



Long time stability of KAM tori for nonlinear wave equation [☆]

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Abstract

It is proved that the KAM tori (thus quasi-periodic solutions) are long time stable for infinite dimensional Hamiltonian systems generated by nonlinear wave equation, by constructing a partial normal form of higher order around the KAM torus and showing that p -tame property persists under KAM iterative procedure and normal form iterative procedure.

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1. Introduction and main results

Since the initial work [13–15,18] of infinite dimensional KAM theory by Kuksin and Wayne, there has been a lot of work about the existence of KAM tori for the nonlinear wave equation (NLW)

$$u_{tt} - u_{xx} + V(x)u + u^3 + h.o.t. = 0 \quad (1.1)$$

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subject to Dirichlet boundary conditions $u(t, 0) = u(t, \pi) = 0$. For examples, Wayne [18] obtains the existence when $V(x)$ does not belong to some set of “bad” potentials; Kuksin [15] considers parameterized potentials $V(x, \xi)$ and shows that there are many quasi-periodic solutions for “most” parameters ξ 's; Pöschel [17] proves that the potential $V = V(x)$ can be replaced by a fixed constant potential $V \equiv m$; Yuan [19] shows the existence of KAM tori for any prescribed non-constant potential. All of these results are obtained by the classic KAM iteration which involves the so-called second Melnikov conditions. Thus, every KAM torus is linearly stable, and around it a normal form of order 2 is obtained. Based on the normal form, one can directly obtain δ^{-1} long time stability of KAM tori, where δ is the distance between initial data of the solutions and KAM tori.

A natural question is whether the KAM tori are stable in a longer time such as $\delta^{-\mathcal{M}}$ for any given $\mathcal{M} \geq 0$. There has been a lot of work on the long time stability of the equilibrium point $u = 0$ and some approximate invariant tori for partial differential equations. For examples, see [1,2,4–6,8,10,12]. It is proposed by Eliasson [11] that whether such stability results can be proved in a neighborhood of a given KAM torus. Also see [7] by Berti and Biasco. Recently, the long time stability of KAM tori for nonlinear Schrödinger equation was obtained in [9]. The basic idea is that, first, define a suitable p -tame property which generalizes the idea in [5]; second, prove that the p -tame property persists under KAM iterative procedure and normal form iterative procedure; third, based on some reasonable non-resonant conditions, one can construct a partial normal form of high order around a KAM torus; finally, combining the p -tame property with the partial normal form of high order, one can deduce that the solutions starting in the neighborhood of a KAM torus still stay in the neighborhood of the KAM torus for a polynomial long time, that is, the KAM tori are stable in a long time.

In this paper, we will prove the long time stability of KAM tori for nonlinear wave equation. The method follows a parallel course as in [9] except two essential differences: (1) more regularity is needed for the Hamiltonian vector field generated by nonlinear wave equation to guarantee that the KAM iterative procedure works (it is necessary for measure estimate and see [16] for the details); (2) since the frequencies of nonlinear wave equation have worse approximations than nonlinear Schrödinger equation, whether the non-resonant conditions hold when constructing the partial normal form of higher order? We will deal with the first problem by modifying the definition of p -tame norm (see (2.5) in Definition 4) and estimate the measure of non-resonant set by the method as in [3] (see Section 5 for the details). The following is our main result:

Theorem 1.1. *Consider the nonlinear wave equation*

$$u_{tt} = u_{xx} - (m + M_\xi)u + \varepsilon u^3, \quad (1.2)$$

subject to Dirichlet boundary conditions $u(t, 0) = u(t, \pi) = 0$, where m is a non-negative constant and M_ξ is a real Fourier multiplier,

$$M_\xi \sin jx = \xi_j \sin jx, \\ \xi \in \Pi := \{ \xi = (\xi_j)_{j \geq 1} \mid \xi_j \in [1, 2]/j, \ j \geq 1 \} \subset \mathbb{R}^{\mathbb{N}}. \quad (1.3)$$

Given an integer $n \geq 1$ and a real number $p \geq 1$, for any sufficiently small $\varepsilon > 0$, there exists a large subset $\tilde{\Pi} \subset \Pi$, such that for each $\xi \in \tilde{\Pi}$ Eq. (1.2) possesses a linearly stable n -dimensional

KAM torus $\mathcal{T}_\xi$ in Sobolev space $H_0^p([0, \pi])$. Moreover, for arbitrarily given $\mathcal{M}$ with $0 \leq \mathcal{M} \leq C(\varepsilon)$ (where $C(\varepsilon)$ is a constant depending on ε and $C(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$) and $p \geq 24(\mathcal{M} + 7)^4 + 1$, there exists a small positive δ_0 depending on n , p and $\mathcal{M}$, and for any $0 < \delta < \delta_0$ and any solution $u(t, x)$ of Eq. (1.2) with the initial datum satisfying

$$d_{H_0^p[0, \pi]}(u(0, x), \mathcal{T}_\xi) := \inf_{w \in \mathcal{T}_\xi} \|u(0, x) - w\|_{H_0^p[0, \pi]} \leq \delta,$$

then

$$d_{H_0^p[0, \pi]}(u(t, x), \mathcal{T}_\xi) := \inf_{w \in \mathcal{T}_\xi} \|u(t, x) - w\|_{H_0^p[0, \pi]} \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}.$$

The rest of the present paper is organized as follows. In Section 2, we give some basic notations and the definition of p -tame norm for a Hamiltonian vector field. In Section 3, we construct a normal form of order 2, which satisfies p -tame property, around the KAM tori based on the standard KAM method (see Theorem 3.1) and a partial normal form of order $\mathcal{M} + 2$ in the δ -neighborhood of the KAM tori (see Theorem 3.2). Since the iterative procedure is parallel to [9], we only prove the measure estimate in detail. Finally, due to the partial normal form of order $\mathcal{M} + 2$ and p -tame property, we show that the KAM tori are stable in a long time (see Theorem 3.3). In Section 4, we finish the proof of Theorem 1.1. In Appendix A, we list some properties of p -tame norm. These properties are used in the proof of Theorem 3.1 and Theorem 3.2 to ensure the p -tame property surviving under KAM iterative procedure and normal form iterative procedure.

2. Definition of p -tame norm for a Hamiltonian vector field

We will define p -tame norm for a Hamiltonian vector field in this section. First we introduce the functional setting and the main notations concerning infinite dimensional Hamiltonian systems. Consider the Hilbert space of complex-valued sequences

$$\ell_p^2 := \left\{ q = (q_1, q_2, \dots) : \|q\|_p^2 := \sum_{j \geq 1} |q_j|^2 j^{2p} < +\infty \right\}$$

with $p > 1/2$, and the symplectic phase space

$$(x, y, z) \in D(s) \times \mathbb{C}^n \times \ell_{b,p}^2 := \mathcal{P}^p, \quad z := (q, \bar{q}) \in \ell_{b,p}^2 := \ell_p^2 \times \ell_p^2,$$

where $D(s) := \{x \in \mathbb{C}^n / (2\pi\mathbb{Z})^n \mid \|\operatorname{Im} x\| < s\}$ is the complex open s -neighborhood of the n -torus $\mathbb{T}^n := \mathbb{R}^n / (2\pi\mathbb{Z})^n$, equipped with the canonic symplectic structure:

$$\sum_{j=1}^n dy_j \wedge dx_j + \sqrt{-1} \sum_{j \geq 1} dq_j \wedge d\bar{q}_j.$$

Let

$$D(s, r_1, r_2) = \{(x, y, z) \in \mathcal{P}^p \mid \|\operatorname{Im} x\| < s, \|y\| < r_1^2, \|z\|_p < r_2\},$$

where $\|\cdot\|$ denotes the sup-norm for complex vectors and

$$\|z\|_p = \|q\|_p + \|\bar{q}\|_p \quad \text{with } z = (q, \bar{q}).$$

Any analytic function $W : D(s, r_1, r_2) \rightarrow \mathbb{C}$ can be developed in a totally convergent power series

$$W(x, y, z) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^{\bar{\mathbb{Z}}}} W^{\alpha\beta}(x) y^\alpha z^\beta, \quad \bar{\mathbb{Z}} = \mathbb{Z} \setminus \{0\}.$$

Note that there is a multilinear, symmetric, bounded map

$$\widetilde{W^{\alpha\beta}(x)} \in \mathcal{L}\left(\overbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}^{|\alpha| \text{-times}} \times \overbrace{\ell_{b,p}^2 \times \cdots \times \ell_{b,p}^2}^{|\beta| \text{-times}}, \mathbb{C}\right)$$

such that

$$\widetilde{W^{\alpha\beta}(x)}(\overbrace{y, \dots, y}^{|\alpha| \text{-times}}, \overbrace{z, \dots, z}^{|\beta| \text{-times}}) = W^{\alpha\beta}(x) y^\alpha z^\beta,$$

where $|\alpha| = \sum_{j=1}^n |\alpha_j|$, $|\beta| = \sum_{j \in \bar{\mathbb{Z}}} |\beta_j|$, and $|\cdot|$ denotes the 1-norm here and below.

