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Regularity of solutions to the liquid crystal flows with rough initial data

Junyu Lin

Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China

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ABSTRACT

In this paper, we are concerned with the regularity of solutions to the liquid crystal flows with rough initial data in R^n. We prove that the solution constructed by Wang (2011) in [23] has higher regularity. Moreover we obtain a decay estimate in time for any space derivative.

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1. Introduction

In this paper, we will study the following equation modeling the hydrodynamic flow of nematic liquid crystals, which has been proposed and investigated by Lin and Liu [15,16] and [17]:

$$v_t + \nabla\cdot(v\otimes v) - \Delta v + \nabla\Pi = -\nabla\cdot(\nabla d\odot\nabla d) \quad \text{in } R^n\times(0,\infty), \tag{1.1}$$

$$\nabla\cdot v = 0 \quad \text{in } R^n\times(0,\infty), \tag{1.2}$$

$$d_t + v\cdot\nabla d = \Delta d + |\nabla d|^2 d \quad \text{in } R^n\times(0,\infty), \tag{1.3}$$

$$v(x,0) = v_0(x), \qquad d(x,0) = d_0(x), \qquad \big|d_0(x)\big| = 1 \quad \text{in } R^n, \tag{1.4}$$

where $v = (v_1,\dots,v_n): R^n\times(0,\infty)\to R^n$ represents the fluid velocity field, $d = (d_1,d_2,d_3): R^n\times(0,\infty)\to S^2$ is the director field of the nematic liquid crystals and $\Pi: R^n\times(0,\infty)\to R$ is the pressure function. $v\otimes v$ denotes a tensor product whose (i,j)-th entry is given by v_iv_j for $1\leqslant i,j\leqslant n$. $\nabla d\odot\nabla d$ denotes the $n\times n$ matrix whose (i,j)-th entry is given by $\nabla_i d\cdot\nabla_j d$ for $1\leqslant i,j\leqslant n$.

E-mail address: linjiny79@gmail.com.

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The system (1.1)–(1.4) is a simplified version of the Ericksen–Leslie model [4,5,13], but still retains most of the interesting mathematical properties of the original Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals. See [15–17] for more discussions on the relations of the two models. In principle, the system (1.1)–(1.4) is a macroscopic continuum description of the time evolutions of these materials under the influence of both the flow field v, and the macroscopic description of the microscopic orientational configurations d, which can be derived from the averaging/coarse graining of the directions of rod-like liquid crystal molecules. Mathematically, (1.1)–(1.4) is a strongly coupled system between the incompressible Navier–Stokes equation and the transported heat flow of harmonic maps.

Lin, Lin and Wang [19] have proved that there is a global Leray–Hopf type weak solution to (1.1)–(1.4) with boundary condition in 2D, which is smooth away from at most finitely many singular times. See also Hong [8] for related work. Authors in [27] have established a global existence and regularity of weak solution for (1.1)–(1.4) in 2D with the large initial velocity and have proved the uniqueness of weak solution by using the Littlewood–Paley analysis.

For higher dimensions, Lin and Wang [20] have established the uniqueness for the class of Leray–Hopf type weak solutions derived in [19] and uniqueness of weak solution $(v, d) \in C([0, T]; L^n(R^n)) \times C([0, T]; W^{1,n}(R^n, S^2))$ to (1.1)–(1.4) in n-dimensions for $n \geqslant 3$ with initial data (v_0, d_0) satisfying $v_0, \nabla d_0 \in L^n(R^n)$. Lin and Ding [14] have established that for $v_0, \nabla d_0 \in L^n(R^n)$, if $(\|v_0\|_{L^n(R^n)} + \|\nabla d_0\|_{L^n(R^n)})$ is small enough, then (1.1)–(1.4) has a unique global solution (v, d) with $v, \nabla d \in C([0, +\infty); L^n(R^n))$.

For rough initial data, Wang [23] have proved that if $(\|v_0\|_{BMO^{-1}(R^n)} + [d_0]_{BMO(R^n)})$ is small enough, there exists a mild solution (v, d) of (1.1)–(1.4) in $Z \times X$. Recently, Lin in [12] established a uniqueness criterion of liquid crystal flows in higher dimensions. He proved that the mild solution (v, d) of liquid crystal flows is unique under some class data in $vmo^{-1}(R^n) \times vmo(R^n)$, where the spaces BMO, BMO^{-1}, vmo and vmo^{-1} will be explained later.

As mentioned above, (1.1)–(1.3) is a coupled system between the incompressible Navier–Stokes equation and the transported heat flow of harmonic maps. It is well known that the weak solutions exist globally and are partial regular for both incompressible Navier–Stokes equation [1,7,11] and heat flow of harmonic maps [2,3,22,24]. Though there are some results on the incompressible flows of liquid crystals (see [25,26]) in dimensions three, but it is still an open problem for the global existence and regularity of weak solution (1.1)–(1.4) in dimensions $n \geqslant 3$ (see [10]). In this paper, we will focus on the regularity of solutions to (1.1)–(1.4) with rough initial data.

To state our result, we recall some definitions and notations. Let

$$K(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

denote the heat kernel and $e^{t\Delta} = K(\cdot, t)*$ denote the heat semigroup.

Definition 1.1. Let $f(x)$ be a measurable function in R^n. For $0 < T \leqslant +\infty$, define

$$\|f\|_{BMO_T^{-1}} := \sup_{x\in R^n,\, 0<r^2<T} \left(r^{-n} \int_{B_r(x)} \int_0^{r^2} \left|e^{t\Delta} f(y)\right|^2 dt\, dy \right)^{\frac{1}{2}},$$

$$[f]_{BMO_T} := \sup_{x\in R^n, 0<r^2<T} \left(r^{-n} \int_{B_r(x)} \int_0^{r^2} \left|\nabla e^{t\Delta} f(y)\right|^2 dt\, dy \right)^{\frac{1}{2}},$$

where $B_r(x)$ is a ball of radius r and centered at x. Denote

$$BMO_T^{-1}(R^n) := \left\{ f(x) \in S'(R^n) \mid \|f\|_{BMO_T^{-1}} < \infty \right\};$$

$$BMO^{-1}(R^n) := \{ f(x) \in S'(R^n) \mid \|f\|_{BMO^{-1}} := \|f\|_{BMO_\infty^{-1}} < \infty \};$$
$$BMO_T(R^n) := \{ f(x) \in S'(R^n) \mid [f]_{BMO_T} < \infty \};$$
$$BMO(R^n) := \{ f(x) \in S'(R^n) \mid [f]_{BMO} < \infty \};$$
$$bmo(R^n) := \{ f(x) \in S'(R^n) \mid \|f\|_{bmo} := [f]_{BMO_1} < \infty \};$$
$$bmo^{-1}(R^n) := \{ f(x) \in S'(R^n) \mid \|f\|_{bmo^{-1}} := \|f\|_{BMO_1^{-1}} < \infty \};$$
$$vmo(R^n) := \Big\{ f(x) \in bmo(R^n) \;\Big|\; \lim_{T\to 0}[f]_{BMO_T} = 0 \Big\};$$
$$vmo^{-1}(R^n) := \Big\{ f(x) \in bmo^{-1}(R^n) \;\Big|\; \lim_{T\to 0}\|f\|_{BMO_T^{-1}} = 0 \Big\}.$$

We also denote

$$BMO_T(R^n, S^2) = \{ f \in BMO_T(R^n) \mid f(x) \in S^2 \text{ for a.e. } x \in R^n \}.$$

Definition 1.2. (i) For a nonnegative integer k, define

$$\big\| |d| \big\|_{X^k} := \|d\|_{X^{-1}} + \|d\|_{X^k}, \quad \|d\|_{X^{-1}} := \sup_{t\in(0,+\infty)} \big\| d(\cdot,t) \big\|_{L^\infty}, \text{ and } \|d\|_{X^k} := \|d\|_{ND_\infty^k} + \|d\|_{ND_c^k},$$

where

$$\|d\|_{ND_\infty^k} = \sup_{\alpha_1+\cdots+\alpha_n=k+1} \sup_t t^{\frac{k+1}{2}} \big\| \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} d \big\|_{L^\infty},$$

$$\|d\|_{ND_C^k} = \sup_{\alpha_1+\cdots+\alpha_n=k+1} \sup_{x_0,r} \left(r^{-n} \int_0^{r^2} \int_{B_r(x_0)} \big| t^{\frac{k}{2}} \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} d(y,t) \big|^2 \,dy\,dt \right)^{\frac{1}{2}}.$$

Define

$$\|v\|_{Z^k} := \|v\|_{NV_\infty^k} + \|v\|_{NV_C^k},$$

where

$$\|v\|_{NV_\infty^k} = \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_t t^{\frac{k+1}{2}} \big\| \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} v \big\|_{L^\infty},$$

$$\|v\|_{NV_C^k} = \sup_{\alpha_1+\cdots+\alpha_n=k} \sup_{x_0,r} \left(r^{-n} \int_0^{r^2} \int_{B_r(x_0)} \big| t^{\frac{k}{2}} \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} v(y,t) \big|^2 \,dy\,dt \right)^{\frac{1}{2}}.$$

(ii) For $k \geqslant 0$, denote by $\widetilde{Z}^k$ the space $\widetilde{Z}^k = \bigcap_{l=0}^k Z^l$ equipped with the norm $\sum_{l=0}^k \|\cdot\|_{Z^l}$. Similarly, denote by $\widetilde{X}^k$ the space $\widetilde{X}^k = \bigcap_{l=0}^k X^l$ equipped with the norm $\sum_{l=0}^k \||\cdot|\|_{X^l}$.

