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The second variation of the stream function energy of water waves with vorticity

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ABSTRACT

We compute the second variation of the stream function energy of two-dimensional steady free surface gravity water waves with vorticity in the stream function formulation. We prove that for nonpositive vorticity the second variation of the stream function energy at extreme waves with Stokes corner asymptotics cannot be nonnegative in any small neighbourhood of a given isolated stagnation point. The particular form of our second variation suggests however the possibility that certain singularities in the case of nonzero vorticity might be constructible as minimizers of the stream function energy.

1. Introduction

The classical problem of traveling two-dimensional gravity water waves with vorticity can be described mathematically as a free boundary problem for a semilinear elliptic equation: given an open connected set Ω in the (x, y) plane and a function γ of one variable, find a nonnegative function ψ in Ω such that

$$\Delta\psi = -\gamma(\psi) \quad \text{in } \Omega \cap \{\psi > 0\}, \tag{1a}$$

$$\left|\nabla\psi(x, y)\right|^2 = -y \quad \text{on } \Omega \cap \partial\{\psi > 0\}. \tag{1b}$$

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For a motivation of (1) as well as a mathematical background concerning existence, regularity, extreme waves and stagnation points see for example [1,2] as well as the introduction of the paper [3]. Note that problem (1) is also relevant for the description of more general steady flow configurations (for example, the fluid domain could have a nonflat bottom, and there could be some further external forcing acting at the boundary of the fluid region which is not in contact with the air region).

In the present paper we derive a formula for the second variation of the stream function energy

$$E(\psi) = \int \left(|\nabla \psi|^2 - 2\Gamma(\psi) - y\chi_{\{\psi>0\}} \right)$$

of (1) at critical points which are allowed to have isolated singularities; here Γ is a primitive of γ. This energy functional resembles the functional introduced in [4], where various variational formulations for steady water waves with vorticity have been investigated. The functional introduced in [4] has been further pursued in [5] with the aim of establishing some formal stability. Although $E(\psi)$ is not the physical energy, the value of the second variation at a given solution can give us a hint as to whether that solution could possibly be constructed minimizing the energy E which could then be done by direct methods in the calculus of variations. Moreover, in the case of zero vorticity, we obtain from our result an expansion formula for

$$\int_{D_\varepsilon} \left| \nabla(\phi + \varepsilon\zeta) \right|^2$$

in ε, where ϕ is the velocity potential and ζ is a harmonic function with homogeneous Neumann boundary values on the boundary of the physical domain Ω, such that $\phi + \varepsilon\zeta$ has homogeneous Neumann boundary values on the perturbed free surface ∂D_ε.

We are particularly interested in solutions with *singularities* arising for example at *stagnation points*, that is points at which the relative fluid velocity $(\psi_y, -\psi_x)$ is the zero vector. Stokes [6] conjectured that, in the irrotational case $\gamma \equiv 0$, at any stagnation point the free surface has a (symmetric) corner of 120°, and formal asymptotics suggest that the same result might be true also in the general case of waves with vorticity $\gamma \not\equiv 0$.

In the irrotational case, the Stokes conjecture was first proved, under isolatedness, symmetry, and monotonicity assumptions, by Amick, Fraenkel and Toland [7] and Plotnikov [8] (see also [9] for a simplification of the proof in [7]), while a geometric proof has recently been given in [10] without any such structural assumptions.

In the case $\gamma \not\equiv 0$, it was proved in [11] that, at stagnation points, a symmetric monotone free boundary has either a corner of 120° or a horizontal tangent. Moreover, it was also shown there that, if $\gamma \geqslant 0$ close to the free surface, then the free surface necessarily has a corner of 120°. The existence of waves with nonzero vorticity, having stagnation points has been obtained in the setting of periodic waves of finite depth over a flat horizontal bottom under certain assumptions in the paper [12]. The result [3] reveals the possibility of *cusp singularities* and excludes horizontally flat singularities in the case that the vorticity is 0 on the free surface and the free surface is an injective curve. In the case of disconnected air components however, the possibility of horizontally flat singularities remains even for zero vorticity (see also [10]).

This variety of possible singularities raises questions of physical stability — which we will not address in the present paper — as well as the question whether certain singularities might be *stable with respect to the stream function energy* E and thus possibly constructible as minimizers of E.

