Unknown input observer design and output feedback stabilization for multi-dimensional wave equation with boundary control matched uncertainty ☆

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Abstract

In this paper, we consider boundary output feedback stabilization for a multi-dimensional wave equation with boundary control matched unknown nonlinear internal uncertainty and external disturbance. A new unknown input type extended state observer is proposed to recover both state and total disturbance which consists of internal uncertainty and external disturbance. A key feature of the proposed observer in this paper is that we do not use the high-gain to estimate the disturbance. By the active disturbance rejection control (ADRC) strategy, the total disturbance is compensated (canceled) in the feedback loop, which together with a collocated stabilizing controller without uncertainty, leads to an output feedback stabilizing feedback control. It is shown that the resulting closed-loop system is well-posed and asymptotically stable under weak assumption on internal uncertainty and external disturbance. The numerical experiments are carried out to show the effectiveness of the proposed scheme.
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1. Introduction

Since the robustness concept was introduced to control theory from the early 1970s ([18]), the capability of dealing with uncertainty has become one of the most important performances for control systems. Many control methods have been developed to cope with internal uncertainty and external disturbance. These include sliding mode control for various uncertainties ([20]); internal model principle for output regulation; adaptive control for unknown parameter identification; and robust control ([3]) which is a remarkable paradigm shift in modern control theory for general plant uncertainty. Most of these methods, among many others, focus however, on the worst case scenario which makes the controller design rather conservative. The worth mentioning methods are internal model principle ([4,19]) and adaptive control ([1,11]) where the uncertainty is estimated and compensated. The estimation/cancellation strategy is an economic way in dealing with uncertainty. In this regard, an emerging control technology named active disturbance rejection control (ADRC) is an epitomized approach to cope with vast uncertainty in control systems ([10]). The uncertainties dealt with by ADRC are much more complicated. For instance, ADRC can deal with the coupling between the external disturbances, the system unmodeled dynamics, and the superadded unknown part of control input. The most remarkable feature of ADRC is that the disturbance is estimated, in real time, through an extended state observer and is canceled in the feedback loop. This reduces the control energy significantly in practice [24]. In the past two decades, there are numerous researches on ADRC from perspectives of both engineering and mathematics. Very recently, we applied ADRC to stabilization for multi-dimensional wave equation with external disturbance in [8] where the full state feedback control was used and a high gain extended state was adopted. The output feedback control for PDEs by ADRC is much complicated. In paper [5], an unknown input observer was designed by variable structure control first and then ADRC was applied to design an observer-based feedback control for 1-d wave equation. But the observer in [5] is very complicated with discontinuous injection of the output and hence is hard to generalize to other PDEs.

The aim of this paper is to design a new extended state observer ([6]) in terms of measurements from the boundary and an interior domain which can be of arbitrarily small measure; and an extended state observer based output feedback stabilizing control law for a multi-dimensional wave equation subject to general internal uncertainty and external boundary disturbance. The uncertainty is hence more general than that considered in [8], and we do not use the high-gain to estimate the state and total disturbance as that in [5,8] to avoid possible peaking value problem.

The system that we are concerned with is a multi-dimensional wave equation with Neumann boundary control and unknown nonlinear internal uncertainty and external disturbance, governed by the following partial differential equation:

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t), \ x \in \Omega, \ t > 0, \\ w(x,t)|_{\Gamma_0} = 0, \ x \in \Gamma_0, \ t \geq 0, \\ \dfrac{\partial w(x,t)}{\partial \nu}\Big|_{\Gamma_1} = f(w(\cdot,t)) + d(x,t) + v(x,t), \ x \in \Gamma_1, \ t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ x \in \Omega, \\ y(x,t) = (w(x,t)|_{\Gamma_1}, w_t(x,t)|_{\Gamma_1}, w_t(x,t)|_\omega), \ t \geq 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded domain with a smooth C^2-boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ where Γ_0 and Γ_1 are subsets of Γ, $\text{int}(\Gamma_0) \neq \emptyset$, $\text{int}(\Gamma_1) \neq \emptyset$, $\Gamma_0 \cap \Gamma_1 = \emptyset$; ν is the unit normal

