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Perturbations of Lane–Emden and Hamilton–Jacobi equations II: Exterior domains

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Abstract

In this article we are interested in the existence of positive classical solutions of

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u + V(x)u = u^p + \gamma u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u + V(x)u = u^p + \gamma|\nabla u|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth exterior domain in $\mathbb{R}^N$ in the case of $N \ge 4$, $p > \frac{N+1}{N-3}$ and $\gamma \in \mathbb{R}$. We assume that V is a smooth nonnegative potential and $a(x)$ is a smooth vector field, both of which satisfy natural decay assumptions. Under suitable assumptions on q we prove the existence of an infinite number of positive classical solutions.

We also consider the case of $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ under further symmetry assumptions on Ω, a and V.

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1. Introduction

In this article we are interested in the following variants of the Lane–Emden and viscous Hamilton–Jacobi equations, on exterior domains, given by

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u + V(x)u = u^p + \gamma u^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

and

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u + V(x)u = u^p + \gamma |\nabla u|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $D \subset \mathbb{R}^N$ is a smooth bounded domain and $\Omega := \mathbb{R}^N \setminus \overline{D}$. We seek positive classical solutions which satisfy $\lim_{|x|\to\infty} u = 0$. The assumptions on a and V are given by

$(A1)$: $a(x)$ is a smooth vector field satisfying $\lim_{R\to\infty} A(R) = 0$ where $A(R) := \sup_{|x|\geq R} |x||a(x)|$,

$(A2)$: $V(x) \geq 0$ is a smooth potential satisfying $\lim_{R\to\infty} V(R) = 0$ where $V(R) := \sup_{|x|\geq R} |x|^2|V(x)|$.

By considering a suitable shift in a and V we can assume that $0 \in D$.

We begin by recalling the bounded domain version of (3) in the case of $a(x) = 0$, $V(x) = 0$ and $\gamma = 0$ given by

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. Define the critical exponent $p_s = \frac{N+2}{N-2}$ and note that it is related to the critical Sobolev imbedding exponent $2^* := \frac{2N}{N-2} = p_s + 1$. For $1 < p < p_s$, $H_0^1(\Omega)$ is compactly imbedded in $L^{p+1}(\Omega)$ and hence standard methods show the existence of a positive minimizer of

$$\min_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

This positive minimizer is a positive solution of (5), see for instance the book [19]. For $p \geq p_s$, $H_0^1(\Omega)$ is no longer compactly imbedded in $L^{p+1}(\Omega)$ and so to find positive solutions of (5) one needs to take other approach. For $p \geq p_s$ the well known Pohozaev identity [18] shows there are no positive solutions of (5) provided Ω is star shaped. For general domains in the critical/supercritical case, $p \geq p_s$, the existence versus nonexistence of positive solutions of (5) is a very delicate question; see for instance [4,9,17,10].

1.1. The full space problem

We now recall (3) in the case of $a(x) = 0$, $V(x) = 0$ and $\gamma = 0$ in the case of $\Omega = \mathbb{R}^N$. There has been much work done on the existence and nonexistence of positive classical solutions of

$$-\Delta w = w^p \qquad \text{in } \mathbb{R}^N. \tag{6}$$

As in the bounded domain case the critical exponent p_s plays a crucial role. For $1 < p < p_s$ there are no positive classical solutions of (6) and for $p \geq p_s$ there exist positive classical solutions, see [2,3,13,12]. The moving plane method shows that all positive classical solutions, satisfying certain assumptions, are radial about a point.

In [5] it was shown that there was a positive classical solution of (3) in the case of $\gamma = 0$ and $\Omega = \mathbb{R}^N$ provided $a(x)$ was smooth divergence free and satisfied a smallness assumption and $N \geq 4$ with $p > \frac{N+1}{N-3}$. Under the further assumption that $p > p_{JL}$ (the so called Joseph–Lundgren exponent; see [16,15,11] regarding p_{JL}) the solution was shown to be stable in some suitable sense. In [1] we considered (3) and (4) in the case of $\Omega = \mathbb{R}^N$ under the same assumptions on $p, a(x)$ and $V(x)$. Our approach in the existence portions of [5] and [1] was to use a linearization argument along with a fixed point argument in various spaces, to obtain positive solutions. Our starting point was the linear theory developed in Dávila–del Pino–Musso [6], see the next section for details. We also mention the work of Dávila–del Pino–Musso–Wei [7] where they examined $-\Delta u + V(x)u = u^p$ on $\mathbb{R}^N$.

The positive radial solution. For the remainder of the paper $w(r)$ will refer to an explicit solution of (6). For $p > \frac{N+2}{N-2}$ let $w = w(r)$ denote the positive radial decreasing solution of (6) with $w(0) = 1$. The asymptotics of w, as $r \to \infty$, are given by

$$w(r) = \beta^{\frac{1}{p-1}} r^{\frac{-2}{p-1}} (1 + o(1)),$$

where

$$\beta = \beta(p, N) = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right) > 0,$$

see [15] for this and for more detailed asymptotics.

1.2. The exterior problem

In Dávila–del Pino–Musso [6] they examined the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $\Omega = \mathbb{R}^N \setminus D$ where D is a bounded open connected domain in $\mathbb{R}^N$. Their interest was in the existence of positive classical solution of (7). They obtained a continuum of positive solutions when $p > \frac{N+1}{N-3}$. For $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ they obtained a similar result but they assumed a symmetry assumption on D. Define the linearized operator $L(\phi) := \Delta\phi + pw^{p-1}\phi$ associated with (6). The starting point for their analysis of (7) was to obtain various mapping properties of L on some weighted L^∞ spaces $\mathbb{R}^N$. They then needed to extend these linear estimates to the exterior space. For this set $L^\lambda(\phi)(x) := \Delta\phi(x) + pw_\lambda(x)^{p-1}\phi(x)$ where $0 < \lambda$ and $w_\lambda(x) := \lambda^\alpha w(\lambda x)$ where $\alpha := \frac{2}{p-1}$; note that w_λ is also a solution of (6). We omit their linear estimates on the full space and only mention their final linear estimates on the exterior domains. For this we first define some spaces. For $0 < \sigma$ we define $Y_\lambda := \{f \in C(\Omega) : \|f\|_{Y_\lambda} < \infty\}$ and $X_{\lambda,0} := \{\phi \in C(\Omega) : \phi = 0 \text{ on } \partial\Omega \text{ with } \|\phi\|_{X_{\lambda,0}} < \infty\}$ where

$$\|f\|_{Y_\lambda} := \lambda^\sigma \sup_{|x|\le\lambda^{-1}} |x|^{\sigma+2}|f(x)| + \lambda^\alpha \sup_{|x|\ge\lambda^{-1}} |x|^{\alpha+2}|f(x)|,$$

$$\|\phi\|_{X_{\lambda,0}} := \lambda^\sigma \sup_{|x|\le\lambda^{-1}} |x|^{\sigma}|\phi(x)| + \lambda^\alpha \sup_{|x|\ge\lambda^{-1}} |x|^{\alpha}|\phi(x)|.$$

Notation. Here and in the rest of the paper all supremums in the various norms are understood to be over $x \in \Omega$ along with the other stated assumptions. In addition recall that we are assuming that $0 \in D$ and hence there are no issues with the weights at the origin. We now come to their linear results.

Theorem A. *(See Dávila–del Pino–Musso [6].)*

1. *Suppose $N \ge 4$, $p > \frac{N+1}{N-3}$ and $0 < \sigma < N-2$. Then there exists some small $\lambda_0 > 0$ and some $C > 0$ such that for all $0 < \lambda < \lambda_0$, $f \in Y_\lambda$ there is some $\phi_\lambda \in X_{\lambda,0}$ such that $L^\lambda(\phi_\lambda) = f$ in Ω with $\phi_\lambda = 0$ on $\partial\Omega$ and $\|\phi_\lambda\|_{X_{\lambda,0}} \le C\|f\|_{Y_\lambda}$.*
2. *Suppose $N \ge 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$, $0 < \sigma < N-2$ and D satisfies $(A3)$ (see the text following Remark 1 for definition of $(A3)$). Then there exists some small $\lambda_0 > 0$ and some $C > 0$ such that for all $0 < \lambda < \lambda_0$, $f \in Y^e_\lambda$ (see Section 4 for definition of Y^e_λ and $X^e_{\lambda,i}$) there is some $\phi_\lambda \in X^e_{\lambda,0}$ such that $L^\lambda(\phi_\lambda) = f$ in Ω with $\phi_\lambda = 0$ on $\partial\Omega$ and $\|\phi_\lambda\|_{X_{\lambda,0}} \le C\|f\|_{Y_\lambda}$.*

To obtain a positive solutions of (7) they then applied a fixed point argument using their linear theory. We also mention the work of Dávila–del Pino–Musso–Wei [8] where they considered the exterior problem and considered both fast and slow decay solutions and they utilized the Lyapunov–Schmidt reduction method to obtain positive solutions of (7), for $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ under no symmetry assumptions on D.

1.3. The main results

We now state our main results.

Theorem 1. *Suppose $N \ge 4$, $p > \frac{N+1}{N-3}$, $q > p$ and $(A1)$, $(A2)$ are satisfied.*

1. *Suppose $\gamma \ge 0$ then there exists an infinite number of positive smooth solutions of (3).*
2. *Suppose $\gamma < 0$ and*

$$\|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\Omega)} < 2S_N,$$

where $(div(a) - 2V)_+$ is the positive part of $div(a) - 2V$ and S_N is the optimal constant in the critical Sobolev imbedding, see Lemma 6. Then there exists an infinite number of smooth positive solutions of (3).

Theorem 2. *Suppose $N \geq 4$, $p > \frac{N+1}{N-3}$, $\frac{2p}{p+1} < q < 2$ and $(A1)$, $(A2)$ are satisfied. Then there exists an infinite number of positive classical solutions of (4).*

Remark 1. We believe the restriction $q > \frac{2p}{p+1}$ in Theorem 2 is somewhat natural and is coming from the equation (4). The other restriction that $q < 2$ we believe is not natural and is mainly an artifact of the choice of function space we are working in. In our prior work [1] we examined (4) on $\mathbb{R}^N$ and in this work we also obtained a positive solution for (4) in the case of $\frac{2p}{p+1} < q < 2$. By considering alternate function spaces we were able to relax the assumption of $q < 2$; we are currently unable to extend these methods to exterior domains.

In our final result we consider (3) and (4) under the assumption that $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$. For our results here we need to impose some conditions on D, which we label $(A3)$: we assume $0 \in D \subset \mathbb{R}^N$ is smooth and bounded and for each $1 \leq i \leq N$ one has $x \in D \iff x^i \in D$ where $x^i := (x_1, x_2, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N)$.

We also define some symmetry assumptions on a and V. Define

$$(A4): \quad V(x^i) = V(x) \quad \forall x \in \Omega, \forall 1 \leq i \leq N.$$

For vector fields a we write $a(x) = (a^1(x), \dots, a^N(x))$ and we consider the symmetry assumption where we require for all $x \in \Omega$ that

$$(A5): \quad \text{for all } x \in \Omega \text{ one has } \; a^j(x^i) = \begin{cases} a^j(x) & i \neq j \\ -a^i(x) & i = j. \end{cases}$$

Theorem 3. *Suppose $N \geq 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and $(A1)$, $(A2)$, $(A3)$, $(A4)$ and $(A5)$ are satisfied.*

1. *Suppose $q > p$ and $\gamma \geq 0$. Then there exists an infinite number of positive smooth solutions of (3).*
2. *Suppose $\gamma < 0$,*

$$\|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\Omega)} < 2S_N,$$

and $q > p$. Then there exists an infinite number of smooth positive solutions of (3).

