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The existence of minimum speed of traveling wave solutions to a non-KPP isothermal diffusion system

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Abstract

The reaction–diffusion system $a_t = a_{xx} - ab^n$, $b_t = Db_{xx} + ab^n$, where $n \geq 1$ and $D > 0$, arises from many real-world chemical reactions. Whereas $n = 1$ is the KPP type nonlinearity, which is much studied and very important results obtained in literature not only in one dimensional spatial domains, but also multi-dimensional spaces, but $n > 1$ proves to be much harder. One of the interesting features of the system is the existence of traveling wave solutions. In particular, for the traveling wave solution $a(x,t) = a(x - vt)$, $b(x,t) = b(x - vt)$, where $v > 0$, if we fix $\lim_{x\to-\infty}(a, b) = (0, 1)$ it was proved by many authors with different bounds $v_*(n, D) > 0$ such that a traveling wave solution exists for any $v \geq v_*$ when $n > 1$. For the latest progress, see [7]. That is, the traveling wave problem exhibits the mono-stable phenomenon for traveling wave of scalar equation $u_t = u_{xx} + f(u)$ with $f(0) = f(1) = 0$, $f(u) > 0$ in $(0, 1)$ and, $u = 0$ is unstable and $u = 1$ is stable. A natural and significant question is whether, like the scalar case, there exists a minimum speed. That is, whether there exists a minimum speed $v_{min} > 0$ such that traveling wave solution of speed v exists iff $v \geq v_{min}$? This is an open question, in spite of many works on traveling wave of the system in last thirty years. This is duo to the reason, unlike the KPP case, the minimum speed cannot be obtained through linear analysis at equilibrium points $(a, b) = (0, 1)$ and $(a, b) = (1, 0)$. In this work, we give an affirmative answer to this question.

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1. Introduction

In this paper we study the reaction–diffusion system

$$(I)\begin{cases} \dfrac{\partial a}{\partial t} = \dfrac{\partial^2 a}{\partial x^2} - ab^n, \\ \dfrac{\partial b}{\partial t} = D\dfrac{\partial^2 b}{\partial x^2} + ab^n, \end{cases}$$

where $n > 1$, $D > 0$ and (a, b) are non-negative smooth functions with continuous initial values (a_0, b_0). In isothermal diffusion, a is the density of material consumed, b is the temperature, see [5,4,15]. In addition, it models, after a simple scaling, a simple autocatalytic chemical reaction of the form

$$A + nB \to (n+1)B \qquad \text{with rate} \quad kab^n,$$

where $k > 0$ is a rate constant, between two chemical species A and B. More importantly, the system arises from many important chemical wave models of excitable media ranging from the idealized Brusselator to real-world clock reactions such as Belousov–Zhabotinsky reaction, the Briggs–Rauscher reaction, the Bray–Liebhafsky reaction and the iodine clock reaction. In that setting, its importance was recognized fairly early, [13,12,22]. Another type of application is that of biological pattern formation of Turing type. In particular, for the purpose of replicating experimental results in early 1990s, two significant models CIMA and Gray–Scott both have the special case of $n = 2$ incorporated in the complete system, see [17,18].

One interesting feature of the system is the existence of traveling waves, which describes the spreading of chemical species B, when locally added to the uniformly distributed A, by consuming chemical species A. This phenomenon was observed in experiment, see [14,22].

By using the simple scaling invariance property of the system and a conservation law, see [7], we can normalize the traveling wave problem, with $a(x,t) = \alpha(x - vt)$, $b(x,t) = \beta(x - vt)$ as follows.

$$\begin{cases} \alpha' + D\beta' = v(1 - \alpha - \beta), \qquad \alpha' \geqslant 0 & \forall z \in \mathbb{R}, \\ D\beta'' + v\beta' = -\alpha\beta^n, & \forall z \in \mathbb{R}, \\ \lim_{z\to\infty}(\alpha, \beta) = (1, 0), \\ \lim_{z\to-\infty}(\alpha, \beta) = (0, 1), \end{cases} \tag{1.1}$$

where $v > 0$ is the speed of traveling wave. Here $\alpha' \geqslant 0$ is not an additional restriction, because for a traveling wave solution, it is direct from (I) that $\alpha'' + c\alpha' = \alpha\beta^n$ and therefore $a' \geqslant 0$.