Let us briefly state our main results and give a plan of the paper:

Main Result A. *(Cf. Proposition 4.3.) Let ψ be a suitable weak solution, and let 0 be an isolated stagnation point of ψ. The second variation in direction ζ, where $0 \leqslant g \in C_0^\infty(B_1 \setminus \{0\})$ and ζ is the harmonic function in $\{\psi > 0\} \cap B_1$ with boundary data g on $\partial(\{\psi > 0\} \cap B_1)$, is given by*

$$\delta^2 J^\psi(\zeta) = \int_{\partial\{\psi>0\}\cap B_1} \zeta^2\left(\frac{\nu_2}{2y} + \kappa - \frac{\gamma(0)}{\sqrt{-y}}\right) d\mathcal{H}^1 + \int_{\{\psi>0\}\cap B_1} \left(|\nabla\zeta|^2 - \gamma'(\psi)\zeta^2\right);$$

here ν is the outward unit normal on $\partial\{\psi > 0\}$ and κ is the curvature on $\partial\{\psi > 0\}$ (nonnegative if $\{\psi = 0\}$ were convex).

The second variation formula is applicable to all isolated singularities, including the more difficult horizontally flat singularities and cusps, however precise estimates for the regularity of the free boundary close to singularities are known only in the case of Stokes corner asymptotics (for a proof of regularity see [3, proof of Theorem 10.1]), we confine ourselves in the present paper to singularities with Stokes corner asymptotics and obtain:

Main Result B. *(Cf. Theorem 5.1.) Let $\gamma \leqslant 0$, let ψ be a suitable weak solution, and let 0 be a stagnation point of ψ with Stokes corner asymptotics. Then ψ is unstable in $B_1(0)$ with respect to the stream function energy in the sense that there exists a test function ζ such that $\delta^2 J^\psi(\zeta) < 0$.*

For zero vorticity we expect the same instability of horizontally flat singularities, but for nonzero vorticity our second variation formula *suggests the possibility of singularities minimizing E in the case that the Rayleigh–Taylor condition is violated*. However, in order to investigate these issues without assuming too much regularity we need regularity theory in the vein of [13] or [14], which we defer to future research.

Plan of the paper. After clarifying our notion of solution and extreme waves in Section 3, we follow in Section 4 the approach in [15] (carried out for the Bernoulli problem) in order to compute the second variation in the water wave problem. Last, we use in Section 5 ideas of [16] in combination with original ideas in order to prove that the second variation is negative in certain directions in the case of Stokes corner asymptotics.

2. Notation

We denote by χ_A the characteristic function of a set A. We denote by $\mathbf{x} = (x, y)$ a point in $\mathbb{R}^2$, by $\mathbf{x}\cdot\mathbf{y}$ the Euclidean inner product in $\mathbb{R}^2 \times \mathbb{R}^2$, by $|\mathbf{x}|$ the Euclidean norm in $\mathbb{R}^2$, by $B_r(\mathbf{x}^0) := \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - \mathbf{x}^0| < r\}$ the ball of center $\mathbf{x}^0$ and radius r and by B_r the ball $B_r(0)$. Also, $\mathcal{L}^n$ shall denote the n-dimensional Lebesgue measure and $\mathcal{H}^s$ the s-dimensional Hausdorff measure. By ν we will always refer to the outer normal on a given surface.

3. Weak solutions

For the sake of convenience we are going to reflect solution etc. at the line $\{y = 0\}$. Let Ω be a bounded domain in $\mathbb{R}^2$ which has a non-empty intersection with the $\{y = 0\}$, in which we consider the combined problem for fluid and air. We study solutions u, in a sense to be specified, of the problem

$$\begin{aligned} \Delta u &= -f(u) \quad \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u|^2 &= y \quad \text{on } \Omega \cap \partial\{u > 0\}. \end{aligned} \tag{2}$$

The nonlinearity f is assumed to be a C^1 function with primitive $F(z) = \int_0^z f(t)\,dt$. Since our results are completely local, we do not specify boundary conditions on $\partial\Omega$.