3. *Suppose $\frac{2p}{p+1} < q < 2$. Then there exists an infinite number of positive classical solutions of (4).*

2. Equation (3); $-\Delta u + a(x) \cdot \nabla u + V(x)u = u^p + \gamma u^q$

For our approach we need to adjust the spaces slightly. Define $X_{2,\lambda} := \{\phi \in C^1(\overline{\Omega}) : \Delta\phi \in C(\Omega)$ with $\phi = 0$ on $\partial\Omega$ and $\|\phi\|_{X_{2,\lambda}} < \infty\}$ where

$$\|\phi\|_{X_{\lambda,2}} := \lambda^\sigma \sup_{|x| \leq \lambda^{-1}} \Big(|x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla\phi(x)| + |x|^{\sigma+2} |\Delta\phi(x)| \Big)$$
$$+ \lambda^\alpha \sup_{|x| \geq \lambda^{-1}} \Big(|x|^\alpha |\phi(x)| + |x|^{\alpha+1} |\nabla\phi(x)| + |x|^{\alpha+2} |\Delta\phi(x)| \Big).$$

The first result we need is to extend the linear theory of Dávila–del Pino–Musso [6] to $X_{\lambda,2}$. This will follow directly from their estimates and a scaling argument.

Lemma 1. *Suppose $N \geq 4$ and $p > \frac{N+1}{N-3}$. Then for $0 < \sigma < N-2$ there exists some small $\lambda_0 > 0$ and some $C > 0$ such that for all $0 < \lambda < \lambda_0$ and $f \in Y_\lambda$ there is some $\phi_\lambda \in X_{\lambda,2}$ such that $L^\lambda(\phi_\lambda) = f$ in Ω with $\phi_\lambda = 0$ on $\partial\Omega$ and $\|\phi_\lambda\|_{X_{\lambda,2}} \leq C\|f\|_{Y_\lambda}$ (recall $L^\lambda(\phi) := \Delta\phi + pw_\lambda(x)^{p-1}\phi$).*

Proof. Fix R big enough such that $D \subset\subset B_R$ and let $0 < \sigma < N-2$, $\lambda_0 > 0$ and $C > 0$ be from the above Theorem A [6]. Fix $N < t < \infty$ and suppose $f \in Y_\lambda$. Then by Theorem A [6] there is some $\phi_\lambda \in X_{\lambda,0}$ such that $L^\lambda(\phi_\lambda) = f$ in Ω and we have $\|\phi_\lambda\|_{X_{\lambda,0}} \leq C\|f\|_{Y_\lambda}$. We will now apply regularity results to obtain the improved estimates. We first obtain gradient estimates and in doing so it will be convenient to introduce the following two regions:

$$(i)\ \Omega_{2R} = \Omega \cap B_{2R}, \qquad (ii)\ \{\,|x| \geq 2R\},$$

where $B_R := \{x \in \mathbb{R}^N : |x| < R\}$.

Before obtaining the estimates in the various regions the following collection of calculations will be helpful. Firstly note that

$$|f(x)| \leq \begin{cases} \frac{\|f\|_{Y_\lambda}}{\lambda^\sigma |x|^{\sigma+2}} & \text{if } |x| \leq \lambda^{-1} \\ \frac{\|f\|_{Y_\lambda}}{\lambda^\alpha |x|^{\alpha+2}} & \text{if } |x| \geq \lambda^{-1}, \end{cases} \quad \text{and } |\phi_\lambda(x)| \leq \begin{cases} \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma |x|^{\sigma}} & \text{if } |x| \leq \lambda^{-1} \\ \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha |x|^{\alpha}} & \text{if } |x| \geq \lambda^{-1}. \end{cases}$$

Also note that $w_\lambda(x) \leq \lambda^\alpha$ for $|x| \leq \lambda^{-1}$ and $w_\lambda(x) \leq C|x|^{-\alpha}$ for $|x| \geq \lambda^{-1}$, where C is independent of λ. We now consider the gradient estimates in the two regions.

Region (i). Using boundary elliptic regularity theory there is some $C = C(t, R) > 0$ such that

$$\sup_{\Omega \cap B_{2R}} |\nabla\phi_\lambda| \leq C \left(\int_{\Omega \cap B_{4R}} |\Delta\phi_\lambda(x)|^t dx \right)^{\frac{1}{t}} + C \int_{\Omega \cap B_{4R}} |\phi_\lambda(x)| dx. \tag{8}$$

By taking $0 < \lambda_0$ smaller, if necessary, we can assume that $4R < \frac{1}{\lambda_0}$ and then note there is some $C = C(R, D)$, where $\Omega = \mathbb{R}^N \backslash \overline{D}$, such that $|f(x)|, |\phi_\lambda(x)| \leq C\lambda^{-\sigma}\|f\|_{Y_\lambda}$ for all $0 < \lambda < \lambda_0$. Recalling that ϕ_λ satisfies $\Delta\phi_\lambda = f(x) - pw_\lambda^{p-1}(x)\phi_\lambda$ we see that $|\Delta\phi_\lambda(x)| \leq C\lambda^{-\sigma}\|f\|_{Y_\lambda}$ in Ω_{4R}. Putting these estimates into (8) we see that $\lambda^\sigma \sup_{\Omega_{2R}} |\nabla\phi_\lambda| \leq C\|f\|_{Y_\lambda}$ and hence we see that $\lambda^\sigma \sup_{\Omega_{2R}} |x|^{\sigma+1}|\nabla\phi_\lambda| \leq C\|f\|_{Y_\lambda}$.

Region (ii). For this region we consider the rescaled functions given by

$$\psi_\lambda(y) = \phi_\lambda(x + |x|y) \text{ where } |y| < \frac{1}{8},$$

which is well-defined (since $x+|x|y \geq 7|x|/4 > R$) and satisfies

$$\Delta\psi_\lambda(y) + p|x|^2 w_\lambda(x+|x|y)^{p-1}\psi_\lambda(y) = |x|^2 f(x+|x|y) \qquad |y| < \frac{1}{8}.$$

Note that if $|x+|x|y| \leq 1/\lambda$, then $7|x|/8 \leq 1/\lambda$ and $|x|^2 w_\lambda(x+|x|y)^{p-1} \leq \lambda^2|x|^2 \leq 64/49$; if $|x+|x|y| \geq 1/\lambda$, then $|x|^2 w_\lambda(x+|x|y)^{p-1} \leq |x|^2/|x+|x|y|^2 \leq 64/49$. The elliptic regularity theory gives

$$\begin{aligned}
\sup_{|y|<\frac{1}{16}} |\nabla\psi_\lambda(y)| &\leq C\left(|x|^2 \int\limits_{|y|<\frac{1}{8}} |f(x+|x|y)|^t dy\right)^{\frac{1}{t}} + C\int\limits_{|y|<\frac{1}{8}} |\psi_\lambda(y)|dy \\
&\leq C\left(\int\limits_{|y|<\frac{1}{8}} \Big(|x+|x|y|^2\,|f(x+|x|y)|\Big)^t dy\right)^{\frac{1}{t}} + C\int\limits_{|y|<\frac{1}{8}} |\phi_\lambda(x+|x|y)|dy.
\end{aligned} \tag{9}$$

Now for each $|x| \geq 2R$, divide $|y| < 1/8$ into two sets A_1 and A_2 such that $A_1 = \{|y| < 1/8 : |x+|x|y| \leq 1/\lambda\}$ and $A_2 = \{|y| < 1/8 : |x+|x|y| > 1/\lambda\}$. Note that the dependence of A_1 and A_2 on x is suppressed, and A_2 can be empty if $|x| \leq 1/\lambda$ and A_1 can be empty if $|x| > 1/\lambda$. Then $|x+|x|y|^2|f(x+|x|y)| \leq \frac{\|f\|_{Y_\lambda}}{\lambda^\sigma|x+|x|y|^\sigma} \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma|x|^\sigma}$ for $y \in A_1$ and $|x+|x|y|^2|f(x+|x|y)| \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha|x+|x|y|^\alpha} \leq \frac{\|f\|_{Y_\lambda}}{\lambda^\alpha|x|^\alpha}$ for $y \in A_2$. Similarly, we get $|\phi_\lambda(x+|x|y)| \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma|x|^\sigma}$ for $y \in A_1$ and $|\phi_\lambda(x+|x|y)| \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha|x|^\alpha}$ for $y \in A_2$. Using these estimates we have, for $2R \leq |x| \leq 1/\lambda$,

$$\begin{aligned}
\left(\int\limits_{|y|<\frac{1}{8}} \Big(|x+|x|y|^2\,|f(x+|x|y)|\Big)^t dy\right)^{1/t} &\leq \left(\int\limits_{A_1} \Big(|x+|x|y|^2\,|f(x+|x|y)|\Big)^t dy\right)^{1/t} \\
&\quad + \left(\int\limits_{A_2} \Big(|x+|x|y|^2\,|f(x+|x|y)|\Big)^t dy\right)^{1/t} \\
&\leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma|x|^\sigma} + \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha|x|^\alpha} \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma|x|^\sigma},
\end{aligned}$$

where in the last equality we used $8/9 \leq |x|\lambda \leq 1$ for $y \in A_2$. Similarly we get $\int_{|y|<\frac{1}{8}} |\phi(x+|x|y)|dy \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma|x|^\sigma}$ for $2R \leq |x| \leq 1/\lambda$. The same argument together with $1 \leq |x|\lambda \leq 8/7$ for $|x| > 1/\lambda$ and $x \in A_1$ yields, for $|x| > 1/\lambda$,

$$\left(\int_{|y|<\frac{1}{8}} \left(|x + |x|y|^2 \, |f(x + |x|y)| \right)^t dy \right)^{1/t} \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha |x|^\alpha}, \qquad \int_{|y|<\frac{1}{8}} |\phi(x + |x|y)| dy \leq \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha |x|^\alpha}.$$

Therefore, it follows from (9) that

$$\sup_{|y|<\frac{1}{16}} |\nabla\psi_\lambda(y)| \leq \begin{cases} \frac{C\|f\|_{Y_\lambda}}{\lambda^\sigma |x|^\sigma} & \text{if } 2R \leq |x| \leq 1/\lambda, \\ \frac{C\|f\|_{Y_\lambda}}{\lambda^\alpha |x|^\alpha} & \text{if } |x| > 1/\lambda. \end{cases}$$

From this we get

$$\lambda^\sigma \sup_{2R<|x|\leq\frac{1}{\lambda}} |x|^{\sigma+1} |\nabla\phi_\lambda(x)| \leq C\|f\|_{Y_\lambda}, \qquad \lambda^\alpha \sup_{|x|>\frac{1}{\lambda}} |x|^{1+\alpha} |\nabla\phi_\lambda(x)| \leq C\|f\|_{Y_\lambda}.$$

Combining the estimates in Regions 1 and 2 gives the desired estimate for $|\nabla\phi_\lambda|$:

$$\lambda^\sigma \sup_{|x|\leq\frac{1}{\lambda}} |x|^{\sigma+1} |\nabla\phi_\lambda(x)| + \lambda^\alpha \sup_{|x|>\frac{1}{\lambda}} |x|^{1+\alpha} |\nabla\phi_\lambda(x)| \leq C\|f\|_{Y_\lambda}.$$

The norm estimates involving the term $|\Delta\phi_\lambda|$ come directly from the equation. □

The right inverse of L^λ. For N, p, σ and λ_0 as in Lemma 1 we define the right inverse of L^λ to be F^λ where $F^\lambda(f) = \phi_\lambda$ where f and ϕ_λ are as in the lemma. Define $\tilde{X}_{\lambda,2} := Ran(F_\lambda)$. Using the continuity of F^λ and L^λ one can easily prove that $\tilde{X}_{\lambda,2}$ is a closed subspace of $X_{\lambda,2}$. Now note that $L^\lambda : \tilde{X}_{\lambda,2} \to Y_\lambda$ is, continuous, one to one and onto and hence its Fredholm index is zero.