There have been many works on the existence of traveling waves of (I) and related models in the last thirty years, [2,7–9,11,16,19–21]. A typical result is there exist two constants $v_1(n, D) > v_2(n, D) > 0$ such that if $v \geq v_1$ there exists a traveling wave with speed v; but if $v < v_2$, there does not exist a traveling wave. The emphasize is on estimating the smallest-speed traveling wave and close the gap between v_1 and v_2. The situation is very different from the case of $n = 1$,

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where the minimum speed is obtained directly from linear analysis of equilibrium points $(1, 0)$ and $(0, 1)$. The underlying reason for getting sharp results on minimum speed is the universal belief that the smallest-speed wave is the most stable one. In particular, when $D = 1$, (1.1) is reduced to a single equation,

$$\begin{cases} \beta'' + v\beta' + (1-\beta)\beta^n = 0, \quad v \geqslant 0 & \forall z \in \mathbb{R}, \\ \lim_{z\to\infty} \beta = 0, \quad \lim_{z\to-\infty} \beta = 1. \end{cases} \tag{1.2}$$

It is well-known that there exists $v_{min} > 0$ such that traveling wave exists iff $v \geq v_{min}$. A classical result on the corresponding parabolic equation

$$\begin{cases} b_t = b_{xx} + (1-b)b^n, \quad \forall x \in \mathbb{R}, t > 0, \\ b(x,0) = b_0(x) \geq 0, \end{cases} \tag{1.3}$$

is that if b_0 has compact support, then as $t \to \infty$, b converges to the minimum speed traveling wave, see [1,10]. For other works of scalar equation on models with non-local operator and discrete lattice structure see [3,6].

The expected result for (I) is that if a_0 is a constant and b_0 has compact support, then the spreading speed of b is that of minimum speed. The case of $n = 1$ is rigorously proved by two authors of this work in [9].

In spite of substantial effort, one outstanding question is still, like the special case of $D = 1$, whether there exists $v_{min} = v_{min}(n, D) > 0$ such that traveling wave solution exists iff $v \geq v_{min}$? We settle this question in the present work.

Theorem 1. *Suppose that $D > 0$ and $n > 1$. Then, there exists a positive constant $v_{\min}$ such that (1.1) admits a solution if and only if $v \geqslant v_{\min}$.*

Remark. The case of $D > 1$ was proved by many authors using the monotone properties of traveling waves in relation to the speed v, see [7]. The argument breaks down when $D < 1$. In this work, we use a totally different approach which works for all values of D. The method we use is a combination of center-manifold theory for the ODE system and reformulation of the system (1.1) into an elliptic-parabolic PDE problem, from which we can construct a traveling wave solution for larger speed using the smaller speed one as a super-solution.

The plan of this work is that we prove the main result in section 2, and put off the proof of some technical lemmas to section 3.

2. Proof of Theorem 1

In this section we prove our main result Theorem 1. The key is to reformulate the system (1.1) into a second order non-autonomous equation for β' as a function of $1 - \beta$ and analyze in details the center-stable manifold at the equilibrium point $(1, 0)$ of (1.1). The basic properties of the system (1.1) are well established and their proof elementary. We shall use it without referring to where they are first derived. Instead, a convenient source is [7].

We carry out the proof after discussion of some preliminary results and put off the proof of some technical results to next section.

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First, as we know, the set of admissible wave speeds is a closed set in $(0, \infty)$. For the existence of a minimum wave speed, we need only to show that if there exists a traveling wave speed $v_0 > 0$, then for any $v > v_0$, there is also a traveling wave of speed v.

Hence, in the sequel we assume that there is a traveling of speed v_0 and $v > v_0$ is any fixed constant. We are going to show that there is a traveling wave of speed v.

Starting from the equations

$$\alpha_z + v\alpha + D\beta_z + v\beta = v, \qquad D\beta_{zz} + v\beta_z = -\alpha\beta^n,$$

for convenience, we use new variable (α, u, p) where $u := 1 - \beta$ and $p := u_y$ with $y := z/\kappa$, $\kappa = D/v$. The traveling wave problem is equivalent to solving

$$\frac{1}{D}\alpha_y = p + u - \alpha, \qquad u_y = p, \qquad p_y = -p + \frac{\kappa^2}{D}\alpha(1-u)^n \tag{2.4}$$

subject to the constraints $\alpha > 0$ and $u < 1$ on $\mathbb{R}$ and the boundary conditions

$$\lim_{y\to-\infty}(\alpha(y), u(y), p(y)) = (0,0,0), \qquad \lim_{y\to\infty}(\alpha(y), u(y), p(y)) = (1,1,0).$$

Notice that $(0, 0, 0)$ is a saddle point of the system (2.4) with a one dimensional unstable manifold, half of which is in the first octant when close to the origin and is a trajectory. We denote this trajectory by $\bar{\gamma} = \{(\bar{\alpha}(y), \bar{u}(y), \bar{p}(y))\}_{y\in(-\infty,Y)}$, where Y is the first point at which $u = 1$. Then

$$\lim_{y\to-\infty}\frac{\bar{\alpha}(y)}{\bar{u}(y)} = \frac{\lambda(\kappa)+1}{\lambda(\kappa)/D+1}, \qquad \lim_{y\to-\infty}\frac{\bar{p}(y)}{\bar{u}(y)} = \lambda(\kappa) := \tfrac{1}{2}\Big(\sqrt{4\kappa^2 + D} - D\Big).$$

Let

$$\bar{y} = \sup\{y \in \mathbb{R} \mid \bar{u} < 1 \ \text{ in } \ (-\infty, y)\}.$$

It is easy to show, see [7], that in $(-\infty, \bar{y})$, $\bar{p} > 0$, $\bar{\alpha} > 0$, and $\bar{\alpha}_y > 0$. We prove that the trajectory is a traveling wave.