vector of Γ pointing the exterior of Ω; $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ is a possibly unknown nonlinear mapping that represents the internal uncertainty; and $d(x, t)$ is the unknown external disturbance which is only supposed to satisfy $d \in L^\infty(0, \infty; L^2(\Gamma_1))$. For the sake of simplicity, we denote $F(w(x, t)) := f(w(\cdot, t)) + d(x, t)$ as the "total disturbance". The $v(x, t)$ is the control input and $y(x, t)$ is the output (measurement), where ω is an open subset of Ω to be supposed to satisfy that there exist open sets $\Omega_j \subset \Omega$ with Lipschitz boundary $\partial\Omega_j$ and points $x_0^j \in \mathbb{R}^n$, $j = 1, 2, \ldots, J$, such that $\Omega_i \cap \Omega_j = \emptyset$ for any $1 \le i < j \le J$ and

$$\omega \supset \Omega \cap \mathcal{N}_\varepsilon \left[(\cup_{j=1}^J \gamma_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j) \right] \tag{1.2}$$

for some $\varepsilon > 0$ where $\mathcal{N}_\varepsilon[S] = \cup_{x \in S}\{y \in \mathbb{R}^n : |y - x| < \varepsilon\}$ for $S \subset \mathbb{R}^n$, $\gamma_j = \{x \in \partial\Omega_j : (x - x_0^j) \cdot v^j(x) > 0\}$ with $v^j(x)$, the unit normal vector of Ω_j at x pointing towards the exterior of Ω_j, defined almost everywhere on $\partial\Omega_j$, and belonging to $L^\infty(\partial\Omega_j; \mathbb{R}^n)$. The geometric condition (1.2) ensures that linear interior damping can stabilize exponentially the wave equation with the measure of ω being sufficiently small ([14]).

The aim of this paper is twofold: We design an unknown input state observer and an observer based output feedback stabilizing control for system (1.1). This is a nontrivial generalization of [8] where full state feedback stabilization was considered. It is also supposed in [8] that $f(\cdot) \equiv 0$; $d_t(x, t)$ is bounded; and $d_x(x, t)$ is Hölder continuous. In this paper, we only suppose that $d \in L^\infty(0, \infty; L^2(\Gamma_1))$. So $d(x, t)$ is allowed to be discontinuous in x. Therefore, the disturbance here is much more general than those in existing works by active disturbance rejection control.

It is well known that when the total disturbance is zero: $F(w(x, t)) = 0$, the collocated feedback control

$$v(x, t) = -k w_t(x, t), \ x \in \Gamma_1, \ t \ge 0, \ k > 0 \tag{1.3}$$

exponentially stabilizes system (1.1) provided that there exists a coercive smooth vector field $h(x) = (h_1(x), \ldots, h_n(x))$ of $C^2(\overline{\Omega})$ on Γ, that is, the following geometric optic condition is satisfied ([13]):

$$\begin{cases} \text{(i)} \ h \cdot v \le 0, \ a.e. \text{ on } \Gamma_0; \\ \text{(ii)} \ h \cdot v \ge \gamma > 0, \ a.e. \text{ on } \Gamma_1; \\ \text{(iii)} \ H(x) + H^\top(x) \text{ is uniformly positive definite on } \overline{\Omega}, \text{ where } H(x) = \{\partial h_i/\partial x_j\}_{i,j=1}^n. \end{cases} \tag{1.4}$$

The assumption (1.4) is satisfied when Ω is "star-complemented-star-shaped" ([2]), that is, there exists a point $x_0 \in \mathbb{R}^n$ such that

$$\begin{cases} (x - x_0) \cdot v \le 0 \text{ on } \Gamma_0, \ (\Gamma_0 \text{ is star complemented with respect to } x_0), \\ (x - x_0) \cdot v > 0 \text{ on } \Gamma_1, \ (\Gamma_1 \text{ is star shaped with respect to } x_0), \end{cases}$$

by setting $H(x) = I_{n \times n}$, $\rho = 1$, and $h(x) = x - x_0$, where $I_{n \times n}$ stands for n-dimensional identity matrix. In this case, the open subset ω in (1.2) can be simply taken as $\omega = \Omega \cap \{y \in \mathbb{R}^n : |y - x| < \varepsilon$ for $x \in \Gamma_1\}$. Notice that ε can be taken arbitrary small, which means that the measurement

$(w_t(x,t)|_{\Gamma_1}, w_t(x,t)|_{\omega})$ is essentially the measurement of $w_t(x,t)|_{\overline{\omega}}$. Therefore, our measurement for system (1.1) is essentially two signals, that is, $y(x,t) = (w(x,t)|_{\Gamma_1}, w_t(x,t)|_{\overline{\omega}})$.