2.1. *The linear theory of $L_\lambda(\phi) := L^\lambda(\phi) - a(x) \cdot \nabla\phi - V(x)\phi : X_{\lambda,2} \to Y_\lambda$*

To examine (3) and (4) we need to obtain a linear theory for L_λ where $L_\lambda(\phi)(x) := \Delta\phi(x) + pw_\lambda(x)^{p-1}\phi(x) - a(x) \cdot \nabla\phi(x) - V(x)\phi(x) = L^\lambda(\phi)(x) - T(x)$. Our approach will be to view L_λ as a compact perturbation of L^λ and then to use Fredholm theory. We begin with showing that T is a compact operator.

Lemma 2. *$T : X_{\lambda,2} \to Y_\lambda$ is a compact operator for each* $0 < \lambda$.

Proof. Fix $0 < \lambda$ and set $T(\phi) = T^1(\phi) + T^2(\phi)$ where $T^1(\phi)(x) = a(x) \cdot \nabla\phi(x)$ and $T^2(\phi)(x) = V(x)\phi(x)$. We show T^1 is compact and the proof that T^2 is compact follows the same approach. Let $\{\phi_m\}_m$ denote a bounded sequence in $X_{\lambda,2}$, bounded by say C_0, and note that elliptic regularity shows that $\{\phi_m\}_m$ is bounded in $C^{1,\frac{3}{4}}_{loc}(\Omega \cup \partial\Omega)$. By a compactness and diagonal argument there is some subsequence $\{\phi_{m_k}\}_k$ which is convergent in $C^{1,\frac{1}{2}}_{loc}(\Omega \cup \partial\Omega)$. Let $R > \frac{1}{\lambda}$ and then note

$$\|T^1(\phi_{m_k}) - T^1(\phi_{m_n})\|_{Y_\lambda} = \lambda^\sigma \sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma} |a(x)\cdot\nabla(\phi_{m_k}(x) - \phi_{m_n}(x))| \\ + \lambda^\alpha \sup_{|x|\ge\lambda^{-1}} |x|^{\alpha+2} |a(x)\cdot\nabla(\phi_{m_k}(x) - \phi_{m_n}(x))|.$$

We now break this second term into a supremum for $\lambda^{-1} \le |x| \le R$ and $|x| \ge R$. We then get an inequality of the form

$$\begin{aligned}\|T^1(\phi_{m_k}) - T^1(\phi_{m_n})\|_{Y_\lambda} &\le C(\lambda) \sup_{|x|\le R} |\nabla(\phi_{m_k}(x) - \phi_{m_n}(x))| \\ &\quad + \sup_{|x|\ge R} (|a(x)||x|) \left(|x|^{\alpha+1}|\nabla\phi_{m_k}(x) - \nabla\phi_{m_n}(x)|\right), \\ &\le C(\lambda) \sup_{|x|\le R} |\nabla(\phi_{m_k}(x) - \phi_{m_n}(x))| \\ &\quad + A(R)\|\phi_{m_k} - \phi_{m_n}\|_{X_{\lambda,2}} \\ &\le C(\lambda) \sup_{|x|\le R} |\nabla(\phi_{m_k}(x) - \phi_{m_n}(x))| \\ &\quad + A(R)2C_0.\end{aligned}$$

Hence we see that $\limsup_{k,n\to\infty} \|T^1(\phi_{m_k}) - T^1(\phi_{m_n})\|_{Y_\lambda} \le 2C_0 A(R)$ and then sending $R \to \infty$ shows that $\{T^1(\phi_k)\}_k$ is Cauchy in Y_λ and hence $T^1 : X_{\lambda,2} \to Y_\lambda$ is compact. □

As noted above $L^\lambda : \tilde{X}_{\lambda,2} \to Y_\lambda$ has Fredholm index zero and since T is compact we can apply Fredholm theory to see that $L_\lambda = L^\lambda - T : \tilde{X}_{\lambda,2} \to Y_\lambda$ is also Fredholm index zero.

The following proposition is the key linear result needed later when we prove existence of solutions to (3) and (4) using a fixed point argument. Additionally our approach for this perturbed linearized operator L_λ theory differs from [7] (where they studied $-\Delta u + V(x)u = u^p$ in $\mathbb{R}^N$) in the sense that we utilize some Liouville theorems (of course we utilize their [6,7] linear theory regarding L^λ as mentioned before).

Proposition 1. *Let $N \ge 4$, $p > \frac{N+1}{N-3}$, $0 < \sigma < N-2$ and suppose (A1) and (A2) are satisfied. Then there are some $\lambda_0 > 0$ small and $C > 0$ such that for all $0 < \lambda < \lambda_0$ and all $f \in Y_\lambda$ there is some $\phi_\lambda \in \tilde{X}_{\lambda,2}$ such that $L_\lambda(\phi_\lambda) = f$ in Ω. Moreover we have $\|\phi_\lambda\|_{X_{\lambda,2}} \le C\|f\|_{Y_\lambda}$.*

Proof. Suppose $N \ge 4$ and $p > \frac{N+1}{N-3}$ and let $0 < \sigma < N-2$ and $\lambda_0 > 0$ be small from Lemma 1.

We now suppose the conclusion of the proposition is false and so there is some $\lambda_m \searrow 0$ such that either kernel $L_{\lambda_m} : \tilde{X}_{\lambda_m,2} \to Y_{\lambda_m}$ is non-empty or its empty but there are some $f_m \in Y_{\lambda_m}$ with $\|f_m\|_{Y_{\lambda_m}} \to 0$ and $\phi_m \in \tilde{X}_{\lambda_m,2}$ such that $L_{\lambda_m}(\phi_m) = f_m$ and $\|\phi_m\|_{X_{\lambda_m,2}} = 1$. So in either case we can assume there is some $f_m \in Y_{\lambda_m}$ with $\|f_m\|_{Y_{\lambda_m}} \to 0$ and some $\phi_m \in \tilde{X}_{\lambda_m,2}$ with $\|\phi_m\|_{X_{\lambda_m,2}} = 1$ such that $L_{\lambda_m}(\phi_m) = f_m$ in Ω with $\phi_m = 0$ on $\partial\Omega$.

So we have

$$\Delta\phi_m(x) + p w_{\lambda_m}(x)^{p-1}\phi_m(x) - a(x)\cdot\nabla\phi_m(x) - V(x)\phi_m(x) = f_m(x) \quad \text{in} \quad \Omega,$$

with $\phi_m = 0$ on $\partial\Omega$. Now set $\hat{f}_m(x) := \lambda_m^\sigma f_m(x)$ and set $\hat{\phi}_m(x) = \lambda_m^\sigma \phi_m(x)$. Then $\hat{f}_m \to 0$ uniformly on any $B_R \cap \Omega$. Also we have

$$|\hat{\phi}_m(x)| \le \frac{\|\phi_m\|_{X_{\lambda_m,0}}}{|x|^\sigma}$$

for all $|x| \le \frac{1}{\lambda_m}$. So we have

$$\begin{cases} \Delta\hat{\phi}_m + p\lambda_m^2 w(\lambda_m x)^{p-1}\hat{\phi}_m - a(x)\cdot\nabla\hat{\phi}_m - V(x)\hat{\phi}_m(x) = \hat{f}_m & \text{in } \Omega, \\ \hat{\phi}_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Let $R_k \nearrow \infty$ and set $\Omega_k := \Omega \cap B_{R_k}$. Using (10) and elliptic boundary regularity shows that $\hat{\phi}_m$ is bounded in $C^{1,\delta}(\Omega_k)$ for all k and large m for some $0 < \delta < 1$. By a compactness and diagonal argument there is some subsequence, which we won't rename, $\{\hat{\phi}_m\}_m$, that converges in $C^{1,\delta}(\Omega_k)$, for each k, to some function $\hat{\phi} : \Omega \to \mathbb{R}$ which satisfies $|\hat{\phi}(x)| \le |x|^{-\sigma}$ on Ω and

$$\begin{cases} \Delta\hat{\phi}(x) - a(x)\cdot\nabla\hat{\phi}(x) - V(x)\hat{\phi}(x) = 0 & \text{in } \Omega, \\ \hat{\phi} = 0 & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty}\hat{\phi}(x) = 0. \end{cases} \tag{11}$$

By the strong maximum principle applied to the subdomain Ω_R, for R large, we can conclude that $\sup_{\Omega_R}|\hat{\phi}| \le \sup_{\partial\Omega_R}|\hat{\phi}|$ but after considering the decay of $\hat{\phi}$ we can conclude that $\hat{\phi} = 0$ in Ω. Hence we can conclude that $\hat{\phi}_m \to 0 = \hat{\phi}$ in $C^{1,\delta}(\Omega_k)$ for each $k \ge 1$.

Now recall that ϕ_m satisfies

$$\begin{cases} L^{\lambda_m}(\phi_m) = f_m + a\cdot\nabla\phi_m + V\phi_m & \text{in } \Omega, \\ \phi_m = 0 & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty}\phi_m = 0. \end{cases} \tag{12}$$

By Lemma 1 there is some $C > 0$ such that for all $0 < \lambda < \lambda_0$ we have

$$C\|\phi_m\|_{X_{\lambda_m,2}} \le \|f_m\|_{Y_{\lambda_m}} + \|a\cdot\nabla\phi_m + V\phi_m\|_{Y_{\lambda_m}}.$$

We now examine this last term. Fix $R > 0$ large, then

$$\begin{aligned} \|a\cdot\nabla\phi_m + V\phi_m\|_{Y_{\lambda_m}} &\le \lambda_m^\sigma \sup_{|x|\le\lambda_m^{-1}} |x|^{\sigma+2}\left(|a(x)||\nabla\phi_m(x)| + V(x)|\phi_m(x)|\right) \\ &\quad + \lambda_m^\alpha \sup_{|x|\ge\lambda_m^{-1}} |x|^{\alpha+2}\left(|a(x)||\nabla\phi_m(x)| + V(x)|\phi_m(x)|\right) \\ &=: I_m + J_m. \end{aligned}$$

A computation shows that

$$\begin{aligned}
J_m &\leq \lambda^\alpha \sup_{|x|\geq\lambda_m^{-1}} |x||a(x)||x|^{\alpha+1}|\nabla\phi_m(x)| \\
&\quad + \lambda^\alpha \sup_{|x|\geq\lambda_m^{-1}} |x|^2 V(x)|x|^\alpha|\phi_m(x)| \\
&\leq \left(A(\lambda_m^{-1}) + V(\lambda_m^{-1})\right)\|\phi_m\|_{X_{\lambda_m,1}} \\
&\leq A(\lambda_m^{-1}) + V(\lambda_m^{-1}) \to 0.
\end{aligned}$$

Fix R big and decompose $I_m = I_m^1 + I_m^2$ where I_m^1 will be the inner portion and I_m^2 the outer. Then

$$\begin{aligned}
I_m^2 &:= \lambda_m^\sigma \sup_{R\leq|x|\leq\lambda_m^{-1}} |x|^{2+\sigma}\left(|a(x)||\nabla\phi_m(x)| + V(x)|\phi_m(x)|\right) \\
&\leq \lambda_m^\sigma \sup_{R\leq|x|\leq\lambda_m^{-1}} \left(|x||a(x)||x|^{1+\sigma}|\nabla\phi_m(x)| + |x|^2V(x)|x|^\sigma|\phi_m(x)|\right) \\
&\leq A(R) + V(R).
\end{aligned}$$

We now come to the I_m^1 term.

$$\begin{aligned}
I_m^1 &= \lambda_m^\sigma \sup_{|x|\leq R} |x|^{2+\sigma}\left(|a(x)||\nabla\phi_m(x)| + V(x)|\phi_m(x)|\right) \\
&\leq \sup_{z\in\mathbb{R}^N} |z||a(z)| \sup_{|x|\leq R} \lambda_m^\sigma|x|^{1+\sigma}|\nabla\phi_m(x)| \\
&\quad + \sup_{z\in\mathbb{R}^N} |z|^2V(z) \sup_{|x|\leq R} |x|^\sigma\lambda_m^\sigma|\phi_m(x)| \\
&\leq CR^{1+\sigma}\sup_{\Omega_R}\left(|\nabla\hat\phi_m(x)| + |\hat\phi_m(x)|\right) \to 0
\end{aligned}$$

for each fixed R big as $m \to \infty$. So combining the above results we have

$$C\|\phi_m\|_{X_{\lambda_m,2}} \leq \|f_m\|_{Y_{\lambda_m}} + A(\lambda_m^{-1}) + V(\lambda_m^{-1}) + A(R) + V(R) + I_m^1,$$

and from this we can contradict the fact that $\|\phi_m\|_{X_{\lambda_m,2}} = 1$ by taking R sufficiently big and then sending $m \to \infty$. $\square$

2.2. *Equation (3); the fixed point argument*

Instead of solving (3) directly we will first find a nonzero solution of

$$\begin{cases} -\Delta u + a(x)\cdot\nabla u + V(x)u = |u|^p + \gamma|u|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

and then argue the solution must be positive.