Definition 3.1 *(Variational solution).* We define $u \in W^{1,2}_{\text{loc}}(\Omega)$ to be a *variational solution* of (2) if $u \in C^0(\Omega) \cap C^2(\Omega \cap \{u > 0\})$, $u \geqslant 0$ in Ω, $u \equiv 0$ in $\Omega \cap \{y \leqslant 0\}$ and the first variation with respect to domain variations of the functional

$$J(v) := \int_\Omega \left(|\nabla v|^2 - 2F(v) + y\chi_{\{v>0\}}\right)$$

vanishes at $v = u$, i.e.

$$\begin{aligned} 0 &= -\frac{d}{d\varepsilon} J\left(u\left(\mathbf{x} + \varepsilon\phi(\mathbf{x})\right)\right)\Big|_{\varepsilon=0} \\ &= \int_\Omega \left(\left(|\nabla u|^2 - 2F(u)\right)\operatorname{div}\phi - 2\nabla u D\phi \nabla u + y\chi_{\{u>0\}}\operatorname{div}\phi + \chi_{\{u>0\}}\phi_2\right) d\mathbf{x} \end{aligned}$$

for any $\phi \in C^1_0(\Omega; \mathbb{R}^2)$.

Since we want to focus in the present paper on the analysis of stagnation points, we will assume that everything is smooth away from $y = 0$, however this assumption may be weakened considerably by using in $\{y > 0\}$ regularity theory for the Bernoulli free boundary problem (see [17] for regularity theory in the case $f = 0$ – which could effortlessly be perturbed to include the case of bounded f – and see [18] for another regularity approach which already includes the perturbation).

Definition 3.2 *(Weak solution).* We define $u \in W^{1,2}_{\text{loc}}(\Omega)$ to be a *weak solution* of (2) if the following are satisfied: u is a *variational solution* of (2) and the topological free boundary $\partial\{u > 0\} \cap \Omega \cap \{y > 0\}$ is locally a $C^{2,\alpha}$-surface.

Remark 3.3. It follows that in $\{y > 0\}$ the solution is a classical solution of (2).

Definition 3.4 *(Stagnation points).* Let u be a variational solution of (2).

We call $S^u := \{(x, y) \in \Omega \mid y = 0 \text{ and } (x, y) \in \partial\{u > 0\}\}$ the set of *stagnation points*.

If $\mathbf{x}^0 \in S^u$ and

$$\frac{u(\mathbf{x}^0 + r\mathbf{x})}{r^{3/2}} \to u_0(\mathbf{x}) = r^{3/2} \cos\left(\frac{3}{2}\left(\min\left(\max\left(\theta, \frac{\pi}{6}\right), \frac{5\pi}{6}\right) - \frac{\pi}{2}\right)\right) \quad \text{in } W^{1,2}_{\text{loc}}(\mathbb{R}^2),$$

then we call $\mathbf{x}$ a *stagnation point with Stokes corner asymptotics*.

Definition 3.5 *(Extreme waves).* If $S^u \neq \emptyset$, the solution u is called an extreme wave.

4. The second variation formula

Let $J(v, B_1) = \int_{\{v>0\}\cap B_1} (|\nabla v|^2 - 2F(v) + y)$, let u be a weak solution in the sense of Definition 3.2, and let 0 be an isolated stagnation point of u.

For $0 \leqslant g \in C^\infty_0(B_1 \setminus \{0\})$, let ζ be the harmonic function in $\{u > 0\} \cap B_1$ with boundary data g on $\partial(\{u > 0\} \cap B_1)$. We define $v_\varepsilon = u - \varepsilon\zeta$ and $D_\varepsilon = \{\mathbf{x} \in B_1 \mid u(\mathbf{x}) < \varepsilon\zeta(\mathbf{x})\}$. Then

$$\begin{aligned} \int_{\{v_\varepsilon>0\}\cap B_1} |\nabla v_\varepsilon|^2 &= \int_{\partial B_1 \cap \{u>0\}} u(u_\nu - \varepsilon\zeta_\nu)\, d\mathcal{H}^1 - \int_{\{v_\varepsilon>0\}\cap B_1} v_\varepsilon \Delta v_\varepsilon \\ &= \int_{\partial B_1 \cap \{u>0\}} u(u_\nu - \varepsilon\zeta_\nu)\, d\mathcal{H}^1 + \int_{\{v_\varepsilon>0\}\cap B_1} f(u) v_\varepsilon \end{aligned}$$