We consider system (1.1) in the energy Hilbert state space $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ where $H^1_{\Gamma_0}(\Omega) = \{\phi \in H^1(\Omega)| \phi|_{\Gamma_0} = 0\}$ with the usual inner product given by

$$\langle (\phi_1, \psi_1)^{\top}, (\phi_2, \psi_2)^{\top} \rangle_{\mathcal{H}} = \int_{\Omega} [\nabla\phi_1(x)\overline{\nabla\phi_2(x)} + \psi_1(x)\overline{\psi_2(x)}]dx, \ \forall\, (\phi_i, \psi_i)^{\top} \in \mathcal{H}, \ i = 1, 2.$$

The control space is $U = L^2(\Gamma_1)$.

This paper adopts the strategy of active disturbance rejection control by estimating and compensating the total disturbance. The first problem we come up with is that weather our measured outputs are sufficient (and necessary in some extent) to achieve this goal. This problem is closely related to observability of uncertain PDEs.

Definition 1.1. Suppose $v(x,t) = 0$. System (1.1) is said to be exactly observable, if
   (i) When $F(w(\cdot,t)) = 0$, system (1.1) is exactly observable;
   (ii) $y(x,t)$ can identify $F(w(\cdot,t))$, that is,

$$y(\cdot,t) = 0, \ t \in [0,T] \Rightarrow F(w(\cdot,t)) = 0, \ t \in [0,T],$$

for any $T > 0$.

Proposition 1.1. *System (1.1) is exactly observable.*

Proof. The condition (i) is satisfied under the geometric condition (1.4) that there exist $T, C_T > 0$ such that ([16, Theorem 2.1])

$$\int_0^T \int_{\Gamma_1} w_t^2(x,t)dxdt \geq C_T \|(w(\cdot,0), w_t(\cdot,0))\|^2_{\mathcal{H}}, \ \forall\, (w(\cdot,0), w_t(\cdot,0)) \in \mathcal{H}.$$

To show the condition (ii), we write (1.1) with $y(x,t) = 0$ as

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t), \ x \in \Omega, \ t > 0, \\ w(x,t) = 0, \ x \in \partial\Omega, \ t \geq 0, \\ w_t(x,t)|_{\Gamma_1} = 0, \ w_t(x,t)|_{\omega} = 0, \ t \geq 0, \\ \dfrac{\partial w(x,t)}{\partial v}\Big|_{\Gamma_1} = f(w(\cdot,t)) + d(x,t), \ x \in \Gamma_1, \ t \geq 0. \end{cases} \tag{1.5}$$

Set

$$E(t) := \frac{1}{2}\int_{\Omega} [|\nabla w(x,t)|^2 + w_t^2(x,t)]dx.$$

Differentiate $E(t)$ with respect to t along the solution of (1.5) to obtain $\dot{E}(t) \equiv 0$ for all $t \in [0, T]$, which yields $E(t) = E(0)$ for all $t \in [0, T]$. By [14, Theorem 2.3 and Theorem 4.1], there exist $T, C_T > 0$ such that the following observability inequality holds:

$$\int_0^T \int_\omega w_t^2(x, t) dx dt \geq C_T \|(w(\cdot, 0), w_t(\cdot, 0))\|_{\mathcal{H}}^2, \quad \forall (w(\cdot, 0), w_t(\cdot, 0)) \in \mathcal{H}.$$

Since $w_t(x, t)|_\omega = 0$, we have $(w(\cdot, 0), w_t(\cdot, 0)) = 0$ and hence $E(t) \equiv 0$, which implies that $(w(\cdot, t), w_t(\cdot, t)) \equiv 0$. By the last equation of (1.5), we obtain $F(w(\cdot, t)) \equiv 0$.   □

Remark 1.1. From Proposition 1.1, we see that one of $w_t(x, t)|_{\Gamma_1}$ and $w_t(x, t)|_\omega$ is almost necessary for exact observability. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 | 1 < x_1^2 + x_2^2 < 4\}$ be a two-dimensional annulus and let $\Gamma_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$, $\Gamma_1 = \Gamma \setminus \Gamma_0$. For this example, we have

(a). The observation $y(x, t) = (w_t(x, t)|_{\Gamma_1}, w_t(x, t)|_\omega)$ is not enough for exact observability. Actually, let $f(w) \equiv 0$, and let $d(x, t) \equiv d$ be a constant. Then, the condition (1.4) is satisfied with $h(x) = x$. System (1.1) admits a solution $(w, w_t) = (d \ln(x_1^2 + x_2^2), 0)$ which makes the output $y(x, t) = (w_t(x, t)|_{\Gamma_1}, w_t(x, t)|_\omega) \equiv 0$.