Remark 1.1. From the definitions of X^k and Z^k, we have

$$\|d\|_{X^k} = \|\nabla d\|_{Z^k}, \quad \text{if } d \in X^k.$$

Let $\mathbb{P}: L^2(R^n) \to PL^2(R^n)$ denote the Leray projection operator. Denote $\Pi_{S^2} \in C^\infty(R^3, R^3)$ is the smooth nearest point projection from $S^2_{\frac{1}{2}} \equiv \{x \in R^3 \mid \frac{1}{2} \leqslant |x| \leqslant \frac{3}{2}\}$ to S^2 (see [23] for more details), that is

$$\Pi_{S^2}(d) = \frac{d}{|d|} : S^2_{\frac{1}{2}} \equiv \left\{ x \in R^3 \,\middle|\, \frac{1}{2} \leqslant |x| \leqslant \frac{3}{2} \right\} \to S^2.$$

Then (1.1)–(1.3) with $v|_{t=0} = v_0$ and $d|_{t=0} = d_0$ are equivalent to the following integral forms

$$v(t) = e^{t\Delta} v_0 - V[v \otimes v + \nabla d \odot \nabla d], \qquad d(t) = e^{t\Delta} d_0 + S\big[-\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d) - v \cdot \nabla d\big],$$

where the operators V and S are defined by

$$Vf(t) := \int_0^t e^{-(t-s)\Delta} \mathbb{P}\nabla \cdot f(s)\, ds; \qquad Sf(t) := \int_0^t e^{-(t-s)\Delta} f(s)\, ds.$$

Remark 1.2. Let $(\alpha_1, \ldots, \alpha_n)$ be a multi-index with $k = \alpha_1 + \cdots + \alpha_n$. Denote $\nabla^k g = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} g$ for simplicity. We also write $\|\nabla^k g\|$ instead of $\sup_{\alpha_1+\cdots+\alpha_n=k} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} g\|$.

Theorem 1.1. *There is $\epsilon > 0$ depending on n, such that if $v_0 \in BMO^{-1}(R^n, R^n)$ with $\nabla \cdot v_0 = 0$ and $d_0 \in BMO(R^n, S^2)$ satisfy*

$$\|v_0\|_{BMO^{-1}} + [d_0]_{BMO} < \epsilon,$$

then the solution (v, d) to (1.1)–(1.4) constructed by Wang in [23] satisfies

$$(v, d) \in Z^k \times X^k, \quad \forall k \geqslant 0. \tag{1.5}$$

Moreover, there is a decay in time of the spaces derivatives as follows

$$\big\|\nabla^k v\big\|_{BMO^{-1}} + \big[\nabla^k d\big]_{BMO} \leqslant Ct^{-\frac{k}{2}}, \quad \textit{for any } t \geqslant 0 \textit{ and any } k \geqslant 0. \tag{1.6}$$

Remark 1.3. From the proof of Theorem 1.1 in Section 3, we also know that the estimate (1.6) can be replaced by the following one

$$\big\|\nabla^{k-1} v\big\|_{L^\infty} + \big\|\nabla^k d\big\|_{L^\infty} \leqslant Ct^{-\frac{k}{2}}, \quad \text{for any } t \geqslant 0 \text{ and any } k \geqslant 0. \tag{1.7}$$

The remaining of the paper is written as follows. In Section 2, we review some preliminary results. In Section 3, we prove Theorem 1.1.

2. Preliminary results

In this section, we will present some preliminary results.

Lemma 2.1 *(Carleson-type estimate). (See [6].) For $N(x,t)$ defined on $R^n \times (0,1)$, let $A(N)$ and $\beta_k(x,t)$ be the quantities*

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$$A(N) = \sup_{x_0 \in R^n} \sup_{t \in (0,1)} t^{\frac{-n}{2}} \int_0^t \int_{B_{\sqrt{t}}(x_0)} \left|N(y,s)\right| dy\, ds,$$

and

$$\beta_k(x,t) = t^{\frac{k}{2}} (-\Delta)^{\frac{k+1}{2}} e^{t\Delta} \int_0^t N(x,s)\, ds.$$

Then there exists a constant $h(k)$ such that the following inequality holds for $\beta_k(x,t)$:

$$\int_0^t \int_{R^n} \left|\beta_k(x,t)\right|^2 dx\, dt \leqslant h(k) A(N) \int_0^1 \int_{R^n} \left|N(y,s)\right| dy\, ds.$$

Lemma 2.2 *(Generalized maximal regularity of the heat kernel). (See [6].) If r is a natural number, then the operators*

$$P_r : f \mapsto \int_0^t e^{(t-s)\Delta} (t-s)^r \Delta^{r+1} f(s)\, ds,$$

and

$$Q_r : f \mapsto \int_0^t e^{(t-s)\Delta} (t-s)^r (\sqrt{t} - \sqrt{s}) \Delta^{r+1} \sqrt{-\Delta} f(s)\, ds$$

are bounded on $L^2([0,T], L^2(R^n))$ for any $T \in [0,+\infty]$ with bounds $p(r)$ and $q(r)$ respectively.

Lemma 2.3 *(Pointwise bound on high derivative estimates for the heat kernel and the Oseen kernel). Both the integral kernels of the operators $\nabla^{k+1} e^{t\Delta}$ and $\nabla^{k+1} \mathbb{P} e^{t\Delta}$ are bounded pointwise by*

$$\frac{C^k k^{k/2}}{t^{k/2}(\sqrt{\frac{t}{k}} + |x|)^{n+1}}.$$

Proof. Miura and Sawada [21] have derived the pointwise bound on high derivative estimates for the Oseen kernel. Similarly, one can derive a similar pointwise bound on high derivative estimates for the heat kernel. We omit the details. □

Remark 2.1. From Proposition 11.1 in [18], we also have that $\nabla^{k+1} e^{t\Delta}$ and $\nabla^{k+1} \mathbb{P} e^{t\Delta}$ are bounded pointwise by

$$\frac{Y(k)}{(\sqrt{t} + |x|)^{n+k+1}},$$

which expresses a slightly different pointwise bound from Lemma 2.3 and where $Y(k)$ is a constant depending on k.

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Lemma 2.4. *Let $v_0 \in BMO^{-1}$ and $d_0 \in BMO$. Then for any $k \geqslant 0$, there exist two positive constants $B_1(k)$ and $B_2(k)$ such that*

$$\left\| e^{t\Delta} v_0 \right\|_{Z^k} \leqslant B_1(k) \| v_0 \|_{BMO^{-1}} \tag{2.1}$$

and

$$\left\| e^{t\Delta} d_0 \right\|_{X^k} \leqslant B_2(k) [d_0]_{BMO}, \tag{2.2}$$

where $B_1(k)$ and $B_2(k)$ depend on k.

Proof. Inequality (2.1) is proved by Miura–Sawada in [6]. By the definitions of BMO and BMO^{-1}, one can derive (2.2) from (2.1). □

In the following, we will recall an estimate for $V(f \otimes g)$ in Z^k.

Proposition 2.1. *(See [6].) If $f, g \in \widetilde{Z}^k$, then $V(f \otimes g) \in Z^k$. Moreover, there holds that for any integer $k \geqslant 1$,*

$$\begin{aligned} \left\| V(f \otimes g) \right\|_{Z^k} \leqslant{} & D_0(k) \| f \|_{Z^0} \| g \|_{Z^0} + D_1 \| f \|_{Z^0} \| g \|_{Z^k} \\ & + D_1 \| g \|_{Z^0} \| f \|_{Z^k} + D(k) \| f \|_{\widetilde{Z}^{k-1}} \| g \|_{\widetilde{Z}^{k-1}}, \end{aligned} \tag{2.3}$$

where the constant D_1 is independent of k while $D_0(k)$ and $D(k)$ depend on k.

Remark 2.2. By using the convention that $\| f \|_{\widetilde{Z}^{-1}} = 0$ and $\| g \|_{\widetilde{Z}^{-1}} = 0$ in (2.3), there holds that

$$\left\| V(f \otimes g) \right\|_{Z^0} \leqslant D \| f \|_{Z^0} \| g \|_{Z^0}, \tag{2.4}$$

which is proved by Koch and Tataru in [9]. Hence (2.4) is included in (2.3) for $k = 0$.

Next, we will prove a critical estimate for the operator S.

Proposition 2.2. *If $d \in \widetilde{X}^k$, then $S(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)) \in X^k$. Moreover, there holds that for any integer $k \geqslant 1$,*

$$\begin{aligned} & \left\| S\left(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\right) \right\|_{X^k} \\ & \quad \leqslant D_0(k) \|d\|_{X^0}^2 \|d\|_{X^{-1}} + D_1 \|d\|_{X^0} \|d\|_{X^k} \|d\|_{X^{-1}} + D(k) \|d\|_{\widetilde{X}^{k-1}}^2 \|d\|_{\widetilde{X}^{k-2}}, \end{aligned} \tag{2.5}$$

where the constant D_1 is independent of k.

Remark 2.3. By using the convention that $\|d\|_{X^{-2}} = 0$ in (2.5), there holds that

$$\left\| S\left(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\right) \right\|_{X^0} \leqslant D_2 \|d\|_{X^0}^2 \|d\|_{X^{-1}}, \tag{2.6}$$

which is proved in [23]. Then (2.6) is included in (2.5) for $k = 0$.

Meanwhile, we also have

$$\left\| S\left(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\right) \right\|_{X^{-1}} \leqslant D_2 \|d\|_{X^0}^2 \|d\|_{X^{-1}}, \tag{2.7}$$

which is proved in Lemma 3.1 of [23].

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Proof of Proposition 2.2. By the definition of the space X^k, we shall prove Proposition 2.2 by combining the following two lemmas.