where ν is the unit outward normal on the boundary. Thus

$$\begin{aligned}
&\int\limits_{\{v_\varepsilon>0\}\cap B_1} |\nabla v_\varepsilon|^2 - \int\limits_{\{u>0\}\cap B_1} |\nabla u|^2 \\
&\quad = -\varepsilon \int\limits_{\partial B_1\cap\{u>0\}} u\zeta_\nu \, d\mathcal{H}^1 + \int\limits_{\{v_\varepsilon>0\}\cap B_1} f(u)(u-\varepsilon\zeta) - \int\limits_{\{u>0\}\cap B_1} f(u)u \\
&\quad = -\varepsilon \int\limits_{\partial B_1\cap\{u>0\}} (u\zeta_\nu - u_\nu\zeta)\, d\mathcal{H}^1 + \int\limits_{\{v_\varepsilon>0\}\cap B_1} f(u)(u-\varepsilon\zeta) - \int\limits_{\{u>0\}\cap B_1} f(u)u \\
&\quad = -\varepsilon \int\limits_{\partial(B_1\cap\{u>0\})} (u\zeta_\nu - u_\nu\zeta)\, d\mathcal{H}^1 - \varepsilon \int\limits_{\partial\{u>0\}\cap B_1} \zeta u_\nu \, d\mathcal{H}^1 \\
&\qquad + \int\limits_{\{v_\varepsilon>0\}\cap B_1} f(u)(u-\varepsilon\zeta) - \int\limits_{\{u>0\}} f(u)u \\
&\quad = \varepsilon \int\limits_{\partial\{u>0\}\cap B_1} \zeta\sqrt{y}\, d\mathcal{H}^1 - \varepsilon \int\limits_{\{v_\varepsilon>0\}\cap B_1} f(u)\zeta - \varepsilon \int\limits_{\{u>0\}\cap B_1} f(u)\zeta - \int\limits_{D_\varepsilon} f(u)u
\end{aligned}$$

because $\zeta = 0$ on $\partial B_1 \cap \{u > 0\}$, $u = 0$ on $\partial\{u > 0\} \cap B_1$ and $u_\nu = -\sqrt{y}$ on $\partial\{u > 0\} \setminus \{0\}$. Moreover

$$\int\limits_{\{v_\varepsilon>0\}\cap B_1} F(v_\varepsilon) - \int\limits_{\{u>0\}} F(u) = -\varepsilon \int\limits_{\{v_\varepsilon>0\}\cap B_1} f(u)\zeta + \varepsilon^2 \int\limits_{\{v_\varepsilon>0\}\cap B_1} \frac{f'(u)}{2}\zeta^2 - \int\limits_{D_\varepsilon} F(u) + o(\varepsilon^2).$$

It follows that

$$\begin{aligned}
J(v_\varepsilon, B_1) - J(u, B_1) &= \varepsilon \int\limits_{\partial\{u>0\}\cap B_1} \zeta\sqrt{y}\, d\mathcal{H}^1 - \int\limits_{D_\varepsilon} y - \varepsilon^2 \int\limits_{\{v_\varepsilon>0\}\cap B_1} f'(u)\zeta^2 \\
&\quad - \varepsilon \int\limits_{D_\varepsilon} f(u)\zeta + 2\int\limits_{D_\varepsilon} F(u) - \int\limits_{D_\varepsilon} f(u)u + o(\varepsilon^2).
\end{aligned}$$

Lemma 4.1. *Let $z : \mathbb{R} \to \mathbb{R}^2$, $z(s) = (s, \eta(s))$. In the coordinate system $\mathbf{x}(s,t) = z(s) - t\nu(s)$ (here ν is the outward unit normal on $\partial\{u > 0\}$) the volume element*

$$d\mathbf{x} = (1 - t\kappa)\, d\mathcal{H}^1\, dt$$

where $\kappa = \frac{\eta''}{(\sqrt{1+(\eta')^2})^3}$ is the curvature of $\partial\{u > 0\}$.