Let $D \subset\subset B_{R_0}$ and let $R_0 \le R \le 4R \le \frac{1}{\lambda}$. Let ζ denote a smooth radial cut-off with $\zeta = 0$ in B_R and $\zeta = 1$ on B_{2R}^c. Then we have $|\nabla\zeta| \le CR^{-1}$ and $|\Delta\zeta| \le CR^{-2}$ where C is independent of R.

We look for solutions to (13) of the form $u = \zeta(x)w_\lambda(x) + \phi(x)$. Then we need ϕ to satisfy

$$\begin{aligned} L_\lambda(\phi) &= |w_\lambda + \phi|^p - |\zeta w_\lambda + \phi|^p \\ &\quad - \left(|w_\lambda + \phi|^p - p w_\lambda^{p-1}\phi - w_\lambda^p\right) \\ &\quad - \gamma|w_\lambda\zeta + \phi|^q \\ &\quad + a(x)\cdot\nabla(\zeta w_\lambda) + V w_\lambda \zeta \\ &\quad + \Delta w_\lambda - \Delta(\zeta w_\lambda) \\ &= I_1(\phi) + I_2(\phi) + I_3(\phi) + I_4(\phi) + I_5(\phi) \quad \text{in } \Omega, \end{aligned} \tag{14}$$

with $\phi = 0$ on $\partial\Omega$. To obtain a solution ϕ we apply a fixed point argument and towards this we define the nonlinear mapping $J_\lambda(\phi) = \psi_\lambda$ where ψ_λ satisfies

$$L_\lambda(\psi_\lambda) = \sum_{k=1}^{5} I_k(\phi) \ \text{ in } \Omega, \qquad \psi_\lambda = 0 \ \text{ on } \partial\Omega. \tag{15}$$

Of course to find a solution ψ_λ we will require N, p, σ, λ_0 to be as in Proposition 1 and $0 < \lambda < \lambda_0$. In addition we will be taking $\sigma > 0$ smaller, if necessary, to ensure various quantities are of a specific sign. Also we will need the right hand side of (15) to belong to Y_λ. In a moment we will apply a fixed point argument on the closed ball B_r of radius r, centered at the origin, in $X_{\lambda,2}$. We will end up taking R, related to the cut off ζ, to be given by $R = \frac{\varepsilon}{\lambda}$ and $r = \beta\lambda^\alpha$ where $\varepsilon, \beta > 0$ will be chosen small to be determined later. Once these parameters are fixed we will take λ small. We now collect the various estimates which will be needed to show that J_λ is a contraction on B_r. We begin with the following lemma.

Lemma 3. ***Into.*** *(Estimates on $\|I_k(\phi)\|_{Y_\lambda}$.) Let $\phi \in B_r \subset X_{\lambda,2}$. Then we have*

$$\|I_1(\phi)\|_{Y_\lambda} \le CR^{\sigma+2}\lambda^{\alpha p+\sigma} + CR^{2+\sigma(2-p)}r^{p-1}\lambda^{\alpha+\sigma(2-p)}, \tag{16}$$

$$\|I_2(\phi)\|_{Y_\lambda} \le Cr^2\lambda^{2-\alpha} + Cr^2\lambda^{-\alpha} + Cr^p\lambda^{-2}, \tag{17}$$

$$\|I_3(\phi)\|_{Y_\lambda} \le C\left(\lambda^{\theta_1} + \frac{r^q}{\lambda^{2+\alpha}}\right)\lambda^\alpha, \tag{18}$$

$$\|I_4(\phi)\|_{Y_\lambda} \le C\left(\frac{A(R)}{R\lambda} + A(R) + V(R) + A(\lambda^{-1}) + V(\lambda^{-1})\right)\lambda^\alpha, \tag{19}$$

$$\|I_5(\phi)\|_{Y_\lambda} \le C\lambda^\sigma\left(R^{2+\sigma}\lambda^2 + R^\sigma\right)\lambda^\alpha. \tag{20}$$

Proof. We begin by listing some computations: $\lambda^\sigma|x|^\sigma|\phi(x)| \le r$ for $|x| \le \lambda^{-1}$, $\lambda^\alpha|x|^\alpha|\phi(x)| \le r$ for $|x| \ge \lambda^{-1}$. Recalling $w_\lambda(x) = \lambda^\alpha w(\lambda x)$ one has: $w_\lambda(x) \le \lambda^\alpha$ for $|x| \le \lambda^{-1}$ and $w_\lambda(x) \le C|x|^{-\alpha}$ for $|x| \ge \lambda^{-1}$.

Estimate of $\|I_1(\phi)\|_{Y_\lambda}$. From part 3 of Lemma 7 we have

$$|I_1(\phi)| := \left| |w_\lambda + \phi|^p - |\zeta w_\lambda + \phi|^p \right| \le C\left(w_\lambda^{p-1} + |\phi|^{p-1} \right)(1-\zeta) w_\lambda.$$

From this we see

$$\sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma}|I_1(\phi)| \le \sup_{|x|\le 2R} C|x|^{\sigma+2} w_\lambda^p + C \sup_{|x|\le 2R} |x|^{\sigma+2} w_\lambda |\phi|^{p-1}.$$

So we have

$$\sup_{|x|\le\lambda^{-1}} |x|^{\sigma+2}|I_1(\phi)| \le C R^{\sigma+2}\lambda^{\alpha p} + C R^{2+\sigma(2-p)} r^{p-1} \lambda^{\alpha-\sigma(p-1)}.$$

Now noting the fact that $I_1(\phi) = 0$ for $|x| \ge 2R$ gives

$$\|I_1(\phi)\|_{Y_\lambda} \le C R^{\sigma+2}\lambda^{\alpha p+\sigma} + C R^{2+\sigma(2-p)} r^{p-1}\lambda^{\alpha+\sigma(2-p)}.$$

Estimate of $\|I_2(\phi)\|_{Y_\lambda}$. From Lemma 7 we have

$$\left| |w_\lambda + \phi|^p - p w_\lambda^{p-1}\phi - w_\lambda^p \right| \le C w_\lambda^{p-2}\phi^2 + C|\phi|^p.$$

A computation shows that

$$\sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma} w_\lambda^{p-2}\phi^2 \le C r^2 \lambda^{\alpha(p-2)-\sigma-2} = C r^2 \lambda^{-\alpha-\sigma}, \quad \text{and}$$

$$\sup_{|x|\ge\lambda^{-1}} |x|^{2+\alpha} w_\lambda^{p-2}\phi^2 \le \frac{C r^2}{\lambda^{2\alpha}}.$$

Hence we have

$$\|w_\lambda^{p-2}\phi^2\|_{Y_\lambda} \le C r^2 \lambda^{-\alpha}.$$

Similarly

$$\sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma}|\phi|^p \le C r^p \lambda^{-2-\sigma} \text{ and } \sup_{|x|\ge\lambda^{-1}} |x|^{2+\alpha}|\phi(x)|^p \le r^p\lambda^{-\alpha p},$$

and hence $\||\phi|^p\|_{Y_\lambda} \le C r^p \lambda^{-2}$. Combining these estimates gives

$$\|I_2(\phi)\|_{Y_\lambda} \le C r^2 \lambda^{\alpha(p-2)} + C r^2 \lambda^{-\alpha} + C r^p \lambda^{-2}.$$

Estimate of $\|I_3(\phi)\|_{Y_\lambda}$. First note that we have $|\zeta w_\lambda + \phi|^q \le C\zeta^q w_\lambda^q + C|\phi|^q$. For $|x| \le \frac{1}{\lambda}$.

$$\sup_{|x|\le\lambda^{-1}} |x|^{\sigma+2} w_\lambda^q |\zeta|^q \le C\lambda^{\alpha q-\sigma-2}.$$

Taking $\sigma > 0$ sufficiently small such that $\sigma + 2 - \sigma q > 0$, we then have $\sup_{|x| \le \lambda^{-1}} |x|^{2+\sigma} |\phi|^q \le \frac{Cr^q}{\lambda^{2+\sigma}}$. We now consider the case of $|x| \ge \frac{1}{\lambda}$. Since $q > p$ we have that $\theta_1 := \alpha q - 2 - \alpha > 0$ and then note that

$$\sup_{|x| \ge \lambda^{-1}} |x|^{2+\alpha} \zeta^q w_\lambda(x)^q \le C\lambda^{\theta_1}.$$

Also we have

$$\sup_{|x| \ge \lambda^{-1}} |x|^{2+\alpha} |\phi|^q \le \sup_{|x| \ge \lambda^{-1}} \frac{r^q}{\lambda^{\alpha q} |x|^{\alpha q - 2 - \alpha}} \le Cr^q \lambda^{-2-\alpha},$$

for $\theta_1 > 0$. So for $\theta_1 > 0$ we have

$$\| I_3(\phi) \|_{Y_\lambda} \le C \left(\lambda^{\theta_1} + \frac{r^q}{\lambda^{2+\alpha}} \right) \lambda^\alpha.$$

Estimate of $\| I_4(\phi) \|_{Y_\lambda}$. First we note that

$$|I_4(\phi)| \le |a| |\nabla \zeta| w_\lambda + |a| \zeta |\nabla w_\lambda| + V w_\lambda \zeta,$$

and we now estimate these three terms individually, but we first recall some estimates: $|\nabla \zeta| \le CR^{-1}$ for $R \le |x| \le 2R$ and $|\nabla w_\lambda(x)| = \lambda^{\alpha+1} |\nabla w(\lambda x)| \le C\lambda^{\alpha+2} |x|$ for all $|x| \le \lambda^{-1}$. In addition we have $|\nabla w_\lambda(x)| \le C|x|^{-\alpha-1}$ for all $|x| \ge \lambda^{-1}$. With these estimates in mind, and after recalling the support of ζ, and the decay estimates on a and V, one sees

$$\sup_{|x| \le \lambda^{-1}} |x|^{2+\sigma} |a| |\nabla \zeta| w_\lambda \le C \frac{A(R)}{R} \lambda^{\alpha - 1 - \sigma}, \quad \text{and} \quad \sup_{|x| \le \lambda^{-1}} |x|^{2+\sigma} |a(x)| \zeta |\nabla w_\lambda| \le CA(R) \lambda^{\alpha - \sigma}.$$

Similarly we see $\sup_{|x| \le \lambda^{-1}} |x|^{2+\sigma} V \zeta w_\lambda \le CV(R) \lambda^{\alpha - \sigma}$. We now consider the case of $|x| \ge \frac{1}{\lambda}$. A computation shows

$$\sup_{|x| \ge \lambda^{-1}} |x|^{2+\alpha} |a| \zeta |\nabla w_\lambda| \le \sup_{|x| \ge \lambda^{-1}} C|x| |a(x)| \le CA(\lambda^{-1}), \text{ and } \sup_{|x| \ge \lambda^{-1}} |x|^{2+\alpha} |a| \zeta V \le CV(\lambda^{-1}).$$