(b). The $y(x, t) = (w(x, t)|_{\Gamma_1}, w_t(x, t)|_{\Gamma_1})$ is also not enough for exact observability. Actually, take initial value $(w(\cdot, 0), w_t(\cdot, 0)) = (\sin(\sqrt{x_1^2 + x_2^2}\pi), 0)$ where $w(\cdot, 0) \in H_0^1(\Omega)$. Then system (1.1) admits a solution $(w, w_t) \in C(0, \infty; H_0^1(\Omega) \times L^2(\Omega))$ for $f \equiv 0$ and $d(x, t) = \frac{\partial w(x,t)}{\partial \nu}|_{\Gamma_1}$. However, $w(\cdot, t)|_{\Gamma_1} \equiv 0$ and hence $w_t(\cdot, t)|_{\Gamma_1} \equiv 0$ makes $y(x, t) = (w(x, t)|_{\Gamma_1}, w_t(x, t)|_{\Gamma_1}) \equiv 0$.

From Remarks 1.1 and 2.2 in section 2, we see that the measurement $y(x, t) = (w(x, t)|_{\Gamma_1}, w_t(x, t)|_{\Gamma_1}, w_t(x, t)|_\omega)$ is almost the minimal signal to make system (1.1) exactly observable and (1.10) later well-posed. This ensures in particular that the output contains all information of the total disturbance, which gives possibility to estimate the total disturbance from the output.

Now, in order to state and prove our results conveniently, we will formulate the system (1.1) into an abstract form.

Define the operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}(\phi, \psi)^\top = (\psi, \Delta\phi)^\top, \ \forall (\phi, \psi)^\top \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ (\phi, \psi)^\top \in \mathcal{H} \cap (H^2(\Omega) \times H^1(\Omega)) \left| \frac{\partial \phi}{\partial \nu} \right|_{\Gamma_1} = \psi|_{\Gamma_0} = 0 \right\}. \end{cases} \tag{1.6}$$

It is easy to verify that $\mathcal{A}^* = -\mathcal{A}$ in $\mathcal{H}$. Define

$$A\phi = -\Delta\phi, \quad D(A) = \left\{ \phi \,\middle|\, \phi \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega), \frac{\partial \phi}{\partial \nu}\Big|_{\Gamma_1} = 0 \right\}.$$

Then A is a positive definite operator in $L^2(\Omega)$. It is easily shown (see e.g., [9]) that $D(A^{1/2}) = H_{\Gamma_0}^1(\Omega)$ and $A^{1/2}$ is a canonical isomorphism from $H_{\Gamma_0}^1(\Omega)$ onto $L^2(\Omega)$. We consider $L^2(\Omega)$ as the pivot space. Then, the following Gelfand triple inclusions are valid:

$$[D(A^{1/2})] \hookrightarrow L^2(\Omega) = (L^2(\Omega))' \hookrightarrow [D(A^{1/2})]',$$

where $[D(A^{1/2})]'$ is the dual space of $D(A^{1/2})$ with the pivot space $L^2(\Omega)$. An extension $\tilde{A} \in \mathcal{L}([D(A^{1/2})], [D(A^{1/2})]')$ of A is defined by

$$\langle \tilde{A}\phi, \psi \rangle_{[D(A^{1/2})]' \times [D(A^{1/2})]} = \langle A^{1/2}\phi, A^{1/2}\psi \rangle_{L^2(\Omega)}, \; \forall \, \phi, \psi \in D(A^{1/2}) = H^1_{\Gamma_0}(\Omega).$$

Define the Neumann map $\Upsilon \in \mathcal{L}(H^s(\Gamma_1), H^{3/2+s}(\Omega))$ ([15, p. 668]), i.e., $\Upsilon q = \hat{v}$ if and only if

$$\begin{cases} \Delta \hat{v} = 0 \text{ in } \Omega, \\ \hat{v}|_{\Gamma_0} = 0, \; \dfrac{\partial \hat{v}}{\partial \nu}\Big|_{\Gamma_1} = q \in H^s(\Gamma_1). \end{cases}$$