Proof. Since $\mathbf{x}(s,t) = (s - t\frac{\eta'(s)}{\sqrt{1+(\eta')^2}}, \eta(s) + t\frac{1}{\sqrt{1+(\eta')^2}})$,

$$D\mathbf{x}(s,t) = \begin{pmatrix} 1 - t(\frac{\eta'}{\sqrt{1+(\eta')^2}})' & \frac{-\eta'}{\sqrt{1+(\eta')^2}} \\ \eta' + t(\frac{1}{\sqrt{1+(\eta')^2}})' & \frac{1}{\sqrt{1+(\eta')^2}} \end{pmatrix}.$$

Thus

$$\det\big(D\mathbf{x}(s,t)\big) = \frac{1}{\sqrt{1+(\eta')^2}} - \frac{1}{\sqrt{1+(\eta')^2}}\frac{t\eta''}{(1+(\eta')^2)^{3/2}} + \frac{(\eta')^2}{\sqrt{1+(\eta')^2}} - \frac{t\eta'\eta''}{(1+(\eta')^2)^{3/2}}\frac{\eta'}{\sqrt{1+(\eta')^2}}$$
$$= \sqrt{1+\left(\eta'\right)^2}\left(1 - t\frac{\eta''}{(1+(\eta')^2)^{3/2}}\right)$$
$$= \sqrt{1+\left(\eta'\right)^2}(1-t\kappa).$$

Therefore

$$d\mathbf{x} = \left|\det\big(D\mathbf{x}(s,t)\big)\right| ds\,dt = (1-t\kappa)\sqrt{1+\left(\eta'\right)^2}\,ds\,dt$$
$$= (1-t\kappa)\,d\mathcal{H}^1\,dt. \qquad \square$$

Similar to [15, Remark 1], we have

Lemma 4.2. *$u_{\nu\nu} = \sqrt{y}\kappa - f(0)$ on $\partial\{u>0\} \cap (B_{r_0} \setminus \{0\})$.*

Proof. Let $\mathbf{x}^0 = (x_0, y_0) \in \partial\{u>0\} \cap (B_{r_0} \setminus \{0\})$. Rotate the coordinate system around $\mathbf{x}^0$ such that $\partial\{u>0\}$ is locally the graph of $y = \eta(x)$ and $\eta' = 0$ at $\mathbf{x}^0$. Since $u(x,\eta(x)) = 0$ near $\mathbf{x}^0$, differentiating with respect to x yields

$$u_x + u_y\eta' = 0$$

and

$$u_{xx} + u_{yx}\eta' + u_y\eta'' = 0.$$

Thus at $\mathbf{x}^0$,

$$u_{xx} = -u_y\eta'' = u_\nu\kappa = -\sqrt{y_0}\kappa.$$

Moreover, since $\Delta u = -f(u)$ and $f(u) = f(0)$ on $\partial\{u>0\}$, we obtain at x_0

$$u_{\nu\nu} = u_{yy} = -u_{xx} - f(0) = \sqrt{y_0}\kappa - f(0). \qquad \square$$

We are now ready to expand the term $\int_{D_\varepsilon} y$ in ε: $z(s) - t\nu(s) \in \partial D_\varepsilon$ is equivalent to

$$u\big(z(s) - t\nu(s)\big) = \varepsilon\zeta\big(z(s) - t\nu(s)\big).$$

Taking Taylor's expansions at $t = 0$, we have

$$t\sqrt{y} + \big(\sqrt{y}\kappa - f(0)\big)\frac{t^2}{2} + O\big(t^3\big) = \varepsilon(\zeta - t\zeta_\nu) + \varepsilon O\big(t^2\big).$$

Let $t = a\varepsilon + b\varepsilon^2 + O(\varepsilon^3)$. We obtain

$$\sqrt{y}a\varepsilon + \sqrt{y}b\varepsilon^2 + \frac{a^2}{2}\varepsilon^2\big(\sqrt{y}\kappa - f(0)\big) + O\big(\varepsilon^3\big) = \varepsilon\zeta - \varepsilon^2 a\zeta_\nu + O\big(\varepsilon^3\big),$$

which implies $a = \frac{\zeta}{\sqrt{y}}$ and $b = \frac{(f(0)-\sqrt{y}\kappa)\zeta^2}{2y^{3/2}} - \frac{\zeta\zeta_\nu}{y}$. Therefore

$$t = \frac{\zeta}{\sqrt{y}}\varepsilon + \left(\frac{(f(0)-\sqrt{y}\kappa)\zeta^2}{2y^{3/2}} - \frac{\zeta\zeta_\nu}{y}\right)\varepsilon^2 + O\big(\varepsilon^3\big).$$

