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# Non-autonomous second order Hamiltonian systems

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**Abstract**

We study the existence of periodic solutions for a second order non-autonomous dynamical system containing variable kinetic energy terms. Our assumptions balance the interaction between the kinetic energy and the potential energy with neither one dominating the other. We study sublinear problems and the existence of non-constant solutions.



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*Keywords:* Critical points; Linking; Dynamical systems; Periodic solutions

## 1. Introduction

We consider the following problem. One wishes to solve

$$-\ddot{x}(t) = B(t)x(t) + \nabla_x V\big(t, x(t)\big), \tag{1}$$

where

$$x(t) = \big(x_1(t), \dots, x_n(t)\big) \tag{2}$$

is a map from $I = [0, T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function in $H^1$ with period $T$, and the function $V(t, x) = V(t, x_1, \dots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t, x) = (\partial V/\partial x_1, \dots, \partial V/\partial x_n) \in C\big(\mathbb{R}^{n+1}, \mathbb{R}^n\big). \tag{3}$$

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For each $x \in \mathbb{R}^n$, the function $V(t,x)$ is periodic in $t$ with period $T$. The elements of the symmetric matrix $B(t)$ are to be real-valued functions $b_{jk}(t) = b_{kj}(t)$, and each function is to be periodic with period $T$. We will consider each function to be defined on the interval $I$.

We shall study this problem under the following assumptions. Our assumption on $B(t)$ is:

$(B1)$ Each component of $B(t)$ is an integrable function on $I$, i.e., for each $j$ and $k$, $b_{jk}(t) \in L^1(I)$.

This assumption implies that there is a linear operator $\mathcal{D}$ depending on $B(t)$ having spectrum consisting only of isolated eigenvalues of finite multiplicities tending to $\infty$. ($\mathcal{D}$ is defined in the next section.)

Concerning the potential $V(t,x)$ we assume:

$(V1)$ There exist functions $W_1, W_2 \in L^1(I)$ and consecutive eigenvalues $\lambda_l, \lambda_{l+1}$ of $\mathcal{D}$ such that for any $t \in I$ and any $x \in \mathbb{R}^n$,

$$\lambda_l |x|^2 - W_1(t) \leqslant 2V(t,x) \leqslant \lambda_{l+1}|x|^2 + W_2(t). \tag{4}$$

$(V2)$ There exists a function $W_0(t) \in L^1(I)$ such that for any $t \in I$ and any $x \in \mathbb{R}^n$,

$$H(t,x) = 2V(t,x) - x \cdot \nabla_x V(t,x) \geqslant -W_0(t).$$

$(V3)$ $H(t,x) \to \infty$ uniformly in $t$ as $|x| \to \infty$.

Assumptions $(V2)$ and $(V3)$ can be replaced by:

$(V2')$ There exists a function $W_0(t) \in L^1(I)$ such that for any $t \in I$ and any $x \in \mathbb{R}^n$,

$$H(t,x) \leqslant W_0(t),$$

and

$(V3')$ $H(t,x) \to -\infty$ uniformly in $t$ as $|x| \to \infty$.

Then we have

**Theorem 1.1.** *If the functions $B(t)$ and $V(t,x)$ satisfy assumptions $(B1)$, $(V1)$, $(V2)$ and $(V3)$, then there exists a $T$-periodic weak solution to (1) whose weak second derivative is an element of $L^1(I)$. If the function $V(t,x)$ satisfies $\nabla_x V(t,\mathbf{0}) \neq \mathbf{0}$, the solution of (1) is not trivial. The conclusions are also valid if we replace assumptions $(V2)$ and $(V3)$ with $(V2')$ and $(V3')$.*

In all of the previous results dealing with the full system (1), the hypotheses cause either the linear terms (kinetic energy) to dominate the nonlinear terms (potential energy) or vice versa. In either case, the subordinate terms become perturbations of the dominant terms. In the present paper each accommodates the other; neither is dominant.

The first to consider the general non-autonomous problem in

$$-\ddot{x}(t) = \nabla_x V\big(t, x(t)\big), \tag{5}$$

was Berger and Schechter [11], who proved the existence of solutions of the periodic system under a continuity assumption on the potential, and under the condition that for almost every $t \in I$, $V(t,x) \to -\infty$ uniformly in $t$ as $|x| \to \infty$. Brezis and Nirenberg [14] showed existence of nonconstant periodic solutions to this system under additional constraints on the potential near the origin. Mawhin and Willem [57] applied a direct variational method to show existence of a solution when the potential satisfies

$$\int_0^T V(t,x) \to -\infty \quad \text{as } |x| \to \infty, \tag{6}$$

and there exists a positive locally integrable function $b(t)$ such that for any $x$, almost everywhere on the interval the gradient of the potential is bounded,

$$\big|\nabla_x V(t,x)\big| \leqslant b(t). \tag{7}$$

By a dual variational method Mawhin and Willem [57] showed existence of a solution when the potential satisfies (6), for almost every $t \in I$, $(-V(t,x))$ is convex in $x$ and there exists a positive continuous function $a(|x|)$ and a positive locally integrable function $b(t)$ such that for any $x$, almost everywhere on the interval the potential satisfies

$$\big|V(t,x)\big| \leqslant a\big(|x|\big)b(t), \qquad \big|\nabla_x V(t,x)\big| \leqslant a\big(|x|\big)b(t). \tag{8}$$

Similar conditions on the potential with some generalizations appear in the work of Tang [68,69, 71,72], Tang and Wu [74,75,77], Cordaro [19], Wu and Zhao [95,97], Chen, Wu and Zhao [16], Yang [99], Cordaro and Rao [20], Deng and Yang [22], Jiang and Tang [42], Meng and Tang [58], Wang and Zhang [87,88] and Aizmahin and An [2]. Brezis and Nirenberg [14], Cordaro [19], Chen, Yang and Zhao [17] and Cordaro and Rao [20] gave conditions for solutions in the case when as $|x| \to \infty$ the sign of the potential is negative and the magnitude is bounded above and below by a multiple of $|x|^2$.

In the case of (5) when the potential is bounded and approaches a constant value as $|x| \to \infty$, Thews [86], Ahmad and Lazer [1], Coti Zelati [21] and Giannoni [35] gave conditions for existence of a nontrivial solution. Schechter [65] gives conditions for existence of solutions when the potential is sufficiently bounded in a neighborhood of the origin, with weaker conditions if the potential also grows quadratically as $|x| \to \infty$. Willem [90], Wu [94], Tang and Wu [79] and Feng and Han [34] have considered the existence of periodic solutions for a potential function which is periodic in at least some of the spatial dimensions. Because the gradient is bounded in these spatial dimensions, the periodic quality generates multiple solutions from a known periodic solution in the remaining dimensions.