We will study the Hamiltonian system

$$(\dot{x}, \dot{y}, \dot{z}) = X_W(x, y, z),$$

where X_W is the Hamiltonian vector field of W ,

$$X_W = (W_y, -W_x, \sqrt{-1} J W_z),$$

and

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Definition 1. Consider a function $W(x; \xi) : D(s) \times \Pi \rightarrow \mathbb{C}$ is analytic in the variable $x \in D(s)$ and C^1 -smooth in the parameter $\xi \in \Pi$ in the Whitney's sense,¹ and the Fourier series of $W(x; \xi)$ is given by

$$W(x; \xi) = \sum_{k \in \mathbb{Z}^n} \widehat{W}(k; \xi) e^{\sqrt{-1} \langle k, x \rangle},$$

where

$$\widehat{W}(k; \xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} W(x; \xi) e^{-\sqrt{-1} \langle k, x \rangle} dx$$

¹ In the whole of this paper, the derivatives with respect to the parameter $\xi \in \Pi$ are understood in the sense of Whitney.

is the k -th Fourier coefficient of $W(x; \xi)$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product, i.e.

$$\langle k, x \rangle = \sum_{j=1}^n k_j x_j.$$

Then define the norm $\| \cdot \|_{D(s) \times \Pi}$ of $W(x; \xi)$ by

$$\|W\|_{D(s) \times \Pi} = \sup_{\xi \in \Pi, j \geq 1} \sum_{k \in \mathbb{Z}^n} (|\widehat{W}(k; \xi)| + |\partial_{\xi_j} \widehat{W}(k; \xi)|) e^{|k|s}. \quad (2.1)$$

Definition 2. Let

$$D(s, r) = \{(x, y) \in D(s) \times \mathbb{C}^n \mid \|\operatorname{Im} x\| < s, \|y\| < r^2\}.$$

Consider a function $W(x, y; \xi) : D(s, r) \times \Pi \rightarrow \mathbb{C}$ is analytic in the variable $(x, y) \in D(s, r)$ and C^1 -smooth in the parameter $\xi \in \Pi$ with the following form

$$W(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} W^\alpha(x; \xi) y^\alpha.$$

Then define the norm $\| \cdot \|_{D(s, r) \times \Pi}$ of $W(x, y; \xi)$ by

$$\|W\|_{D(s, r) \times \Pi} = \sum_{\alpha \in \mathbb{N}^n} \|\widetilde{\mathcal{W}}^\alpha\| r^{2|\alpha|}, \quad (2.2)$$

where $\mathcal{W}^\alpha = \|W^\alpha(x; \xi)\|_{D(s) \times \Pi}$, $\widetilde{\mathcal{W}}^\alpha \in \mathcal{L}(\overbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}^{|\alpha| \text{-times}}, \mathbb{C})$ is an $|\alpha|$ -linear symmetric bounded map such that

$$\widetilde{\mathcal{W}}^\alpha(\overbrace{y, \dots, y}^{|\alpha| \text{-times}}) = \mathcal{W}^\alpha y^\alpha,$$

and $\|\cdot\|$ is the operator norm of multilinear symmetric bounded maps.

Definition 3. Consider a function $W(x, y, z; \xi) : D(s, r, r) \times \Pi \rightarrow \mathbb{C}$ is analytic in the variable $(x, y, z) \in D(s, r, r)$ and C^1 -smooth in the parameter $\xi \in \Pi$ with the following form

$$W(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\bar{\mathbb{Z}}}} W^\beta(x, y; \xi) z^\beta.$$

Define the modulus $[W]_{D(s, r) \times \Pi}(z)$ of $W(x, y, z; \xi)$ by

$$[W]_{D(s, r) \times \Pi}(z) := \sum_{\beta \in \mathbb{N}^{\bar{\mathbb{Z}}}} \|W^\beta\|_{D(s, r) \times \Pi} z^\beta. \quad (2.3)$$

Definition 4. Let

$$W(x, y, z; \xi) := W_h(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\tilde{z}}, |\beta|=h} W_h^\beta(x, y; \xi) z^\beta$$

be a function analytic in the variable $(x, y, z) \in D(s, r, r)$ and C^1 -smooth in the parameter $\xi \in \Pi$, and let

$$\|z^h\|_{p,1} := \frac{1}{h} \sum_{j=1}^h \|z^{(1)}\|_1 \cdots \|z^{(j-1)}\|_1 \|z^{(j)}\|_p \|z^{(j+1)}\|_1 \cdots \|z^{(h)}\|_1. \quad (2.4)$$

Define the p -tame operator norm for W_z by

$$\|W_z\|_{p,D(s,r) \times \Pi}^T := \sup_{0 \neq z^{(j)} \in \ell_{b,p}^2, 1 \leq j \leq h-1} \frac{\|[\widetilde{JW_z}]_{D(s,r) \times \Pi}(z^{(1)}, \dots, z^{(h-1)})\|_{p+1}}{\|(z^{h-1})\|_{p,1}}, \quad (2.5)$$

define the 1-operator norm for W_z by

$$\|W_z\|_{1,D(s,r) \times \Pi} := \sup_{0 \neq z^{(j)} \in \ell_{b,1}^2, 1 \leq j \leq h-1} \frac{\|[\widetilde{JW_z}]_{D(s,r) \times \Pi}(z^{(1)}, \dots, z^{(h-1)})\|_1}{\|(z^{h-1})\|_{1,1}}, \quad (2.6)$$

and define the operator norm for W_v ($v = x$ or y) by

$$\|W_v\|_{D(s,r) \times \Pi} := \sup_{0 \neq z^{(j)} \in \ell_{b,1}^2, 1 \leq j \leq h} \frac{\|[\widetilde{W_v}]_{D(s,r) \times \Pi}(z^{(1)}, \dots, z^{(h)})\|}{\|(z^h)\|_{1,1}}. \quad (2.7)$$

Finally define the p -tame norm of the Hamiltonian vector field X_W as follows,

$$\|X_W\|_{p,D(s,r,r) \times \Pi}^T = \|W_y\|_{D(s,r,r) \times \Pi} + \frac{1}{r^2} \|W_x\|_{D(s,r,r) \times \Pi} + \frac{1}{r} \|W_z\|_{p,D(s,r,r) \times \Pi}^T, \quad (2.8)$$

where

$$\|W_v\|_{D(s,r,r) \times \Pi} := \|W_v\|_{D(s,r) \times \Pi} r^h, \quad v = x \text{ or } y, \quad (2.9)$$

and

$$\|W_z\|_{p,D(s,r,r) \times \Pi}^T = \max\{\|W_z\|_{p,D(s,r) \times \Pi}^T, \|W_z\|_{1,D(s,r) \times \Pi}\} r^{h-1}. \quad (2.10)$$

Remark 1. In view of (2.5), $[JW_z]_{D(s,r) \times \Pi}$ is required as a bounded map from $\ell_{b,p}^2$ to $\ell_{b,p+1}^2$ instead of a bounded map from $\ell_{b,p}^2$ to $\ell_{b,p}^2$ as in [9].