Using the Neumann map, one can write (1.1) in $[D(A^{1/2})]'$ as

$$\ddot{w} + \tilde{A}(w - \Upsilon(v + f(w) + d)) = 0,$$

which is further written as

$$\ddot{w} = -\tilde{A}w + B(v + f(w) + d), \tag{1.7}$$

where $B \in \mathcal{L}(U, [D(A^{1/2})]')$ is given by

$$Bu_0 = \tilde{A}\Upsilon u_0, \; \forall \, u_0 \in U.$$

Define $B^* \in \mathcal{L}([D(A^{1/2})], U)$, the adjoint of B, by

$$\langle B^*\phi, u_0 \rangle_U = \langle \phi, Bu_0 \rangle_{[D(A^{1/2})] \times [D(A^{1/2})]'}, \; \forall \, \phi \in D(A^{1/2}), \; u_0 \in U.$$

Then, for any $\phi \in D(A)$ and $u_0 \in C^\infty_0(\Gamma_1)$, by Green's formula,

$$\langle \phi, Bu_0 \rangle_{[D(A^{1/2})] \times [D(A^{1/2})]'} = \langle A\phi, \tilde{A}^{-1}Bu_0 \rangle_{L^2(\Omega)}$$

$$= \langle A\phi, \Upsilon u_0 \rangle_{L^2(\Omega)} = \langle A\phi, v_0 \rangle_{L^2(\Omega)} = -\int_\Omega \Delta\phi(x) v_0(x) dx$$

$$= -\int_\Omega \phi \Delta v_0(x) dx - \int_{\Gamma_0 \cup \Gamma_1} \frac{\partial \phi(x)}{\partial \nu} v_0(x) dx + \int_{\Gamma_0 \cup \Gamma_1} \phi(x) \frac{\partial v_0(x)}{\partial \nu} dx = \int_{\Gamma_1} \phi(x) u_0(x) dx$$

$$= \langle \phi, u_0 \rangle_U,$$

where $v_0 = \Upsilon u_0$. Since $C^\infty_0(\Gamma_1)$ is dense in $L^2(\Gamma_1)$, we obtain

$$B^*\phi = \phi|_{\Gamma_1}.$$

Therefore, system (1.1) can be written as

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}[f(w) + v + d], \qquad (1.8)$$

where $\mathcal{B} = (0, B)^\top$ and $\mathcal{B}^*$, the adjoint of $\mathcal{B}$, is given by

$$\mathcal{B}^*(\phi, \psi)^\top = \psi|_{\Gamma_1}, \ \forall\, (\phi, \psi) \in (H^1_{\Gamma_0}(\Omega))^2.$$

However, since $\mathcal{B}$ is not admissible for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ on $\mathcal{H}$ (see [23] and [15, p. 669]), system (1.8) does not always admit a unique solution in $\mathcal{H}$ for general $v, d \in L^2_{loc}(0, \infty, U)$.

To overcome this difficulty, we first introduce a damping on the control boundary by designing

$$v(x, t) = -kw_t(x, t) + u(x, t), \ k > 0, \ \forall\, x \in \Gamma_1, \ t \geq 0, \qquad (1.9)$$

where the gain k is a positive constant and $u(x, t)$ is the new control. Under (1.9), system (1.1) becomes

$$\begin{cases} w_{tt}(x, t) = \Delta w(x, t), \ x \in \Omega, \ t > 0, \\ w(x, t)|_{\Gamma_0} = 0, \ t \geq 0, \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Gamma_1} = -kw_t(x, t) + f(w(x, t)) + d(x, t) + u(x, t), \ t > 0, \\ w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x), \ x \in \Omega. \end{cases} \qquad (1.10)$$

Exactly the same as from (1.1) to (1.7), we can write (1.10) as

$$\ddot{w} = -\tilde{A}w - kBB^*\dot{w} + B(f(w) + u + d) \text{ in } [D(A^{1/2})]' \qquad (1.11)$$

or in the first order form

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathbb{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathbb{B}(f(w) + u + d) \text{ in } [D(A^{1/2})] \times [D(A^{1/2})]',$$

where the operators $\mathbb{A}$ and $\mathbb{B}$ are given by

$$\begin{cases} \mathbb{A}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ -\tilde{A}\phi - kBB^*\psi \end{pmatrix}, \ \forall\, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in D(\mathbb{A}), \\ D(\mathbb{A}) = \{(\phi, \psi)^\top \,|\, \phi, \psi \in D(A^{1/2}), \ \tilde{A}\phi + kBB^*\psi \in L^2(\Omega)\}, \\ \mathbb{B} = \mathcal{B}. \end{cases} \qquad (1.12)$$

The Proposition 1.2 presents a sufficient condition for well-posedness of system (1.10).