Combining the estimates gives

$$\| I_4(\phi) \|_{Y_\lambda} \le C \left(\frac{A(R)}{R\lambda} + A(R) + V(R) + A(\lambda^{-1}) + V(\lambda^{-1}) \right) \lambda^\alpha.$$

Estimate of $\| I_5(\phi) \|_{Y_\lambda}$.

$$I_5(\phi) = \Delta w_\lambda - (\Delta \zeta) w_\lambda - 2 \nabla \zeta \cdot \nabla w_\lambda - \zeta \Delta w_\lambda$$

and so

$$I_5(\phi) = (\zeta - 1) w_\lambda^p - (\Delta \zeta) w_\lambda - 2 \nabla \zeta \cdot \nabla w_\lambda.$$

First consider $|x| \le \frac{1}{\lambda}$. A computation shows

$$\sup_{|x|\le\lambda^{-1}} |x|^{\sigma+2}|\zeta-1|w_\lambda^p \le CR^{2+\sigma}\lambda^{\alpha p}, \text{ and } \sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma}|\Delta\zeta|w_\lambda \le CR^\sigma\lambda^\alpha.$$

Similarly we show $\sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma}|\nabla\zeta||\nabla w_\lambda| \le CR^{2+\sigma}\lambda^{\alpha+2}$. Note that $I_5(\phi)=0$ for $|x| \ge \frac{1}{\lambda}$ after the considering the support of ζ. Combining the estimates gives

$$\|I_5(\phi)\|_{Y_\lambda} \le C\lambda^\sigma \left(R^{2+\sigma}\lambda^{\alpha p-\alpha} + R^\sigma + R^{2+\sigma}\lambda^2\right)\lambda^\alpha. \qquad \square$$

We now collect the various facts for showing that J_λ is a contraction on $B_r \subset X_{\lambda,2}$.

Lemma 4. ***Contraction.*** *(Estimates on $\|I_k(\hat\phi)-I_k(\phi)\|_{Y_\lambda}$.) Let $\hat\phi, \phi \in B_r \subset X_{\lambda,2}$. Then we have*

$$\|I_1(\hat\phi)-I_1(\phi)\|_{Y_\lambda} \le C\left(2\lambda^{-\sigma(p-1)}r^{p-1}R^{2-\sigma(p-1)} + 4R^2\lambda^2\right)\|\hat\phi-\phi\|_{X_{\lambda,2}}, \tag{21}$$

$$\|I_2(\hat\phi)-I_2(\phi)\|_{Y_\lambda} \le C(\lambda^{-\alpha}r + \lambda^{-2}r^{p-1})\|\hat\phi-\phi\|_{X_{\lambda,2}}, \tag{22}$$

$$\|I_3(\hat\phi)-I_3(\phi)\|_{Y_\lambda} \le C(\lambda^{\theta_1} + \lambda^{-2}r^{q-1})\|\hat\phi-\phi\|_{X_{\lambda,2}}. \tag{23}$$

Proof. Let $\hat\phi, \phi \in B_r \subset X_{\lambda,2}$. Then as in the proof of the previous lemma we have $\lambda^\sigma|x|^\sigma|\phi(x)| \le r$ for $|x| \le \lambda^{-1}$, $\lambda^\alpha|x|^\alpha|\phi(x)| \le r$ for $|x| \ge \lambda^{-1}$ along with the analogous statement of $\hat\phi$. Additionally we have $w_\lambda(x) \le \lambda^\alpha$ for $|x| \le \lambda^{-1}$ and $w_\lambda(x) \le C|x|^{-\alpha}$ for $|x| \ge \lambda^{-1}$.

Estimate of $\|I_1(\hat\phi)-I_1(\phi)\|_{Y_\lambda}$. Note we can write

$$I_1(\hat\phi)-I_1(\phi) = |w_\lambda+\hat\phi|^p - |w_\lambda+\phi|^p + |\zeta w_\lambda+\phi|^p - |\zeta w_\lambda+\hat\phi|^p,$$

and note for $|x| \ge 2R$ this quantity is zero. So we can estimate

$$\begin{aligned}\|I_1(\hat\phi)-I_1(\phi)\|_{Y_\lambda} \le{}& \lambda^\sigma \sup_{|x|\le 2R} |x|^{2+\sigma}\left||w_\lambda+\hat\phi|^p - |w_\lambda+\phi|^p\right| \\ &+ \lambda^\sigma \sup_{|x|\le 2R} |x|^{2+\sigma}\left||\zeta w_\lambda+\hat\phi|^p - |\zeta w_\lambda+\phi|^p\right|.\end{aligned}$$

By Lemma 7 we have

$$\left||w_\lambda+\hat\phi|^p - |w_\lambda+\phi|^p\right| \le C\left(w_\lambda^{p-1} + |\phi|^{p-1} + |\hat\phi|^{p-1}\right)|\hat\phi-\phi|.$$

From this we see

$$\lambda^\sigma \sup_{|x|\le 2R} |x|^{2+\sigma}\left||w_\lambda+\hat\phi|^p - |w_\lambda+\phi|^p\right| \le \sup_{|x|\le 2R} |x|^2\left(w_\lambda^{p-1} + |\phi|^{p-1} + |\hat\phi|^{p-1}\right)\|\hat\phi-\phi\|_{X_{\lambda,2}}.$$

A computation shows that $\sup_{|x|\le 2R}|x|^2|\phi|^{p-1}\le \lambda^{-\sigma(p-1)}r^{p-1}R^{2-\sigma(p-1)}$ and similarly for the $\hat\phi$ term. A computation also shows that $\sup_{|x|\le 2R}|x|^2 w_\lambda^{p-1}\le 4R^2\lambda^2$. Hence we can conclude that

$$\lambda^\sigma \sup_{|x|\le 2R}|x|^{2+\sigma}\left||w_\lambda+\hat\phi|^p-|w_\lambda+\phi|^p\right|\le\left(2\lambda^{-\sigma(p-1)}r^{p-1}R^{2-\sigma(p-1)}+4R^2\lambda^2\right)\|\hat\phi-\phi\|_{X_{\lambda,2}}.$$

The term involving the cut-off gives a similar estimate and hence we see that

$$\|I_1(\hat\phi)-I_1(\phi)\|_{Y_\lambda}\le C\left(2\lambda^{-\sigma(p-1)}r^{p-1}R^{2-\sigma(p-1)}+4R^2\lambda^2\right)\|\hat\phi-\phi\|_{X_{\lambda,2}}.$$

Estimate of $\|I_2(\hat\phi)-I_2(\phi)\|_{Y_\lambda}$. Using Lemma 7 we have

$$|I_2(\hat\phi)-I_2(\phi)|\le C\left(w_\lambda^{p-2}(|\phi|+|\hat\phi|)+|\phi|^{p-1}+|\hat\phi|^{p-1}\right)|\hat\phi-\phi|.$$

From this we have

$$\lambda^\sigma\sup_{|x|\le\lambda^{-1}}|x|^{\sigma+2}|I_2(\hat\phi)-I_2(\phi)|\le CK_2^{in}\|\hat\phi-\phi\|_{X_{\lambda,2}},$$

where

$$K_2^{in}:=\sup_{|x|\le\lambda^{-1}}|x|^2\left(w_\lambda^{p-2}(|\phi|+|\hat\phi|)+|\phi|^{p-1}+|\hat\phi|^{p-1}\right).$$

We now estimate these terms in K_2^{in}. A computation shows $\sup_{|x|\le\lambda^{-1}}|x|^2w_\lambda^{p-1}|\phi|\le\lambda^{\alpha(p-2)-2}r=\lambda^{-\alpha}r$. A similar calculation shows that $\sup_{|x|\le\lambda^{-1}}|x|^2|\phi|^{p-1}\le r^{p-1}\lambda^{-2}$. Hence we see that

$$K_2^{in}\le 2\lambda^{-\alpha}r+2r^{p-1}\lambda^{-2}.$$

We now estimate the portion where $|x|\ge\frac{1}{\lambda}$. An identical argument shows that

$$\lambda^\alpha\sup_{|x|\ge\lambda^{-1}}|x|^{2+\alpha}|I_2(\hat\phi)-I_2(\phi)|\le CK_2^{out}\|\hat\phi-\phi\|_{X_{\lambda,2}},$$

where K_2^{out} is defined exactly as K_2^{in}, except the supremum is now over $|x|\ge\frac{1}{\lambda}$, i.e.

$$K_2^{out}:=\sup_{|x|\ge\lambda^{-1}}|x|^2\left(w_\lambda^{p-2}(|\phi|+|\hat\phi|)+|\phi|^{p-1}+|\hat\phi|^{p-1}\right).$$

We now estimate the individual terms of K_2^{out}. First note that

$$\sup_{|x|\ge\lambda^{-1}}|x|^2w_\lambda^{p-2}|\phi|\le\sup_{|x|\ge\lambda^{-1}}\frac{|x|^2Cr}{|x|^{\alpha(p-2)}\lambda^\alpha|x|^\alpha}\le C\lambda^{-\alpha}r.$$

Similarly one sees that $\sup_{|x|\ge\lambda^{-1}} |x|^2|\phi|^{p-1} \le \lambda^{-2}r^{p-1}$. Combining these estimates gives $K_2^{out} \le C(\lambda^{-\alpha}r + \lambda^{-2}r^{p-1})$. From this we see that

$$\|I_2(\hat{\phi}) - I_2(\phi)\|_{Y_\lambda} \le C(\lambda^{-\alpha}r + \lambda^{-2}r^{p-1})\|\hat{\phi} - \phi\|_{X_{\lambda,2}}.$$

Estimate of $\|I_3(\hat{\phi}) - I_3(\phi)\|_{Y_\lambda}$. Recall $I_3(\phi) = -\gamma|\zeta w_\lambda + \phi|^q$. So we have

$$|\gamma|^{-1}|I_3(\hat{\phi}) - I_3(\phi)| = \Big||\zeta w_\lambda + \hat{\phi}|^q - |\zeta w_\lambda + \phi|^q\Big|.$$

By Lemma 7 we have

$$\Big||\zeta w_\lambda + \hat{\phi}|^q - |\zeta w_\lambda + \phi|^q\Big| \le C\left(\zeta^{q-1}w_\lambda^{q-1} + |\phi|^{q-1} + |\hat{\phi}|^{q-1}\right)|\hat{\phi} - \phi|.$$