We shall work both on this trajectory in the phase space (α, u, p) and on the projection of it in the (u, p)-plane. For the latter, we shall write p as a function of u. Since solutions of interest has the property $u_y = p > 0$, there is an inverse $y = Y(u)$ of the function $u = u(y)$. Then we can define $P(u) = p(Y(u))$. Hence we can express $\bar{\gamma} \cap \{u \leqslant 1\}$ as $\bar{p}(y) = \bar{P}(\bar{u}(y))$, which satisfies

$$\bar{P}(u) > 0 \quad \forall u \in (0,1), \qquad \lim_{u\searrow 0}\frac{\bar{P}(u)}{u} = \lambda(\kappa) := \tfrac{1}{2}\Big(\sqrt{4\kappa^2 + D^2} - D\Big).$$

To show that there exists a traveling wave, it suffices to show that $\bar{P}(1) = 0$, as was shown in [7]. To be self-contained, we give a brief explanation in what follows.

If $\bar{P}(1) = 0$, then $(\bar{p} + \bar{u})_y = \kappa^2\bar{\alpha}(1-\bar{u})^n/D > 0$ in $(-\infty, \bar{y})$ so that $\bar{p} + \bar{u} < 1$ in $(-\infty, \bar{y})$. The equation for $\bar{\alpha}$ then implies that $\bar{\alpha}_y/D = \bar{p} + \bar{u} - \bar{\alpha} < 1 - \bar{\alpha}$ in $(-\infty, \bar{y})$, therefore $\bar{\alpha} < 1$ in $(-\infty, \bar{y})$. This implies that $\bar{y} = \infty$, since otherwise, the trajectory would be the trajectory

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passes through $(\bar{\alpha}(\bar{y}), 1, 0)$ whose solution is given by $(1 - [1 - \bar{\alpha}(\bar{y})]e^{-(z-\bar{y})/D}, 1, 0)$, which is impossible. Thus $\bar{y} = \infty$ and $\lim_{y\to\infty}(\bar{\alpha}, \bar{u}, \bar{p}) = (1, 1, 0)$.

Next, to show that $\bar{P}(1) = 0$, we shall work on the equation for P, derive as follows. Since solutions of interest satisfy $u < 1$ on $\mathbb{R}$,

$$\alpha = \frac{D}{\kappa^2}(1-u)^{-n}(u_{yy} + u_y), \qquad \alpha_y = \frac{D}{\kappa^2}(1-u)^{-n}\Big\{[u_{yyy} + u_{yy}] + \frac{nu_y(u_{yy} + u_y)}{1-u}\Big\}.$$

Substituting these two expressions into the equation $\kappa^2(\frac{1}{D}\alpha_y + \alpha - u_y - u)(1-u)^n = 0$, we conclude that, as far as traveling wave solution is concerned, the original traveling wave problem is reduced to a single equation

$$[u_{yyy} + u_{yy}] + [u_{yy} + u_y][\tfrac{nu_y}{1-u} + D] - \kappa^2[u_y + u][1-u]^n = 0.$$

On the (u, p)-phase plane the equation then becomes a second order non-autonomous equation, denoting $p(y) = P(u(y))$, $P' = \frac{dp}{du}$ and $P'' = \frac{d^2p}{du^2}$,

$$\mathcal{N}[\kappa, P] := P^2P'' + P[P' + 1]\Big\{P' + D + \frac{nP}{1-u}\Big\} - \kappa^2(P + u)(1-u)^n = 0. \tag{2.5}$$

Note that if $P(\cdot) \in C^2((c, d))$, $(c, d) \subset (0, 1)$ is a **positive** solution to (2.5) in an interval $u \in (c, d)$, then it provides a trajectory via the transformation, for any fixed $u_0 \in [c, d]$,

$$y = \int_{u_0}^{u(y)} \frac{ds}{P(s)}, \qquad p(y) := P(u(y)), \qquad \alpha(y) = \frac{D}{\kappa^2}(1-u)^{-n}(u_{yy} + u_y). \tag{2.6}$$

Since we have assumed that there exists a traveling wave of speed v_0, the traveling wave solution provides a solution of (2.5) with $\kappa = \kappa_0 := D/v_0$. Denote the solution by $P_0(\cdot)$. It has the following properties

$$P_0(u) > 0 \quad \forall u \in (0, 1), \quad \lim_{u\searrow 0} \frac{P_0(u)}{u} = \lambda(\kappa_0) := \tfrac{1}{2}\Big(\sqrt{4\kappa_0^2 + D^2} - D\Big),$$

$$\text{either} \quad \lim_{u\nearrow 1} \frac{P_0(u)}{1-u} = 1 \qquad \text{or} \quad \lim_{u\nearrow 1} \frac{P_0(u)}{(1-u)^n} = \frac{\kappa_0^2}{D}.$$

These properties can be derived from local analysis of the equilibria $(0, 0, 0)$ and $(1, 1, 0)$ of the system (2.4), we give a proof of the last statement in Appendix.