Proposition 1.2. *The operator $\mathbb{A}$ defined in (1.12) generates a C_0-semigroup of contractions $e^{\mathbb{A}t}$ on $\mathcal{H}$ and $\mathbb{B}$ is admissible for the semigroup $e^{\mathbb{A}t}$. Suppose that $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ is continuous with $f(0) = 0$ and satisfies the local Lipschitz condition in $H^1_{\Gamma_0}(\Omega)$. Then, for any initial value $(w_0, w_1)^\top \in \mathcal{H}$, control $u \in L^2_{loc}(0, \infty; L^2(\Gamma_1))$, and disturbance $d \in L^2_{loc}(0, \infty; L^2(\Gamma_1))$,*

system (1.10) admits a unique local solution $(w, w_t)^\top \in C(0, T; \mathcal{H})$ for some $T > 0$ such that for $t \in [0, T)$,

$$\begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = e^{\mathbb{A}t} \begin{pmatrix} w_0(\cdot) \\ w_1(\cdot) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}[u(\cdot, s) + d(\cdot, s)] ds + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B} f(w(\cdot, s)) ds.$$

(1.13)

Moreover, if $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ satisfies the global Lipschitz condition:

$$\|f(\phi_1) - f(\phi_2)\|_{L^2(\Gamma_1)} \leq L \|\phi_1 - \phi_2\|_{H^1_{\Gamma_0}(\Omega)}, \ \forall \phi_1, \phi_2 \in H^1_{\Gamma_0}(\Omega)$$

for some $L > 0$, then (1.10) admits a unique global solution $(w, w_t)^\top \in C(0, \infty; \mathcal{H})$ satisfying (1.13) with $T = +\infty$.

Proof. By [8], we know that $\mathbb{A}$ generates a C_0-semigroup $e^{\mathbb{A}t}$ of contractions on $\mathcal{H}$ and $\mathbb{B}$ is admissible for the semigroup $e^{\mathbb{A}t}$. Therefore, for any fixed $T > 0$, and any given $u, d \in L^2_{loc}(0, \infty; L^2(\Gamma_1))$,

$$\int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}[u(\cdot, s) + d(\cdot, s)] ds \in C(0, T; \mathcal{H}).$$

For initial value $(w_0, w_1)^\top$, let $(\eta_1(t), \eta_2(t))^\top = e^{\mathbb{A}t}(w_0, w_1)^\top$. For any given $\sigma > \max_{0 \leq t \leq 1} \|\eta_1(t)\|_{H^1_{\Gamma_0}(\Omega)} > 0$ and $t \in [0, 1]$, define a set Λ_t by

$$\Lambda_t = \{\phi : (\phi, \psi) \in \mathcal{H}, \ \|\phi - \eta_1(t)\|_{H^1_{\Gamma_0}(\Omega)} \leq \sigma\}.$$

Since $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ satisfies the local Lipschitz condition, there exists a constant $L_\sigma > 0$ independent of t such that

$$\|f(\phi_1) - f(\phi_2)\|_{L^2(\Gamma_1)} \leq L_\sigma \|\phi_1 - \phi_2\|_{H^1_{\Gamma_0}(\Omega)}, \ \forall \phi_1, \phi_2 \in \Lambda_t.$$

The admissibility of $\mathbb{B}$ implies that for all $t > 0$, and $\zeta \in L^\infty(0, t; L^2(\Gamma_1))$,

$$\left\| \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B} \zeta(\cdot, s) ds \right\|_{\mathcal{H}} \leq C_t \|\zeta(\cdot, s)\|_{L^2(0,t; L^2(\Gamma_1))} \leq C_t \sqrt{t} \|\zeta(\cdot, s)\|_{L^\infty(0,t; L^2(\Gamma_1))}$$

(1.14)

for some constant C_t that is independent of ζ. By [23, Proposition 2.3], we know that C_t is nondecreasing in t. Let $\tau \leq 1$. Then $C_\tau \leq C_1$. Choose τ so that $C_1 \sqrt{\tau} L_\sigma < 1$ and

$$C_1 \sqrt{\tau} L_\sigma \left(\sigma + \left\| e^{\mathbb{A}t} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}[u(\cdot, s) + d(\cdot, s)] ds \right\|_{C(0,1; \mathcal{H})} \right) < \sigma.$$

(1.15)