We first consider $|x| \le \frac{1}{\lambda}$. A computation gives

$$\lambda^\sigma \sup_{|x|\le\lambda^{-1}} |x|^{2+\sigma}\Big||\zeta w_\lambda + \hat{\phi}|^q - |\zeta w_\lambda + \phi|^q\Big| \le CK_3^{in}\|\hat{\phi} - \phi\|_{X_{\lambda,2}},$$

where $K_3^{in} = \sup_{|x|\le\lambda^{-1}} |x|^2\left(\zeta^{q-1}w_\lambda^{q-1} + |\phi|^{q-1} + |\hat{\phi}|^{q-1}\right)$. We now estimate these terms individually. First note that $\sup_{|x|\le\lambda^{-1}} |x|^2\zeta^{q-1}w_\lambda^{q-1} \le |x|^2\lambda^{\alpha(q-1)} \le \lambda^{\theta_1}$ where, as before, $\theta_1 = \alpha(q-1) - 2$ and this is positive provided $q > p$. A similar calculation shows that $\sup_{|x|\le\lambda^{-1}} |x|^2|\phi|^{q-1} \le \lambda^{-2}r^{q-1}$. From this we see that $K_3^{in} \le \lambda^{\theta_1} + 2\lambda^{-2}r^{q-1}$. Now consider $|x| \ge \frac{1}{\lambda}$. A computation shows

$$\lambda^\alpha \sup_{|x|\ge\lambda^{-1}} |x|^{2+\alpha}\Big||\zeta w_\lambda + \hat{\phi}|^q - |\zeta w_\lambda + \phi|^q\Big| \le CK_3^{out}\|\hat{\phi} - \phi\|_{X_{\lambda,2}}$$

where, as before, we are defining K_3^{out} exactly as K_3^{in} except the supremum is now over $|x| \ge \frac{1}{\lambda}$, i.e. $K_3^{out} = \sup_{|x|\ge\lambda^{-1}} |x|^2\left(\zeta^{q-1}w_\lambda^{q-1} + |\phi|^{q-1} + |\hat{\phi}|^{q-1}\right)$. A computation shows $\sup_{|x|\ge\lambda^{-1}} |x|^2 w_\lambda^{q-1} \le C\lambda^{\theta_1}$. Similarly we have $\sup_{|x|\ge\lambda^{-1}} |x|^2|\phi|^{q-1} \le r^{q-1}\lambda^{-2}$. So we have $K_3^{out} \le C\lambda^{\theta_1} + 2r^{q-1}\lambda^{-2}$. Combining with the above estimates gives

$$\|I_3(\hat{\phi} - I_3(\phi)\|_{Y_\lambda} \le C(\lambda^{\theta_1} + \lambda^{-2}r^{q-1})\|\hat{\phi} - \phi\|_{X_{\lambda,2}}. \qquad \square$$

Proof of Theorem 1. We begin by finding a nonzero solution u of (13) and for this we don't need to distinguish the cases of γ positive or negative. Fix N, p, σ, λ_0 as Proposition 1. Take $0 < \lambda < \lambda_0$ and given $\phi \in B_r \subset X_{\lambda,2}$ define $\psi_\lambda = J_\lambda(\phi)$ as defined in (15). We will now show that J_λ is a contraction on B_r. Set $r := \beta\lambda^\alpha$ and $R := \frac{\varepsilon}{\lambda}$ where $\beta, \varepsilon > 0$ will be chosen later; and recall that R is related to the cut off ζ.

Into. Let $\phi \in B_r \subset X_{\lambda,2}$. Then by Proposition 1 we have

$$C\|J_\lambda(\phi)\|_{X_{\lambda,2}} = C\|\psi_\lambda\|_{X_{\lambda,2}} \le \sum_{k=1}^{5} \|I_k(\phi)\|_{Y_\lambda}.$$

We now compute each of these terms with these choices of r and R. By Lemma 3 we have

$$\frac{\|I_1(\phi)\|_{Y_\lambda}}{Cr} \le \varepsilon^{\sigma+2}\beta^{-1} + \varepsilon^{2+\sigma(2-p)}\beta^{p-2}, \quad \frac{\|I_2(\phi)\|_{Y_\lambda}}{Cr} \le \beta\lambda^2 + \beta + \beta^{p-1},$$
$$\frac{\|I_3(\phi)\|_{Y_\lambda}}{Cr} \le \lambda^{\theta_1}(\beta^{-1} + \beta^{q-1}), \quad \frac{\|I_5(\phi)\|_{Y_\lambda}}{Cr} \le (2\varepsilon^{2+\sigma} + \varepsilon^{\sigma})\beta^{-1},$$

and

$$\frac{\|I_4(\phi)\|_{Y_\lambda}}{Cr} \le \Big((1+\varepsilon^{-1})A(\lambda^{-1}\varepsilon) + A(\lambda^{-1}) + V(\lambda^{-1}\varepsilon) + V(\lambda^{-1})\Big)\beta^{-1}.$$

Using these estimates one sees that $J_\lambda(B_r) \subset B_r$ provided we first fix $\beta > 0$ sufficiently small, then fix $\varepsilon > 0$ sufficiently small and then take $\lambda > 0$ small.

Contraction. Let $\hat{\phi}, \phi \in B_r$ and we let $\hat{\psi}_\lambda = J_\lambda(\hat{\phi})$ and $\psi_\lambda = J_\lambda(\phi)$. Then by (14) we have

$$\|J_\lambda(\hat{\phi}) - J_\lambda(\phi)\|_{X_{\lambda,2}} \le C\sum_{k=1}^{3} \|I_k(\hat{\phi}) - I_k(\phi)\|_{Y_\lambda}.$$

We now take $R = \frac{\varepsilon}{\lambda}$ and $r = \beta\lambda^\alpha$ and use Lemma 4 to see

$$\frac{\|J_\lambda(\hat{\phi}) - J_\lambda(\phi)\|_{X_{\lambda,2}}}{C\|\hat{\phi} - \phi\|_{X_{\lambda,2}}} \le \beta^{p-1}\varepsilon^{2-\sigma(p-1)} + \varepsilon^2 + \beta + \beta^{p-1} + \lambda^{\theta_1}(1+\beta^{q-1}). \tag{24}$$

Note that the same procedure for picking $\beta, \varepsilon, \lambda$ that we used to show that $J_\lambda(B_r) \subset B_r$ also shows that J_λ a contraction on B_r. Hence we can apply Banach's fixed point theorem to see there is some $\phi \in B_r = B_{\beta\lambda^\alpha}$ such that $J_\lambda(\phi) = \phi$ in Ω with $\phi = 0$ on $\partial\Omega$. Hence we have $u = \zeta w_\lambda + \phi$ satisfies (13). Also note that there is some $\beta_0 > 0$ such that $w(\lambda x) \ge \frac{\beta_0}{\lambda^\alpha |x|^\alpha}$ for all $\lambda|x| \ge 1$. Also recall that for all $|x| \ge \frac{1}{\lambda}$ we have $\lambda^\alpha |x|^\alpha |\phi(x)| \le \beta\lambda^\alpha$ and hence we have

$$|x|^\alpha u(x) \ge \beta_0 - \beta,$$

for $|x| \ge \frac{1}{\lambda}$. Hence by taking $\beta > 0$ small we see that $u > 0$ for $|x| \ge \frac{1}{\lambda}$. We now separate the cases of positive and negative γ.

Case 1: $\gamma \ge 0$. In this case we have

$$-\Delta u + a(x)\cdot\nabla u + V(x)u = |u|^p + \gamma|u|^q \quad \text{in } \Omega$$

with $u = 0$ on $\partial\Omega$ and $u > 0$ for large $|x|$. We can then apply the maximum principle and the strong maximum principle to see that $u > 0$ in Ω.

Case 2: $\gamma < 0$. Recall how we picked the parameters. We fixed $\beta > 0$ small and then took $\varepsilon > 0$ small and then were able to take $\lambda > 0$ as small as we wish. With this in mind let $\lambda_m \searrow 0$ and let $u_m = \zeta_m w_{\lambda_m} + \phi_m$ denote a solution of (13) and as mentioned above we have $u_m > 0$ for $|x| \ge \frac{1}{\lambda_m}$. Note ζ_m is just the cut off from before but we are indicating the dependence on m. Our goal is to show that for large enough m that $u_m \ge 0$ in Ω. So towards a contradiction suppose that for all large m we have $\{x \in \Omega : u_m(x) < 0\}$ is non-empty and let Ω_m denote a maximal connected component. So we have $\Omega_m \subset \Omega \cap B_{\lambda_m^{-1}}$. So u_m is a negative solution of

$$\begin{cases} -\Delta u_m + a(x) \cdot \nabla u_m + \left(V(x) - |\gamma||u_m|^{q-1}\right) u_m = |u_m|^p & \text{in } \Omega_m \\ u_m = 0 & \text{on } \partial\Omega_m. \end{cases} \tag{25}$$

We now use a slight variation of the maximum principle given in Lemma 6, to show $u_m \geq 0$. Multiply (25) by $(u_m)_- \in H_0^1(\Omega_m)$ and integrate by parts to arrive at

$$2\int_{\Omega_m} |\nabla (u_m)_-|^2 \leq \int_{\Omega_m} \left(div(a) - 2V + 2\gamma |u_m|^{q-1}\right)(u_m)_-^2 \leq \int_{\Omega} C_m(x)(u_m)_-^2 dx$$

where $C_m(x) := (div(a) - 2V)_+ + 2|\gamma||u_m|^{q-1}$. We apply Hölder's inequality on the right to see the right hand side is bounded above by $\|C_m\|_{L^{\frac{N}{2}}(\Omega_m)} \|(u_m)_-\|^2_{L^{2^*}(\Omega_m)}$. We apply the critical Sobolev inequality, $S_N \|\psi\|^2_{L^{2^*}(\Omega_m)} \leq \|\nabla \psi\|^2_{L^2(\Omega_m)}$, on the left with $\psi = (u_m)_-$, and regroup to see

$$\left(2S_N - \|C_m\|_{L^{\frac{N}{2}}(\Omega_m)}\right) \|(u_m)_-\|^2_{L^{2^*}(\Omega_m)} \leq 0.$$

If we can show $2S_N - \|C_m\|_{L^{\frac{N}{2}}(\Omega_m)} > 0$ then we see that $u_m \geq 0$ in Ω_m giving us the desired contradiction. We now examine this term in more detail. Note

$$\begin{aligned} \|C_m\|_{L^{\frac{N}{2}}(\Omega_m)} &\leq \|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\Omega_m)} + 2|\gamma| \|u_m\|^{q-1}_{L^{\frac{N(q-1)}{2}}(\Omega_m)} \\ &\leq \|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\Omega)} + 2|\gamma| \|u_m\|^{q-1}_{L^{\frac{N(q-1)}{2}}(\Omega_m)}. \end{aligned}$$

Now recall that we are assuming $\|(div(a) - 2V)_+\|_{L^{\frac{N}{2}}(\Omega)} < 2S_N$ and hence it will be sufficient to show that $\|u_m\|_{L^{\frac{N(q-1)}{2}}(\Omega_m)} \to 0$ as $m \to \infty$. Recall that $u_m = \zeta_m w_{\lambda_m} + \phi_m$ and $\Omega_m \subset B_{\lambda_m^{-1}}$ in $\mathbb{R}^N$ and hence we have $w_{\lambda_m} \leq \lambda_m^\alpha$ in Ω_m. Hence we have

$$\|u_m\|_{L^{\frac{N(q-1)}{2}}(\Omega_m)} \leq \lambda_m^\alpha |\Omega_m|^{\frac{2}{N(q-1)}} + \|\phi_m\|_{L^{\frac{N(q-1)}{2}}(\Omega_m)},$$

and note $\lambda_m^\alpha |\Omega_m|^{\frac{2}{N(q-1)}} \leq C_N \lambda_m^{\frac{2}{p-1} - \frac{2}{q-1}} \to 0$ since $q > p$. Now since $\phi_m \in B_{\beta\lambda_m^\alpha} \subset X_{\lambda_m,2}$ we see $|\phi_m(x)| \leq \beta |x|^{-\sigma} \lambda_m^{\alpha-\sigma}$ in Ω_m and hence

$$\int_{\Omega_m} |\phi_m(x)|^{\frac{N(q-1)}{2}} dx \leq C(N, \beta, \sigma) \lambda_m^{\frac{N(\alpha-\sigma)(q-1)}{2}} \int_\rho^{\lambda_m^{-1}} s^{N-1-\frac{\sigma N(q-1)}{2}} ds$$

where $\rho > 0$ is sufficiently small such that $B_\rho \subset D$. By taking $\sigma > 0$ sufficiently small and since $q > p$ we see then that $\int_{\Omega_m} |\phi_m(x)|^{\frac{N(q-1)}{2}} dx \to 0$ which gives us the desired conclusion. Hence by contradiction we have $u_m \geq 0$ is a $C^{2,\delta}$ nonzero solution of (13), for sufficiently large m, and hence we can apply the strong maximum principle to see that $u_m > 0$ in Ω.