In the following we use a sub-super solution technique from theory of elliptic and parabolic PDEs to find a solution P^* to (2.5) that is sandwiched between $P_0(u)$ and $1 - u$.

Lemma 1. *Suppose that there is a traveling wave of speed $v_0 > 0$. Let $\kappa \in (0, \kappa_0)$. There exists $P^* \in C^0([0, 1]) \cap C^2((0, 1))$ such that*

$$\mathcal{N}[\kappa, P^*](u) = 0, \quad 0 < P^*(u) < 1 - u \quad \forall u \in (0, 1), \qquad P^*(0) = P^*(1) = 0.$$

In addition, denoting the corresponding solution to (2.4) by $\gamma^* = (\alpha^*, u^*, p^*)$ *and write* $\alpha^*(y) = A^*(u^*(y))$, *there holds*

$$\begin{aligned} &A^*(0) > 0, \\ &P^*(u) \sim \sqrt{\frac{2\kappa^2 A^*(0)}{D}}\sqrt{u} \qquad \textit{as } u \searrow 0, \\ &P^*(u) \sim (1-u) - \frac{\kappa^2}{nD}(1-u)^n \quad \textit{as } u \nearrow 1. \end{aligned}$$

This lemma will be proven later.

Proof of Theorem 1. The local behavior of solutions to (2.4) near the equilibrium, see [2] and Proposition 2.1 in [7], is as follows. The equilibrium $(1, 1, 0)$ has a two dimensional stable manifold whose tangent space at $(1, 1, 0)$ is spanned by the vector $(1, 0, 0)$ and $(0, 1, -1)$. Also, the equilibrium $(1, 1, 0)$ has a center manifold, with tangent vector $(1, 1, 0)$. In addition, there exists a positive constant $r > 0$ and a continuously differentiable function P_s defined on $B_r := \{(\alpha, u) \mid (\alpha - 1)^2 + (u - 1)^2 \leqslant r^2\}$ such that

(i) The local stable manifold of (2.4) associated with $(1, 1, 0)$ is given by the surface $p = P_s(\alpha, u)$, $(\alpha, u) \in B_r$, and the function P_s has the expansion

$$P_s(\alpha, u) = 1 - u + \frac{\kappa^2}{nD}\alpha(1-u)^n + o(1)(1-u)^n.$$

(ii) For given $(\alpha_1, u_1) \in B_r \cap \{u < 1\}$ and each $p_1 \in [0, P_s(\alpha_1, u_1))$, the trajectory of (2.4) through (α_1, u_1, p_1) is on the center-stable manifold associated with the equilibrium $(1, 1, 0)$ and has the property

$$\lim_{y\to\infty}(\alpha(y), u(y), p(y)) = (1, 1, 0), \qquad \lim_{y\to\infty}\frac{p(y)}{(1-u(y))^n} = \frac{\kappa^2}{D}.$$

To show that $\bar{P}(1) = 0$, we use a contradiction argument. Suppose to the contrary that $\bar{P}(1) \neq 0$. Then $\bar{y} < \infty$ and $\bar{P}(1) > 0$.

Fix a $u_0 \in (1 - \frac{1}{2}r, 1)$ and denote $(\alpha_0, p_0) = (A^*(u_0), P^*(u_0))$. Let L be the line segment in the phase space (α, u, p) that connects $(\alpha_0, u_0, 0)$ and (α_0, u_0, p_0):

$$L = \{(\alpha_0, u_0, p) \mid 0 \leqslant p \leqslant p_0\}.$$

For each $s \in [0, 1]$, we denote by $\gamma(s) = \{(\alpha(s, y), u(s, y), p(s, y) \mid y \in \mathbb{R}\}$ the trajectory to the system

$$\frac{1}{D}\dot{\alpha} = p + u - \alpha, \quad \dot{u} = p, \quad \dot{p} = -p + \frac{\kappa^2}{D}|\alpha|(1-u)^n, \qquad (\alpha, u, p)|_{y=0} = (\alpha_0, u_0, sp_0). \tag{2.7}$$

Here the replacement of α by $|\alpha|$ in the last equation will be explained later.