Remark 2. Based on (2.5) and (2.7) in Definition 4, for each $(x, y, z) \in \mathcal{P}^p$ and $\xi \in \Pi$, the following estimates hold

$$\begin{aligned} \|(W_h)_z(x, y, z; \xi)\|_p &\leq \|(W_h)_z(x, y, z; \xi)\|_{p+1} \\ &\leq \| (W_h)_z \|_{p, D(s, r) \times \Pi}^T \|z\|_p \|z\|_1^{\max\{h-2, 0\}}, \end{aligned} \quad (2.11)$$

$$\|(W_h)_x(x, y, z; \xi)\| \leq \| (W_h)_x \|_{D(s, r) \times \Pi} \|z\|_1^h, \quad (2.12)$$

and

$$\|(W_h)_y(x, y, z; \xi)\| \leq \| (W_h)_y \|_{D(s, r) \times \Pi} \|z\|_1^h. \quad (2.13)$$

Definition 5. Let $W(x, y, z; \xi) = \sum_{h \geq 0} W_h(x, y, z; \xi)$ be a Hamiltonian analytic in the variable $(x, y, z) \in D(s, r, r)$ and C^1 -smooth in the parameter $\xi \in \Pi$, where

$$W_h(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\tilde{Z}}, |\beta|=h} W_h^\beta(x, y; \xi) z^\beta.$$

Then define the p -tame norm of the Hamiltonian vector field X_W by

$$\|X_W\|_{p, D(s, r, r) \times \Pi}^T := \sum_{h \geq 0} \|X_{W_h}\|_{p, D(s, r, r) \times \Pi}^T. \quad (2.14)$$

Moreover, we say that a Hamiltonian vector field X_W (or a Hamiltonian $W(x, y, z; \xi)$) has p -tame property on the domain $D(s, r, r) \times \Pi$, if and only if $\|X_W\|_{p, D(s, r, r) \times \Pi}^T < \infty$.

3. The abstract results

As in [16], define

$$\|w\|_{\mathcal{P}^p, D(s, r, r)} = \|x\| + \frac{1}{r^2} \|y\| + \frac{1}{r} \|z\|_p \quad (3.1)$$

for each $w = (x, y, z) \in D(s, r, r)$, and define the usual weighted norm of Hamiltonian vector field X_U on the domain $D(s, r, r) \times \Pi$ by

$$\|X_U\|_{\mathcal{P}^p, D(s, r, r) \times \Pi} = \sup_{(x, y, z; \xi) \in D(s, r, r) \times \Pi} \left(\|U_y\| + \frac{1}{r^2} \|U_x\| + \frac{1}{r} \|U_z\|_{p+1} \right). \quad (3.2)$$

Theorem 3.1 (Normal form of order 2). Consider a perturbation of the integrable Hamiltonian

$$H(x, y, q, \bar{q}; \xi) = N(y, q, \bar{q}; \xi) + R(x, y, q, \bar{q}; \xi) \quad (3.3)$$

defined on the domain $D(s_0, r_0, r_0) \times \Pi$ with $s_0, r_0 \in (0, 1]$, where

$$N(y, q, \bar{q}; \xi) = \sum_{j=1}^n \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) q_j \bar{q}_j$$

is a family of parameter dependent integrable Hamiltonian and

$$R(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^{\mathbb{N}}} R^{\alpha\beta\gamma}(x; \xi) y^\alpha q^\beta \bar{q}^\gamma$$

is the perturbation. Suppose the tangent frequency and normal frequency satisfy the following assumption:

Frequency asymptotics.

$$\omega_j(\xi) = \sqrt{j^2 + m + \xi_j}, \quad \text{for } 1 \leq j \leq n \quad (3.4)$$

and

$$\Omega_j(\xi) = \sqrt{(j+n)^2 + m + \xi_{j+n}} \quad \text{for } j \geq 1, \quad (3.5)$$

where $\xi = (\xi_j)_{j \geq 1} \in \Pi$. The perturbation $R(x, y, q, \bar{q}; \xi)$ has p -tame property on the domain $D(s_0, r_0, r_0) \times \Pi$ and satisfies the small assumption:

$$\varepsilon := \|X_R\|_{p, D(s_0, r_0, r_0) \times \Pi}^T \leq \eta^{12} \epsilon, \quad \text{for some } \eta \in (0, 1),$$

where ϵ is a positive constant depending on s_0, r_0 and n . Then there exists a subset $\Pi_\eta \subset \Pi$ with the estimate

$$\text{Meas } \Pi_\eta \geq (\text{Meas } \Pi)(1 - O(\eta^{1/2})).$$

For each $\xi \in \Pi_\eta$, there is a symplectic map

$$\Psi : D(s_0/2, r_0/2, r_0/2) \rightarrow D(s_0, r_0, r_0),$$

such that

$$\check{H}(x, y, q, \bar{q}; \xi) := H \circ \Psi = \check{N}(y, q, \bar{q}; \xi) + \check{R}(x, y, q, \bar{q}; \xi), \quad (3.6)$$

where

$$\check{N}(y, q, \bar{q}; \xi) = \sum_{j=1}^n \check{\omega}_j(\xi) y_j + \sum_{j \geq 1} \check{\Omega}_j(\xi) q_j \bar{q}_j \quad (3.7)$$

and

$$\check{R}(x, y, q, \bar{q}; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta, \gamma \in \mathbb{N}^{\mathbb{N}}, 2|\alpha| + |\beta| + |\gamma| \geq 3} \check{R}^{\alpha\beta\gamma}(x; \xi) y^\alpha q^\beta \bar{q}^\gamma. \quad (3.8)$$

Moreover, the following estimates hold:

(1) for each $\xi \in \Pi_\eta$, the symplectic map $\Psi : D(s_0/2, r_0/2, r_0/2) \rightarrow D(s_0, r_0, r_0)$ satisfies

$$\|\Psi - id\|_{p, D(s_0/2, r_0/2, r_0/2)} \leq c\eta^6 \epsilon \quad (3.9)$$

and

$$\|D\Psi - Id\|_{p, D(s_0/2, r_0/2, r_0/2)} \leq c\eta^6 \epsilon, \quad (3.10)$$

where on the left-hand side we use the operator norm²

$$\|D\Psi - Id\|_{p, D(s_0/2, r_0/2, r_0/2)} = \sup_{0 \neq w \in D(s_0/2, r_0/2, r_0/2)} \frac{\|(D\Psi - Id)w\|_{\mathcal{P}^p, D(s_0, r_0, r_0)}}{\|w\|_{\mathcal{P}^p, D(s_0/2, r_0/2, r_0/2)}};$$

(2) the frequencies $\check{\omega}(\xi)$ and $\check{\Omega}(\xi)$ satisfy

$$\|\check{\omega}(\xi) - \omega(\xi)\| + \sup_{j \geq 1} \|\partial_{\xi_j}(\check{\omega}(\xi) - \omega(\xi))\| \leq c\eta^8 \epsilon \quad (3.11)$$

and

$$\|\check{\Omega}(\xi) - \Omega(\xi)\|_{-1} + \sup_{j \geq 1} \|\partial_{\xi_j}(\check{\Omega}(\xi) - \Omega(\xi))\|_{-1} \leq c\eta^8 \epsilon, \quad (3.12)$$

where

$$\|w = (w_i)_{i \geq 1}\|_{-1} := \sup_{i \geq 1} |w_i(\xi)|; \quad (3.13)$$

(3) the Hamiltonian vector field $X_{\check{R}}$ of the new perturbed Hamiltonian $\check{R}(x, y, q, \bar{q}; \xi)$ satisfies

$$\|X_{\check{R}}\|_{p, D(s_0/2, r_0/2, r_0/2) \times \Pi_\eta}^T \leq \epsilon(1 + c\eta^6 \epsilon), \quad (3.14)$$

where $c > 0$ is a constant depending on s_0, r_0 and n .

Remark 3. This theorem is parallel to Theorem 2.9 in [9] and is essentially due to a standard KAM proof. The same as in [9], the tame property (3.14) of $X_{\check{R}}$ can be verified explicitly in view of Lemmas A.1–A.5. Moreover, as a corollary of this theorem, the existence and time δ^{-1} stability can be obtained directly (p -tame property is not necessary here).