Lemma 2.5. *There holds that for any integer $k \geqslant 1$,*

$$\begin{aligned} &\big\| S\big(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\big)\big\|_{ND^k_\infty} \\ &\quad \leqslant a_0(k)\|d\|^2_{X^0}\|d\|_{X^{-1}} + 2a_1\|d\|_{X^{-1}}\|d\|_{X^0}\|d\|_{X^k} \\ &\qquad + a_1\|d\|_{X^{k-1}}\|d\|^2_{X^0} + a(k)\sum_{i=1}^{k-1}\sum_{j=1}^{k-i-1}\|d\|_{ND^{i-1}_\infty}\|d\|_{ND^j_\infty}\|d\|_{ND^{k-i-j}_\infty}, \end{aligned} \tag{2.8}$$

where the constant a_1 is independent of k.

Lemma 2.6. *There holds that for any integer $k \geqslant 1$,*

$$\begin{aligned} &\big\| S\big(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\big)\big\|_{ND^k_C} \\ &\quad \leqslant b_0(k)\|d\|^2_{ND^0_C}\|d\|_{X^{-1}} + b_1\|d\|_{ND^0_\infty}\|d\|_{ND^k_C}\|d\|_{X^{-1}} + b(k)\|d\|^2_{\widetilde{X}^{k-1}}\|d\|_{\widetilde{X}^{k-2}}, \end{aligned} \tag{2.9}$$

where the constant b_1 is independent of k.

Proof of Lemma 2.5. Let

$$m = m(k) = C^{-\frac{4}{n+k+1}} k^{\frac{k-3}{n+k+1}},$$

where C is a constant from Lemma 2.3. We split $(0,t)$ as $(0,t) = (0, t(1-\frac{1}{m})) \cup [t(1-\frac{1}{m}), t)$.
Then

$$\begin{aligned} \nabla^{k+1} S\big(\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\big) &= \Bigg(\int_0^{t(1-\frac{1}{m})} + \int_{t(1-\frac{1}{m})}^{t}\Bigg)\nabla^{k+1} e^{(t-s)\Delta}\big[\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\big]\,ds \\ &:= I_{11} + I_{12}. \end{aligned}$$

By the definition of Π_{S^2}, we have

$$\big|\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)\big| \leqslant C|\nabla d|^2|d|,$$

where the constant $C > 0$ depends on S^2. Then by Lemma 2.3, we have

$$\begin{aligned} I_{11} &\leqslant C^k k^{\frac{k}{2}} \int_0^{t(1-\frac{1}{m})}\int_{R^n} (t-s)^{-\frac{k}{2}}\Bigg(\sqrt{\frac{t-s}{k}} + |x-y|\Bigg)^{-n-1}|\nabla d|^2|d|(y,s)\,dy\,ds \\ &\leqslant C^k k^{\frac{k}{2}}\Big(\frac{m}{t}\Big)^{(n+k+1)/2}\int_0^{t(1-\frac{1}{m})}\int_{R^n}\Bigg(\sqrt{\frac{1}{k}} + \frac{|x-y|}{\sqrt{t-s}}\Bigg)^{-n-1}|\nabla d|^2|d|(y,s)\,dy\,ds \end{aligned}$$

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$$\leqslant C^k k^{\frac{k}{2}} \left(\frac{m}{t}\right)^{\frac{n+k+1}{2}} \int_0^{t(1-\frac{1}{m})} \sum_{q\in Z^n} \left[\left(\sqrt{\frac{1}{k}}+|q|\right)^{-n-1} \int_{\{y|x-y\in\sqrt{t}(q+[0,1]^n)\}} |\nabla d|^2 |d|\,dy\right] ds. \tag{2.10}$$

Meanwhile, by

$$\left(\sqrt{\frac{1}{k}}+|q|\right)^{-n-1} = \left(\sqrt{\frac{1}{k}}+|q|\right)^{-1}\left(\sqrt{\frac{1}{k}}+|q|\right)^{-n} \leqslant \sqrt{k}|q|^{-n}$$

and the convergence of series $\sum_{q\in Z^n}|q|^{-n}$, we have

$$\sum_{q\in Z^n}\left(\sqrt{\frac{1}{k}}+|q|\right)^{-n-1} \leqslant C\sqrt{k}. \tag{2.11}$$

Therefore, combining (2.10) with (2.11), we have

$$\left|\int_0^{t(1-\frac{1}{m})} \nabla^{k+1} e^{(t-s)\Delta}\big[\nabla^2 \Pi_{S^2}(d)(\nabla d,\nabla d)\big](x,s)\,ds\right| \leqslant t^{-\frac{k+1}{2}} a_0(k)\|d\|_{X^0}^2 \|d\|_{X^{-1}}, \tag{2.12}$$

where

$$a_0(k) = C^k k^{\frac{k+1}{2}} m^{\frac{n+k+1}{2}} = C^{k-2}k^{k-1}.$$

For I_{12}, we have firstly

$$\big|\nabla^k\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big| \leqslant C\sum_{i=0}^{k}\sum_{j=0}^{k-i} C_i^k C_j^{k-i}\big|\nabla^i d\big|\big|\nabla^{j+1}d\big|\big|\nabla^{k-i-j+1}d\big|,$$

where $C>0$ depends on S^2 and $C_i^k = \frac{k!}{i!(k-i)!}$.

Then

$$\begin{aligned}
I_{12} &\leqslant \left|\int_{t(1-\frac{1}{m})}^{t} \nabla e^{(t-s)\Delta}\right|\nabla^k\big[\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big](x,s)|ds| \\
&\leqslant \int_{t(1-\frac{1}{m})}^{t} (t-s)^{-\frac{1}{2}} \sum_{i=0}^{k}\sum_{j=0}^{k-i} C_i^k C_j^{k-i}\big\|\nabla^i d\big\|_{L^\infty}\big\|\nabla^{j+1}d\big\|_{L^\infty}\big\|\nabla^{k-i-j+1}d\big\|_{L^\infty}\,ds \\
&\leqslant \int_{t(1-\frac{1}{m})}^{t} (t-s)^{-\frac{1}{2}} s^{-(k+2)}\,ds \sum_{i=0}^{k}\sum_{j=0}^{k-i} C_i^k C_j^{k-i}\|d\|_{ND_\infty^{i-1}}\|d\|_{ND_\infty^{j}}\|d\|_{ND_\infty^{k-i-j}} \\
&\leqslant \int_{t(1-\frac{1}{m})}^{t} (t-s)^{-\frac{1}{2}} s^{-(k+2)}ds\cdot\Big(\|d\|_{X^{-1}}\|d\|_{ND_\infty^0}\|d\|_{ND_\infty^k} + \|d\|_{X^{-1}}\|d\|_{ND_\infty^k}\|d\|_{ND_\infty^0}
\end{aligned}$$

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$$+\|d\|_{ND_\infty^{k-1}}\|d\|_{ND_\infty^0}^2+\sum_{i=1}^{k-1}\sum_{j=1}^{k-i-1}C_i^{k-1}C_j^{k-i-1}\|d\|_{ND_\infty^{i-1}}\|d\|_{ND_\infty^j}\|d\|_{ND_\infty^{k-i-j}}\bigg). \tag{2.13}$$

From [6], there exists a constant $a_1>0$, independent of k, such that for any integer $k\geqslant 1$ there holds

$$\int\limits_{t(1-\frac{1}{m})}^{t}(t-s)^{-\frac{1}{2}}s^{-(k+2)}\,ds\leqslant a_1t^{-\frac{k+1}{2}}. \tag{2.14}$$

Combining (2.13) with (2.14), we have

$$\begin{aligned}
&\left|t^{\frac{k+1}{2}}\int\limits_{t(1-\frac{1}{m})}^{t}\nabla^{k+1}e^{(t-s)\Delta}\big[\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big](x,s)\,ds\right|\\
&\quad\leqslant a_1\|d\|_{X^{-1}}\|d\|_{X^0}\|d\|_{X^k}+a_1\|d\|_{X^{-1}}\|d\|_{X^k}\|d\|_{X^0}\\
&\qquad+a_1\|d\|_{X^{k-1}}\|d\|_{X^0}\|d\|_{X^0}+a(k)\sum_{i=1}^{k-1}\sum_{j=1}^{k-i-1}\|d\|_{ND_\infty^{i-1}}\|d\|_{ND_\infty^j}\|d\|_{ND_\infty^{k-i-j}},
\end{aligned} \tag{2.15}$$

where $a(k)$ is a constant depending on k.

Combining (2.12) with (2.15), we have proved Lemma 2.5. □

Proof of Lemma 2.6. Let $\chi\in C^\infty(R^n)$ be a smooth function such that

$$\chi(x)=1\quad\text{for } x\in B_1(0);\qquad \chi(x)=0\quad\text{for } x\in B_2(0)\text{ and } 0\leqslant\chi(x)\leqslant 1.$$

Denote $\chi_{x_0,r}(x)=\chi(\frac{x-x_0}{r})$. We rewrite

$$\int\limits_0^t\nabla e^{(t-s)\Delta}\big[\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big](x,s)\,ds=I_{21}+I_{22}+I_{23}$$

with I_{2i} $(i=1,2,3)$ being defined by

$$I_{21}=\int\limits_0^t e^{(t-s)\Delta}(1-\chi_{x_0,r})\big[\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big](x,s)\,ds;$$

$$I_{22}=\frac{1}{\sqrt{-\Delta}}\sqrt{-\Delta}e^{t\Delta}\int\limits_0^t\big(\chi_{x_0,r}\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big)(x,s)\,ds;$$

$$I_{23}=\frac{1}{\sqrt{-\Delta}}\int\limits_0^t e^{(t-s)\Delta}\frac{\Delta}{\sqrt{-\Delta}}\big(Id-e^{s\Delta}\big)\big(\chi_{x_0,r}\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)\big)(x,s)\,ds.$$