Let

$$
\Theta = \left\{
\begin{array}{l}
(\varphi(\cdot,t), \varphi_t(\cdot,t))^\top \in C(0,\tau;\mathcal{H}) : \varphi(\cdot,0) = w_0(\cdot),\ \varphi_t(\cdot,0) = w_1(\cdot) \\[2mm]
\left\| \begin{pmatrix} \varphi(\cdot,t) \\ \varphi_t(\cdot,t) \end{pmatrix} - e^{\mathbb{A}t} \begin{pmatrix} w_0(\cdot) \\ w_1(\cdot) \end{pmatrix} - \displaystyle\int_0^t e^{\mathbb{A}(t-s)} \mathbb{B}[u(\cdot,s) + d(\cdot,s)]ds \right\|_{\mathcal{H}} \leq \sigma
\end{array}
\right\}
$$

be a closed subset of $C(0,\tau;\mathcal{H})$. Define the nonlinear map $\mathcal{F}$ from Θ to $C(0,\tau;\mathcal{H})$ by

$$
\mathcal{F}\begin{pmatrix} \varphi(\cdot,t) \\ \varphi_t(\cdot,t) \end{pmatrix} = e^{\mathbb{A}t}\begin{pmatrix} w_0(\cdot) \\ w_1(\cdot) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}[u(\cdot,s)+d(\cdot,s)]ds + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}f(\varphi(\cdot,s))ds.
$$

$$(1.16)$$

It follows from (1.14) and (1.16) that for any $(\varphi_1, \varphi_{1t})^\top, (\varphi_2, \varphi_{2t})^\top \in \Theta$,

$$
\begin{aligned}
\left\| \mathcal{F}\begin{pmatrix} \varphi_1(\cdot,t) \\ \varphi_{1t}(\cdot,t) \end{pmatrix} - \mathcal{F}\begin{pmatrix} \varphi_2(\cdot,t) \\ \varphi_{2t}(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} &= \left\| \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}[f(\varphi_1(\cdot,s)) - f(\varphi_2(\cdot,s))]ds \right\|_{\mathcal{H}} \\
&\leq C_t\sqrt{t}\|f(\varphi_1(\cdot,s)) - f(\varphi_2(\cdot,s))\|_{L^\infty(0,t;L^2(\Gamma_1))} \\
&\leq C_1\sqrt{\tau}\|f(\varphi_1(\cdot,s)) - f(\varphi_2(\cdot,s))\|_{L^\infty(0,\tau;L^2(\Gamma_1))} \\
&\leq C_1\sqrt{\tau}L_\sigma\|\varphi_1(\cdot,s) - \varphi_2(\cdot,s)\|_{L^\infty(0,\tau;H^1_{\Gamma_1}(\Omega))} \\
&\leq C_1\sqrt{\tau}L_\sigma\left\| \begin{pmatrix} \varphi_1(\cdot,t) \\ \varphi_{1t}(\cdot,t) \end{pmatrix} - \begin{pmatrix} \varphi_2(\cdot,t) \\ \varphi_{2t}(\cdot,t) \end{pmatrix} \right\|_{C(0,\tau;\mathcal{H})},
\end{aligned}
$$

$$(1.17)$$

from which and (1.15), we can see that $\mathcal{F}\Theta \subset \Theta$. Thus, $\mathcal{F}$ is strictly contraction on Θ. By the contraction mapping theorem, (1.16) admits a unique fixed point $(w, w_t)^\top \in C(0,\tau;\mathcal{H})$ which satisfies (1.13).

Now, we prove the second assertion. Let $[0,T)$ be the maximal interval of existence of the solution of (1.10). If $T < \infty$, then it follows from (1.14) that for $t \in [0,T)$,

$$
\begin{aligned}
\left\| \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}f(w(\cdot,s))ds \right\|_{\mathcal{H}}^2 &\leq C_t^2\|f(w(\cdot,s))\|_{L^2(0,t;L^2(\Gamma_1))}^2 \\
&\leq C_T^2\|f(w(\cdot,s))\|_{L^2(0,t;L^2(\Gamma_1))}^2 \leq C_T^2 L^2\|w(\cdot,s)\|_{L^2(0,t;H^1_{\Gamma_0}(\Omega))}^2 \\
&= C_T^2 L^2\int_0^t \|w(\cdot,s)\|_{H^1_{\Gamma_0}(\Omega)}^2 ds \leq C_T^2 L^2\int_0^t \left\| \begin{pmatrix} w(\cdot,s) \\ w_t(\cdot,s) \end{pmatrix} \right\|_{\mathcal{H}}^2 ds.
\end{aligned}
$$

$$(1.18)$$

Since the solution over $[0,T)$ satisfies (1.13), by (1.18), it follows that

$$\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}\right\|^2_{\mathcal{H}} \le 2\left\|e^{\mathbb{A}t}\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}[u(\cdot,s)+d(\cdot,s)]ds\right\|^2_{\mathcal{H}}$$