Starting from the normal form of order 2 obtained in Theorem 3.1, we will further construct a partial normal form of order $\mathcal{M} + 2$ through $\mathcal{M}$ -times symplectic transformations under some non-resonant conditions. To this end, some notations are given first. Given a large $\mathcal{N} \in \mathbb{N}$, split the normal frequency $\Omega(\xi)$ and normal variable $(q, \bar{q})$ into two parts respectively, i.e.

² There id denotes the identity map from $\mathcal{P}^p \rightarrow \mathcal{P}^p$ and Id denotes its tangent map.

$$\Omega(\xi) = (\tilde{\Omega}(\xi), \hat{\Omega}(\xi)), \quad q = (\tilde{q}, \hat{q}), \quad \bar{q} = (\bar{\tilde{q}}, \bar{\hat{q}}),$$

where

$$\tilde{\Omega}(\xi) = (\Omega_1(\xi), \dots, \Omega_N(\xi)), \quad \tilde{q} = (q_1, \dots, q_N), \quad \bar{\tilde{q}} = (\bar{q}_1, \dots, \bar{q}_N)$$

are the low frequencies and

$$\hat{\Omega}(\xi) = (\Omega_{N+1}(\xi), \Omega_{N+2}(\xi), \dots), \quad \hat{q} = (q_{N+1}, q_{N+2}, \dots), \quad \bar{\hat{q}} = (\bar{q}_{N+1}, \bar{q}_{N+2}, \dots)$$

are the high frequencies. Given $0 < \tilde{\eta} < 1$, and $\tau > 2n + 5$, define the resonant sets $\mathcal{R}_{k\tilde{l}\bar{l}}$ by

$$\mathcal{R}_{k\tilde{l}\bar{l}} = \left\{ \xi \in \Pi_\eta : \left| \langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \bar{l}, \hat{\Omega}(\xi) \rangle \right| \leq \frac{\tilde{\eta}}{4^3 \mathcal{M}(|k| + 1)^\tau C(\mathcal{N}, \tilde{l})} \right\}. \quad (3.15)$$

Let

$$\mathcal{R} = \bigcup_{|k| + |\tilde{l}| + |\bar{l}| \neq 0, |\tilde{l}| + |\bar{l}| \leq \mathcal{M} + 2, |\bar{l}| \leq 2} \mathcal{R}_{k\tilde{l}\bar{l}} \quad (3.16)$$

and

$$\tilde{\Pi}_{\tilde{\eta}} = \Pi_\eta \setminus \mathcal{R}, \quad (3.17)$$

where Π_η is defined in [Theorem 3.1](#).

Theorem 3.2 (Partial normal form of order $\mathcal{M} + 2$). Consider the normal form of order 2

$$\check{H}(x, y, q, \bar{q}; \xi) = \check{N}(y, q, \bar{q}; \xi) + \check{R}(x, y, q, \bar{q}; \xi)$$

obtained in [Theorem 3.1](#). Given any positive integer $\mathcal{M}$ and $0 < \tilde{\eta} < 1$, there exist a small $\rho_0 > 0$ and a large positive integer $\mathcal{N}_0$ depending on s_0 , r_0 , n and $\mathcal{M}$, such that for each $0 < \rho < \rho_0$, any integer $\mathcal{N}$ satisfying

$$\mathcal{N}_0 < \mathcal{N} < \left(\frac{\tilde{\eta}^2}{2\rho} \right)^{\frac{1}{6(\mathcal{M}+7)^2}}, \quad (3.18)$$

the non-resonant set $\tilde{\Pi}_{\tilde{\eta}}$ fulfills the estimate

$$\text{Meas } \tilde{\Pi}_{\tilde{\eta}} \geq (\text{Meas } \Pi_\eta) (1 - c\tilde{\eta}^{1/2}), \quad (3.19)$$

and for any $\xi \in \tilde{\Pi}_{\tilde{\eta}}$, there is a symplectic map

$$\Phi : D(s_0/4, 4\rho, 4\rho) \rightarrow D(s_0/2, 5\rho, 5\rho),$$

such that

$$\begin{aligned}\check{\check{H}}(x, y, q, \bar{q}; \xi) &:= \check{H} \circ \Phi \\ &= \check{N}(y, q, \bar{q}; \xi) + Z(y, q, \bar{q}; \xi) + P(x, y, q, \bar{q}; \xi) + Q(x, y, q, \bar{q}; \xi) \quad (3.20)\end{aligned}$$

is a partial normal form of order $\mathcal{M} + 2$, where

$$Z(y, q, \bar{q}; \xi) = \sum_{4 \leq 2|\alpha| + 2|\beta| + 2|\mu| \leq \mathcal{M} + 2, |\mu| \leq 1} Z^{\alpha\beta\mu\mu}(\xi) y^\alpha \tilde{q}^\beta \bar{\tilde{q}}^{\bar{\beta}} \hat{q}^\mu \bar{\hat{q}}^\mu$$

is the integrable term depending only on y and $I_j = |q_j|^2$, $j \geq 1$, and where

$$P(x, y, q, \bar{q}; \xi) = \sum_{2|\alpha| + |\beta| + |\gamma| + |\mu| + |v| \geq \mathcal{M} + 3, |\mu| + |v| \leq 2} P^{\alpha\beta\gamma\mu\nu}(x; \xi) y^\alpha \tilde{q}^\beta \bar{\tilde{q}}^{\bar{\beta}} \hat{q}^\mu \bar{\hat{q}}^v$$

and

$$Q(x, y, q, \bar{q}; \xi) = \sum_{|\mu| + |v| \geq 3} Q^{\alpha\beta\gamma\mu\nu}(x; \xi) y^\alpha \tilde{q}^\beta \bar{\tilde{q}}^{\bar{\beta}} \hat{q}^\mu \bar{\hat{q}}^v.$$

Moreover, we have the following estimates:

(1) the symplectic map Φ satisfies

$$\|\Phi - id\|_{p, D(s_0/4, 4\rho, 4\rho)} \leq \frac{c\mathcal{N}^{294}\rho}{\tilde{\eta}^2} \quad (3.21)$$

and

$$\|D\Phi - Id\|_{p, D(s_0/4, 4\rho, 4\rho)} \leq \frac{c\mathcal{N}^{294}}{\tilde{\eta}^2}; \quad (3.22)$$

(2) the Hamiltonian vector fields X_Z , X_P and X_Q satisfy

$$\begin{aligned}\|X_Z\|_{p, D(s_0/4, 4\rho, 4\rho) \times \Pi_\eta}^T &\leq c\rho \left(\frac{1}{\tilde{\eta}^2} \mathcal{N}^{6(\mathcal{M}+6)^2} \rho \right), \\ \|X_P\|_{p, D(s_0/4, 4\rho, 4\rho) \times \Pi_\eta}^T &\leq c\rho \left(\frac{1}{\tilde{\eta}^2} \mathcal{N}^{6(\mathcal{M}+7)^2} \rho \right)^\mathcal{M}\end{aligned} \quad (3.23)$$

and

$$\|X_Q\|_{p, D(s_0/4, 4\rho, 4\rho) \times \Pi_\eta}^T \leq c\rho,$$

where $c > 0$ is a constant depending on s_0 , r_0 , n and $\mathcal{M}$.

Proof. In this proof, Lemmas A.1–A.5 are used, and the normal form iterative procedure is the same as Theorem 5.1 in [9]. Thus, we prove the measure estimate (3.19) in detail while omit the other parts of proof. Firstly, we will estimate the measure of the resonant sets $\mathcal{R}_{k\tilde{l}\tilde{l}}$.