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For I_{21}, we have from Remark 2.1

$$
\begin{aligned}
& r^{-n}\int_0^{r^2}\int_{B_r(x_0)}\left|t^{\frac{k}{2}}\nabla^{k+1}I_{21}\right|^2 dx\,dt \\
&\leqslant r^{-n}\int_0^{r^2}\int_{B_r(x_0)}\left|\int_0^t\int_{B^c_{2r}(x_0)}\frac{M(k)t^{\frac{k}{2}}}{(\sqrt{t-s}+|x-y|)^{n+k+1}}|\nabla d|^2|d|\,dy\,ds\right|^2 dx\,dt \\
&\leqslant r^{-n+k}Y^2(k)\int_0^{r^2}\int_{B_r(x_0)}\left|\int_0^{r^2}\int_{B^c_{2r}(x_0)}\frac{1}{|x-y|^{n+k+1}}|\nabla d|^2|d|\,dy\,ds\right|^2 dx\,dt \\
&\leqslant r^{-n+k}Y^2(k)C(k)\|d\|^2_{X^{-1}}\int_0^{r^2}\int_{B_r(x_0)}\left|\int_0^{r^2}\int_{\bigcup_{q\in\mathbb{Z}^n_+}B_r(x_q)}\frac{1}{(|q|r)^{n+k+1}}|\nabla d|^2\,dy\,ds\right|^2 dx\,dt \\
&\leqslant Y^2(k)C(k)\|d\|^2_{X^{-1}}\left(\sum_{q\in\mathbb{Z}^n,|k|\geqslant 1}|q|^{-n-1}\sup_{x_k\in R^n}\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_q)}|\nabla d|^2\,dy\,ds\right)^2 \\
&\leqslant Y^2(k)C(k)\|d\|^2_{X^{-1}}\left(\sup_{x_k\in R^n}\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_q)}|\nabla d|^2\,dy\,ds\right)^2 \\
&\leqslant Y^2(k)C(k)\|d\|^2_{X^{-1}}\|d\|^4_{ND^0_C},
\end{aligned}
\tag{2.16}
$$

where the constant $Y(k)$ comes from Remark 2.1 and $C(k)$ is a constant dependent of k.

For I_{22}, denote $M(x,s) := \chi_{x_0,r}\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)(x,s)$. Then by the boundedness of Riesz transform on L^2, we get

$$
\begin{aligned}
& r^{-n}\int_0^{r^2}\int_{B_r(x_0)}\left|t^{\frac{k}{2}}\nabla^{k+1}I_{22}\right|^2 dx\,dt \\
&\leqslant r^{-n}\int_0^{r^2}\int_{B_r(x_0)}\left|t^{\frac{k}{2}}(-\Delta)^{\frac{k+1}{2}}e^{t\Delta}\int_0^t M(x,s)\,ds\right|^2 dx\,dt \\
&= r^{-n}\int_0^1\int_{R^n}\left|\left(r^2\tau\right)^{\frac{k+1}{2}}\left(-\frac{1}{r^2}\Delta_z\right)^{\frac{k+1}{2}}e^{\tau\Delta_z}\int_0^\tau M\left(rz,r^2\rho\right)r^2\,d\rho\right|^2 r^n\,dz\,r^2\,d\tau \\
&= r^4\int_0^1\int_{R^n}\left|\tau^{\frac{k}{2}}(-\Delta_z)^{\frac{k+1}{2}}e^{\tau\Delta_z}\int_0^\tau N(z,\rho)\,d\rho\right|^2 dz\,d\tau \\
&\leqslant r^4h(k)A(N)\int_0^1\int_{R^n}\left|N(z,\tau)\right|dz\,d\tau,
\end{aligned}
\tag{2.17}
$$

where we have used a change of variables

$$t = r^2\tau, \qquad s = r^2\rho, \qquad x = rz$$

in the first equality, and Lemma 2.1 in the last inequality and have used notations as follows

$$N(z,\rho) := M\big(rz, r^2\rho\big), \qquad A(N) := \sup_{x_0\in R^n,\,\tau\in(0,1)} \tau^{-\frac{n}{2}} \int_0^\tau \int_{B_\tau(x_0)} \big|N(z,\rho)\big|\,dz\,d\rho.$$

In the following, we will estimate an upper bound of the L^1-norm of $N(z,\rho)$ on $(0,1)\times R^n$ and $A(N)$.

$$\begin{aligned}
\int_0^1 \int_{R^n} \big|N(z,\tau)\big|\,dz\,d\tau &= \int_0^1 \int_{R^n} \big|M\big(rz, r^2\tau\big)\big|\,dz\,d\tau \\
&= \frac{1}{r^{n+2}} \int_0^{r^2} \int_{R^n} \big|M(x,t)\big|\,dx\,dt \\
&= \frac{1}{r^{n+2}} \int_0^{r^2} \int_{R^n} \chi_{x_0,r} \big|\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)(x,s)\big|\,dx\,dt \\
&\leqslant C \frac{1}{r^{n+2}} \int_0^{r^2} \int_{R^n} \chi_{x_0,r} \big|\nabla d(x,s)\big|^2 \big|d(x,s)\big|\,dx\,dt \\
&\leqslant C r^{-2} \|d\|_{X^{-1}} \|d\|^2_{ND_C^0},
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
A(N) &= \sup_{x_0\in R^n,\,\tau\in(0,1)} \tau^{-\frac{n}{2}} \int_0^\tau \int_{B_\tau(x_0)} \big|N(z,\rho)\big|\,dz\,d\rho \\
&= \sup_{x_0\in R^n,\,\tau\in(0,1)} \tau^{-\frac{n}{2}} \frac{1}{r^{n+2}} \int_0^{r^2\tau} \int_{|\frac{x}{r}-x_0|<\tau} \left|N\left(\frac{x}{r}, \frac{s}{r^2}\right)\right|\,dx\,ds \\
&= \sup_{\widetilde{x_0}\in R^n,\,\tau\in(0,1)} \tau^{-\frac{n}{2}} \frac{1}{r^{n+2}} \int_0^{r^2\tau} \int_{|x-\widetilde{x_0}|<r\tau} \big|M(x,s)\big|\,dx\,ds \\
&= \sup_{\widetilde{x_0}\in R^n,\,\tau\in(0,1)} \tau^{-\frac{n}{2}} \frac{1}{r^{n+2}} \int_0^{r^2\tau} \int_{|x-\widetilde{x_0}|<r\tau} \chi_{x_0,r} \big|\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)(x,s)\big|\,dx\,ds
\end{aligned}$$

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$$\leqslant Cr^{-2}\|d\|_{X^{-1}} \sup_{\widetilde{x_0}\in R^n,\tau\in(0,1)} \frac{1}{(r\tau^{\frac{1}{2}})^n} \int_0^{r^2\tau} \int_{B_{r\tau}(\widetilde{x_0})} |\nabla d|^2\,dx\,ds$$

$$\leqslant Cr^{-2}\|d\|_{X^{-1}}\|d\|^2_{ND_C^0}. \tag{2.19}$$

Then (2.17)–(2.19) imply that

$$r^{-n}\int_0^{r^2}\int_{B_r(x_0)} \left|t^{\frac{k}{2}}\nabla^{k+1}I_{22}\right|^2 dx\,dt \leqslant Ch(k)\|d\|^2_{X^{-1}}\|d\|^4_{ND_C^0}. \tag{2.20}$$

For I_{23}, the cases k odd and k even are slightly different. Here we will argue for the case k odd, which is a little more difficult. Let $k = 2K+1$. Decompose $t^{\frac{k}{2}}$ as

$$t^{\frac{k}{2}} = (t-s+s)^K(\sqrt{t}-\sqrt{s}+\sqrt{s}) = \sum_0^K C_l^K\left[s^l(t-s)^{K-l}(\sqrt{t}-\sqrt{s}) + s^l\sqrt{s}(t-s)^{K-l}\right].$$

Denote $M(s) := \chi_{x_0,r}\nabla^2\Pi_{S^2}(d)(\nabla d,\nabla d)(s)$. Then

$$\begin{aligned}
&t^{\frac{k}{2}}\nabla^{k+1}I_{23}\\
&= \sum_{l=0}^K C_l^K \frac{\nabla}{\sqrt{-\Delta}}\int_0^t (t-s)^{K-l}(\sqrt{t}-\sqrt{s})\Delta\nabla^{2K-2l+1}e^{(t-s)\Delta}\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\nabla^{2l}M(s)\,ds\\
&\quad + \sum_{l=0}^K C_l^K \frac{\nabla}{\sqrt{-\Delta}}\int_0^t (t-s)^{K-l}\Delta\nabla^{2K-2l}e^{(t-s)\Delta}\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\,ds\\
&= \sum_{l=0}^{K-1} C_l^K \frac{\nabla}{\sqrt{-\Delta}}Q_{K-l}\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\nabla^{2l}M(s)\right) + \frac{\nabla}{\sqrt{-\Delta}}Q_0\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\nabla^{2l}M(s)\right)\\
&\quad + \sum_{l=0}^{K-1} C_l^K \frac{\nabla}{\sqrt{-\Delta}}P_{K-l}\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right) + \frac{\nabla}{\sqrt{-\Delta}}P_0\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right).
\end{aligned}$$

From Lemma 2.2, *i.e.* the operator P_{K-l} is bounded on $L^2(0,T;L^2(R^n))$ for every $T\in(0,+\infty]$, we have that for $t\leqslant r^2$,

$$\begin{aligned}
&\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|P_{K-l}\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right)\right|^2 dx\,ds\\
&\quad \leqslant p(K-l)\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right)\right|^2 dx\,ds
\end{aligned}$$

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$$\leqslant Cp(K-l)\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|s^{l+1}\nabla^{2l+1}M(s)\right|^2 dx\,ds,$$

where we have used that the operator $\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}$ is bounded by $C\sqrt{s}$ on L^2, since $\frac{1-e^{s|\xi|^2}}{|\xi|} \leqslant C\sqrt{s}$. One can see that $s^{l+1}\nabla^{2l+1}M(s)$ is a sum of terms of the type

$$\left(s^{\frac{m+1}{2}}\nabla^{m+1}d\right)\cdot\left(s^{\frac{\rho+1}{2}}\nabla^{\rho+1}d\right)\cdot\left(s^{\frac{j}{2}}\nabla^{j}d\right)\cdot\left(s^{\frac{2l-m-\rho-j}{2}}\nabla^{2l+1-m-\rho-j}\chi_{x_0,r}\right)$$

with $m, \rho, j \geqslant 0$ and $m+\rho+j \leqslant 2l+1$.