Benci [10] showed existence of solutions to the second order Hamiltonian system in (5) when the potential function satisfies the Ambrosetti–Rabinowitz condition and additional conditions near the origin. These conditions are generalized by Bahri and Berestycki [7], Li [51], Ekeland and Ghoussoub [25] and Faraci [27]. Second order systems which satisfy a superquadratic growth condition in (5) but violate the Ambrosetti–Rabinowitz condition are studied by Felmer [32], Felmer and Silva [33], Faraci [26], Fei [30], Tang and Tao [73], Luan and Mao [55], Tao, Wu and Yan [85], He and Wu [40], Xiao [98], Hu and Papageorgiou [41], Tang and Ye [83], Wang, Zhang and Zhang [89], Kyritsi and Papageorgiou [47], Tang and Zhang [84], Chen and Ma [15] and Schechter [65].

The potential in (5) is quadratically bounded if it grows to infinity and there exist constants $\alpha > 0$ and $c_0$ such that the potential satisfies

$$V(t,x) \leqslant \alpha |x|^2 + c_0. \tag{9}$$

By using variational methods Clarke and Ekeland [18] showed existence of solutions to (5) under the condition that the potential was convex, had superlinear growth and was quadratically bounded. Willem [91] used a variational method to show existence of nontrivial solutions in a particular case when the potential is quadratically bounded and satisfies the bi-even symmetry condition

$$\nabla_x V(-t,-x) = -\nabla_x V(t,x). \tag{10}$$

Long [54] omitted the lower bound but gave similar conditions near $x = \mathbf{0}$, and Tang and Wu [78] showed existence of solutions for the bi-even potential with the regularity condition in (8) but without the conditions near $x = \mathbf{0}$. Schechter [61–63] showed existence of a solution under both a lower and upper quadratic bound on the potential without convexity or symmetry assumptions. Schechter [65] showed existence of a solution under the condition that the average value of the potential on the interval $I$ increases to $\infty$ as $|x| \to \infty$, and the upper quadratic bound is of the form,

$$V(t,x) \leqslant \alpha |x|^2 + W_2(t). \tag{11}$$

Dong [24] gives conditions for existence of solutions in (5) under the condition that the potential is asymptotically quadratic at infinity, has continuous second derivatives such that for sufficiently large $|x|$ the Hessian matrix is bounded above by a bounded matrix $A(t)$ such that no solutions exist to the linear system $\ddot{x}(t) + A(t)x(t) = 0$.

Another category of quadratically bounded second order Hamiltonian system has a potential function $V(t,x)$ which grows to $\infty$ as $|x| \to \infty$ and is the sum of an autonomous potential $V_1(x)$ and a non-autonomous potential $V_2(t,x)$. The autonomous term satisfies a directional bound on the gradient,

$$\big\langle \nabla V_1(x) - \nabla V_1(y), x - y \big\rangle_{\mathbb{R}^n} \leqslant r|x-y|^2. \tag{12}$$

Ahmad and Lazer [1] showed existence of solutions in such system under the condition that the gradient of the non-autonomous term is bounded. Ma and Tang [56] generalized these results to show existence of solutions under weaker bounds on the gradient of the non-autonomous term and the growth of the potential function as $|x| \to \infty$. Tang and Ye [82] further generalized these results to show existence of solutions under a weaker bound on the growth of the potential function.

In the second order Hamiltonian system in (5), Rabinowitz [59] showed existence of solutions when the potential $V(t,x)$ satisfies a subquadratic growth condition, and also the existence of solutions for the case of a potential which grows subquadratically and satisfies a strict convexity condition but may violate the above condition. Willem [92,93], Tang [70] and Tang and Wu [80] generalized the results of Rabinowitz for a convex potential with subquadratic growth. Ahmad and Lazer [1] showed existence of a solution in (5) when the potential is the sum of a time invariant potential with subquadratic growth and a linearly bounded time dependent perturbation, provided $V(t,x) \to \infty$ uniformly as $|x| \to \infty$.

A potential which grows to $+\infty$ as $|x| \to \infty$ satisfies a linearly bounded growth condition if for some integrable function $b(t)$ it is bounded by $b(t)|x|$. Giannoni [35] and Mawhin and Willem [57] applied saddle point theorems to show that a solution exists for the system in (5) with a potential which grows to $+\infty$. These results were generalized by Fei, Kim and Wang [31]. The method of Mawhin and Willem was adapted by Tang [71], Zhang and Zhou [100,101], Meng and Tang [58] and Tang and Wu [80] to find conditions for existence of solutions to a system with a subquadratic growth condition, including subadditive and subconvex potentials. The method was adapted by Ma and Tang [56], Wu and Zhao [96,97], Jiang and Tang [46], Tang and Ye [82], and Tang and Wu [80] to find conditions for existence of solutions to a system with a quadratically bounded potential.

Another category of second order dynamical systems has a forcing function of the form $f(t,x) = b(t)\nabla W(x) + f(t)$, where $b(t)$ changes sign. This dynamical system has applications to radio frequency confinement of quantum particles, and Lassoued [48,49] gave conditions for the existence of solutions by using a dual variational approach in cases where the function $W(x)$ is convex and subquadratic or quadratically bounded. When the forcing function $f(t)$ is zero and the potential satisfies a superquadratic growth condition, existence of solutions was shown by Lassoued [48,50], who assumed homogeneity and convexity conditions, and Girardi and Matzeu [36,37] who assumed a stricter growth conditions without homogeneity or convexity. Avila and Felmer [6] and Jiang [43] considered the case when the sign-changing potential determines the dynamic behavior near the origin but is dominated by a larger fixed sign term as $|x| \to \infty$. Ben Naoum, Troestler and Willem [9] considered a more general sign-changing potential, showing the existence of a solution to (5) when the potential is continuous in $x$, is homogeneous from the unit sphere with $|x|^\alpha$ for $\alpha > 1$ and $\alpha \neq 2$, and at some point on the unit sphere periodically changes sign.