So we have shown the existence of a solution of (3) of the form $u_\lambda(x) = \zeta_\lambda(x) w_\lambda(x) + \phi_\lambda(x)$ for sufficiently small λ; here $\zeta = \zeta_\lambda$ was the appropriate cut off that depended on R (and recall R now depends on λ). As pointed out in [6] and (more details were given in [8]) one has $\sup_\Omega u_\lambda \to 0$ as $\lambda \searrow 0$ and recall that $u_\lambda > 0$ and so we see this implies that there is an infinite number of solutions of (3). We now give some details. First note that for all $x \in \Omega$ we have $0 < u_\lambda(x) \leq \lambda^\alpha w(\lambda x) + |\phi_\lambda(x)| \leq \lambda^\alpha + |\phi_\lambda(x)|$. Now recall that $\phi_\lambda \in B_r = B_{\beta\lambda^\alpha}$ in $X_{\lambda,2}$ we have

$$\sup_{x \in \Omega, |x| \geq \lambda^{-1}} |\phi_\lambda(x)| \leq \beta\lambda^\alpha, \qquad \sup_{x \in \Omega, |x| \leq \lambda^{-1}} |\phi_\lambda(x)| \leq C(\Omega, \sigma)\beta\lambda^{\alpha-\sigma}$$

where $C(\Omega, \sigma)$ is some positive constant. Note we need to take $\sigma > 0$ small enough such that $\alpha - \sigma > 0$. Combining these computations shows that $\sup_\Omega |\phi_\lambda| \to 0$ as $\lambda \searrow 0$. So from this we see $\sup_\Omega u_\lambda \to 0$ as $\lambda \searrow 0$. $\square$

3. Equation (4); $-\Delta u + a(x) \cdot \nabla u + V(x)u = u^p + \gamma|\nabla u|^q$

We now find a positive solution of (4), but as usual, we instead will find a positive classical solution of

$$\begin{cases} -\Delta u + a(x) \cdot \nabla u + V(x)u = |u|^p + \gamma|\nabla u|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

and then argue the solution must be positive. The approach we take is exactly the same as in the previous section: let $D \subset\subset B_{R_0}$ and $R_0 \leq R \leq 4R \leq \frac{1}{\lambda}$ and ζ denote a smooth radial cut-off with $\zeta = 0$ in B_R and $\zeta = 1$ on B_{2R}^c. Then we have $|\nabla\zeta| \leq CR^{-1}$ and $|\Delta\zeta| \leq CR^{-2}$ where C is independent of R.

We look for solutions to (26) of the form $u = \zeta(x) w_\lambda(x) + \phi(x)$. Then we need ϕ to satisfy

$$\begin{aligned} L_\lambda(\phi) &= |w_\lambda + \phi|^p - |\zeta w_\lambda + \phi|^p \\ &\quad - \left(|w_\lambda + \phi|^p - p w_\lambda^{p-1}\phi - w_\lambda^p\right) \\ &\quad - \gamma|\nabla(w_\lambda\zeta) + \nabla\phi|^q \\ &\quad + a(x) \cdot \nabla(\zeta w_\lambda) + V w_\lambda \zeta \\ &\quad + \Delta w_\lambda - \Delta(\zeta w_\lambda) \\ &= I_1(\phi) + I_2(\phi) + I_3(\phi) + I_4(\phi) + I_5(\phi) \quad \text{in } \Omega, \end{aligned} \tag{27}$$

with $\phi = 0$ on $\partial\Omega$. Note that each term I_k agrees with the previous section except the term I_3. To obtain a solution ϕ we apply a fixed point argument and towards this we define the nonlinear mapping $J_\lambda(\phi) = \psi_\lambda$ where ψ_λ satisfies

$$\begin{aligned} L_\lambda(\psi_\lambda) &= |w_\lambda + \phi|^p - |\zeta w_\lambda + \phi|^p \\ &\quad - \left(|w_\lambda + \phi|^p - p w_\lambda^{p-1}\phi - w_\lambda^p\right) \end{aligned}$$

$$\begin{aligned}&-\gamma|\nabla(w_\lambda\zeta)+\nabla\phi|^q\\&+a(x)\cdot\nabla(\zeta w_\lambda)+Vw_\lambda\zeta\\&+\Delta w_\lambda-\Delta(\zeta w_\lambda)\\=\;&I_1(\phi)+I_2(\phi)+I_3(\phi)+I_4(\phi)+I_5(\phi)\quad\text{in }\Omega.\end{aligned}\tag{28}$$

So we first find a positive classical solution of (26) by showing J_λ is a contraction on $B_r \subset X_{\lambda,2}$ for suitable $r > 0$ and small $0 < \lambda$, as was the earlier approach. Let $0 < \lambda_0$ be small and $C > 0$ such that $L_\lambda : X_{\lambda,2} \to Y_\lambda$ has a right inverse bounded by C for all $0 < \lambda < \lambda_0$.

Into. Let $0 < \lambda < \lambda_0$ and $\phi \in B_r$. Then we have

$$\|J_\lambda(\phi)\|_{X_{\lambda,2}} \le C\sum_{k=1}^{5}\|I_k(\phi)\|_{Y_\lambda},$$

and now recall that Lemma 3 gives the estimates

$$\|I_1(\phi)\|_{Y_\lambda} \le CR^{\sigma+2}\lambda^{\alpha p+\sigma}+CR^{2+\sigma(2-p)}r^{p-1}\lambda^{\alpha+\sigma(2-p)},\tag{29}$$

$$\|I_2(\phi)\|_{Y_\lambda} \le Cr^2\lambda^{2-\alpha}+Cr^2\lambda^{-\alpha}+Cr^p\lambda^{-2},\tag{30}$$

$$\|I_4(\phi)\|_{Y_\lambda} \le C\left(\frac{A(R)}{R\lambda}+A(R)+V(R)+A(\lambda^{-1})+V(\lambda^{-1})\right)\lambda^\alpha,\tag{31}$$

$$\|I_5(\phi)\|_{Y_\lambda} \le C\lambda^\sigma\left(R^{2+\sigma}\lambda^2+R^\sigma\right)\lambda^\alpha.\tag{32}$$

We now calculate the I_3 estimates. By Lemma 7 there is some $C > 0$ such that

$$|I_3(\phi)| \le C\zeta^q|\nabla w_\lambda|^q+Cw_\lambda^q|\nabla\zeta|^q+C|\nabla\phi|^q,\tag{33}$$

and hence we have

$$\|I_3(\phi)\|_{Y_\lambda} \le C\|\zeta^q|\nabla w_\lambda|^q\|_{Y_\lambda}+C\|w_\lambda^q|\nabla\zeta|^q\|_{Y_\lambda}+C\||\nabla\phi|^q\|_{Y_\lambda}.\tag{34}$$

Taking $0 < \sigma$ small enough we have $\sigma + 2 > q(\sigma + 1)$. Computations show that $\||\nabla\phi|^q\|_{Y_\lambda} \le 2r^q\lambda^{q-2}$, $\|w_\lambda^q|\nabla\zeta|^q\|_{Y_\lambda} \le CR^{\sigma+2-q}\lambda^{\sigma+\alpha q}$ and $\|\zeta^q|\nabla w_\lambda|^q\|_{Y_\lambda} \le C\lambda^{q(\alpha+1)-2}$.

Combining the results gives

$$\|I_3(\phi)\|_{Y_\lambda} \le C\lambda^{q(\alpha+1)-2}+CR^{\sigma+2-q}\lambda^{\sigma+\alpha q}+Cr^q\lambda^{q-2}.\tag{35}$$

Contraction. Let $\phi, \hat\phi \in B_r$ and $\hat\psi_\lambda = J_\lambda(\hat\phi)$ and $\psi_\lambda = J_\lambda(\phi)$. Then we have

$$L_\lambda(\hat\psi_\lambda-\psi_\lambda)=\sum_{k=1}^{3}(I_k(\hat\phi)-I_k(\phi)),$$

where I_1, I_2 and I_3 are as above. From Lemma 4 we have

$$\|I_1(\hat{\phi}) - I_1(\phi)\|_{Y_\lambda} \le C\left(2\lambda^{-\sigma(p-1)} r^{p-1} R^{2-\sigma(p-1)} + 4R^2\lambda^2\right)\|\hat{\phi} - \phi\|_{X_{\lambda,2}}, \tag{36}$$

$$\|I_2(\hat{\phi}) - I_2(\phi)\|_{Y_\lambda} \le C(\lambda^{-\alpha} r + \lambda^{-2} r^{p-1})\|\hat{\phi} - \phi\|_{X_{\lambda,2}}. \tag{37}$$

We now need to examine I_3 term. By Lemma 7 we have

$$|I_3(\hat{\phi}) - I_3(\phi)| \le C\left(|\nabla(w_\lambda \zeta)|^{q-1} + |\nabla\phi|^{q-1} + |\nabla\hat{\phi}|^{q-1}\right)|\nabla\hat{\phi} - \nabla\phi|.$$

Using this we can rearrange it to see that

$$\|I_3(\hat{\phi}) - I_3(\phi)\|_{Y_\lambda} \le C(K_{in} + K_{out})\|\hat{\phi} - \phi\|_{X_{\lambda,2}},$$

where

$$K_{in} := \sup_{|x| \le \lambda^{-1}} |x|\left(|\nabla(w_\lambda \zeta)|^{q-1} + |\nabla\phi|^{q-1} + |\nabla\hat{\phi}|^{q-1}\right),$$

and where K_{out} is K_{in} but with the supremum taken over $|x| \ge \lambda^{-1}$. We now estimate K_{in} and K_{out}. Using the support of ζ and estimates for $|\nabla w_\lambda|$ we see

$$\frac{K_{out}}{C} \le \sup_{|x| \ge \lambda^{-1}} |x|\left(\frac{1}{|x|^{(\alpha+1)(q-1)}} + |\nabla\phi|^{q-1} + |\nabla\hat{\phi}|^{q-1}\right).$$

A computation shows that

$$K_{out} \le C\lambda^{(\alpha+1)(q-1)-1} + Cr^{q-1}\lambda^{q-2}.$$

A further computation shows that

$$\frac{K_{in}}{C} \le \lambda^{(\alpha+2)(q-1)-q} + \frac{\lambda^{\alpha(q-1)}}{R^{q-2}} + r^{q-1}\lambda^{q-2}.$$

Combining the above results shows that

$$\frac{\|I_3(\hat{\phi}) - I_3(\phi)\|_{Y_\lambda}}{C\|\hat{\phi} - \phi\|_{X_{\lambda,2}}} \le \lambda^{(\alpha+1)(q-1)-1} + r^{q-1}\lambda^{q-2} + \frac{\lambda^{\alpha(q-1)}}{R^{q-2}}.$$

Now let $\hat{\phi}, \phi \in B_r \subset X_{\lambda,2}$ and set $\hat{\psi}_\lambda = J_\lambda(\hat{\phi})$ and $\psi_\lambda = J_\lambda(\phi)$. After considering (29)–(35) one sees that for $J_\lambda(B_r) \subset B_r$ it is sufficient that

$$\begin{aligned} R^{\sigma+2}\lambda^{\alpha p+\sigma} + R^{2+\sigma(2-p)} r^{p-1}\lambda^{\alpha+\sigma(2-p)} + r^2\lambda^{2-\alpha} + r^2\lambda^{-\alpha} + r^p\lambda^{-2} \\ + \lambda^{q(\alpha+1)-2} + R^{\sigma+2-q}\lambda^{\sigma+\alpha q} + r^q\lambda^{q-2} \\ + \left(\frac{A(R)}{R\lambda} + A(R) + V(R) + A(\lambda^{-1}) + V(\lambda^{-1})\right)\lambda^\alpha \\ + \lambda^\sigma\left(R^{2+\sigma}\lambda^2 + R^\sigma\right)\lambda^\alpha \le \frac{r}{C^2+1}. \end{aligned} \tag{38}$$

For J_λ to be a contraction on B_r with Lipschitz constant at most $\frac{3}{4}$ it is sufficient

$$\lambda^{-\sigma(p-1)} r^{p-1} R^{2-\sigma(p-1)} + 4R^2\lambda^2 + \lambda^{-\alpha} r + \lambda^{-2} r^{p-1} \\ + \lambda^{(\alpha+1)(q-1)-1} + r^{q-1}\lambda^{q-2} + \frac{\lambda^{\alpha(q-1)}}{R^{q-2}} \leq \frac{3}{4(C+1)}. \tag{39}$$

We are now in a position to pick the parameters. As before we take $\varepsilon, \beta > 0$ (to be determined later) and we take $r := \beta\lambda^\alpha$ and $R := \frac{\varepsilon}{\lambda}$ and eventually we will take $\lambda > 0$ small.