It follows that

$$\begin{aligned}
&\int_{\{u>0\}} y - \int_{\{v_\varepsilon>0\}} y \\
&\quad = \int_{D_\varepsilon} y \\
&\quad = O\big(\varepsilon^3\big) + \int_{\partial\{u>0\}\cap B_1} \int_0^{\frac{\zeta}{\sqrt{y}}\varepsilon + (\frac{(f(0)-\sqrt{y}\kappa)\zeta^2}{2y^{3/2}} - \frac{\zeta\zeta_\nu}{y})\varepsilon^2} (1-t\kappa)(y - t\nu_2)\,dt\,d\mathcal{H}^1 \\
&\quad = O\big(\varepsilon^3\big) + \int_{\partial\{u>0\}\cap B_1} \int_0^{\frac{\zeta}{\sqrt{y}}\varepsilon + (\frac{(f(0)-\sqrt{y}\kappa)\zeta^2}{2y^{3/2}} - \frac{\zeta\zeta_\nu}{y})\varepsilon^2} \big(y - t(y\kappa + \nu_2) + O\big(t^2\big)\big)\,dt\,d\mathcal{H}^1 \\
&\quad = \int_{\partial\{u>0\}\cap B_1} \left(\varepsilon\zeta\sqrt{y} + \frac{(f(0)-\sqrt{y}\kappa)\zeta^2}{2\sqrt{y}}\varepsilon^2 - \zeta\zeta_\nu\varepsilon^2 - \frac{y\kappa+\nu_2}{2}\frac{\zeta^2\varepsilon^2}{y}\right) d\mathcal{H}^1 + O\big(\varepsilon^3\big) \\
&\quad = \varepsilon \int_{\partial\{u>0\}\cap B_1} \zeta\sqrt{y} - \varepsilon^2 \int_{\partial\{u>0\}\cap B_1} \left(\kappa\zeta^2 - \frac{f(0)\zeta^2}{2\sqrt{y}} + \frac{\nu_2\zeta^2}{2y}\right) d\mathcal{H}^1 - \varepsilon^2 \int_{\{u>0\}\cap B_1} |\nabla\zeta|^2 + O\big(\varepsilon^3\big)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon\int_{D_\varepsilon} f(u)\zeta &= \varepsilon \int_{\partial\{u>0\}\cap B_1} \int_0^{\frac{\varepsilon\zeta}{\sqrt{y}}} \big(f(0)\zeta + O(t)\big)(1-t\kappa)\,dt\,d\mathcal{H}^1 + O\big(\varepsilon^3\big) \\
&= \varepsilon^2 \int_{\partial\{u>0\}\cap B_1} \frac{f(0)\zeta^2}{\sqrt{y}}\,d\mathcal{H}^1 + O\big(\varepsilon^3\big).
\end{aligned}$$

Moreover, we have $\varepsilon^2\int_{\{v_\varepsilon>0\}\cap B_1} f'(u)\zeta^2 = \varepsilon^2\int_{\{u>0\}\cap B_1} f'(u)\zeta^2 + O(\varepsilon^3)$,

$$\int_{D_\varepsilon} F(u) = \int_{\partial\{u>0\}\cap B_1} \int_0^{\frac{\varepsilon\zeta}{\sqrt{y}}+(\frac{f(0)-\sqrt{y}\kappa}{2y^{3/2}}-\frac{\zeta\zeta_\nu}{y})\varepsilon^2} \big(-f(0)(\nabla u\cdot\nu)t + O\big(t^2\big)\big)(1-t\kappa)\,dt\,d\mathcal{H}^1 + O\big(\varepsilon^3\big)$$

$$= \varepsilon^2 \int_{\partial\{u>0\}\cap B_1} \frac{f(0)\zeta^2}{2\sqrt{y}}\,d\mathcal{H}^1 + O\big(\varepsilon^3\big),$$

and

$$\int_{D_\varepsilon} f(u)u = \int_{\partial\{u>0\}\cap B_1} \int_0^{\frac{\varepsilon\zeta}{\sqrt{y}}+(\frac{f(0)-\sqrt{y}\kappa}{2y^{3/2}}-\frac{\zeta\zeta_\nu}{y})\varepsilon^2} \big(f(0)\sqrt{y}t + O\big(t^2\big)\big)(1-\kappa t)\,dt\,d\mathcal{H}^1 + O\big(\varepsilon^3\big)$$

$$= \varepsilon^2 \int_{\partial\{u>0\}\cap B_1} \frac{f(0)\zeta^2}{2\sqrt{y}}\,d\mathcal{H}^1 + O\big(\varepsilon^3\big).$$