Case 1. For $|k| \neq 0$, without loss of generality, we assume

$$|k_1| = \max_{1 \leq i \leq n} \{|k_1|, \dots, |k_n|\}. \quad (3.24)$$

Then

$$\begin{aligned} & |\partial_{\xi_1}(\langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle)| \\ & \geq |k_1| |\partial_{\xi_1} \check{\omega}_1(\xi)| - \left| \partial_{\xi_1} \left(\sum_{i=2}^n k_i \check{\omega}_i(\xi) + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle \right) \right| \\ & \geq |k_1| (1 - c\eta^8 \epsilon) - \left(\sum_{i=2}^n |k_i| + |\tilde{l}| + |\hat{l}| \right) c\eta^8 \epsilon \quad (\text{in view of (3.4), (3.5), (3.11) and (3.12)}) \\ & \geq |k_1| - (|k| + \mathcal{M} + 2) c\eta^8 \epsilon \quad (\text{in view of } |\tilde{l}| + |\hat{l}| \leq \mathcal{M} + 2) \\ & \geq \frac{1}{4} |k_1| \quad (\text{by (3.24) and } \mathcal{M} \leq (2c\eta^8 \epsilon)^{-1}) \\ & \geq \frac{1}{4}. \end{aligned}$$

Hence,

$$\text{Meas } \mathcal{R}_{k\tilde{l}\tilde{l}} \leq \frac{4\tilde{\eta}}{4^{3\mathcal{M}}(|k| + 1)^\tau C(\mathcal{N}, \tilde{l})} \cdot \text{Meas } \Pi_\eta. \quad (3.25)$$

Case 2. If $|k| = 0$ and $|\tilde{l}| \neq 0$, without loss of generality, we assume

$$|\tilde{l}_j| \neq 0$$

and

$$\tilde{l}_i = 0, \quad 1 \leq i \leq j-1.$$

Then

$$\begin{aligned} & |\partial_{\xi_{n+j}}(\langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle)| \\ & \geq |\tilde{l}_j| |\partial_{\xi_{n+j}} \tilde{\Omega}_j(\xi)| - |\partial_{\xi_{n+j}}(\langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle - \tilde{l}_j \tilde{\Omega}_j(\xi))| \\ & \geq (\sqrt{(j+n)^2 + m + \xi_j})^{-1} \left(|\tilde{l}_j| (1 - c\eta^8 \epsilon) - \left(\sum_{i=j+1}^{\mathcal{N}} |\tilde{l}_i| + |\hat{l}| \right) c\eta^8 \epsilon \right) \quad (\text{by (3.5) and (3.12)}) \\ & \geq (\sqrt{(j+n)^2 + m + \xi_j})^{-1} (|\tilde{l}_j| - (\mathcal{M} + 2) c\eta^8 \epsilon) \quad (\text{in view of } |\tilde{l}| + |\hat{l}| \leq \mathcal{M} + 2) \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{\mathcal{N}} |\tilde{l}_j| \quad (\text{in view of } \mathcal{M} \leq (2c\eta^8\epsilon)^{-1} \text{ and } j \leq \mathcal{N}) \\ &\geq \frac{1}{\mathcal{N}}. \end{aligned}$$

Hence,

$$\text{Meas } \mathcal{R}_{0\tilde{l}} \leq \frac{4\tilde{\eta}\mathcal{N}}{4^{3\mathcal{M}}C(\mathcal{N}, \tilde{l})} \cdot \text{Meas } \Pi_\eta. \quad (3.26)$$

Case 3. If $|k| = 0$, $|\tilde{l}| = 0$ and $1 \leq |\hat{l}| \leq 2$, then it is easy to see that $|\langle \hat{l}, \hat{\Omega}(\xi) \rangle|$ is not small, i.e.

$$\text{the sets } \mathcal{R}_{k\tilde{l}} \text{ are empty for } |k| = 0, |\tilde{l}| = 0 \text{ and } 1 \leq |\hat{l}| \leq 2. \quad (3.27)$$

Now we would like to estimate the measure of $\mathcal{R}$ (see (3.16)). Following the notations in [3], we define the set

$$\mathcal{Z}_{n,\mathcal{N}} := \{(k, \tilde{l}, \hat{l}) \in \mathbb{Z}^n \times \mathbb{Z}^{\mathcal{N}} \times \mathbb{Z}^{\mathbb{N}} \setminus (0, 0, 0) : |\hat{l}| \leq 2\},$$

and we split

$$\mathcal{L} := \{\hat{l} \in \mathbb{Z}^{\mathbb{N}} : |\hat{l}| \leq 2\}$$

as the union of the following four disjoint sets

$$\begin{aligned} \mathcal{L}_0 &= \{\hat{l} = 0\}, \\ \mathcal{L}_1 &= \{\hat{l} = e_j\}, \\ \mathcal{L}_{2+} &= \{\hat{l} = e_i + e_j\}, \\ \mathcal{L}_{2-} &= \{\hat{l} = e_i - e_j, i \neq j\}, \end{aligned}$$

where

$$e_j = (0, \dots, 0, \overbrace{1}^{j\text{-th}}, 0, \dots)$$

and $i, j \geq n + \mathcal{N} + 1$.

Let $|\hat{l}| = 2$ and $\hat{l} = e_i + e_j \in \mathcal{L}_{2+}$ for some $i, j \geq n + \mathcal{N} + 1$. If

$$\min\{i, j\} \geq |k| \cdot \|\check{\omega}(\xi)\| + 2(\mathcal{M} + 2)\mathcal{N} + 1,$$

then it is easy to see that

$$|\langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \check{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle| \geq 1,$$

which is not small. Namely, the resonant set $\mathcal{R}_{k\tilde{l}\hat{l}}$ is empty. So it is sufficient to consider

$$\max\{i, j\} < |k| \cdot \|\check{\omega}(\xi)\| + 2(\mathcal{M} + 2)\mathcal{N} + 1,$$

when the estimate (3.28) is given below. In fact, we obtain

$$\begin{aligned} & Meas \bigcup_{(k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_{2+}} \mathcal{R}_{k\tilde{l}\hat{l}} \\ & \leq \sum_{k \neq 0, (k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_{2+}} \frac{4\tilde{\eta}}{4^{3\mathcal{M}}(|k| + 1)^\tau C(\mathcal{N}, \tilde{l})} \cdot Meas \Pi_\eta \\ & \quad + \sum_{k=0, (k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_{2+}} \frac{4\tilde{\eta}\mathcal{N}}{4^{3\mathcal{M}}C(\mathcal{N}, \tilde{l})} \cdot Meas \Pi_\eta \\ & \leq c\tilde{\eta} \cdot Meas \Pi_\eta, \end{aligned} \quad (3.28)$$

where $c_1 > 0$ is a constant depending on n and τ .

Similarly we obtain

$$Meas \bigcup_{(k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_0} \mathcal{R}_{k\tilde{l}\hat{l}} \leq c_2\tilde{\eta} \cdot Meas \Pi_\eta \quad (3.29)$$

and

$$Meas \bigcup_{(k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_1} \mathcal{R}_{k\tilde{l}\hat{l}} \leq c_2\tilde{\eta} \cdot Meas \Pi_\eta, \quad (3.30)$$

where $c_2 > 0$ is a constant depending on n and τ . Now let

$$(k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_{2-},$$

and assume $i > j$ without loss of generality. In view of (3.5) and (3.12), there is a constant $C > 0$ such that

$$\left| \frac{\check{\Omega}_i(\xi) - \check{\Omega}_j(\xi)}{i - j} - 1 \right| \leq \frac{C}{j}.$$

Hence,

$$\langle \hat{l}, \hat{\Omega}(\xi) \rangle = \check{\Omega}_i(\xi) - \check{\Omega}_j(\xi) = i - j + r_{ij},$$

with

$$|r_{ij}| \leq \frac{Cm}{j}$$

and $m = i - j$. Then we have

$$|\langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + \langle \hat{l}, \hat{\Omega}(\xi) \rangle| \geq |\langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + m| - |r_{ij}|.$$

Therefore,

$$\mathcal{R}_{k\tilde{l}\hat{l}} \subset \mathcal{Q}_{k\tilde{l}mj} := \left\{ |\langle k, \check{\omega}(\xi) \rangle + \langle \tilde{l}, \tilde{\Omega}(\xi) \rangle + m| \leq \frac{\tilde{\eta}}{4^{3\mathcal{M}}(|k|+1)^\tau C(\mathcal{N}, \tilde{l})} + \frac{Cm}{j} \right\}.$$

For $j \geq j_0$, we have

$$\mathcal{Q}_{k\tilde{l}mj} \subset \mathcal{Q}_{k\tilde{l}mj_0}.$$

Then it is sufficient to consider

$$m \leq |k| \cdot \|\check{\omega}(\xi)\| + 2(\mathcal{M} + 2)\mathcal{N} + 1,$$

and let

$$j_0 = \tilde{\eta}^{-1/2} 4^{\mathcal{M}} (|k| + 1)^{\tau/2} C(\mathcal{N}, \tilde{l})^{1/2}.$$

