} \|\partial_k a_n\|_{L^q(\mathbb{R}^n)} \\
&\quad + |\!|\!| R_n |\!|\!| \left\|\nabla' e G_t\right\|_{L^{q'}(\mathbb{R}^n)} \|\partial_n a_n\|_{L^q(\mathbb{R}^n)} \Big\} \\
&\quad + \left\| R_n R_n \nabla' \left(\nabla'\Lambda^{-1} \cdot \partial_n e E(t) a'\right)\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leqslant C t^{-\frac{1}{2}-\frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)} + C\left\| R_n R_n \nabla' \left(\nabla'\Lambda^{-1} \cdot \partial_n e E(t) a'\right)\right\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned} \tag{3.39}
$$

It follows from (3.39) that for all $1 \leqslant k \leqslant n$ and $t > 0$

$$
\begin{aligned}
\left\|\partial_k \partial_n \bar{u}'\right\|_{L^\infty(\mathbb{R}^n)} &\leqslant \left\|\partial_k \partial_n \bar{u}' - \delta_{kn}\partial_n\partial_n E(t) a'\right\|_{L^\infty(\mathbb{R}^n)} + \left\|\partial_n E(t) \partial_n a'\right\|_{L^\infty(\mathbb{R}^n)} \\
&\quad + \left\|\partial_n\partial_n\big[E(t) - F(t)\big] a'\right\|_{L^\infty(\mathbb{R}^n_+)} \\
&\leqslant C t^{-\frac{1}{2}-\frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)} + C\left\| R_n R_n \nabla' \left(\nabla'\Lambda^{-1} \cdot \partial_n e E(t) a'\right)\right\|_{L^\infty(\mathbb{R}^n)} \\
&\quad + \left\|\partial_n\partial_n\big[E(t) - F(t)\big] a'\right\|_{L^\infty(\mathbb{R}^n_+)}.
\end{aligned} \tag{3.40}
$$

Note that for any $1 < q < \infty$ and $t > 0$

$$
\begin{aligned}
\left\|\partial_n\partial_n\big[E(t) - F(t)\big] a'\right\|_{L^\infty(\mathbb{R}^n_+)} &\leqslant 2\left\| \int_{\mathbb{R}^n_+} G_t^{(n-1)}\left(x' - y'\right) \left|\partial_{x_n}\partial_{x_n} G_t^{(1)}(x_n + y_n)\right| \left|a'(y)\right| dy \right\|_{L^\infty(\mathbb{R}^n_+)} \\
&\leqslant C t^{-1+\frac{\theta}{2}} \left\| \int_{\mathbb{R}^n_+} G_t^{(n-1)}\left(x' - y'\right) \left(1 + \left(\frac{x_n + y_n}{2\sqrt{t}}\right)^2\right) \left(\frac{x_n + y_n}{2\sqrt{t}}\right)^\theta \right.
\end{aligned}
$$

$$\times G_t^{(1)}(x_n+y_n)y_n^{-\theta}\left|a'(y)\right|dy\Bigg\|_{L^\infty(\mathbb{R}^n_+)}$$
$$\leqslant Ct^{-1+\frac{\theta}{2}-\frac{n}{2q}}\left\|y_n^{-\theta}a'\right\|_{L^q(\mathbb{R}^n_+)},\quad \forall 0<\theta<1; \tag{3.41}$$

and for any $\varphi\in C_0^\infty(\mathbb{R}^n)$ and $t>0$

$$\left\langle\partial_n eE(t)a',\varphi\right\rangle=\left\langle e\partial_nE(t)a',\varphi\right\rangle=\left\langle e\partial_nF(t)a',\varphi\right\rangle+\left\langle e\partial_n\big(E(t)-F(t)\big)a',\varphi\right\rangle,$$

which implies that for any $t>0$

$$\left[\partial_n eE(t)a'\right](x)=\left[eE(t)\partial_na'\right](x)-2\int_{\mathbb{R}^{n-1}}\int_0^\infty G_t^{(n-1)}\big(x'-y'\big)\theta(x_n)\partial_{x_n}G_t^{(1)}(x_n+y_n)a'(y)\,dy. \tag{3.42}$$