Since $\nabla^{2l+1-m-\rho-j}\chi_{x_0,r} \leqslant Cr^{-(2l+1-m-\rho-j)}$, we have for $s \leqslant r^2$,

$$s^{\frac{2l-m-\rho-j}{2}}\nabla^{2l+1-m-\rho-j}\chi_{x_0,r} \leqslant Cr^{2l-m-\rho-j}r^{-(2l+1-m-\rho-j)} \leqslant \frac{C}{r}.$$

Then we have

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|s^{l+1}\nabla^{2l+1}M(s)\right|^2 dx\,ds \leqslant C\frac{1}{r^n}\sum_{m+\rho+j\leqslant 2l+1}\|d\|^2_{ND^m_\infty}\|d\|^2_{ND^\rho_C}\|d\|^2_{ND^{j-1}_\infty}.$$

Hence

$$\begin{aligned}&\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|P_{K-l}\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right)\right|^2 dx\,ds\\ &\quad\leqslant Cp(K-l)\sum_{m+\rho+j\leqslant 2l+1}\|d\|^2_{ND^m_\infty}\|d\|^2_{ND^\rho_C}\|d\|^2_{ND^{j-1}_\infty}.\end{aligned}\tag{2.21}$$

Similarly, by Lemma 2.3, we have

$$\begin{aligned}&\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|P_0\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right)\right|^2 dx\,ds\\ &\quad\leqslant p(0)\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^K\sqrt{s}\nabla^{2K+1}M(s)\right)\right|^2 dx\,ds\\ &\quad\leqslant Cp(0)\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)} \left|s^{K+1}\nabla^{2K+1}M(s))\right|^2 dx\,ds\\ &\quad\leqslant Cp(0)\left(\|d\|^2_{ND^0_\infty}\|d\|^2_{ND^{2K+1}_C}\|d\|^2_{X^{-1}}\right)+C(K)\|d\|^4_{\tilde{X}^{2K}}\|d\|^2_{\tilde{X}^{2K-1}}.\end{aligned}\tag{2.22}$$

Similarly, we have

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$$\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)}\left|Q_{K-l}\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right)\right|^2 dx\,ds$$
$$\leqslant Cq(K-l)\sum_{m+\rho+j\leqslant 2l+1}\|d\|^2_{ND^m_\infty}\|d\|^2_{ND^\rho_C}\|d\|^2_{ND^{j-1}_\infty}, \tag{2.23}$$

and

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)}\left|Q_0\left(\frac{Id-e^{s\Delta}}{\sqrt{-\Delta}}s^l\sqrt{s}\nabla^{2l+1}M(s)\right)\right|^2 dx\,ds$$
$$\leqslant Cq(0)\left(\|d\|^2_{ND^0_\infty}\|d\|^2_{ND^{2K+1}_C}\|d\|^2_{X^{-1}}\right)+C(K)\|d\|^4_{\widetilde{X}^{2K}}\|d\|^2_{\widetilde{X}^{2K-1}}. \tag{2.24}$$

Therefore by the boundedness of the Riesz $\frac{\nabla}{\sqrt{-\Delta}}$ on L^2, we get from (2.21) to (2.24) and $k=2K+1$ that

$$\frac{1}{r^n}\int_0^{r^2}\int_{B_r(x_0)}\left|t^{\frac{k}{2}}\nabla^{k+1}I_{23}\right|^2 dx\,ds$$
$$\leqslant Cq(0)\left(\|d\|^2_{ND^0_\infty}\|d\|^2_{ND^k_C}\|d\|^2_{X^{-1}}\right)+C(K)\|d\|^4_{\widetilde{X}^{k-1}}\|d\|^2_{\widetilde{X}^{k-2}}. \tag{2.25}$$

Therefore we prove Lemma 2.6 by putting (2.16), (2.20) and (2.25) together. □

Proposition 2.3. *If $f,g,h\in\widetilde{X}^k$, then $S(|\nabla f||\nabla g||h|)\in X^k$. Moreover, there holds that for any integer $k\geqslant 1$,*

$$\begin{aligned}\left\|S\left(|\nabla f||\nabla g||h|\right)\right\|_{X^k}\leqslant{}& D_0(k)\|f\|_{X^0}\|g\|_{X^0}\|h\|_{X^{-1}}+D(k)\|f\|_{\widetilde{X}^{k-1}}\|g\|_{\widetilde{X}^{k-1}}\|h\|_{\widetilde{X}^{k-2}}\\ &+D_1\|f\|_{X^0}\|g\|_{X^k}\|h\|_{X^{-1}}+D_1\|g\|_{X^0}\|f\|_{X^k}\|h\|_{X^{-1}},\end{aligned} \tag{2.26}$$

where the constant D_1 is independent of k.

Proof. One can prove this proposition by using the inequality $|\nabla|f||\leqslant|\nabla f|$ and a method similar to Proposition 2.1. □

Remark 2.4. By using the convention that $\|f\|_{X^{-2}}=\|g\|_{X^{-2}}=\|h\|_{X^{-2}}=0$ in (2.26), there holds that

$$\left\|S\left(|\nabla f||\nabla g||h|\right)\right\|_{X^0}\leqslant D_2\|f\|_{X^0}\|g\|_{X^0}\|h\|_{X^{-1}}. \tag{2.27}$$

Then (2.27) is included in (2.26) for $k=0$.

By a similar argument as (2.7), we also have

$$\left\|S\left(|\nabla f||\nabla g||h|\right)\right\|_{X^{-1}}\leqslant D_2\|f\|_{X^0}\|g\|_{X^0}\|h\|_{X^{-1}}. \tag{2.28}$$

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Similarly, we have

Proposition 2.4. *If $v \in \widetilde{Z}^k$ and $d \in \widetilde{X}^k$, then $S(v \cdot \nabla d) \in X^k$. Moreover, there holds that for any integer $k \geqslant 1$,*

$$\begin{aligned}\|S(v \cdot \nabla d)\|_{X^k} &\leqslant D_0(k)\|d\|_{X^0}\|v\|_{Z^0} \\ &\quad + D_1\|v\|_{Z^0}\|d\|_{X^k} + D_1\|d\|_{X^0}\|v\|_{Z^k} + D(k)\|v\|_{\widetilde{Z}^{k-1}}\|d\|_{\widetilde{X}^{k-1}} \qquad (2.29)\end{aligned}$$

where the constant D_1 is independent of k.

Remark 2.5. By using the convention that $\|v\|_{Z^{-1}} = 0$ in (2.29), there holds that

$$\|S(v \cdot \nabla d)\|_{X^0} \leqslant D\|d\|_{X^0}\|v\|_{Z^0}, \qquad (2.30)$$

which is proved in [23]. Then (2.30) is included in (2.29) for $k = 0$.

Meanwhile we also have

$$\|S(v \cdot \nabla d)\|_{X^{-1}} \leqslant D\|d\|_{X^0}\|v\|_{Z^0}, \qquad (2.31)$$

which is proved in [23].

Remark 2.6. For simplicity, we use the same constants $D_0(k)$, D_1 and $D(k)$ in Propositions 2.1–2.4, where D_1 is independent of k, while $D_0(k)$ and $D(k)$ depend on k.

3. Proof of the main result

In this section, we will prove our main result, *i.e.* Theorem 1.1.

Proof of Theorem 1.1. Define an approximating sequence $\{v^j, d^j\}$ by

$$\begin{aligned} &v^{-1} = 0, \qquad d^{-1} = 0, \\ &v^0 = e^{t\Delta}v_0, \qquad d^0 = e^{t\Delta}d_0, \\ &v^{j+1} = v^0 + V\big[v^j \otimes v^j + \nabla d^j \odot \nabla d^j\big], \qquad (3.1) \\ &d^{j+1} = d^0 + S\big[-\nabla^2 \Pi_{S^2}\big(d^j\big)\big(\nabla d^j, \nabla d^j\big) - v^j \cdot \nabla d^j\big]. \qquad (3.2)\end{aligned}$$

We will prove that if (v_0, d_0) is small enough in $BMO^{-1} \times BMO$, then $\{v^j, d^j\}$ converge to $\{v, d\}$ in $\widetilde{Z}^k \times \widetilde{X}^k$ for any integer $k \geqslant 0$ as $j \to +\infty$ with finite norm $\|v\|_{\widetilde{Z}^k} + \||d|\|_{\widetilde{X}^k}$. Then by a similar argument as in [23], we also have $d(x,t) \in S^2$ for all $(x,t) \in R^n \times [0, +\infty)$. Hence (1.3) of Theorem 1.1 is proved, if the following claim is proved. (1.4) of Theorem 1.1 follows easily by using (1.3).

Claim. *For any integer $k \geqslant 0$, there are positive constants E, F_k and G_k such that*

$$\|d^j\|_{X^{-1}} \leqslant E, \qquad (3.3)$$

$$\|v^j\|_{\widetilde{Z}^k} + \|d^j\|_{\widetilde{X}^k} \leqslant F_k, \qquad (3.4)$$

$$\|v^{j+1} - v^j\|_{\widetilde{Z}^k} + \|d^{j+1} - d^j\|_{\widetilde{X}^k} \leqslant G_k\left(\frac{2}{3}\right)^j. \qquad (3.5)$$

Proof. We prove this claim by induction.