Then following the proof of Lemma 5 in [3], we obtain

$$\text{Meas} \bigcup_{(k, \tilde{l}, \hat{l}) \in \mathcal{Z}_{n, \mathcal{N}} \cap \mathcal{L}_{2-}} \mathcal{R}_{k\tilde{l}\hat{l}} \leq c_3 \tilde{\eta}^{1/2} \cdot \text{Meas} \Pi_\eta, \quad (3.31)$$

where $c_3 > 0$ is a constant depending on n and τ . Finally, in view of (3.28)–(3.31) and (3.16), we obtain

$$\text{Meas} \mathcal{R} \leq c \tilde{\eta}^{1/2} \cdot \text{Meas} \Pi_\eta, \quad (3.32)$$

where c is a constant depending on n and τ . Then combining (3.17) with (3.32), we finish the proof of (3.19). $\square$

Based on the partial normal form of order $\mathcal{M} + 2$ and p -tame property, we obtain the long time stability of KAM tori as follows.

Theorem 3.3 (The long time stability of KAM tori). *Based on the partial normal form (3.20), for any $p \geq 24(\mathcal{M} + 7)^4 + 1$ and $0 < \delta < \rho$, the KAM tori $\mathcal{T}$ are stable in long time, i.e. if $w(t)$ is a solution of Hamiltonian vector field X_H with the initial datum $w(0) = (w_x(0), w_y(0), w_q(0), w_{\bar{q}}(0))$ satisfying*

$$d_p(w(0), \mathcal{T}) \leq \delta,$$

then

$$d_p(w(t), \mathcal{T}) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}. \quad (3.33)$$

Proof. Take $\delta < \rho$ and $\delta^{1/10} < \tilde{\eta} < 1$. Let

$$\delta^{-\frac{\mathcal{M}+1}{p-1}} \leq \mathcal{N} + 1 < \delta^{-\frac{\mathcal{M}+1}{p-1}} + 1. \quad (3.34)$$

Then based on [Theorem 3.2](#), we obtain a partial normal form of order $\mathcal{M} + 2$

$$\check{H}(x, y, q, \bar{q}; \xi) := \check{H} \circ \Phi = \check{N}(y, q, \bar{q}; \xi) + Z(y, q, \bar{q}; \xi) + P(x, y, q, \bar{q}; \xi) + Q(x, y, q, \bar{q}; \xi).$$

To obtain the estimate [\(3.33\)](#), it is sufficient to prove that

$$\|X_P\|_{\mathcal{P}^p, D(s_0/4, 4\delta, 4\delta) \times \Pi_\eta} \leq \delta^{\mathcal{M} + \frac{1}{2}} \quad (3.35)$$

and

$$\|X_Q\|_{\mathcal{P}^p, D(s_0/4, 4\delta, 4\delta) \times \Pi_\eta} \leq \delta^{\mathcal{M} + \frac{1}{2}}. \quad (3.36)$$

By a direct calculation,

$$\begin{aligned} & c\delta \left(\frac{1}{\tilde{\eta}^2} \mathcal{N}^{6(\mathcal{M}+7)^2} \delta \right)^{\mathcal{M}} \\ &= \delta^{\mathcal{M}+1} \left(\frac{c\mathcal{N}^{6(\mathcal{M}+7)^2} \mathcal{M}}{\tilde{\eta}^{2\mathcal{M}}} \right) \\ &\leq \delta^{\mathcal{M}+1} \left(\frac{c}{\tilde{\eta}^{2\mathcal{M}}} \delta^{-\frac{6(\mathcal{M}+7)^2 \mathcal{M}(\mathcal{M}+1)}{p-1}} \right) \quad (\text{in view of the inequality (3.34)}) \\ &\leq \delta^{\mathcal{M}+1} \left(\frac{c}{\tilde{\eta}^{2\mathcal{M}}} \delta^{-\frac{1}{4}} \right) \quad (\text{in view of } p \geq 24(\mathcal{M}+7)^4 + 1) \\ &\leq \delta^{\mathcal{M} + \frac{5}{8}} \quad (\text{by assuming } \delta \text{ is very small}). \end{aligned} \quad (3.37)$$

In view of the inequalities [\(3.23\)](#) and [\(3.37\)](#), we have

$$\|X_P\|_{p, D(s_0/4, 4\delta, 4\delta) \times \Pi_\eta}^T \leq \delta^{\mathcal{M} + \frac{1}{2}}. \quad (3.38)$$

Moreover, by the inequalities [\(A.4\)](#) and [\(3.38\)](#), we finish the proof of [\(3.35\)](#).

On the other hand,

$$\begin{aligned} \|\hat{z}\|_1 &= \sqrt{\sum_{|j| \geq \mathcal{N}+1} |z_j|^2 j^2} \\ &= \sqrt{\sum_{|j| \geq \mathcal{N}+1} |z_j|^2 j^{2p} / j^{2(p-1)}} \\ &\leq \frac{\|\hat{z}\|_p}{(\mathcal{N}+1)^{p-1}} \\ &\leq \delta^{\mathcal{M}+1} \|\hat{z}\|_p \quad (\text{by (3.34)}). \end{aligned} \quad (3.39)$$

Note that $Q(x, y, q, \bar{q}; \xi) = O(\|\hat{q}\|_p^3)$. Then in view of (2.11)–(2.13) and (3.39), we finish the proof of (3.36). $\square$

4. Proof of Theorem 1.1

Step 1. Write Eq. (1.2) as an infinite dimensional Hamiltonian system. Introduce coordinates $q, \bar{q} \in \ell_p^2$ by

$$u = \sum_{j \geq 1} \frac{q_j}{\sqrt{\lambda_j}} \phi_j, \quad v := u_t = \sum_{j \geq 1} \tilde{q}_j \sqrt{\lambda_j} \phi_j, \quad \lambda_j(\xi) = \sqrt{\mu_j + \xi_j},$$

where $\mu_j = j^2 + m$ and $\phi_j(x) = \sqrt{2/\pi} \sin jx$ are respectively the simple eigenvalues and eigenvectors of $-\partial_{xx} + m$ with Dirichlet boundary conditions. The Hamiltonian of (1.2) is

$$\begin{aligned} H_{NLW} &= \int_0^\pi \left(\frac{v^2}{2} + \frac{1}{2} (u_x^2 + (m + M_\xi)u) + \frac{1}{4} u^4 \right) dx \\ &= \frac{1}{2} \sum_{j \geq 1} \lambda_j (q_j^2 + \tilde{q}_j^2) + G(q), \end{aligned}$$

where

$$G(q) = \frac{1}{4} \sum_{i \pm j \pm k \pm l = 0} G_{ijkl} q_i q_j q_k q_l$$

with

$$G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_0^\pi \phi_i \phi_j \phi_k \phi_l dx.$$

Introduce complex coordinates

$$w_j := \frac{1}{\sqrt{2}} (q_j + \sqrt{-1} \tilde{q}_j), \quad \bar{w}_j := \frac{1}{\sqrt{2}} (q_j - \sqrt{-1} \tilde{q}_j),$$

and let $\underline{z} = ((w_j)_{j \geq 1}, (\bar{w}_j)_{j \geq 1})$. Following Example 3.2 in [5], it is proved that there exists a constant $c_p > 0$ such that

$$\|\widetilde{X}_G(\underline{z}^{(1)}, \underline{z}^{(2)}, \underline{z}^{(3)})\|_{p+1} \leq c_p \|\underline{z}^{(3)}\|_{p,1}. \quad (4.1)$$

In particular, when $p = 1$, the inequality (4.1) reads

$$\|\widetilde{X}_G(\underline{z}^{(1)}, \underline{z}^{(2)}, \underline{z}^{(3)})\|_1 \leq \|\widetilde{X}_G(\underline{z}^{(1)}, \underline{z}^{(2)}, \underline{z}^{(3)})\|_2 \leq c_1 \|\underline{z}^{(3)}\|_{1,1}. \quad (4.2)$$

The inequalities (4.1) and (4.2) show that the Hamiltonian vector field $X_G(\underline{z})$ has p -tame property.