It follows from (3.42) that for all $1\leqslant j,k\leqslant n-1$ and $t>0$

$$\left\|R_nR_n\partial_j\partial_k\Lambda^{-1}\int_{\mathbb{R}^{n-1}}\int_0^\infty G_t^{(n-1)}\big(x'-y'\big)\theta(x_n)\theta(y_n)\partial_nG_t^{(1)}(x_n+y_n)a'(y)\,dy\right\|_{L^\infty(\mathbb{R}^n)}$$
$$\leqslant C\left\|y_n^{-\theta}a'\right\|_{L^q(\mathbb{R}^n_+)}|||R_n|||^2$$
$$\times\sup_{y=(y',y_n)\in\mathbb{R}^n}\left\|\partial_j\partial_k\Lambda^{-1}G_t^{(n-1)}\big(x'-y'\big)\theta(x_n)\theta(y_n)y_n^\theta\partial_nG_t^{(1)}(x_n+y_n)\right\|_{L^{q'}(\mathbb{R}^n)}$$
$$\leqslant Ct^{-\frac{1-\theta}{2}}\left\|y_n^{-\theta}a'\right\|_{L^q(\mathbb{R}^n_+)}\sup_{y=(y',y_n)\in\mathbb{R}^n}\Bigg\|\partial_j\partial_k\Lambda^{-1}G_t^{(n-1)}\big(x'-y'\big)$$
$$\times\theta(x_n)\theta(y_n)\left(\frac{x_n+y_n}{2\sqrt{t}}\right)^{1+\theta}G_t^{(1)}(x_n+y_n)\Bigg\|_{L^{q'}(\mathbb{R}^n)}$$
$$\leqslant Ct^{-1+\frac{\theta}{2}-\frac{n}{2q}}\left\|y_n^{-\theta}a'\right\|_{L^q(\mathbb{R}^n_+)},\quad\forall 0<\theta<1. \tag{3.43}$$

Here we have used the estimate: Let $1\leqslant m,j\leqslant n-1$ and $0\leqslant\ell\leqslant n$. Then (see Lemma 2.2)

$$\left|\Lambda G_t^{(n-1)}\big(x'\big)\right|+\left|\partial_m\partial_j\Lambda^{-1}G_t^{(n-1)}\big(x'\big)\right|\leqslant C_\ell t^{\frac{\ell-n}{2}}\left|x'\right|^{-\ell},\quad\forall x'\in\mathbb{R}^{n-1}.$$

Whence, from (3.42) and (3.43), we conclude that for all $1<q<\infty$ and $t>0$

$$\left\|R_nR_n\nabla'\big(\nabla'\Lambda^{-1}\cdot\partial_neE(t)a'\big)\right\|_{L^\infty(\mathbb{R}^n)}$$
$$\leqslant Ct^{-\frac{1}{2}-\frac{n}{2q}}\|\nabla a\|_{L^q(\mathbb{R}^n_+)}+Ct^{-1+\frac{\theta}{2}-\frac{n}{2q}}\left\|y_n^{-\theta}a'\right\|_{L^q(\mathbb{R}^n_+)},\quad\forall 0<\theta<1. \tag{3.44}$$

From (3.40), (3.41) and (3.44), we conclude for all $1\leqslant k\leqslant n$, $1<q<\infty$ and $t>0$

$$\left\|\partial_k\partial_n\bar{u}'\right\|_{L^\infty(\mathbb{R}^n_+)}\leqslant Ct^{-\frac{1}{2}-\frac{n}{2q}}\|\nabla a\|_{L^q(\mathbb{R}^n_+)}+Ct^{-1+\frac{\theta}{2}-\frac{n}{2q}}\left\|y_n^{-\theta}a'\right\|_{L^q(\mathbb{R}^n_+)},\quad\forall 0<\theta<1. \tag{3.45}$$