One case in which a separate matrix term is used in a second order system is when the dynamic behavior near the origin is largely determined by the matrix $B(t)$, but as $|x|$ increases the gradient of the potential increases superlinearly and dominates the dynamic behavior. Li [51] considered the system in (1) in which $B(t)$ has a constant value and $V(t,x)$ satisfies the Ambrosetti–Rabinowitz growth condition and showed existence of a solution under conditions on the potential near the origin. The existence conditions are less restrictive conditions in the case that the linear part of the system does not satisfy the resonance condition. Li and Willem [52] generalized this work to find solutions in the case where the components of $B(t)$ are continuous functions. Second order systems of the form of (1) in which the potential function $V(t,x)$ satisfies a more general superquadratic growth condition are studied by Luan and Mao [55], Faraci [27], He and Wu [40], Xiao [98], Hu and Papageorgiou [41], Tang and Ye [83], Wang, Zhang and Zhang [89], Kyritsi and Papageorgiou [47], Tang and Zhang [84] and Chen and Ma [15].

Ding and Girardi [23] considered the case of (1) when the potential oscillates in magnitude and sign,

$$-\ddot{x}(t) = B(t)x(t) + b(t)\nabla W\big(x(t)\big), \tag{13}$$

and found conditions for solutions when the matrix $B(t)$ is symmetric and negative definite and the function $W(x)$ grows superquadratically and satisfies a homogeneity condition. Antonacci [3,4] gave conditions for existence of solutions with stronger constraints on the potential but without the homogeneity condition, and without the negative definite condition on the matrix. Generalizations of the above results are given by Antonacci and Magrone [5], Barletta and Livrea [8], Guo and Xu [38], Li and Zou [53], Faraci and Livrea [29], Bonanno and Livrea [12,13], Jiang [44,45], Shilgba [66,67], Faraci and Iannizzotto [28] and Tang and Xiao [81].

Last we consider the second order system in (1) where the potential function $V(t,x)$ is quadratically bounded as $|x| \to \infty$. Berger and Schechter [11] considered the case of (1) where $B(t)$ is a constant symmetric matrix that is positive definite, and showed existence of solutions when the magnitude of $\nabla_x V(t,x)$ is uniformly bounded, the potential is strictly convex, and if $y(t)$ is a $T$-periodic solutions of the linear system $-\ddot{y} = Ay$, then there exists a function $x(t)$ which is weakly differentiable with $\dot{x} \in L^2(I, \mathbb{R}^n)$ and satisfies

$$\int_0^T \bigl\langle \nabla_x V\bigl(t, x(t)\bigr), y(t)\bigr\rangle_{\mathbb{R}^n}\, dt = 0.$$

Han [39] gave conditions for existence of solutions when $B(t)$ was a multiple of the identity matrix, the system satisfies the resonance condition, and the potential has upper and lower subquadratic bounds. Li and Zou [53] considered the case where $B(t)$ is continuous and nonconstant and the system satisfies the resonance condition, and showed existence of solutions when the potential is even and grows no faster than linearly. Tang and Wu [76] required the function that satisfies the resonance condition to pass through the zero vector, and gave upper and lower conditions for subquadratic growth of the magnitude of $V(t,x)$ without the requirement that the potential be even. Faraci [27] considered the case where for each $t \in I$, $B(t)$ is negative definite with elements that are bounded but not necessarily continuous and the potential has an upper quadratic bound as $|x| \to \infty$, showing existence of a solution when the gradient of the potential is bounded near the origin and exceeds the matrix product in at least one direction.

## 2. Some lemmas

In proving Theorem 1.1 we shall make use of the following considerations.

We define a bilinear form $a(\cdot,\cdot)$ on the set $L^2(I,\mathbb{R}^n) \times L^2(I,\mathbb{R}^n)$,

$$a(u,v) = (\dot{u}, \dot{v}) + (u,v). \tag{14}$$

The domain of the bilinear form is the set $D(a) = H$, consisting of those periodic $x(t) = (x_1(t), \dots, x_n(t)) \in L^2(I,\mathbb{R}^n)$ having weak derivatives in $L^2(I,\mathbb{R}^n)$. $H$ is a dense subset of $L^2(I,\mathbb{R}^n)$. Note that $H$ is a Hilbert space. Thus we can define an operator $\mathcal{A}$ such that $u \in D(\mathcal{A})$ if and only if $u \in D(a)$ and there exists $g \in L^2(I,\mathbb{R}^n)$ such that

$$a(u,v) = (g,v), \quad v \in D(a). \tag{15}$$

If $u$ and $g$ satisfy this condition we say $\mathcal{A}u = g$.

**Lemma 2.1.** *The operator $\mathcal{A}$ is a self-adjoint Fredholm operator from $L^2(I, \mathbb{R}^n)$ to $L^2(I, \mathbb{R}^n)$. It is one-to-one and onto.*

**Proof.** Let $f \in L^2(I, \mathbb{R}^n)$. Then

$$(v, f) \leqslant \|v\| \cdot \|f\| \leqslant \|v\|_H \|f\|, \quad v \in H.$$

Thus $(v, f)$ is a bounded linear functional on $H$. Since $H$ is complete, there is a $u \in H$ such that

$$(u, v)_H = (f, v), \quad v \in H.$$

Consequently, $u \in D(\mathcal{A})$ and $\mathcal{A}u = f$. Moreover, if $\mathcal{A}u = 0$, then

$$(u, v)_H = 0, \quad v \in H.$$

Thus, $u = 0$. Hence, $\mathcal{A}$ is one-to-one and onto.

For any two functions $x, y \in D(\mathcal{A})$,

$$(\mathcal{A}x, y) = (\dot{x}, \dot{y}) + (x, y) = (x, \mathcal{A}y). \tag{16}$$

Thus, $\mathcal{A}$ is symmetric. It is now easy to show that $D(\mathcal{A}) \subset D(a)$ is also a dense subset of $L^2(I, \mathbb{R}^n)$. In fact, if $f \in L^2(I, \mathbb{R}^n)$ satisfies $(f, v) = 0\ \forall v \in D(\mathcal{A})$, then $w = \mathcal{A}^{-1} f$ satisfies $(w, \mathcal{A}v) = (\mathcal{A}w, v) = 0\ \forall v \in D(\mathcal{A})$. Since $\mathcal{A}$ is onto, $w = 0$. Hence, $f = \mathcal{A}w = 0$.