Step 2. Introduce the action-angle variables. Without loss of generality, we choose $\phi_1, \phi_2, \dots, \phi_n$ as tangent direction and the other as normal direction. Let

$$\tilde{w} = (w_1, \dots, w_n) \quad \text{and} \quad \hat{w} = (w_j)_{j \geq n+1} \quad (4.3)$$

be the tangent variable and normal variable, respectively. Then rewrite $G(w, \bar{w})$ in the multiple-index as

$$G(w, \bar{w}) = \sum_{|\mu|+|\nu|+|\beta|=4} G^{\mu\nu\beta} \tilde{w}^\mu \bar{\tilde{w}}^\nu z^\beta, \quad \mu, \nu \in \mathbb{N}^n, \quad \beta \in \mathbb{N}^N, \quad (4.4)$$

where $z = (\hat{w}, \bar{\hat{w}})$ and $G^{\mu\nu\beta} = G_{ijkl}$ for some corresponding i, j, k, l .

Introduce action-coordinates on the first n modes

$$w_j := \sqrt{I_j + y_j} e^{\sqrt{-1}x_j}, \quad 1 \leq j \leq n$$

with

$$I_j \in \left(\frac{r^{2\theta}}{2}, r^{2\theta} \right], \quad \theta \in (0, 1).$$

Then the symplectic structure is $dy \wedge dx + \sqrt{-1}d\hat{w} \wedge d\bar{\hat{w}}$, where $y = (y_1, \dots, y_n)$ and $x = (x_1, \dots, x_n)$. Hence, (4.4) is changed into

$$\begin{aligned} G(w, \bar{w}) &= G(x, y, z) \\ &= \sum_{|\mu|+|\nu|+|\beta|=4} 2^{\frac{1}{2}(|\mu|+|\nu|)} G^{\mu\nu\beta} \sqrt{(\zeta + y)^\mu} e^{\sqrt{-1}\langle \mu, x \rangle} \sqrt{(\zeta + y)^\nu} e^{-\sqrt{-1}\langle \nu, x \rangle} z^\beta \\ &= \sum_{|\mu|+|\nu|+|\beta|=4} (2^{\frac{1}{2}(|\mu|+|\nu|)} G^{\mu\nu\beta} e^{\sqrt{-1}\langle \mu - \nu, x \rangle}) \sqrt{(\zeta + y)^{\mu + \nu}} z^\beta \\ &= \sum_{|\beta| \leq 4} G^\beta(x, y) z^\beta, \end{aligned} \quad (4.5)$$

where

$$\zeta = (\zeta_1, \dots, \zeta_n), \quad G^\beta(x, y) = \sum_{|\mu|+|\nu|=4-|\beta|} 2^{\frac{1}{2}(|\mu|+|\nu|)} G^{\mu\nu\beta} e^{\sqrt{-1}\langle \mu - \nu, x \rangle} \sqrt{(\zeta + y)^{\mu + \nu}}.$$

Step 3. Show that the Hamiltonian vector field X_G has p -tame property. Note G has p -tame property in the coordinate $\underline{z}$, and introducing action-angle variables is a coordinate transformation of finite variables, so G has p -tame property in the coordinate (x, y, z) . More precisely, assume $G(x, y, z) = \sum_{|\beta| \leq 4} G^\beta(x, y) z^\beta$ is defined on the domain $(x, y, z) \in D(s, r, r)$ for some $0 < s, r \leq 1$. In this step, we will show that the Hamiltonian vector field $X_G(x, y, z)$ has p -tame property on the domain $D(s, r, r)$.

Firstly, we would like to show

$$\|G_z\|_{p,D(s,r,r)\times\Pi}^T < \infty.$$

In view of (4.4),

$$\begin{aligned} G_z(x, y, z) &= G_z(w, \bar{w}) \\ &= \sum_{|\mu|+|v|+|\beta|=4} \beta G^{\mu\nu\beta} \tilde{w}^\mu \bar{\tilde{w}}^\nu z^{\beta-1}. \end{aligned}$$

Let

$$G_z^{\beta-1}(x, y) = G_z^{\beta-1}(\tilde{w}, \bar{\tilde{w}}) = \sum_{|\mu|+|v|=4-|\beta|} \beta G^{\mu\nu\beta} \tilde{w}^\mu \bar{\tilde{w}}^\nu,$$

and then

$$G_z(x, y, z) = \sum_{|\beta|\leq 4} G_z^{\beta-1}(x, y) z^{\beta-1}.$$

Hence, we obtain

$$\|\widetilde{[G_z]}_{D(s,r)\times\Pi}\|_{p+1} \leq c \left(\sum_{1\leq i\leq 3} \|z^{(i)}\|_p + \sum_{1\leq i\neq j\leq 3} \|z^{(i)}\|_1 \|z^{(j)}\|_p + \|(z^3)\|_{p,1} \right), \quad (4.6)$$

where $c > 0$ is a constant depending on s, r, n and p , and the above inequality is based on the inequality (4.1) and the definition of $[\cdot]_{D(s,r)\times\Pi}$ (see (2.2) for the details), and noting that $\underline{z} = (\tilde{w}, \hat{w}, \bar{\tilde{w}}, \bar{\hat{w}})$ and $z = (\hat{w}, \bar{\tilde{w}})$. In particular, when $p = 1$, the inequality (4.6) reads

$$\|\widetilde{[G_z]}_{D(s,r)\times\Pi}\|_1 \leq \check{c} \left(\sum_{1\leq i\leq 3} \|z^{(i)}\|_1 + \sum_{1\leq i\neq j\leq 3} \|z^{(i)}\|_1 \|z^{(j)}\|_1 + \|(z^3)\|_{1,1} \right), \quad (4.7)$$

where $\check{c} > 0$ is a constant depending on s, r and n . Based on the inequalities (4.6) and (4.7), we obtain

$$\|G_z\|_{p,D(s,r,r)\times\Pi}^T < \infty. \quad (4.8)$$

Similarly, we obtain

$$\|G_x\|_{D(s,r,r)\times\Pi} < \infty \quad (4.9)$$

and

$$\|G_y\|_{D(s,r,r)\times\Pi} < \infty. \quad (4.10)$$

In view of the inequalities (4.8), (4.9) and (4.10), we get

$$\|X_G\|_{p,D(s,r,r)\times\Pi}^T < \infty. \quad (4.11)$$

Finally, we obtain a Hamiltonian $H(x, y, q, \bar{q}; \xi)$ having the following form

$$H(x, y, q, \bar{q}; \xi) = N(x, y, q, \bar{q}; \xi) + R(x, y, q, \bar{q}; \xi), \quad (4.12)$$

where

$$N(x, y, q, \bar{q}; \xi) = H_0(w, \bar{w}) = \sum_{j=1}^n \omega_j(\xi) y_j + \sum_{j \geq 1} \Omega_j(\xi) q_j \bar{q}_j, \quad (4.13)$$

with the tangent frequency

$$\omega(\xi) = (\omega_j(\xi))_{1 \leq j \leq n}, \quad \omega_j = \sqrt{j^2 + m + \xi_j}, \quad (4.14)$$

the normal frequency

$$\Omega(\xi) = (\Omega_1(\xi), \Omega_2(\xi), \dots), \quad \Omega_j(\xi) = \sqrt{(j+n)^2 + m + \xi_{j+n}}, \quad j \geq 1 \quad (4.15)$$

and the perturbation

$$R(x, y, q, \bar{q}; \xi) = \varepsilon G(x, y, q, \bar{q}; \xi). \quad (4.16)$$

In view of (4.11), (4.14)–(4.16), all assumptions in Theorem 3.1 hold. According to Theorem 3.1, we obtain a KAM normal form of order 2 where the nonlinear terms satisfy p -tame property.

Furthermore, let δ be given in the statement of Theorem 3.2 and $\mathcal{N}$ be given in (3.34). Take η satisfying $\delta^{1/6} < \tilde{\eta} < \mathcal{N}^{-3}$. Then we obtain a KAM partial normal form of order $\mathcal{M} + 2$ where the nonlinear terms satisfy p -tame property.