Note that when $s = 1$, the trajectory $\gamma(1)$ is exactly γ^*, which is a trajectory for all $y \in (-\infty, \infty)$.

Since $(a_0, u_0) \in B(r)$, we see that for each $s \in [0, 1)$, $\gamma(s)$ is on center manifold associated with $(1, 1, 0)$, in particular, $\alpha(s, y) > 0$, $p(s, y) > 0$ for all $y \in [0, \infty)$ and that

$$\lim_{y\to\infty} (\alpha(s, y), u(0, y), p(s, y)) = (1, 1, 0), \qquad \lim_{y\to\infty} \frac{p(s, y)}{(1 - u(s, y))^n} = \frac{\kappa^2}{D}.$$

Lemma 2. *For each $s \in [0, 1]$, there exists a unique $T(s) \leqslant 0$ such that the unique solution to (2.7) satisfies*

$$p(s, T(s)) = 0, \qquad p(s, z) > 0 \quad \forall z \in (T(s), \infty).$$

We put off the proof to next section.

We continue our proof. Now in $(T(s), \infty)$, we denote by $p(s, z)$ as $P(s, u)$ in the sense that $p(s, y) = P(s, u(s, y))$.

When $s = 0$, $T(s) = 0$. Also when u_0 is sufficiently close to 1, $P(0, u) = O((1 - u)^n)$ so

$$P^*(u) > P(0, u) \quad \forall u \in [u_0, 1).$$

We denote by

$$U(s) = u(s, T(s)), \quad A(s) = \alpha(s, T(s)).$$

Then $\{(U(s), A(s)\}_{s\in[0,1]}$ is a smooth curve on the $\{P = 0\}$ plane with end point $(A^*(0), 0)$ and (α_0, u_0). We define

$$\hat{s} = \min\{s \in [0, 1] \mid U(s) \leqslant 0\}.$$

Since $U(0) > 0$, we see that $\hat{s} \in (0, 1]$, $U(\hat{s}) = 0$ and $0 < U(s) < 1$ for every $s \in (0, \hat{s})$.

Now we can complete our proof as follows.

Suppose $\bar{P}(1) \neq 0$. Then $\bar{P}(1) > 0$. Let

$$P_*(u) := \min\{\bar{P}(u), P^*(u)\} \qquad \forall u \in [0, 1].$$

In the notion of elliptic PDEs, P_* is a supersolution to the operator $\mathcal{N}(\kappa, \cdot)$. As $\bar{P}(1) > 0$, there exists $u_1 \in (0, 1)$ such that $P_*(u) = P^*(u) < \bar{P}(u)$ for all $u \in (u_1, 1)$. We assume that u_0 is close to 1 enough such that $u_0 > u_1$.

Note that when $s = 0$, $P(0, u) < P_*(u)$ for all $u \in (u_0, 1)$. Hence, the following is well-defined:

$$\tilde{s} = \sup\{s \in (0, \hat{s}] \mid P(s, u) \leqslant P_*(u) \quad \forall u \in (U(s), u_0)\}.$$

First of all, we must have $\tilde{s} < \hat{s}$. Indeed, if $\tilde{s} = \hat{s}$, then $U(\hat{s}, T(\hat{s})) = 0$. As $\bar{\gamma}$ is not on the stable manifold of $(1, 1, 0)$, we cannot have $|\alpha(\hat{s}, T(\hat{s}))| = 0$. Hence, $\dot{p}(\hat{s}, T(\hat{s})) = |\alpha(\hat{s}, T(\hat{s}))| =: m > 0$ so that when t is close to $T = T(\hat{s})$,

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$$p(\hat{s},t) \approx m(z-T), \qquad u(\hat{s},t) \approx \tfrac{1}{2}m(z-T)^2$$

so that $P(\hat{s},u) \approx \sqrt{2mu}$ when u is small. However, when u is small, $P_*(u) = \bar{P}(u) \approx \lambda(\kappa)u$, and it is impossible to have $P(\hat{s},u) < P_*(u)$ for all $u \in (0,u_0)$.

Thus, $\tilde{s} \in (0,\hat{s})$, namely, $U(\tilde{s}) > 0$. Since $P_* > 0$ in $(0,1)$, we have $P_*(u) > P(\tilde{s},u)$ in $[U(\tilde{s}), U(\tilde{s})+\varepsilon]$ for some small $\varepsilon > 0$.

On the other hand, we know that

$$\lim_{u\to 1} \frac{P(\tilde{s},u)}{1-u} = 0 < \lim_{u\searrow 1} \frac{P_*(u)}{1-u}.$$

We see that $P_*(u) > P(\tilde{s},u)$ when u is closed to 1.

Now by continuity we cannot have $P(s,u) < P_*(u)$ for all $u \in (U(\tilde{s}),u_0)$ since it would implies the existence of a small positive ε such that $P(\tilde{s}+\varepsilon,u) < P_*(u)$ for $u \in [U(\tilde{s}+\varepsilon),u_0)$ which contradicts the maximality of $\tilde{s}$.