Next, we show that $\mathcal{A}$ is self-adjoint. Consider any $u, f \in L^2(I, \mathbb{R}^n)$, and suppose for any $v \in D(\mathcal{A})$,

$$(u, \mathcal{A}v) = (f, v). \tag{17}$$

Since $\mathcal{A}$ is onto and $f \in L^2(I, \mathbb{R}^n)$, there exists $w \in D(\mathcal{A})$ such that $\mathcal{A}w = f$. Then using (16),

$$(u - w, \mathcal{A}v) = (f, v) - (\mathcal{A}w, v) = 0.$$

Since $u - w \in L^2(I, \mathbb{R}^n)$, we can find a $v \in D(\mathcal{A})$ such that $\mathcal{A}v = u - w$, and

$$\|u - w\|^2 = 0.$$

This implies $u = w$ in the space $L^2(I, \mathbb{R}^n)$, and therefore $u \in D(\mathcal{A})$. Hence, $\mathcal{A}u = \mathcal{A}w = f$. $\square$

**Lemma 2.2.** *The essential spectrum of $\mathcal{A}$ is the null set.*

**Proof.** By Lemma 2.1, $\mathcal{A}$ is linear, self-adjoint, and onto $L^2(I, \mathbb{R}^n)$.

Next, we note that

$$\left\|\mathcal{A}^{-1} u\right\| \leqslant \|u\|.$$

To see this, let $f = \mathcal{A}u$. Then $u = \mathcal{A}^{-1} f$, and

$$(u, v)_H = (f, v), \quad v \in H.$$

Thus,

$$\|u\|_H^2 \leqslant \|f\| \cdot \|u\| \leqslant \|f\| \cdot \|u\|_H.$$

Hence, $\|u\| \leqslant \|f\|$.

Now we show that the inverse operator $\mathcal{A}^{-1}$ is compact on $L^2(I, \mathbb{R}^n)$. Let $(u_k)$ be a bounded sequence in $L^2(I, \mathbb{R}^n)$, and let $C > 0$ satisfy for each $k$, $\|u_k\| \leqslant C$. By applying the inverse operator, let $(x_k)$ be the sequence such that for each $k$, $\mathcal{A}x_k = u_k$. From the above statements, for each $k$, $\|x_k\| \leqslant C$. From the definition of the operator $\mathcal{A}$, for any $v \in H$,

$$(\mathcal{A}x_k, v) = (\dot{x}_k, \dot{v}) + (x_k, v).$$

By setting $v = x_k$, we can use Schwartz' inequality on the norm of $x_k$ with respect to $H$,

$$\|x_k\|_H^2 = \|x_k\|^2 + \|\dot{x}_k\|^2 = (\mathcal{A}x_k, x_k) = (u_k, x_k) \leqslant C^2.$$

Then $(x_k)$ is a bounded sequence in $H$, and we can find a subsequence which converges in $L^2(I, \mathbb{R}^n)$. Since $(\mathcal{A}^{-1}u_k)$ has a subsequence which converges in $L^2(I, \mathbb{R}^n)$, the operator is compact. Also $\mathcal{A}$ is a positive operator, since for any $x \in D(\mathcal{A})$,

$$(\mathcal{A}x, x) = (\dot{x}, \dot{x}) + (x, x) = \|x\|_H^2 \geqslant 0.$$

Hence, $\mathcal{K} = \mathcal{A}^{-1}$ is a positive compact operator, and the eigenvalues $\mu_k$ of $\mathcal{K}$ are denumerable and have 0 as their only possible limit point. The eigenfunctions $\phi_k$ of $\mathcal{K}$ are also eigenfunctions of $\mathcal{K}^{-1} = \mathcal{A}$ and satisfy

$$\mathcal{A}\phi_k = \frac{1}{\mu_k}\phi_k.$$

Since the values $\mu_k$ are bounded and have no limit point except 0, there are no limit points of the set $(1/\mu_k)$ and the essential spectrum of $\mathcal{A}$ is the null set. □

We will use two theorems of Schechter [60] on bilinear forms to prove Lemma 2.5.

**Theorem 2.3.** *Let $a(\cdot,\cdot)$ be a closed Hermitian bilinear form with dense domain in $L^2(I, \mathbb{R}^n)$. If for some real number $N$,*

$$a(u, u) + N\|u\|^2 \geqslant 0, \tag{18}$$

*then the operator $\mathcal{A}$ associated with $a(\cdot,\cdot)$ is self-adjoint and $\sigma(A) \subset [-N, \infty)$.*

**Theorem 2.4.** *Suppose $a(\cdot,\cdot)$ is a bilinear form satisfying the hypotheses of Theorem 2.3. Let $b(\cdot,\cdot)$ be a Hermitian bilinear form such that $D(a) \subset D(b)$ and for some positive real number $K$, for any $u \in D(a)$,*

$$\left|b(u,u)\right| \leqslant K a(u,u). \tag{19}$$

*Assume that every sequence $(u_k) \subset D(a)$ which satisfies*

$$\|u_k\|^2 + a(u_k,u_k) \leqslant C \tag{20}$$

*has a subsequence $(v_j)$ such that*

$$b(v_j - v_k, v_j - v_k) \to 0. \tag{21}$$

*Assume also that if (20), (21) hold and $v_j \to 0$ in the $L^2(I,\mathbb{R}^n)$ norm, then $b(v_j,v_j) \to 0$. Set*

$$c(u,v) = a(u,v) + b(u,v), \tag{22}$$

*and let $\mathcal{A}$, $\mathcal{C}$ be the operators associated with $a$, $c$, respectively. Then*

$$\sigma_e(\mathcal{A}) = \sigma_e(\mathcal{C}).$$

Let

$$b(u,v) = -\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T \big(b_{jk}(t) + \delta_{jk}\big)u_k(t)v_j(t)\,dt \tag{23}$$

and

$$d(u,v) = a(u,v) + b(u,v). \tag{24}$$

We shall prove

**Lemma 2.5.** *The operator $\mathcal{D}$ associated with the bilinear form $d(\cdot,\cdot)$ under assumption $(B1)$ is self-adjoint. Its essential spectrum is the null set and there exists a finite real value L such that $\sigma(\mathcal{D}) \subset [-L,\infty)$. $\mathcal{D}$ has a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound $-L$*

$$-\infty < -L \leqslant \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_l < \cdots. \tag{25}$$