Thus

$$J(v_\varepsilon, B_1) - J(u, B_1) = \varepsilon^2 \int_{\partial\{u>0\}\cap B_1} \left(\frac{\nu_2\zeta^2}{2y} + \kappa\zeta^2 - \frac{f(0)\zeta^2}{\sqrt{y}}\right) d\mathcal{H}^1$$
$$+ \varepsilon^2 \int_{\{u>0\}\cap B_1} |\nabla\zeta|^2 - \varepsilon^2 \int_{\{u>0\}\cap B_1} f'(u)\zeta^2 + o\big(\varepsilon^2\big).$$

We obtain:

Proposition 4.3. *Let u be a weak solution in the sense of Definition 3.2, and let 0 be an isolated stagnation point of u. The second variation in direction ζ, where $0 \leqslant g \in C_0^\infty(B_1 \setminus \{0\})$ and ζ is the harmonic function in $\{u>0\}\cap B_1$ with boundary data g on $\partial(\{u>0\}\cap B_1)$, is given by*

$$\delta^2 J(\zeta) = \int_{\partial\{u>0\}\cap B_1} \zeta^2\left(\frac{\nu_2}{2y} + \kappa - \frac{f(0)}{\sqrt{y}}\right) d\mathcal{H}^1 + \int_{\{u>0\}\cap B_1} \big(|\nabla\zeta|^2 - f'(u)\zeta^2\big).$$

5. The second variation of extreme waves

Theorem 5.1. *Let $f \leqslant 0$, let u be a weak solution in the sense of Definition 3.2 satisfying in addition $u_y > 0$ in $\Omega \cap \{u>0\}$, and let 0 be a stagnation point of u with Stokes corner asymptotics. Then u is unstable in $B_1(0)$ with respect to the stream function energy in the sense that there exists a test function ζ such that $\delta^2 J(\zeta) < 0$.*

Proof. First, by the proof of [3, Theorem 10.1], 0 is an isolated stagnation point, $\{u>0\}$ is in B_ρ the supergraph of a function η in the y-direction, $\eta \in C^1((-\rho/2, 0]) \cap C^1([0, -\rho/2))$. It follows that there are positive constants $0 < c_1 < C_2 < +\infty$ such that $c_1 \leqslant |\eta'| \leqslant C_2$ in $(-\rho/2, \rho/2)$. The outward normal is given by $\nu = (\frac{\eta'(x)}{\sqrt{1+\eta'(x)^2}}, \frac{-1}{\sqrt{1+\eta'(x)^2}})$. From the higher regularity in [19, Theorem 2] we know also that for $\eta_r(x) := \eta(rx)/r$

$$\|\eta_r\|_{C^2([-1,-1/2]\cup[1/2,1])} \leqslant C_3.$$

It follows that $|\eta''(x)| \leqslant C_3/|x|$ in $(-\rho/2, \rho/2) \setminus \{0\}$ and that $|\kappa(\mathbf{x})| \leqslant C_4/|x|$ in $B_{\rho/4}$. Last, we obtain from [20, (3.15)] that the differential inequality $\frac{\nu_2}{2y} + \kappa \leqslant 0$ is in this graph case always satisfied.

By Proposition 4.3, we have

$$\delta^2 J(\zeta) = \int\limits_{\partial\{u>0\}\cap B_1} \zeta^2 \left(\frac{\nu_2}{2y} + \kappa - \frac{f(0)}{\sqrt{y}}\right) d\mathcal{H}^1 + \int\limits_{\{u>0\}\cap B_1} \left(|\nabla \zeta|^2 - f'(u)\zeta^2\right).$$

Towards a contradiction we assume that $\delta^2 J(\zeta) \geqslant 0$ for all ζ.

For $r \in (0, \rho)$, we define $A_r = \{x \in [-r, -r/2) \mid \kappa = \frac{\eta''(x)}{(\sqrt{1+\eta'(x)^2})^3} \leqslant -\frac{\nu_2}{4y}\}$, $D(r) = 2r^{-1}\mathcal{H}^1(A_r)$ and $D(0+) = \limsup_{r\to 0+} D(r)$.