Since $e^{-tA}a = \bar{u}|_{\mathbb{R}^n_+}$, from (3.37), (3.38) and (3.45), we derive for all $1 < q < \infty$ and $t > 0$

$$\|\nabla^2 e^{-tA} a\|_{L^\infty(\mathbb{R}^n_+)} \leqslant C t^{-\frac{1}{2}-\frac{n}{2q}} \|\nabla a\|_{L^q(\mathbb{R}^n_+)} + C t^{-1+\frac{\theta}{2}-\frac{n}{2q}} \|y_n^{-\theta} a\|_{L^q(\mathbb{R}^n_+)}, \quad \forall 0 < \theta < 1. \qquad \square$$

To proceed, we need the following known result (see [19]): Let $1 \leqslant k, m \leqslant n$, then for all $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$

$$\left\| \sum_{i,j=1}^{n} \partial_k \partial_m \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \leqslant C\big(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|_{L^{2q_1}(\mathbb{R}^n_+)} \|\nabla^2 u\|_{L^{2q_2}(\mathbb{R}^n_+)}\big), \tag{3.46}$$

for any $1 \leqslant q \leqslant \infty$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, $1 \leqslant q_1, q_2 \leqslant \infty$.

Proof of Theorem 1.3. Let u be the strong solution of (1.1). Then for any $0 < \theta < 1$, $1 < r < \infty$ and $t > 0$

$$\begin{aligned} &\|x_n^{-\theta} u(t) \cdot \nabla u(t)\|_{L^r(\mathbb{R}^n_+)} \\ &\quad \leqslant \|x_n^{-\theta} u(t)\|_{L^{2r}(\mathbb{R}^{n-1}\times(0,1))} \|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)} + \|u(t)\|_{L^{2r}(\mathbb{R}^n_+)} \|\nabla u(t)\|_{L^{2r}(\mathbb{R}^n_+)}. \end{aligned} \tag{3.47}$$

Note that for any $0 < \theta < 1$, $1 < r < \infty$ and $t > 0$

$$\begin{aligned} \|x_n^{-\theta} u(t)\|^{2r}_{L^{2r}(\mathbb{R}^{n-1}\times(0,1))} &\leqslant \int_0^1 \int_{\mathbb{R}^{n-1}} x_n^{-2\theta r} \big|u(x', x_n, t) - u(x', 0, t)\big|^{2r}\, dx'\, dx_n \\ &\leqslant \int_0^1 \int_{\mathbb{R}^{n-1}} x_n^{2r-1-2\theta r} \int_0^{x_n} \big|\partial_n u(x', z_n, t)\big|^{2r}\, dz_n\, dx'\, dx_n \\ &\leqslant \frac{1}{2r(1-\theta)} \int_{\mathbb{R}^{n-1}} \int_0^1 \big|\partial_n u(x', z_n, t)\big|^{2r}\, dz_n\, dx' \\ &\leqslant C \|\nabla u(t)\|^{2r}_{L^{2r}(\mathbb{R}^n_+)}. \end{aligned}$$

Whence, from (3.47), we obtain for any $1 < r < \infty$, $0 < \theta < 1$ and $t > 0$

$$\|x_n^{-\theta} u(t) \cdot \nabla u(t)\|_{L^r(\mathbb{R}^n_+)} \leqslant C\big(\|u(t)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla u(t)\|^2_{L^{2r}(\mathbb{R}^n_+)}\big). \tag{3.48}$$