**Proof.** Using assumption $(B1)$, since each diagonal element $b_{jj}(t)$ of the matrix is in $L^1(I,\mathbb{R}^n)$ and $I$ has finite measure, the diagonal elements of the matrix $B(t) + I$, $b_{jj}(t) + 1$, are also in $L^1(I,\mathbb{R}^n)$ and we can find a constant which bounds the magnitude of $b(u,u)$ on the set $D(a) = H$,

$$\begin{aligned}\left|b(u,u)\right| &= \left|-\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T \big(b_{jk}(t)+\delta_{jk}\big)u_k(t)u_j(t)\,dt\right| \\ &\leqslant \|u\|^2_{L^\infty(I,\mathbb{R}^n)}\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T \left|b_{jk}(t)+\delta_{jk}\right|dt \\ &\leqslant K_B\|u\|^2_{L^\infty(I,\mathbb{R}^n)}. \end{aligned}\tag{26}$$

To show the bilinear form $b(\cdot,\cdot)$ is Hermitian, we can use the symmetry of the matrix $B(t)+I$ to rearrange the order of the finite summation,

$$\begin{aligned} b(u,v) &= -\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T \big(b_{jk}(t)+\delta_{jk}\big)u_k(t)v_j(t)\,dt \\ &= -\sum_{k=1}^{n}\sum_{j=1}^{n}\int_0^T \big(b_{jk}(t)+\delta_{jk}\big)v_j(t)u_k(t)\,dt \\ &= -\sum_{k=1}^{n}\sum_{j=1}^{n}\int_0^T \big(b_{kj}(t)+\delta_{kj}\big)v_j(t)u_k(t)\,dt \\ &= b(v,u). \end{aligned}$$

Also the magnitude of $b(u,u)$ is bounded by a multiple of the bilinear form $a(\cdot,\cdot)$ and satisfies (19),

$$\begin{aligned}\left|b(u,u)\right| &\leqslant K_B\|u\|^2_{L^\infty(I,\mathbb{R}^n)} \\ &\leqslant K_B\big(M\|u\|_H\big)^2 \\ &\leqslant K_B\cdot M^2\|u\|_H^2 = Ka(u,u). \end{aligned}\tag{27}$$

Consider a sequence $(x_k)\subset D(\mathcal{A})$ which is bounded by a constant $C$ in the $H$ norm. Then each term of the sequence satisfies

$$\|x_k\|^2 + a(x_k,x_k) = 2(x_k,x_k) + (\dot{x}_k,\dot{x}_k) \leqslant 2\|x_k\|_H^2 \leqslant 4C^2.$$

Since,

$$\|u\|_{L^\infty(I,\mathbb{R}^n)} \leqslant C\|u\|_H, \quad u\in H, \tag{28}$$

we can find a subsequence $(x_{\bar{k}})$ which converges weakly in $H$ and strongly in $L^\infty(I,\mathbb{R}^n)$ and $L^2(I,\mathbb{R}^n)$ to some function $x\in H$. Because the subsequence is convergent in $L^\infty(I,\mathbb{R}^n)$ it is also Cauchy under this norm. As $\bar{j},\bar{k}\to\infty$ we can apply (26) to show this subsequence satisfies (21),

$$\left|b(x_{\bar{j}}-x_{\bar{k}},x_{\bar{j}}-x_{\bar{k}})\right| \leqslant K_B\|x_{\bar{j}}-x_{\bar{k}}\|^2_{L^\infty(I,\mathbb{R}^n)} \to 0. \tag{29}$$

If in addition the subsequence $(x_{\bar{k}})$ converges to zero in $L^2(I,\mathbb{R}^n)$, the subsequence must also converge in $L^\infty(I,\mathbb{R}^n)$ to the zero function, and by (26),

$$b(x_{\bar{k}}, x_{\bar{k}}) \to 0.$$

Then the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy the conditions of Theorem 2.4. The bilinear form $d(\cdot,\cdot)$ is the sum of these two bilinear forms as in (22). By this theorem, the operator $\mathcal{D}$ associated with this bilinear form has the same essential spectrum as the operator $\mathcal{A}$ associated with the bilinear form $a(\cdot,\cdot)$.

Now we show that for any constant $\epsilon > 0$ there exists a positive constant $K_\epsilon$ such that

$$\left|b(x,x)\right| \leqslant \epsilon\|\dot{x}\|^2 + K_\epsilon\|x\|^2, \quad x \in D(\mathcal{A}). \tag{30}$$

We can use (26) to find a constant $K_B$, and for any $\epsilon > 0$, let $\xi = \epsilon/K_B$. Then there is a constant $C_\xi$ which satisfies

$$\begin{aligned}\left|b(x,x)\right| &\leqslant K_B\|x\|^2_{L^\infty(I,\mathbb{R}^n)} \\ &\leqslant K_B\left(\frac{\epsilon}{K_B}\|\dot{x}\|^2 + C_\xi\|x\|^2\right) \\ &\leqslant \epsilon\|\dot{x}\|^2 + (K_B \cdot C_\xi)\|x\|^2.\end{aligned}$$

Setting $K_\epsilon = K_B \cdot C_\xi$ gives the stated inequality.

To show $d(\cdot,\cdot)$ is closed, first apply (30) with $\epsilon = 1/2$. Thus there is a constant $C_0$ such that

$$\left|b(u,u)\right| \leqslant \frac{1}{2}a(u,u) + C_0\|u\|^2. \tag{31}$$

Now suppose a sequence $(u_k) \subset D(d)$ satisfies

$$d(u_j - u_k, u_j - u_k) \to 0, \tag{32}$$

and $(u_k) \to u$ in $L^2(I,\mathbb{R}^n)$. The sequence is Cauchy in $L^2(I,\mathbb{R}^n)$ and as $j,k$ increase

$$\|u_j - u_k\|^2 \to 0.$$

Suppose that $u \notin D(d)$. Because the domains of $d(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are the same, $u \notin D(a)$. We have shown above that $a(\cdot,\cdot)$ is closed, so the sequence cannot be Cauchy and as $j,k$ increase $a(u_j - u_k, u_j - u_k)$ does not approach zero. But by (32),

$$a(u_j - u_k, u_j - u_k) - b(u_j - u_k, u_j - u_k) \to 0.$$

Applying the inequality in (31) bounds the magnitude of each $b(\cdot,\cdot)$ term, and since $a(u,u) \geqslant 0$, the following inequality is satisfied,

$$a(u_j - u_k, u_j - u_k) - b(u_j - u_k u_j - u_k) \geqslant \frac{1}{2}a(u_j - u_k, u_j - u_k) - C_0\|u_j - u_k\|^2.$$

Adding the last term to both sides leaves only the positive bilinear form on the right side,

$$a(u_j-u_k,u_j-u_k)-b(u_j-u_ku_j-u_k)+C_0\|u_j-u_k\|^2\geqslant\frac{1}{2}a(u_j-u_k,u_j-u_k)$$
$$\geqslant 0.$$

As $j$, $k$ increase the left side of this equation approaches zero so the center term must also approach zero, a contradiction to the statement above. Therefore, $u\in D(a)=D(d)$, and $d(\cdot,\cdot)$ is also a closed bilinear form.