Thus, we must have $P(\tilde{s},\hat{u}) = P_*(\hat{u})$ and $\partial P(\tilde{s},\hat{u})/\partial u = \partial P_*(\hat{u})/\partial u$ at some $\hat{u} \in (U(\tilde{s}),1)$. As $P(\tilde{s},\cdot)$ is twice differentiable at $\hat{u}$ and the derivative of P_* has negative jump at every intersection point of $p = P^*$ and $p = \bar{P}$, we cannot have $P^*(\hat{u}) = \bar{P}(\hat{u})$. Hence one of the following holds:

$$\begin{aligned} \text{either (i) } \bar{P}(\hat{u}) &= P(\tilde{s},\hat{u}), \qquad \bar{P}_u(\hat{u}) = P_u(\tilde{s},\hat{u}), \\ \text{or (ii) } P^*(\hat{u}) &= P(\tilde{s},\hat{u}), \qquad P^*_u(\hat{u}) = P_u(\tilde{,}\hat{u}). \end{aligned}$$

However, the former case implies that $\gamma(s) = \bar{\gamma}$ and the latter case $\gamma(s) = \gamma^*$ since they represent the same solution to (2.4) with the same initial data. We note that under the change of variable $y \to u = u(s,y)$, in case (i),

$$\begin{aligned} A(s) &= \frac{D}{\kappa^2}(1-T(s))^{-n} P(s,T(s))[P_u(s,T(s))+1] \\ &= \frac{D}{\kappa^2}(1-T(s))^{-n} P^*(T(s))[P^*_u(T(s))+1] = \bar{A}(T(s)) > 0, \end{aligned}$$

whereas in case (ii), $A(s,T(s)) = A^*(T(s)) > 0$.

This is a clear contradiction. Hence, $\bar{\gamma}$ is a heteroclinic orbit, completing the proof of Theorem. □

3. Proof of two lemmas

In this section we prove the technical lemmas used in the proof of our main result. We start by proving Lemma 2.

Proof of Lemma 2. We define

$$T(s) := \inf\{z \mid p(s,\cdot) > 0 \text{ in } (z,\infty)\}.$$

We need only show that $T(s)$ is finite. Suppose on the contrary that $T(s) = -\infty$. Then $p > 0$ on $\mathbb{R}$, so $u(s,\cdot)$ is strictly increasing on $\mathbb{R}$. In addition,

$$\frac{d}{dz}(p+u) = |\alpha|(1-u)^n \geqslant 0$$

so $p+u$ is increasing and $p+u<1$ on $\mathbb{R}$. Since $(0,0,0)$ is the only equilibrium in the phase space of interest and there is no heteroclinic. By the contradiction assumption, we must have either $\lim_{y\to-\infty}(p+u) = -\infty$ or $\lim_{y\to\infty}|\alpha| = \infty$. From the equation $\frac{1}{D}\dot{\alpha} = p+u-|\alpha|$, we conclude that $\lim_{y\to-\infty}\alpha = \infty$.

Now from $\frac{d}{dy}p = -(p+u)+u+|\alpha|(1-u)^n \geq -1+u+|\alpha|(1-u)^n \to \infty$ as $y\to-\infty$. But this contradicts $p>0$ in $\mathbb{R}$. Hence, $T(s)$ is finite. $\square$

Next, we prove Lemma 1.

Proof of Lemma 1. Denote $\kappa_0 = D/v_0$. The traveling wave with speed v_0 gives a unique solution $P_0 \in C^2([0,1])$ to

$$\mathcal{N}[\kappa_0, P_0] = 0 < P_0 \quad \text{in } (0,1), \qquad P_0(0) = P_0(1) = 0.$$

By the same derivation as we did for $\bar{\gamma}$ we conclude that

$$0 < P_0(u) < 1-u, \qquad P_0'(u)+1>0 \quad \forall u\in(0,1).$$

The key here is that P_0 is a sub-solution and $P_1(u) := 1-u$ is a super-solution of (2.5) with $0<\kappa<\kappa_0$ in the sense that

$$\mathcal{N}[\kappa, P_0] = (\kappa_0^2-\kappa^2)(P_0+u)(1-u)^n > 0 \quad \forall u\in(0,1),$$
$$\mathcal{N}[\kappa, P_1] = -\kappa^2(1-u)^n < 0 \quad \forall u\in(0,1).$$