Step 1. We prove that (3.4) and (3.5) are right for $k=0$ firstly.

From the definition of d^0, we have

$$\|d^0\|_{X^{-1}} = \sup_{t\in[0,+\infty)} \|e^{t\Delta}d_0\|_{L^\infty} = \sup_{t\in[0,+\infty)} \left| \int_{R^n} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|y|^2}{2}} d_0(x-\sqrt{t}y)\,dy \right| \leqslant \|d_0\|_{L^\infty}. \tag{3.6}$$

From Lemma 2.4 for $k=0$, we have

$$\|d^0\|_{X^0} \leqslant B_2[d_0]_{BMO} \quad \text{and} \quad \|v^0\|_{Z^0} \leqslant B_1[v_0]_{BMO^{-1}}.$$

From Remark (2.7) and (2.28), there exists a constant $D > \max\{4B_1, 4B_2\} > 0$ such that

$$\begin{aligned}
\|d^1\|_{X^{-1}} &\leqslant \|d^1 - d^0\|_{X^{-1}} + \|d^0\|_{X^{-1}} \\
&= \|S[-\nabla^2\pi_{S^2}(d^0)(\nabla d^0, \nabla d^0) - v^0\cdot\nabla d^0]\|_{X^{-1}} + \|d^0\|_{X^{-1}} \\
&\leqslant D_0\|d^0\|_{X^{-1}}\|d^0\|_{X^0}^2 + D_0\|v^0\|_{Z^0}^2 + D_0\|d^0\|_{X^0}^2 + \|d_0\|_{L^\infty} \\
&\leqslant D(1+\|d_0\|_{L^\infty})[d_0]_{BMO}^2 + D\|v_0\|_{BMO^{-1}}^2 + \|d_0\|_{L^\infty} \\
&\leqslant D(1+\|d_0\|_{L^\infty})\epsilon^2 + D\epsilon^2 + \|d_0\|_{L^\infty}.
\end{aligned}$$

Then choose ϵ so small and E so large that

$$\|d^1\|_{X^{-1}} \leqslant E. \tag{3.7}$$

Meanwhile, from (3.2), (2.6) and (2.27)

$$\begin{aligned}
\|d^1\|_{X^0} &= \|d^0\|_{X^0} + \|S[-\nabla^2\pi_{S^2}(d^0)(\nabla d^0, \nabla d^0) - v^0\cdot\nabla d^0]\|_{X^0} \\
&\leqslant B_2[d_0]_{BMO} + D_0\|d^0\|_{X^{-1}}\|d^0\|_{X^0}^2 + D_0\|v^0\|_{Z^0}^2 + D_0\|d^0\|_{X^0}^2 \\
&\leqslant B_2[d_0]_{BMO} + D(1+\|d_0\|_{L^\infty})[d_0]_{BMO}^2 + D\|v_0\|_{BMO^{-1}}^2 \\
&\leqslant \frac{1}{4}D\epsilon + D(1+\|d_0\|_{L^\infty})\epsilon^2 + D\epsilon^2,
\end{aligned}$$

where we have used $B_2 < \frac{1}{4}D$.

Then we choose ϵ small enough such that

$$\|d^1\|_{X^0} \leqslant \frac{1}{2}D\epsilon. \tag{3.8}$$

For v^1, we have

$$\|v^1\|_{Z^0} \leqslant B_1\|v_0\|_{BMO^{-1}} + D_0\|v^0\|_{Z^0}^2 + D_0\|d^0\|_{X^0}^2 \leqslant B_1\epsilon + D\epsilon^2 \leqslant \frac{1}{4}D\epsilon,$$

where we have used $B_1 < \frac{1}{4}D$. Then one can choose ϵ small enough such that

$$\|v^1\|_{Z^0} \leqslant \frac{1}{2}D\epsilon. \tag{3.9}$$

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Combining (3.8) with (3.9), we have

$$\left\|v^1\right\|_{Z^0} + \left\|d^1\right\|_{X^0} \leqslant D\epsilon. \tag{3.10}$$

Now we estimate $\|d^2\|_{X^{-1}}$. In fact,

$$\begin{aligned}
\left\|d^2\right\|_{X^{-1}} &\leqslant \left\|d^2 - d^0\right\|_{X^{-1}} + \left\|d^0\right\|_{X^{-1}} \\
&= \left\|S\left[-\nabla^2\pi_{S^2}\left(d^1\right)\left(\nabla d^1, \nabla d^1\right) - v^1 \cdot \nabla d^1\right]\right\|_{X^{-1}} + \left\|d^0\right\|_{X^{-1}} \\
&\leqslant D_0\left\|d^1\right\|_{X^{-1}}\left\|d^1\right\|_{X^0}^2 + D_0\left\|v^1\right\|_{Z^0}^2 + D_0\left\|d^1\right\|_{X^0}^2 + \|d_0\|_{L^\infty} \\
&\leqslant D\left(1 + \left\|d^1\right\|_{X^{-1}}\right)\left\|d^1\right\|_{X^0}^2 + D\|v_1\|_{Z^0}^2 + \|d_0\|_{L^\infty} \\
&\leqslant D(1+E)\left\|d^1\right\|_{X^0}^2 + D\|v_1\|_{Z^0}^2 + \|d_0\|_{L^\infty} \\
&\leqslant D^3(1+E)\epsilon^2 + D^3\epsilon^2 + \|d_0\|_{L^\infty}.
\end{aligned}$$

In above last inequality, we have used (3.10).
Then we choose ϵ small enough that

$$\left\|d^2\right\|_{X^{-1}} \leqslant E. \tag{3.11}$$

For $\|d^2\|_{X^0}$, we have

$$\begin{aligned}
\left\|d^2\right\|_{X^0} &= \left\|d^0\right\|_{X^0} + \left\|S\left[-\nabla^2\pi_{S^2}\left(d^1\right)\left(\nabla d^1, \nabla d^1\right) - v^1 \cdot \nabla d^1\right]\right\|_{X^0} \\
&\leqslant B_2[d_0]_{BMO} + D_0\left\|d^1\right\|_{X^{-1}}\left\|d^1\right\|_{X^0}^2 + D_0\left\|v^1\right\|_{Z^0}^2 + D_0\left\|d^1\right\|_{X^0}^2 \\
&\leqslant B_2[d_0]_{BMO} + D(1+E)\left\|d^1\right\|_{X^0}^2 + D\left\|v^1\right\|_{Z^0}^2 \\
&\leqslant B_2\epsilon + D^3(1+E)\epsilon^2 + D^3\epsilon^2 \\
&\leqslant \frac{1}{4}D\epsilon + D^3(1+E)\epsilon^2 + D^3\epsilon^2.
\end{aligned}$$

Then one can choose ϵ small enough such that

$$\left\|d^2\right\|_{X^0} \leqslant \frac{1}{2}D\epsilon. \tag{3.12}$$

For $\|v^2\|_{Z^0}$, we have

$$\left\|v^2\right\|_{Z^0} \leqslant B_1[v_0]_{BMO^{-1}} + D_0\left\|v^1\right\|_{Z^0}^2 + D_0\left\|d^1\right\|_{X^0}^2 \leqslant B_1\epsilon + D^2\epsilon^2 \leqslant \frac{1}{4}D\epsilon.$$

Then one can choose ϵ small enough such that

$$\left\|v^2\right\|_{Z^0} \leqslant \frac{1}{2}D\epsilon. \tag{3.13}$$

Combining (3.12) with (3.13), we have

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$$\|v^2\|_{Z^0} + \|d^2\|_{X^0} \leqslant D\epsilon. \tag{3.14}$$

Hence, by choosing d_0 small enough in BMO and v_0 small enough in BMO^{-1}, we can ensure that for any integer $j \geqslant 0$, there are constants $D > 0$ and $E > 0$ such that (3.3) holds and

$$\|v^j\|_{Z^0} + \|d^j\|_{X^0} \leqslant D\epsilon, \tag{3.15}$$

which means that (3.4) is true for $k = 0$.

Now we will prove that (3.5) is true for $k = 0$.

From the proof of (3.10), we also have

$$\|v^1 - v^0\|_{Z^0} \leqslant \frac{1}{2}D\epsilon \quad \text{and} \quad \|d^1 - d^0\|_{X^0} \leqslant \frac{1}{2}D\epsilon. \tag{3.16}$$

From the definition of $S[\nabla^2 \Pi_{S^2}(d)(\nabla d, \nabla d)]$, for any integer $j \geqslant 1$, we have

$$\begin{aligned} &\big|S\big[\nabla^2 \Pi_{S^2}(d^j)(\nabla d^j, \nabla d^j)\big] - S\big[\nabla^2 \Pi_{S^2}(d^{j-1})(\nabla d^{j-1}, \nabla d^{j-1})\big]\big| \\ &\quad \leqslant S\big[\big(|\nabla d^j| + |\nabla d^{j-1}|\big)|\nabla d^j - \nabla d^{j-1}||d^j|\big] + S\big[|\nabla d^j|^2|d^j - d^{j-1}|\big]. \end{aligned} \tag{3.17}$$