Next we show that there exists a positive constant $N$ such that for any $x\in D(a)$,

$$d(x,x)+N\|x\|^2\geqslant 0. \tag{33}$$

For any positive constant $\epsilon>0$ we can find $K_\epsilon$ which satisfies (30) and thereby find a lower bound for $b(x,x)$,

$$a(x,x)+b(x,x)+N\|x\|^2\geqslant a(x,x)-\epsilon\|\dot{x}\|^2-K_\epsilon\|x_15\|.$$

We have shown that $d(\cdot,\cdot)$ is closed, and as the sum of two Hermitian bilinear forms, $d(\cdot,\cdot)$ is clearly Hermitian. Its domain is dense in $L^2(I,\mathbb{R}^n)$ and the $N$ in (33) satisfies the conditions of Theorem 2.3, so the operator $\mathcal{D}$ associated with this bilinear form is self-adjoint and has its spectrum bounded below by $-N$. We have shown that the essential spectrum of this operator is the null set, so the spectrum is discrete and we can number the eigenvalues in increasing order, and each eigenvalue is of finite multiplicity. □

Finding a solution to (1) is equivalent to finding a solution to the following operator equation in $L^2(I,\mathbb{R}^n)$,

$$\mathcal{D}x=\nabla_x V\big(t,x(t)\big). \tag{34}$$

To prove the existence of such a solution, we will use a linking theorem of Schechter [64].

**Theorem 2.6.** *Let $N$ be a closed subspace of a Hilbert space $H$ and let $M=N^\perp$. Assume that at least one of the subspaces $M,N$ is finite dimensional. Let $G$ be a $C^1$ functional on $H$ such that*

$$m_0=\inf_{u\in M}G(u)\neq-\infty\quad\text{and}\quad m_1=\sup_{v\in N}G(v)\neq\infty. \tag{35}$$

*Then for any sequence $(R_k)\subset\mathbb{R}^+$ which is increasing to $\infty$, there exists a constant $\overline{m}\in\mathbb{R}$, $m_0\leqslant\overline{m}\leqslant m_1$, and a sequence $(u_k)\subset H$ such that*

$$G(u_k)\to\overline{m}\quad\text{and}\quad\big(R_k+\|u_k\|\big)\big\|G'(u_k)\big\|\leqslant\frac{m_1-m_0}{\ln(4/3)}. \tag{36}$$

## 3. Proof of Theorem 1.1

**Proof.** Define the real-valued functional $G$ on the function space $H$

$$G(x) = d(x,x) - 2\int_0^T V\big(t, x(t)\big)\,dt, \tag{37}$$

where

$$d(x,x) = (\dot{x}, \dot{x}) - \int_0^T \big(B(t)x(t), x(t)\big)_{\mathbb{R}^n}\,dt. \tag{38}$$

The first term on the right side of (38) is bounded by the square of the $H$ norm of $x$, and because the norm on the space $H$ is an upper bound for the $L^\infty$ norm, the matrix term is also bounded by the square of the $H$ norm. In the potential term in (37), the potential $V$ is continuously differentiable in $x$, so this term is a continuous map from $H$ to $\mathbb{R}$. Consequently, the functional $G$ is a continuously differentiable map from $H$ to $\mathbb{R}$.

For any $y \in H$,

$$\frac{1}{2}\big(G'(x), y\big) = d(x,y) - \big(\nabla_x V(t,x), y\big). \tag{39}$$

Let $\lambda_l$ satisfy assumption $(V4)$ for the potential function $V$, with integrable functions $W_1(t)$ and $W_2(t)$. Let $C_1$ and $C_2$ be real-valued constants such that

$$C_1 = \int_0^T W_1(t), \qquad C_2 = \int_0^T W_2(t). \tag{40}$$

For any eigenvalue $\lambda_k$ in the spectrum of $\mathcal{D}$, let $E(\lambda_k)$ be the null space of the operator $\mathcal{D} - \lambda_k$. Given a non-negative integer $l$, define the subspaces $M$ and $N$ of $H$,

$$N = \bigoplus_{k \leqslant l} E(\lambda_k), \qquad M = N^\perp, \qquad H = M \oplus N.$$

For any element $x \in H$, we can apply assumption $(V4)$ to obtain the following two bounds on $G(x)$,

$$d(x,x) - \lambda_{l+1}\int_0^T |x|^2\,dt - C_2 \leqslant G(x), \tag{41}$$

$$G(x) \leqslant d(x,x) - \lambda_l \int_0^T |x|^2\,dt + C_1. \tag{42}$$

For any element $u \in N$, $u$ is in the direct sum of the eigenspaces of eigenvalues less than or equal to $\lambda_l$. Then $d(u,u) \leqslant \lambda_l \|u\|^2$, and we use inequality (42) to obtain

$$G(u) \leqslant \lambda_l \|u\|^2 - \lambda_l \|u\|^2 + C_1 = C_1, \quad u \in N.$$

For any element $v \in M$, $v$ is in the direct sum of the eigenspaces of eigenvalues strictly greater than or equal to $\lambda_{l+1}$. Then $d(v,v) \geqslant \lambda_{l+1} \|v\|^2$ and we use inequality (41) to obtain

$$G(v) \geqslant \lambda_{l+1} \|v\|^2 - \lambda_{l+1} \|v\|^2 - C_2 = -C_2, \quad v \in M.$$