Case 1. $D(0+) > 0$. Then there exist $\delta > 0$ and a sequence $\{r_i\}$ such that $r_{i+1} \leqslant r_i/2$ and $D(r_i) > \delta$ for all $i \in \mathbb{N}$. We denote $A_i = A_{r_i}$, whereupon $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Following [16, proof of Theorem 8.1], we define $g_\tau(\mathbf{x}) = \phi(\mathbf{x}) - \phi(\frac{2\mathbf{x}}{\tau})$ for $0 < \tau < 1$, where $0 \leqslant \phi \in C_0^\infty(B_1)$, $\phi \equiv 1$ in $B_{\frac{1}{2}}$ and ϕ is a non-increasing function of $|x|$. Let ζ_τ be the harmonic function in $\{u > 0\} \cap B_1$ with boundary data g_τ. Then we have

$$\int\limits_{\{u>0\}\cap B_1} |\nabla \zeta_\tau|^2 \leqslant \int\limits_{\{u>0\}\cap B_1} |\nabla g_\tau|^2 < C_5,$$

where C_5 is independent of τ. On the other hand we obtain for $\zeta_n = \zeta_{r_n}$ that

$$\begin{aligned}
\int\limits_{\partial\{u>0\}\cap B_1} \left(\frac{\nu_2}{2y} + \kappa\right)\zeta_n^2 \, d\mathcal{H}^1 &\leqslant \sum_{i=1}^n \int\limits_{A_i} \left(\frac{\nu_2}{2y} + \kappa\right)\sqrt{1 + (\eta')^2}\, dx \\
&\leqslant -\sum_{i=1}^n \int\limits_{A_i} \frac{1}{4\eta(x)}\, dx \leqslant -\sum_{i=1}^n \int\limits_{A_i} \frac{1}{4C_2|x|}\, dx \\
&\leqslant -\sum_{i=1}^n \frac{\delta}{4C_2} \to -\infty
\end{aligned}$$

as $n \to \infty$.

We also know that

$$\int\limits_{B_1\cap\partial\{u>0\}} \frac{|f(0)|\zeta_n^2}{\sqrt{y}}\, d\mathcal{H}^1 \leqslant \int_{-1}^{1} \frac{|f(0)|}{\sqrt{c_1|x|}}(1 + C_2)\, dx < +\infty$$

and

$$\int\limits_{\{u>0\}\cap B_1} \left(|\nabla \zeta_n|^2 - f'(u)\zeta_n^2\right) \leqslant C_6,$$

where C_6 is independent of n. Thus $\delta^2 J(\zeta_n) \to -\infty$ as $n \to \infty$. So we obtain a contradiction.

Case 2. $D(0+) = 0$. For every $\delta > 0$, there exists an $r_0 \in (0, \rho)$ such that $D(r) < \delta$ for all $r \leqslant r_0$. Let $r_i = \frac{r_0}{2^i}$, $A_i = A_{r_i}$ and $B_i = (-r_i, -r_i/2) \setminus A_i$ for $i \in \mathbb{N}$. Then we obtain

$$\left| \int_{A_i} \kappa \sqrt{1 + \eta'(x)^2}\, dx \right| \leqslant \left| \int_{A_i} \frac{(1 + C_2)C_4}{x}\, dx \right| \leqslant (1 + C_2)C_4\delta$$

and

$$\int_{B_i} \kappa \sqrt{1 + \eta'(x)^2}\, dx \geqslant \int_{B_i} \frac{1}{4\eta}\, dx \geqslant \frac{r_i}{2}(1 - \delta)\frac{1}{4C_2 r_i} = (1 - \delta)(8C_2)^{-1}.$$

We choose a $\delta \in (0, r_0)$ such that $(1 - \delta)(8C_1)^{-1} - (1 + C_2)C_4\delta > \tau > 0$. Then we have

$$\begin{aligned} \left| \int_{r_{n+1}}^{r_1} \kappa \sqrt{1 + \eta'(x)^2}\, dx \right| &\geqslant \sum_{i=1}^{n} \int_{B_i} \kappa \sqrt{1 + \eta'(x)^2}\, dx \\ &\quad - \sum_{i=1}^{n} \left| \int_{A_i} \kappa \sqrt{1 + \eta'(x)^2}\, dx \right| \geqslant n\tau \to +\infty \end{aligned}$$

as $n \to +\infty$, which contradicts the fact that — using the fact that the free boundary is a graph — $|\int_D \kappa\, d\mathcal{H}^1| \leqslant 2\pi$ for all connected subsets D of $\partial\{u > 0\} \setminus \{0\}$. $\square$

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