Let $0 < \theta < 1$ and $\frac{n}{\theta} < r < \infty$. From (2.28), (3.46), (3.48) and Lemmata 3.1, 3.2, 3.4, we deduce for any $t > 0$

$$\|\nabla^2 u(t)\|_{L^\infty(\mathbb{R}^n_+)} \leqslant \left\|\nabla^2 e^{-\frac{t}{2}A} u\left(\frac{t}{2}\right)\right\|_{L^\infty(\mathbb{R}^n_+)} + \int_{\frac{t}{2}}^{t} \|\nabla^2 e^{-(t-s)A} P u(s) \cdot \nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)}\, ds$$

$$
\begin{aligned}
&\leqslant Ct^{-1-\frac{n}{4}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)} + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\left\|\nabla Pu(s)\cdot\nabla u(s)\right\|_{L^r(\mathbb{R}^n_+)}\,ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\theta}{2}-\frac{n}{2r}}\left\|x_n^{-\theta}Pu(s)\cdot\nabla u(s)\right\|_{L^r(\mathbb{R}^n_+)}\,ds \\
&\leqslant Ct^{-1-\frac{n}{4}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)} + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\Biggl(\left\|\nabla\bigl(u(s)\cdot\nabla u(s)\bigr)\right\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + \left\|\nabla\left(\sum_{i,j=1}^{n}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\right)\right\|_{L^r(\mathbb{R}^n_+)}\Biggr)\,ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\theta}{2}-\frac{n}{2r}}\Biggl(\left\|x_n^{-\theta}u(s)\cdot\nabla u(s)\right\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + \left\|\sum_{i,j=1}^{n}x_n^{-\theta}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\right\|_{L^r(\mathbb{R}^n_+)}\Biggr)\,ds \\
&\leqslant Ct^{-1-\frac{n}{4}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)} \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\left(\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\right)ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\theta}{2}-\frac{n}{2r}}\left(\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\right)ds \\
&\leqslant Ct^{-\frac{n+2}{2}} + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\Bigl((1+s)^{-n(1-\frac{1}{2r})} + s^{-1-n(1-\frac{1}{2r})} \\
&\quad + s^{-2-n(1-\frac{1}{2r})}\bigl(1+s^{-2n+2}\bigr)\Bigr)\,ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\theta}{2}-\frac{n}{2r}}\left((1+s)^{-n(1-\frac{1}{2r})} + s^{-1-n(1-\frac{1}{2r})}\right)ds \\
&\leqslant \begin{cases} Ct^{-\frac{3}{2}}(1+t^{-4}) & \text{if } n=2; \\ Ct^{-\frac{n+2}{2}}(1+t^{-\frac{5n-3}{2}}) & \text{if } n\geqslant 3. \end{cases}
\end{aligned}
\tag{3.49}
$$

Assume that the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$ holds. Let $0<\theta<1$ and $\frac{n}{\theta} < r < \infty$. Following the proof of (3.49), we conclude that for any $t>0$

$$
\begin{aligned}
\left\|\nabla^2 u(t)\right\|_{L^\infty(\mathbb{R}^n_+)} \leqslant{}& Ct^{-1-\frac{n}{4}}\left\|u\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R}^n_+)} \\
&+ C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\left(\left\|u(s)\right\|^2_{L^{2r}(\mathbb{R}^n_+)}+\left\|\nabla u(s)\right\|^2_{L^{2r}(\mathbb{R}^n_+)}+\left\|\nabla^2 u(s)\right\|^2_{L^{2r}(\mathbb{R}^n_+)}\right)ds \\
&+ C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\theta}{2}-\frac{n}{2r}}\left(\left\|u(s)\right\|^2_{L^{2r}(\mathbb{R}^n_+)}+\left\|\nabla u(s)\right\|^2_{L^{2r}(\mathbb{R}^n_+)}\right)ds \\
\leqslant{}& Ct^{-\frac{n+3}{2}} + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\Big((1+s)^{-1-n(1-\frac{1}{2r})}+s^{-2-n(1-\frac{1}{2r})} \\
&+ s^{-3-n(1-\frac{1}{2r})}\left(1+s^{-2n}\right)\Big)ds \\
&+ C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\theta}{2}-\frac{n}{2r}}\left((1+s)^{-1-n(1-\frac{1}{2r})}+s^{-2-n(1-\frac{1}{2r})}\right)ds \\
\leqslant{}& Ct^{-\frac{n+3}{2}}\left(1+t^{-\frac{5n+2}{2}}\right). \qquad \square
\end{aligned}
$$

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