Now we can apply Theorem 2.6 with $m_0 \geqslant -C_2$ being the infimum over the subspace $M$ and $m_1 \leqslant C_1$ being the supremum over the finite dimensional subspace $N$. Let $(R_k)$ be a sequence of positive real numbers such that $R_k \to \infty$. There exists a real constant $\overline{m}$, $m_0 \leqslant \overline{m} \leqslant m_1$, and a sequence $(x_k) \subset H$ which satisfy the conditions of this theorem for the sequence $(R_k)$. Let $C$ be the positive constant on the right side of (36). Then $G(x_k) \to \overline{m}$ and

$$\big(R_k + \|x_k\|_H\big)\big\|G'(x_k)\big\|_H \leqslant C. \tag{43}$$

Suppose this sequence $(x_k)$ is bounded in $H$. Let $K > 0$ be such that for any $k$ we have $\|x_k\|_H \leqslant K$. Then there must exist a subsequence $(x_\kappa)$ which converges weakly in $H$ and strongly in $L^\infty(I, \mathbb{R}^n)$ and $L^2(I, \mathbb{R}^n)$ to a function $x \in H$. By taking a subsequence of $(R_k)$ we see the subsequence also satisfies (43). We have

$$\begin{aligned}\frac{1}{2}\big(G'(x_\kappa), y\big) &= d(x_\kappa, y) - \big(\nabla_x V(t, x_\kappa), y\big)\\ &= (x_\kappa, y)_H - (x_\kappa, y) - \int_0^T \big(B(t)x_\kappa(t), y(t)\big)_{\mathbb{R}^n}\,dt\\ &\quad - \int_0^T \big(\nabla_x V(t, x_\kappa), y\big)_{\mathbb{R}^n}\,dt, \quad y \in H.\end{aligned}$$

By weak convergence of $(x_\kappa)$ in $H$, the first term on the right side of the above equation converges in $\mathbb{R}$. By convergence of $(x_\kappa)$ in $L^2(I, \mathbb{R}^n)$ and the Schwarz inequality the second term converges. By convergence of $(x_\kappa)$ in $L^\infty(I, \mathbb{R}^n)$ the last two terms converge. Thus,

$$\frac{1}{2}\big(G'(x), y\big) = d(x, y) - \big(\nabla_x V(t, x), y\big) = 0, \quad y \in H. \tag{44}$$

Hence, $x$ is a weak solution of (1).

Next we will show that our assumptions require that the sequence given by Theorem 2.6 be bounded. Consider the case where the sequence $(x_k)$ is unbounded in $H$, and without loss of generality assume $\|x_k\|_H \to \infty$. Using the definition of the functional $G$, as $k \to \infty$,

$$d(x_k, x_k) - 2\int_0^T V(t, x_k)\,dt - \overline{m} \to 0. \tag{45}$$

We apply the definition of $G'(x)$ and the inequality in (43),

$$\left|d(x_k, x_k) - \int_0^T \big(x_k, \nabla_x V(t, x_k)\big)_{\mathbb{R}^n}\, dt\right| = \frac{1}{2}\big|\big(G'(x_k), x_k\big)\big| \leqslant \frac{1}{2}\big\|G'(x_k)\big\|_H \|x_k\|_H \leqslant \frac{1}{2}C. \tag{46}$$

Thus we have

$$\left|\int_0^T H(t, x_k)\, dt\right| \leqslant C'. \tag{47}$$

For each $k$, let $\rho_k = \|x_k\|_H$ and define the sequence $(u_k)$ such that for each $k$, $u_k = x_k/\rho_k$. Because $(u_k)$ is a bounded sequence in $H$, there is a subsequence $(u_\kappa)$ and an element $u \in H$ such that $(u_\kappa)$ converges to $u$ weakly in $H$, strongly in $L^\infty(I, \mathbb{R}^n)$ and $L^2(I, \mathbb{R}^n)$.

To show that the sequence $(u_k)$ cannot converge to the zero function, let $(x_\kappa)$ be the subsequence of $(x_k)$ with corresponding indices. The value of the functional $G(x_\kappa)$ converges to a finite value $\overline{m}$. Using the definition of the operator $\mathcal{D}$ and the $L^2(I, \mathbb{R}^n)$ norm the following sequence converges in $\mathbb{R}$,

$$\|\dot{x}_\kappa\|^2 - \int_0^T \big(B(t)x_\kappa(t), x_\kappa(t)\big)_{\mathbb{R}^n}\, dt - \int_0^T 2V\big(t, x_\kappa(t)\big)\, dt \to \overline{m}.$$

Scaling each term by $(1/\rho_\kappa^2)$ and using the linearity of the matrix multiplication gives a sequence which converges to zero in $\mathbb{R}$,

$$\|\dot{u}_\kappa\|^2 - \int_0^T \big(B(t)u_\kappa(t), u_\kappa(t)\big)_{\mathbb{R}^n}\, dt - \int_0^T \frac{2V(t, x_\kappa(t))}{\rho_\kappa^2}\, dt \to 0.$$

Adding and subtracting the $L^2(I, \mathbb{R}^n)$ norm of $u_\kappa$ to this expression,

$$\|\dot{u}_\kappa\|^2 + \|u_\kappa\|^2 - \int_0^T \big(B(t)u_\kappa(t), u_\kappa(t)\big)_{\mathbb{R}^n}\, dt - \int_0^T \left[|u_\kappa|^2 + \frac{2V(t, x_\kappa(t))}{\rho_\kappa^2}\right] dt \to 0. \tag{48}$$

For each $\kappa$, the two terms outside the integrals give the $H$ norm of $u_\kappa$, equal to 1. By the properties of the subsequence, $(u_\kappa)$ converges to $u$ in the $L^\infty(I, \mathbb{R}^n)$ norm. Each matrix element $b_{jk}(t)$ is integrable, so the sequence of functions in the first integral converges in $L^1(I, \mathbb{R}^n)$ and

$$\|\dot{u}_\kappa\|^2 + \|u_\kappa\|^2 - \int_0^T \big(B(t)u_\kappa(t), u_\kappa(t)\big)_{\mathbb{R}^n}\, dt \to 1 - \int_0^T \big(B(t)u(t), u(t)\big)_{\mathbb{R}^n}\, dt. \tag{49}$$

Because the sums in (48) and (49) both converge, for sufficiently large $\kappa$ we can find a lower bound for the remaining integral, and using the upper inequality in assumption $(V4)$,

$$\begin{aligned}\frac{1}{2} - \int_0^T \big(B(t)u(t), u(t)\big)_{\mathbb{R}^n}\,dt &\leqslant \int_0^T \left[|u_\kappa|^2 + \frac{2V(t, x_\kappa(t))}{\rho_\kappa^2}\right] dt \\ &\leqslant \int_0^T \left[|u_\kappa|^2 + \frac{\lambda_{l+1}|x_\kappa|^2}{\rho_\kappa^2} + \frac{W_2(t)}{\rho_\kappa^2}\right] dt \\ &= (1+\lambda_{l+1})\|u_\kappa\|^2 + \frac{C_2}{\rho_\kappa^2}.\end{aligned}$$

If $\|u\| = 0$, then $u$ equals the zero vector almost everywhere on the interval $I$ and the left side of the inequality above equals the positive constant $1/2$. Because $(u_\kappa)$ converges to 0 in the $L^2(I, \mathbb{R}^n)$ norm and $\rho_\kappa$ approaches $\infty$ as $\kappa$ increases, the right side approaches zero and will eventually violate the inequality. Therefore the limit $u$ cannot be the zero function in $L^2(I, \mathbb{R}^n)$.