Fix any $\varepsilon,\delta\in(0,1/2)$. Set $I_{\varepsilon,\delta} = (\varepsilon, 1-\delta)$ and consider the initial boundary value problem for the parabolic equation, for $P_{\varepsilon,\delta} = P_{\varepsilon,\delta}(u,t)$, $(u,t)\in \bar{I}_{\varepsilon,\delta}\times[0,\infty)$:

$$\begin{cases} \dfrac{\partial P_{\varepsilon,\delta}}{\partial t} = \mathcal{N}[\kappa, P_{\varepsilon,\delta}] & \text{in } I_{\varepsilon,\delta}\times(0,\infty), \\ P_{\varepsilon,\delta} = p_0 & \text{on } \Big(\partial I_{\varepsilon,\delta}\times[0,\infty)\Big)\cup\Big(I_{\varepsilon,\delta}\times\{0\}\Big). \end{cases}$$

Since $P_0>0$ on $\bar{I}_{\varepsilon,\delta} = [\varepsilon, 1-\delta]$, local existence in time of a positive solution follows from a standard parabolic equation theory. Also, by comparison, $P_0 \leqslant P_{\varepsilon,\delta} \leqslant P_1$. Hence, there exists a global positive solution in time satisfying

$$P_0 < P_{\varepsilon,\delta} < P_1 \quad \text{in } I_{\varepsilon,\delta}\times(0,\infty).$$

Furthermore, one notices that $Q := \frac{\partial}{\partial t}P_{\varepsilon,\delta}$ satisfies

$$Q = 0 \quad \text{on } \partial I_{\varepsilon,\delta}\times[0,\infty), \qquad Q|_{t=0} = \mathcal{N}(\kappa, p_0) > 0.$$

Therefore, by maximum principle, $Q = \frac{\partial}{\partial t}P_{\varepsilon,\delta} > 0$ in $I_{\varepsilon,\delta}\times(0,\infty)$. Consequently,

$$\hat{P}_{\varepsilon,\delta}(u) := \lim_{t\to\infty} P_{\varepsilon,\delta}(u,t) \quad \forall u \in [\varepsilon, 1-\delta]$$

exists. Since $P_{\varepsilon,\delta}$ is uniformly positive on $\bar{I}_{\varepsilon,\delta} \times [0,\infty)$, classical regularity theory shows that the above limit is in $C^m(\bar{I})$ for any positive integer m. Hence, $\mathcal{N}[\kappa, \hat{P}_{\varepsilon,\delta}] = 0$.

Next, consider the family $\{\hat{P}_{\varepsilon,\delta}\}_{0<\varepsilon,\delta<1/2}$. From the construction and comparison,

$$\hat{P}_{\varepsilon,\delta} > \hat{P}_{\hat{\varepsilon},\hat{\delta}} \quad \text{in } \bar{I}_{\hat{\varepsilon},\hat{\delta}} \times (0,\infty) \qquad \text{if } 0<\varepsilon<\hat{\varepsilon}<1/2,\ 0<\delta<\hat{\delta}<1/2.$$

It then follows that $\hat{P}_{\varepsilon,\delta} \geqslant \hat{P}_{\hat{\varepsilon},\hat{\delta}}$ in $\bar{I}_{\hat{\varepsilon},\hat{\delta}}$. Hence, there exists the limit

$$\hat{P}_{\varepsilon}(u) = \lim_{\delta\searrow 0} \hat{P}_{\varepsilon,\delta}(u) \qquad \forall u \in [\varepsilon, 1),$$

and $\hat{P}_\varepsilon$ is non-increasing in ε. Note that $P_0 \leqslant \hat{P}_\varepsilon \leqslant 1-u$. It is then not very difficult to show that $\mathcal{N}(\kappa, \hat{P}_\varepsilon) = 0$ in $(\varepsilon, 1)$. In addition, since $\hat{P}_\varepsilon(\varepsilon) = P_0(\varepsilon)$ and $\hat{P}_\varepsilon \geqslant p_0$ on $(\varepsilon, 1)$, we have $\hat{P}'_\varepsilon(\varepsilon) \geq P'_0(\varepsilon) \geq -1$. Thus, the corresponding solution $(\alpha_\varepsilon, u_\varepsilon)$ (via (2.6)) satisfies $\alpha_\varepsilon > 0$ in $[0,\infty)$.

Let

$$P^*(u) = \lim_{\varepsilon\searrow 0} \hat{P}_\varepsilon(u) \quad \forall u \in (0, 1]$$