Finally, based on Corollary 3.3, for each $\xi \in \Pi_\eta \subset \Pi_\infty$, the KAM torus $\mathcal{T}_\xi$ for Eq. (1.2) is stable in a long time, i.e. for any solution $u(t, x)$ of Eq. (1.2) with the initial datum satisfying

$$d_{H_0^p[0, \pi]}(u(0, x), \mathcal{T}_\xi) \leq \delta,$$

then

$$d_{H_0^p[0, \pi]}(u(t, x), \mathcal{T}_\xi) \leq 2\delta, \quad \text{for all } |t| \leq \delta^{-\mathcal{M}}.$$

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Appendix A. Properties of the Hamiltonian with p -tame property

The following lemma shows that p -tame property persists under Poisson brackets, which can be parallel proved following the proof of Theorem 3.1 in [9]:

Lemma A.1. *Suppose that both Hamiltonian functions*

$$U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\bar{z}}} U^{\beta}(x, y; \xi) z^{\beta}$$

and

$$V(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\bar{z}}} V^{\beta}(x, y; \xi) z^{\beta},$$

satisfy p -tame property on the domain $D(s, r, r) \times \Pi$, where

$$U^{\beta}(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} U^{\alpha\beta}(x; \xi) y^{\alpha}$$

and

$$V^{\beta}(x, y; \xi) = \sum_{\alpha \in \mathbb{N}^n} V^{\alpha\beta}(x; \xi) y^{\alpha}.$$

Then the Poisson bracket $\{U, V\}(x, y, z; \xi)$ of $U(x, y, z; \xi)$ and $V(x, y, z; \xi)$ with respect to the symplectic structure $\sum_{1 \leq j \leq n} dy_j \wedge dx_j + \sqrt{-1} \sum_{j \geq 1} dz_{-j} \wedge dz_j$ has p -tame property on the domain $D(s - \sigma, r - \sigma', r - \sigma') \times \Pi$ for $0 < \sigma < s$, $0 < \sigma' < r/2$. Moreover, the following inequality holds

$$\begin{aligned} & \|X_{\{U, V\}}\|_{p, D(s-\sigma, r-\sigma', r-\sigma') \times \Pi}^T \\ & \leq C \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\} \|X_U\|_{p, D(s, r, r) \times \Pi}^T \|X_V\|_{p, D(s, r, r) \times \Pi}^T, \end{aligned} \quad (\text{A.1})$$

where $C > 0$ is a constant depending on n .

Denote by X_U^t the flow of the Hamiltonian vector field of $U(x, y, z; \xi)$. It follows from Taylor's formula that

$$V \circ X_U^t(x, y, z; \xi) = \sum_{j \geq 0} \frac{t^j}{j!} V^{(j)}(x, y, z; \xi), \quad (\text{A.2})$$

where

$$V^{(0)}(x, y, z; \xi) := V(x, y, z; \xi), \quad V^{(j)}(x, y, z; \xi) := \{V^{(j-1)}, U\}(x, y, z; \xi).$$

Then based on (A.1) in Lemma A.1 and (A.2), we have the following estimate of symplectic transformation, which can be parallel proved following the proof of Theorem 3.3 in [9]:

Lemma A.2. Consider two Hamiltonians $U(x, y, z; \xi)$ and $V(x, y, z; \xi)$ satisfying p -tame property on the domain $D(s, r, r) \times \Pi$ for some $0 < s, r \leq 1$. Given $0 < \sigma < s$, $0 < \sigma' < r/2$, suppose

$$\|X_U\|_{p, D(s, r, r) \times \Pi}^T \leq \frac{1}{2A},$$

where

$$A = 4Ce \max \left\{ \frac{1}{\sigma}, \frac{r}{\sigma'} \right\}$$

and $C > 0$ is the constant given in (A.1) in Lemma A.1. Then for each $|t| \leq 1$, we have

$$\|X_{V \circ X_U^t}\|_{p, D(s-\sigma, r-\sigma', r-\sigma') \times \Pi}^T \leq 2\|X_V\|_{p, D(s, r, r) \times \Pi}^T.$$

The following lemma estimates p -tame norm of the solution of homological equation during KAM iterative procedure and normal form iterative procedure, which can be parallel proved following the proof of Theorem 3.4 in [9]:

Lemma A.3. Consider two Hamiltonians

$$U(x, y, z; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^{\bar{z}}} U^{\alpha\beta}(x; \xi) y^\alpha z^\beta$$

and

$$V(x, y, z; \xi) = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^{\bar{z}}} V^{\alpha\beta}(x; \xi) y^\alpha z^\beta.$$

Suppose $V(x, y, z; \xi)$ has p -tame property on the domain $D(s, r, r) \times \Pi$, i.e.

$$\|X_V\|_{p, D(s, r, r) \times \Pi}^T < \infty.$$

For each $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{\bar{\mathbb{Z}}}$, $k \in \mathbb{Z}^n$, $j \geq 1$ and some fixed constant $\tau > 0$, assume the following inequality holds

$$|\widehat{U^{\alpha\beta}}(k; \xi)| + |\partial_{\xi_j} \widehat{U^{\alpha\beta}}(k; \xi)| \leq (|k| + 1)^\tau (|\widehat{V^{\alpha\beta}}(k; \xi)| + |\partial_{\xi_j} \widehat{V^{\alpha\beta}}(k; \xi)|),$$

where $\widehat{U^{\alpha\beta}}(k; \xi)$ and $\widehat{V^{\alpha\beta}}(k; \xi)$ are the k -th Fourier coefficients of $U^{\alpha\beta}(x; \xi)$ and $V^{\alpha\beta}(x; \xi)$, respectively. Then, $U(x, y, z; \xi)$ has p -tame property on the domain $D(s - \sigma, r, r) \times \Pi$ for $0 < \sigma < s$. Moreover, we have

$$\|X_U\|_{p, D(s-\sigma, r, r) \times \Pi}^T \leq \frac{c}{\sigma^\tau} \|X_V\|_{p, D(s, r, r) \times \Pi}^T, \quad (\text{A.3})$$

where $c > 0$ is a constant depending on s and τ .

The following lemma compares p -tame norm with the usual weighted norm for Hamiltonian vector field, which can be parallel proved following the proof of Theorem 3.5 in [9]:

Lemma A.4. *Given a Hamiltonian*

$$U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\bar{\mathbb{Z}}}} U^\beta(x, y; \xi) z^\beta$$

satisfying p -tame property on the domain $D(s, r, r) \times \Pi$ for some $0 < s, r \leq 1$, then we have

$$\|X_U\|_{p, D(s, r, r) \times \Pi} \leq \|X_U\|_{p, D(s, r, r) \times \Pi}^T. \quad (\text{A.4})$$

Then based on (A.4) in Lemma A.4 and the proof of Lemma A.4 in [16], we have the following lemma:

Lemma A.5. *Suppose the Hamiltonian*

$$U(x, y, z; \xi) = \sum_{\beta \in \mathbb{N}^{\bar{\mathbb{Z}}}} U^\beta(x, y; \xi) z^\beta$$

has p -tame property on the domain $D(s, r, r) \times \Pi$ for some $0 < s, r \leq 1$. Let X_U^t be the phase flow generalized by the Hamiltonian vector field X_U . Given $0 < \sigma < s$ and $0 < \sigma' < r/2$, assume

$$\|X_U\|_{p, D(s, r, r) \times \Pi}^T < \min\{\sigma, \sigma'\}.$$

Then, for each $\xi \in \Pi$ and each $|t| \leq 1$, one has

$$\|X_U^t - id\|_{p, D(s-\sigma, r-\sigma', r-\sigma')} \leq \|X_U\|_{p, D(s, r, r) \times \Pi}^T, \quad (\text{A.5})$$

where

$$\|X_U^t - id\|_{p, D(s-\sigma, r-\sigma', r-\sigma')} = \sup_{w \in D(s-\sigma, r-\sigma', r-\sigma')} \|(X_U^t - id)w\|_{\mathcal{P}, D(s, r, r)}. \quad (\text{A.6})$$

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