Because $u$ is nonzero in $L^2(I, \mathbb{R}^n)$, there exists a real number $\epsilon > 0$ such that on a set of nonzero measure $\Omega \subset I$, $|u(t)| \geqslant \epsilon$. Thus, the corresponding sequence $(|x_\kappa(t)|)$ uniformly approaches infinity on this set, and we can give a lower bound for the integral in (47),

$$\begin{aligned}\int_0^T H(t, x_\kappa)\,dt &= \int_{I\setminus\Omega} H(t, x_\kappa)\,dt + \int_\Omega H(t, x_\kappa)\,dt \\ &\geqslant -\int_{I\setminus\Omega} \big|W_0(t)\big|\,dt + \int_\Omega H(t, x_\kappa)\,dt. \end{aligned} \tag{50}$$

Because $W_0(t)$ is integrable, the first term is finite. As $\kappa \to \infty$, $(|x_\kappa(t)|)$ uniformly approaches infinity on $\Omega$. Since the function $H(t, x)$ uniformly approaches infinity as $|x| \to \infty$, the second integral is unbounded and the sum of the two integrals will approach infinity. For sufficiently large $\kappa$ the inequality in (50) will contradict (47). The sequence $(x_k)$ must be bounded and a subsequence converges weakly to the element $x \in H$.

Thus, $x$ satisfies

$$\int_0^T \big(\dot{x}(t), \dot{v}(t)\big)_{\mathbb{R}^n}\,dt = \int_0^T \big(B(t)x(t) + \nabla_x V\big(t, x(t)\big), v(t)\big)_{\mathbb{R}^n}\,dt, \quad v \in H, \tag{51}$$

and $x$ is a weak solution of (1). If $\nabla_x V(t, \mathbf{0}) \neq \mathbf{0}$, the solution $x$ cannot be the constant function $x = \mathbf{0}$, and the solution is nontrivial.

In the case that $V(t,x)$ satisfies assumptions $(V2')$ and $(V3')$, the definition of $H(t,x)$ is unchanged and for $k$ sufficiently large the bound on the integral of $H(t,x_k)$ over $t$ in (47) is still valid,

$$\left|\int_0^T H(t,x_k)\,dt\right| \leqslant C+\overline{m}.$$

We then give an upper bound for the integral of $H(t,x_k)$ over $t$ which replaces the lower bound in (50),

$$\begin{aligned}\int_0^T H(t,x_\kappa)\,dt &= \int_{I\setminus\Omega} H(t,x_\kappa)\,dt+\int_\Omega H(t,x_\kappa)\,dt\\ &\leqslant \int_{I\setminus\Omega} \left|W_0(t)\right|dt+\int_\Omega H(t,x_\kappa)\,dt.\end{aligned}\tag{52}$$

As before, $W_0(t)$ is integrable and the first term is finite. As $\kappa\to\infty$, $(|x_\kappa(t)|)$ uniformly approaches infinity. Since the function $H(t,x)$ uniformly approaches negative infinity as $|x|\to\infty$, the second integral is unbounded and the sum of the two integrals will approach negative infinity. For sufficiently large $\kappa$ the inequality in (50) will contradict (47). Therefore, under assumptions $(V2')$ and $(V3')$ the sequence $(x_k)$ must be bounded. By the same steps as above, there exists a subsequence which converges to a function which is twice weakly differentiable, and if $\nabla_x V(t,\mathbf{0})\neq\mathbf{0}$ this solution is nontrivial. □

Next we consider the case of (1) when $B(t)=0$ and the term $\nabla V_x(t,x)$ is nonlinear as $|x|\to\infty$. The operator $\mathcal{B}$ above then satisfies

$$b(u,v)=-(u,v),$$

and the eigenvalues of the operator $\mathcal{D}$ on $L^2(I,\mathbb{R}^n)$ can be found exactly.

**Lemma 3.1.** *When $B(t)=0$, the spectrum of the operator $\mathcal{D}$ is the set of squares of integers multiplied by $\omega^2=4\pi^2/T^2$,*

$$\sigma(\mathcal{D})=\{0,\omega^2,4\omega^2,9\omega^2,\dots,l^2\omega^2,\dots\}.$$

We rewrite condition $(V1)$ to reflect the fixed eigenvalue of zero for the operator $\mathcal{D}$.

$(V4)$ There exist functions $W_1,W_2\in L^1(I,\mathbb{R}^n)$ such that for any $t\in I$ and any $x\in\mathbb{R}^n$,

$$l^2\omega^2|x|^2-W_1(t)\leqslant 2V(t,x)\leqslant (l+1)^2\omega^2|x|^2+W_2(t).\tag{53}$$

**Theorem 3.2.** *If $B(t)=0$ and the function $V(t,x)$ satisfies assumptions $(V2)$, $(V3)$ and $(V4)$, there exists a $T$-periodic solution to (1) whose weak second derivative is an element of $L^1(I,\mathbb{R})$.*

*If the function $V(t,x)$ satisfies $\nabla_x V(t,\mathbf{0}) \neq V(t,\mathbf{0})$, the solution of (1) is not trivial. The conclusions are also valid if we replace assumptions $(V2)$ and $(V3)$ with $(V2')$ and $(V3')$.*

**Proof.** The zero matrix $B(t)$ satisfies assumption $(B1)$, and by Lemma 3.1, the conditions on the potential $V(t,x)$ comprise a quadratic bound on the potential function which satisfies assumption $(V4)$. We can then apply Theorem 1.1 with $B(t)$ equal to the zero matrix, with the same condition for existence of a nontrivial solution. In the case that $V(t,x)$ satisfies assumptions $(V2')$ and $(V3')$, we can similarly apply Theorem 1.1 with these assumptions. □

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