Then $1-u \geqslant P^*(u) \geqslant P_0(u)$ for all $u \in (0,1]$. It is then easy to show that $\mathcal{N}(\kappa, P_*) = 0$ in $(0,1]$. Let $(\alpha^*(0), u^*(0), q^*(0)) = \lim_{\varepsilon\searrow 0}(\alpha_\varepsilon(0), u_\varepsilon(0), p_\varepsilon(\varepsilon))$. Clearly, $u^*(0) = 0$, $q^*(0) = 0$, and $\alpha^*(0) \geqslant 0$. It is easy to exclude the case $\alpha^*(0) = \infty$. If $\alpha^*(0) = 0$, we have a heteroclinic connection. However, this would imply the asymptotic $P^*(u) = [\lambda(\kappa) + o(1)]u$ as $u \searrow 0$ where $\lambda(\kappa) = \frac{1}{2}(\sqrt{D^2 + 4\kappa^2} - D)$. Since we know $\kappa_0 > \kappa$, $P_0 = [\lambda(\kappa_0) + o(1)]u$, and $P^* \geqslant P_0$, we obtain a contradiction. Hence, $\alpha^*(0) > 0$ and for small positive y,

$$u^*_{yy}(0) = \frac{\kappa^2\alpha^*(0)}{D}, \quad u^*_y(y) \approx \frac{\kappa^2\alpha^*(0)}{D}\, y, \quad u^*(y) \approx \frac{\kappa^2\alpha^*(0)}{2D}\, y^2, \qquad P^*(u) \approx \sqrt{\frac{2\kappa^2\alpha^*(0)}{D}}\sqrt{u}.$$

Finally, as $u \nearrow 1$, either $P^*(u) \sim \frac{\kappa^2}{D}(1-u)^n$ or $P^*(u) \sim 1-u$, and either $P_0 \sim \frac{\kappa_0^2}{D}(1-u)^n > \frac{\kappa^2}{D}(1-u)^n$ or $P_0 \sim 1-u$. But, since $\kappa < \kappa_0$, $P^* > P_0$, $P^*(u) \sim \frac{\kappa^2}{D}(1-u)^n$ would lead to $P_0(u) \sim \frac{\kappa_0^2}{D}(1-u)^n > P^*(u)$. A contradiction. The only possibility is that $P^*(u) \sim 1 - u - \frac{\kappa^2}{nD}(1-u)^n$. This completes the proof of the Lemma. □

Appendix A

We prove the following proposition.

Proposition 1. *Suppose (α, u) is a traveling wave solution of (2.5). Then,*

$$\textit{either} \ \lim_{u\nearrow 1} \frac{P(u)}{1-u} = 1 \qquad \textit{or} \ \lim_{u\nearrow 1} \frac{P(u)}{(1-u)^n} = \frac{\kappa^2}{D}.$$

Proof. For any $\varepsilon_0 \in (0, 1)$, consider

$$\mu(u) \equiv P(u) - \varepsilon_0(1-u).$$

It is easy to verify that

$$\begin{aligned}
P\mu' &= (P' + \varepsilon_0)P \\
&= (\varepsilon_0 - 1)P + \frac{\kappa^2}{D}\alpha(1-u)^n \\
&= -(1-\varepsilon_0)\mu - \varepsilon_0(1-\varepsilon_0)(1-u) + \frac{\kappa^2}{D}\alpha(1-u)^n.
\end{aligned}$$

It is clear that as u is close to 1, $\mu(u)$ cannot change sign.

If $\mu(u) > 0$ for all u close to 1, then, using the fact that $P'(u) > -1$, we get

$$P' + 1 < \frac{\kappa^2}{D\varepsilon_0}\alpha(1-u)^{n-1} < \frac{\kappa^2}{D\varepsilon_0}(1-u)^{n-1}.$$

An integration on $[u, 1]$ then gives

$$-P(u) + 1 - u < \frac{\kappa^2}{Dn\varepsilon_0}\alpha(1-u)^n.$$

Hence,

$$P(u) > 1 - u - \frac{\kappa^2}{Dn\varepsilon_0}\alpha(1-u)^n.$$

This, together with $P(u) < 1 - u$, yields

$$\lim_{u\nearrow 1}\frac{P_0(u)}{1-u} = 1.$$

If $\mu(u) < 0$ for all $\varepsilon_0 > 0$ when u close to 1, say for $u \geq u_0$, choose $L \gg 1$ such that

$$L(1-u_0)^n > \varepsilon_0(1-u_0) \qquad \text{and} \quad Ln(1-u_0)^{n-1} < 1$$

for some $u_0 \in (0, 1)$. It is easy to compute

$$P(P - L(1-u)^n)` = -P + Ln(1-u)^{n-1}P < 0,$$

and therefore $P(u) < L(1-u)^n$ on $[u_0, 1)$. Next, by using

$$P' = -1 + \frac{\kappa^2}{DP}\alpha(1-u)^n,$$

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we find

$$P'' = \frac{\kappa^2}{DP^2}\Big((\alpha'(1-u)^n - n\alpha(1-u)^{n-1})P - \alpha(1-u)^n P'\Big),$$

which yields that P' cannot change sign if u is sufficiently close to 1. It follows that

$$\lim_{u\nearrow 1} P'(u) = 0.$$

Therefore,

$$\lim_{u\nearrow 1} \frac{P(u)}{(1-u)^n} = \frac{\kappa^2}{D}. \quad \square$$

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