Then from (2.4), (2.27) and (2.30), we have

$$\begin{aligned} &\|d^{j+1} - d^j\|_{X^0} + \|v^{j+1} - v^j\|_{Z^0} \\ &\quad \leqslant \big\|S\big[\big(|\nabla d^j| + |\nabla d^{j-1}|\big)|\nabla(d^j - d^{j-1})||d^j|\big]\big\|_{X^0} + \big\|S\big[|\nabla d^{j-1}|^2|d^j - d^{j-1}|\big]\big\|_{X^0} \\ &\qquad + \big\|S\big[(v^j - v^{j-1}) \cdot \nabla d^j\big]\big\|_{X^0} + \big\|S\big[v^{j-1} \cdot (\nabla d^j - \nabla d^{j-1})\big]\big\|_{X^0} \\ &\qquad + \big\|V\big[(v^j - v^{j-1}) \otimes v^j\big]\big\|_{Z^0} + \big\|V\big[v^{j-1} \otimes (v^j - v^{j-1})\big]\big\|_{Z^0} \\ &\qquad + \big\|V\big[(\nabla d^j - \nabla d^{j-1}) \odot \nabla d^j\big]\big\|_{Z^0} + \big\|V\big[\nabla d^{j-1} \odot (\nabla d^j - \nabla d^{j-1})\big]\big\|_{Z^0} \\ &\quad \leqslant D\|d^j\|_{X^{-1}}\big(\|d^j\|_{X^0} + \|d^{j-1}\|_{X^0}\big)\|d^j - d^{j-1}\|_{X^0} + D\|d^{j-1}\|_{X^0}^2\|d^j - d^{j-1}\|_{X^{-1}} \\ &\qquad + D\|v^j - v^{j-1}\|_{Z^0}\|d^j\|_{X^0} + D\|v^{j-1}\|_{Z^0}\|d^j - d^{j-1}\|_{X^0} + D\|v^j - v^{j-1}\|_{Z^0}\|v^j\|_{Z^0} \\ &\qquad + D\|v^{j-1}\|_{Z^0}\|v^j - v^{j-1}\|_{Z^0} + D\|d^j - d^{j-1}\|_{X^0}\|d^j\|_{X^0} + D\|d^{j-1}\|_{X^0}\|d^j - d^{j-1}\|_{X^0} \\ &\quad \leqslant \big(2D^2E + 3D^2\big)\epsilon\|d^j - d^{j-1}\|_{X^0} + D^3\epsilon^2\|d^j - d^{j-1}\|_{X^{-1}} + 3D^2\epsilon\|v^j - v^{j-1}\|_{Z^0}. \end{aligned} \tag{3.18}$$

Specially, for $j = 1$, by using (3.3) and (3.16) we have

$$\|d^2 - d^1\|_{X^0} + \|v^2 - v^1\|_{Z^0} \leqslant \big(4D^3E + 6D^3\big)\epsilon^2.$$

Let $\epsilon^2 \leqslant \frac{1}{4D^3E+6D^3}(\frac{2}{3})^2$, then we have

$$\|d^2 - d^1\|_{X^0} + \|v^2 - v^1\|_{Z^0} \leqslant \left(\frac{2}{3}\right)^2.$$

Moreover, by (3.18) we obtain that (3.5) is true for $k = 0$.

Step 2. We will prove that (3.4) and (3.5) are right for any integer $k \geqslant 0$. Assume that (3.4) and (3.5) are true for $k - 1$. We will prove that (3.4) and (3.5) are true for k. We first prove (3.4) is true for k.

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By (2.3) we have

$$\begin{aligned}\|v^j\|_{Z^k} &\leqslant \|v^0\|_{Z^k} + \big\|V\big[v^{j-1}\otimes v^{j-1} + \nabla d^{j-1}\odot \nabla d^{j-1}\big]\big\|_{Z^k} \\ &\leqslant \|v^0\|_{Z^k} + 2D_1\|v^{j-1}\|_{Z^0}\|v^{j-1}\|_{Z^k} + J_k\|v^{j-1}\|^2_{\widetilde{Z}^{k-1}} \\ &\quad + 2D_1\|d^{j-1}\|_{X^0}\|d^{j-1}\|_{X^k} + J_k\|d^{j-1}\|^2_{\widetilde{X}^{k-1}},\end{aligned} \tag{3.19}$$

where $J_k = D_0(k) + D(k)$.

By (2.5) and (2.29) we have

$$\begin{aligned}\|d^j\|_{X^k} &\leqslant \|d^0\|_{X^k} + \big\|S\big[\nabla^2\Pi_{S^2}(d^{j-1})(\nabla d^{j-1}, \nabla d^{j-1}) - v^{j-1}\cdot\nabla d^{j-1}\big]\big\|_{X^k} \\ &\leqslant \|d^0\|_{X^k} + D_1\|d^{j-1}\|_{X^0}\|d^{j-1}\|_{X^k}\|d^{j-1}\|_{X^{-1}} + J_k\|d^{j-1}\|^3_{\widetilde{X}^{k-1}} \\ &\quad + D_1\big(\|v^{j-1}\|_{Z^0}\|d^{j-1}\|_{X^k} + \|v^{j-1}\|_{Z^k}\|d^{j-1}\|_{X^0}\big) + J_k\|v^{j-1}\|_{\widetilde{Z}^{k-1}}\|d^{j-1}\|_{\widetilde{X}^{k-1}} \\ &\leqslant \|d^0\|_{X^k} + D_1E\|d^{j-1}\|_{X^0}\|d^{j-1}\|_{X^k} + J_k\|d^{j-1}\|^3_{\widetilde{X}^{k-1}} \\ &\quad + D_1\big(\|v^{j-1}\|_{Z^0}\|d^{j-1}\|_{X^k} + \|v^{j-1}\|_{Z^k}\|d^{j-1}\|_{X^0}\big) + J_k\|v^{j-1}\|^2_{\widetilde{Z}^{k-1}} + J_k\|d^{j-1}\|^2_{\widetilde{X}^{k-1}}\end{aligned} \tag{3.20}$$

where $J_k = D_0(k) + D(k)$.

Combining (3.19) together with (3.20) and choosing ϵ small enough yields

$$\begin{aligned}\|d^j\|_{X^k} + \|v^j\|_{Z^k} &\leqslant \|d^0\|_{X^k} + \|v^0\|_{Z^k} + \big(2D_1\|v^{j-1}\|_{Z^0} + D_1\|d^{j-1}\|_{X^0}\big)\|v^{j-1}\|_{Z^k} \\ &\quad + \big(2D_1\|d^{j-1}\|_{X^0} + D_1E\|d^{j-1}\|_{X^0} + D_1\|v^{j-1}\|_{Z^0}\big)\|d^{j-1}\|_{X^k} \\ &\quad + 2J_k\|v^{j-1}\|^2_{\widetilde{Z}^{k-1}} + J_k\|d^{j-1}\|^2_{\widetilde{X}^{k-1}} + J_k\|d^{j-1}\|^3_{\widetilde{X}^{k-1}} \\ &\leqslant \|d^0\|_{X^k} + \|v^0\|_{Z^k} + \frac{1}{2}\big(\|d^{j-1}\|_{X^k} + \|v^{j-1}\|_{Z^k}\big) \\ &\quad + J_k\big(\|d^{j-1}\|^2_{\widetilde{X}^{k-1}} + \|v^{j-1}\|^2_{\widetilde{Z}^{k-1}}\big) + 2J_k\|d^{j-1}\|^3_{\widetilde{X}^{k-1}} \\ &\leqslant \frac{1}{2}\big(\|d^{j-1}\|_{X^k} + \|v^{j-1}\|_{Z^k}\big) + H_k\end{aligned} \tag{3.21}$$

where we have used the induction hypothesis and Lemma 2.4 to obtain the last inequality for some constant H_k.

Then from (3.21) we have

$$\begin{aligned}\|d^j\|_{X^k} + \|v^j\|_{Z^k} &\leqslant \frac{1}{2}\left[\frac{1}{2}\big(\|d^{j-2}\|_{X^k} + \|v^{j-2}\|_{Z^k}\big) + H_k\right] + H_k \\ &\leqslant \cdots \leqslant \frac{1}{2^j}\big(\|v^0\|_{Z^k} + \|d^0\|_{X^k}\big) + H_k\sum_{l=0}^{j-1}\left(\frac{1}{2}\right)^l \\ &\leqslant \big(\|v^0\|_{Z^k} + \|d^0\|_{X^k}\big) + H_k\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^l,\end{aligned}$$

which implies that $(\|d^j\|_{X^k} + \|v^j\|_{Z^k})$ is uniform bounded in j. Then there exists $F_k > 0$ such that

$$\|d^j\|_{X^k} + \|v^j\|_{Z^k} \leqslant F_k,$$

which implies that (3.4) is right for k. Hence (3.4) is proved.

Now we will prove (3.5) for k if (3.5) is true for $k-1$.

By Proposition 2.1 we have

$$\begin{aligned}
&\|v^{j+1} - v^j\|_{Z^k} \\
&\quad\leqslant \|V[(v^j - v^{j-1}) \otimes v^j]\|_{Z^k} + \|V[v^j \otimes (v^j - v^{j-1})]\|_{Z^k} \\
&\qquad + \|V[(\nabla d^j - \nabla v^{j-1}) \odot \nabla d^j]\|_{Z^k} + \|V[\nabla d^j \odot (\nabla d^j - \nabla d^{j-1})]\|_{Z^k} \\
&\quad\leqslant D_1\|v^j - v^{j-1}\|_{Z^k}(\|v^j\|_{Z^0} + \|v^{j-1}\|_{Z^0}) + D_1\|v^j - v^{j-1}\|_{Z^0}(\|v^j\|_{Z^k} + \|v^{j-1}\|_{Z^k}) \\
&\qquad + D_1\|d^j - d^{j-1}\|_{X^k}(\|d^j\|_{X^0} + \|d^{j-1}\|_{X^0}) + D_1\|d^j - d^{j-1}\|_{X^0}(\|d^j\|_{X^k} + \|d^{j-1}\|_{X^k}) \\
&\qquad + J_k[\|v^j - v^{j-1}\|_{\widetilde{Z}^{k-1}}(\|v^j\|_{\widetilde{Z}^{k-1}} + \|v^{j-1}\|_{\widetilde{Z}^{k-1}})] \\
&\qquad + J_k\|d^j - d^{j-1}\|_{\widetilde{X}^{k-1}}(\|d^j\|_{\widetilde{X}^{k-1}} + \|d^{j-1}\|_{\widetilde{X}^{k-1}}) \\
&\quad\leqslant D_1(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k})(\|v^j\|_{Z^0} + \|v^{j-1}\|_{Z^0} + \|d^j\|_{X^0} + \|d^{j-1}\|_{X^0}) \\
&\qquad + D_1(\|v^j - v^{j-1}\|_{Z^0} + \|d^j - d^{j-1}\|_{X^0})(\|v^j\|_{Z^k} + \|v^{j-1}\|_{Z^k} + \|d^j\|_{X^k} + \|d^{j-1}\|_{X^k}) \\
&\qquad + J_k(\|v^j - v^{j-1}\|_{\widetilde{Z}^{k-1}} + \|d^j - d^{j-1}\|_{\widetilde{X}^{k-1}})(\|v^j\|_{\widetilde{Z}^{k-1}} + \|v^{j-1}\|_{\widetilde{Z}^{k-1}} + \|d^j\|_{\widetilde{X}^{k-1}} + \|d^{j-1}\|_{\widetilde{X}^{k-1}}),
\end{aligned}$$

where $J_k = D_0(k) + D(k)$.