Substituting these values in shows that to satisfy (38) it is sufficient that

$$\varepsilon^{\sigma+2} + \varepsilon^\sigma + \varepsilon^{2+\sigma(2-p)}\beta^{p-1} + \beta^2(\lambda^2+1) + \beta^p + \lambda^{q(\alpha+1)-2-\alpha}(1 + \varepsilon^{\sigma+2-q} + \beta^q) \\ + \left(\frac{A(\varepsilon\lambda^{-1})}{\varepsilon} + A(\varepsilon\lambda^{-1}) + V(\varepsilon\lambda^{-1}) + A(\lambda^{-1}) + V(\lambda^{-1}) \right) \leq \frac{\beta}{C^2+1}. \tag{40}$$

Also note the left hand side of (39) is controlled by a constant times

$$\beta^{p-1}\varepsilon^{2-\sigma(p-1)} + \varepsilon^2 + \beta + \beta^{p-1} + \lambda^{q(\alpha+1)-\alpha-2}\left(1 + \varepsilon^{2-q} + \beta^{q-1}\right), \tag{41}$$

and hence (39) is satisfied provided this can be made arbitrarily small.

To satisfy (40) and to make (41) sufficiently small one first fixes $\beta > 0$ small, then fixes $\varepsilon > 0$ small and finally takes $\lambda > 0$ sufficiently small. One can then apply the contraction mapping principle to obtain a solution $\phi \in B_r = B_{\beta\lambda^\alpha} \subset X_{\lambda,2}$ of (27). We then have $u(x) = \zeta(x)w_\lambda(x) + \phi(x)$ is a solution of (26). To see u is not identically zero note that for $|x| \geq \lambda^{-1}$ we have $u(x) = \lambda^\alpha w(\lambda x) + \phi(x) \geq \lambda^\alpha w(\lambda x) - \beta|x|^{-\alpha}$. Recall there is some $\beta_0 > 0$ such that $|x|^\alpha w(x) \geq \beta_0$ for all $|x| \geq 1$. So for fixed λ and sufficiently large $|x|$ we have

$$u(x) \geq \frac{\lambda^\alpha \beta_0}{2\lambda^\alpha |x|^\alpha} - \frac{\beta}{|x|^\alpha},$$

so hence for sufficiently small β and large $|x|$ we see $u(x) > 0$. We can then apply the maximum principle to see that u is a positive solution of (26) and hence a solution of (4). To obtain an infinite number of solutions we argue as in the previous section.

4. Equations (3) and (4) for $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$

In this section we prove Theorem 3. Since the approach is very similar to the case of $p > \frac{N+1}{N-3}$ we will be fairly brief. We will always assume that D satisfies $(A3)$ (see the text following Remark 1 for definition of $(A3)$). We define the subspace Y_λ^e of Y_λ by

$$Y_\lambda^e := \left\{ f \in Y_\lambda : f(x^i) = f(x) \text{ for all } x \in \Omega \text{ and } 1 \leq i \leq N \right\},$$

where, as before, $\Omega := \mathbb{R}^N \setminus \overline{D}$. It is clear that Y_λ^e is a closed subspace of Y_λ. We similarly define $X_{\lambda,0}^e$ and $X_{\lambda,2}^e$ to be the closed subspaces of $X_{\lambda,0}$ and $X_{\lambda,2}$ (respectively) which contain functions ϕ with the same symmetries as functions in Y_λ^e. We now recall the definitions $L^\lambda(\phi)(x) := \Delta\phi(x) + pw_\lambda(x)^{p-1}\phi(x)$ and $L_\lambda(\phi)(x) := L^\lambda(\phi)(x) - a(x)\cdot\nabla\phi(x) - V(x)\phi(x)$.

We now need to develop the needed linear theory on these spaces of symmetric functions. Firstly recall Theorem A, 2) gives us the existence of a continuous right inverse for L^λ as a mapping on $X^e_{\lambda,0}$ to Y^e_λ. Using the same approach as we did in the proof of Lemma 1 one is able to show the analogue of Lemma 1, for the symmetric functions, given by

Lemma 5. *Suppose $N \geq 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and D satisfies $(A3)$. Then for $0 < \sigma < N-2$ there exists some small $\lambda_0 > 0$ and some $C > 0$ such that for all $0 < \lambda < \lambda_0$ and $f \in Y^e_\lambda$ there is some $\phi_\lambda \in X^e_{\lambda,2}$ such that $L^\lambda(\phi_\lambda) = f$ in Ω with $\phi_\lambda = 0$ on $\partial\Omega$ and $\|\phi_\lambda\|_{X_{\lambda,2}} \leq C\|f\|_{Y_\lambda}$.*

We can now construct the right inverse of L^λ exactly as we did following the proof of Lemma 1. So there is some closed subspace $\tilde{X}^e_{\lambda,2}$ of $X^e_{\lambda,2}$ such that $L^\lambda : \tilde{X}^e_{\lambda,2} \to Y^e_\lambda$ is continuous, one to one and onto and hence its Fredholm index is zero.

We now would like to extend the above linear theory to the operator L_λ. A computation shows that the symmetry assumptions $(A4)$ and $(A5)$ imposed on V and a (along with the decay assumptions $(A1)$ and $(A2)$) show that $L_\lambda(X^e_{\lambda,2}) \subset Y^e_\lambda$. So we have $L_\lambda : X^e_{\lambda,2} \to Y^e_\lambda$ is a continuous linear operator. From this we see that $L_\lambda : \tilde{X}^e_{\lambda,2} \to Y^e_\lambda$ is a Fredholm index zero linear map. We can now argue exactly as in the proof of Proposition 1 to obtain the analogues result given by:

Proposition 2. *Suppose $N \geq 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and $(A1)$, $(A2)$, $(A3)$, $(A4)$, $(A5)$ are satisfied. Then for $0 < \sigma < N-2$ there exists some small $\lambda_0 > 0$ and some $C > 0$ such that for all $0 < \lambda < \lambda_0$ and $f \in Y^e_\lambda$ there is some $\phi_\lambda \in X^e_{\lambda,2}$ such that $L_\lambda(\phi_\lambda) = f$ in Ω with $\phi_\lambda = 0$ on $\partial\Omega$ and $\|\phi_\lambda\|_{X_{\lambda,2}} \leq C\|f\|_{Y_\lambda}$.*

This gives us all the needed linear theory and we now would like to apply fixed point arguments to solve the nonlinear problems. The main difference now will be that we will replace $X_{\lambda,2}$ with $X^e_{\lambda,2}$ in the various fixed point arguments.

Proof of Theorem 3, 1) and 2). We begin by considering (3). Given $\phi \in X^e_{\lambda,2}$ consider $J_\lambda(\phi) := \psi_\lambda \in X^e_{\lambda,2}$ where ψ_λ satisfies (15); to see this is possible note that the right hand side of (15) is an element of Y^e_λ. We can now argue exactly as before to obtain a fixed point of J_λ, on a suitable closed ball in $X^e_{\lambda,2}$, provided $q > p$, and again we need to split up the cases of $\gamma \geq 0$ and $\gamma < 0$. Omitting the details one obtains a positive solution of (3) and we then argue as before to obtain an infinite number of solutions. □

The proof of Theorem 3, 3) is very similar to part 1) and 2) and so we omit the details.

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Appendix A

We now recall the particular maximum principle but this requires we recall the best constant S_N associated with the critical Sobolev imbedding $H^1_0 \subset L^{2^*}$ which is independent of the domain; $S_N\|\phi\|^2_{L^{2^*}} \leq \|\nabla\phi\|^2_{L^2}$ for all $\phi \in H^1_0$.

Lemma 6 *(Maximum principle). (See [14].) Suppose $w \in H_0^1(\Omega)$ is a weak solution of $-\Delta w(x) - C(x)w = f(x) \geq 0$ in Ω where $\|C_+\|_{L^{\frac{N}{2}}(\Omega)} < S_N$. Then $w \geq 0$ in Ω.*

Proof. Their proof involves multiplying the equation by w_- (the negative part of w) and integrating by parts and applying Hölder's inequality. □

Lemma 7. *Suppose $p > 1$. There exists a constant $C > 0$ such that the following hold:*

1. *For all numbers $w > 0$, $\phi \in \mathbb{R}$, and $\hat{\phi}$,*

$$\Big||w+\phi|^p - pw^{p-1}\phi - w^p\Big| \leq C\Big(w^{p-2}\phi^2 + |\phi|^p\Big),$$

and

$$\Big||w+\hat{\phi}|^p - |w+\phi|^p - pw^{p-1}(\hat{\phi}-\phi)\Big| \leq C\Big(w^{p-2}(|\phi|+|\hat{\phi}|) + |\phi|^{p-1} + |\hat{\phi}|^{p-1}\Big)|\hat{\phi}-\phi|;$$

2. *For all $x, y, z \in \mathbb{R}^n$,*

$$\Big||x+y|^p - |x+z|^p\Big| \leq C\Big(|x|^{p-1} + |y|^{p-1} + |z|^{p-1}\Big)|y-z|.$$

References

[1] S. Ai, C. Cowan, Perturbations of Lane–Emden and Hamilton–Jacobi equations I: the full space case, preprint, 2015.
[2] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989) 271–297.
[3] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991) 615–622.
[4] J.M. Coron, Topologie et cas limite des injections de Sobolev, C. R. Math. Acad. Sci. Paris Ser. I 299 (1984) 209–212.
[5] C. Cowan, Stability of entire solutions to supercritical elliptic problems involving advection, Nonlinear Anal. 104 (2014) 1–11.
[6] J. Dávila, M. del Pino, M. Musso, The supercritical Lane–Emden–Fowler equation in exterior domains, Comm. Partial Differential Equations 32 (2007) 1225–1243.
[7] J. Dávila, M. del Pino, M. Musso, J. Wei, Standing waves for supercritical nonlinear Schrödinger equations, J. Differential Equations 236 (2007) 164–198.
[8] J. Dávila, M. del Pino, M. Musso, J. Wei, Fast and slow decay solutions for supercritical elliptic problems in exterior domains, Calc. Var. Partial Differential Equations 32 (2008) 453–480.
[9] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri–Coron's problem, Calc. Var. Partial Differential Equations (2003) 113–145.
[10] M. del Pino, M. Musso, F. Pacard, Bubbling along geodesics for some semilinear supercritical elliptic problem in bounded domains, J. Eur. Math. Soc. (JEMS) 12 (2010) 1553–1605.
[11] A. Farina, On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$, J. Math. Pures Appl. 87 (2007) 537–561.
[12] B. Gidas, W. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 525–598.
[13] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981) 525–598.
[14] M. Grossi, R. Molle, On the shape of the solutions of some semilinear elliptic problems, Commun. Contemp. Math. 5 (2003) 85–99.

[15] C. Gui, W.M. Ni, X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$, Comm. Pure Appl. Math. 45 (1992) 1153–1181.
[16] D.D. Joseph, T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal. 49 (1973) 241–269.
[17] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains, J. Funct. Anal. 114 (1993) 97–105.
[18] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Sov. Math. Dokl. 6 (1965) 1408–1411.
[19] M. Struwe, Variational Methods – Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, Berlin, 1990.