Then by (3.16), (3.4) and the induction hypothesis and choosing $\epsilon < \frac{1}{32DD_1}$, we have

$$\begin{aligned}
&\|v^{j+1} - v^j\|_{Z^k} \\
&\quad\leqslant \frac{1}{4}(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}) + 2F_kD_1\left(\frac{2}{3}\right)^{j-1} + 2F_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j-1} \\
&\quad= \frac{1}{4}(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}) + 3D_1F_k\left(\frac{2}{3}\right)^{j} + 3F_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j}.
\end{aligned} \tag{3.22}$$

By Propositions 2.3–2.4, we have

$$\begin{aligned}
\|d^{j+1} - d^j\|_{Z^k} &\leqslant \|S[(|\nabla d^j| + |\nabla d^{j-1}|)|\nabla(d^j - d^{j-1})||d^j|]\|_{X^k} + \|S[|\nabla d^{j-1}|^2|d^j - d^{j-1}|]\|_{X^k} \\
&\quad + \|S[(v^j - v^{j-1}) \cdot \nabla d^j]\|_{X^k} + \|S[v^{j-1} \cdot (\nabla d^j - \nabla d^{j-1})]\|_{X^k} \\
&\leqslant D_1\|d^j - d^{j-1}\|_{X^k}(\|d^j\|_{X^0} + \|d^{j-1}\|_{X^0})\|d^j\|_{X^{-1}} \\
&\quad + D_1\|d^j - d^{j-1}\|_{X^0}(\|d^j\|_{X^k} + \|d^{j-1}\|_{X^k})\|d^j\|_{X^{-1}} \\
&\quad + D_1\|d^j - d^{j-1}\|_{X^k}(\|d^j\|_{X^0} + \|d^{j-1}\|_{X^0}) + D_1\|d^j - d^{j-1}\|_{X^0}(\|d^j\|_{X^k} + \|d^{j-1}\|_{X^k}) \\
&\quad + J_k[(\|d^j\|_{\widetilde{X}^{k-1}} + \|d^{j-1}\|_{\widetilde{X}^{k-1}})\|d^j - d^{j-1}\|_{\widetilde{X}^{k-1}}\|d^{j-1}\|_{\widetilde{X}^{k-1}}] \\
&\quad + D_1\|d^{j-1}\|_{X^0}^2\|d^j - d^{j-1}\|_{X^{-1}} + J_k[\|d^j\|_{\widetilde{X}^{k-1}}^2\|d^j - d^{j-1}\|_{\widetilde{X}^{k-1}}] \\
&\quad + D_1\|v^j - v^{j-1}\|_{Z^k}\|d^j\|_{X^0} + D_1\|v^j - v^{j-1}\|_{Z^0}\|d^j\|_{X^k} \\
&\quad + D_1\|d^j - d^{j-1}\|_{X^k}\|v^j\|_{Z^0} + D_1\|d^j - d^{j-1}\|_{X^0}\|v^j\|_{X^k}
\end{aligned}$$

$$
\begin{aligned}
&+ J_k\big[\|d^j - d^{j-1}\|_{\widetilde{X}^{k-1}}\|v^j\|_{\widetilde{Z}^{k-1}} + \|v^j - v^{j-1}\|_{\widetilde{Z}^{k-1}}\|d^j\|_{\widetilde{X}^{k-1}}\big] \\
&\leqslant (ED_1 + D_1)\big(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}\big)\big(\|v^j\|_{Z^0} + \|v^{j-1}\|_{Z^0} + \|d^j\|_{X^0} + \|d^{j-1}\|_{X^0}\big) \\
&\quad + (ED_1 + D_1)\big(\|v^j - v^{j-1}\|_{Z^0} + \|d^j - d^{j-1}\|_{X^0}\big)\big(\|v^j\|_{Z^k} + \|v^{j-1}\|_{Z^k} + \|d^j\|_{X^k} + \|d^{j-1}\|_{X^k}\big) \\
&\quad + J_k\big(\|v^j - v^{j-1}\|_{\widetilde{Z}^{k-1}} + \|d^j - d^{j-1}\|_{\widetilde{X}^{k-1}}\big)\big[\|v^{j-1}\|_{\widetilde{Z}^{k-1}} + \|d^j\|_{\widetilde{X}^{k-1}} \\
&\quad + \|d^{j-1}\|^2_{\widetilde{X}^{k-1}} + \big(\|d^j\|_{\widetilde{X}^{k-1}} + \|d^{j-1}\|_{\widetilde{X}^{k-1}}\big)\|d^j\|_{\widetilde{X}^{k-1}}\big].
\end{aligned}
$$

Then by (3.16), (3.4) and the induction hypothesis and choosing $\epsilon < \frac{1}{32D(ED_1+D_1)}$, we have

$$
\begin{aligned}
\|d^{j+1} - d^j\|_{X^k} &\leqslant \frac{1}{4}\big(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}\big) + 2F_k(ED_1 + D_1)\left(\frac{2}{3}\right)^{j-1} \\
&\quad + 2F_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j-1} + 2F^2_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j-1} \\
&= \frac{1}{4}\big(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}\big) + 3(ED_1 + D_1)F_k\left(\frac{2}{3}\right)^{j} \\
&\quad + 3F_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j} + \frac{9}{2}F^2_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j}.
\end{aligned}
\tag{3.23}
$$

Combining (3.22) with (3.23), we get

$$
\begin{aligned}
&\|v^{j+1} - v^j\|_{X^k} + \|d^{j+1} - d^j\|_{X^k} \\
&\quad \leqslant \frac{1}{2}\big(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}\big) + 3(2D_1 + ED_1)F_k\left(\frac{2}{3}\right)^{j} \\
&\quad + 6F_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j} + \frac{9}{2}F^2_{k-1}J_kG_{k-1}\left(\frac{2}{3}\right)^{j}.
\end{aligned}
\tag{3.24}
$$

We can choose a constant $L_k > 0$ such that

$$
3(ED_1 + 2D_1)F_k + 6F_{k-1}J_kG_{k-1} + \frac{9}{2}F^2_{k-1}J_kG_{k-1} \leqslant L_k.
$$

Then (3.24) implies that

$$
\begin{aligned}
&\|v^{j+1} - v^j\|_{X^k} + \|d^{j+1} - d^j\|_{X^k} \\
&\quad \leqslant \frac{1}{2}\big(\|v^j - v^{j-1}\|_{Z^k} + \|d^j - d^{j-1}\|_{X^k}\big) + L_k\left(\frac{2}{3}\right)^{j} \\
&\quad \leqslant \cdots \\
&\quad \leqslant \frac{1}{2^j}\big(\|v^1 - v^0\|_{Z^k} + \|d^1 - d^0\|_{X^k}\big) + L_k\sum_{l=0}^{j-1}\left(\frac{2}{3}\right)^{j-l}\left(\frac{1}{2}\right)^{l}
\end{aligned}
$$

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$$\leqslant \left(\frac{2}{3}\right)^j \left(\|v^1 - v^0\|_{Z^k} + \|d^1 - d^0\|_{X^k}\right) + L_k \left(\frac{2}{3}\right)^j \sum_{l=0}^{j} \left(\frac{3}{4}\right)^l$$

$$\leqslant G_k \left(\frac{2}{3}\right)^j,$$

which implies that (3.5) is true for k. Hence (3.5) is proved. Therefore the claim is proved. □

Now we are in the position to prove (1.4) of Theorem 1.1.

When $k = 0$, from the definitions of BMO and BMO^{-1} in Definition 1.1, we obtain (1.4) of Theorem 1.1 by using Lemma 3.1 in [23], Corollary 2.3 in [6]. We omit the details.

When $k \geqslant 1$, we have for any integer $i \geqslant 0$,

$$\|\nabla^i v\|_{L^\infty} \leqslant Ct^{-\frac{i+1}{2}} \quad \text{and} \quad \|\nabla^{i+1} d\|_{L^\infty} \leqslant Ct^{-\frac{i+1}{2}}.$$

Then from the fact that $L^\infty \hookrightarrow BMO$, we have $i \geqslant 0$,

$$[\nabla^i v]_{BMO} \leqslant Ct^{-\frac{i+1}{2}} \quad \text{and} \quad [\nabla^{i+1} d]_{BMO} \leqslant Ct^{-\frac{i+1}{2}}. \tag{3.25}$$

Let $i = k - 1$. From (3.25), we have $k \geqslant 1$,

$$\|\nabla^k v\|_{BMO^{-1}} \leqslant Ct^{-\frac{k}{2}} \quad \text{and} \quad [\nabla^k d]_{BMO} \leqslant Ct^{-\frac{k}{2}}. \tag{3.26}$$

Hence (1.4) of Theorem 1.1 is right for $k \geqslant 1$.

Therefore Theorem 1.1 is proved. □

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