$$+2\left\|\int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}f(w(\cdot,s))ds\right\|^2_{\mathcal{H}}$$

$$\le 2\max_{t\in[0,T]}\left\|e^{\mathbb{A}t}\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}[u(\cdot,s)+d(\cdot,s)]ds\right\|^2_{\mathcal{H}}$$

$$+2C_T^2 L^2\int_0^t\left\|\begin{pmatrix} w(\cdot,s) \\ w_t(\cdot,s) \end{pmatrix}\right\|^2_{\mathcal{H}}ds,$$

which, by using Gronwall's inequality, yields

$$\left\|\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}\right\|^2_{\mathcal{H}} \le 2\max_{t\in[0,T]}\left\|e^{\mathbb{A}t}\begin{pmatrix} w(\cdot,0) \\ w_t(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}[u(\cdot,s)+d(\cdot,s)]ds\right\|^2_{\mathcal{H}}e^{2C_T^2 L^2 T},$$

that is, $(w,\dot{w})^\top$ is bounded on $\mathcal{H}$ over $[0,T]$. Since $T<\infty$, similarly to the proof of the existence of local solution, we can prove that (1.10) has a unique solution over $[0,T+\delta_0)$ for some $\delta_0 > 0$. This is a contradiction. This proves that (1.10) admits a unique global solution. □

Let us briefly indicate the main contributions of this paper. First, we design an extended state observer from which the state of system can be recovered asymptotically and the total disturbance can be estimated. Second, the disturbance is canceled in the feedback loop and a stabilizing output feedback control is then designed. The closed-loop system is shown to be asymptotically stable by guaranteeing that all subsystems are uniformly bounded.

We proceed as follows. In Section 2, we state the main results. These include, the procedure of design of an extended state observer that not only estimates the system state but also the total disturbance, the well-posedness of the extended state observer, and the stability of the closed-loop system. Section 3 is devoted to the proof of the main results. Some numerical simulations are presented in Section 4 for illustration. Finally, to end this section, we stipulate that in the rest of the paper, all obvious domains both for time and spatial variables are dropped without confusion.

2. The main results

2.1. Extended state observer design

In this subsection, we list the outline of design of an extended state observer in terms of the input and output of system (1.10). The extended state observer is a special unknown input observer served for estimation of total disturbance as well. The mathematical justification follows up in next section. We shall achieve this goal through the following steps.

Step 1: Separate the total disturbance from the original system to an exponentially stable system. Actually, design an auxiliary system "$z(x, t)$" as follows:

$$
\begin{cases}
z_{tt}(x, t) = \Delta z(x, t) - a(x)[z_t(x, t) - w_t(x, t)], \\
z(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial z(x, t)}{\partial \nu}\Big|_{\Gamma_1} = -k z_t(x, t) + u(x, t), \\
z(x, 0) = z_0(x), \; z_t(x, 0) = z_1(x),
\end{cases}
\tag{2.1}
$$

which is completely determined by output $w_t(x, t)|_\omega$ and input $u(x, t)|_{\Gamma_1}$, and $a : \Omega \to [0, +\infty)$ in (2.1) is chosen to be continuous with $\mathrm{supp}(a) \subset \omega$ and there exists an open set $\omega_o \subset\subset \omega$ such that $a(x) > 0$ for $x \in \omega_o$, where ω_o also satisfies the condition (1.2). In other words, system (2.1) is a completely known system. Let

$$
\widehat{z}(x, t) = z(x, t) - w(x, t)
\tag{2.2}
$$

be the error of z-system (2.1) and original system (1.1). Then $\widehat{z}(x, t)$ satisfies

$$
\begin{cases}
\widehat{z}_{tt}(x, t) = \Delta \widehat{z}(x, t) - a(x)\widehat{z}_t(x, t), \\
\widehat{z}(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widehat{z}(x, t)}{\partial \nu}\Big|_{\Gamma_1} = -k \widehat{z}_t(x, t) - f(w(\cdot, t)) - d(x, t), \\
\widehat{z}(x, 0) = z_0(x) - w_0(x), \; \widehat{z}_t(x, 0) = z_1(x) - w_1(x).
\end{cases}
\tag{2.3}
$$

The system (2.3) is just the system that we are looking for because its linear part is exponentially stable and the inhomogeneous part of (2.3) is just the total disturbance. System (2.3) is our starting point to estimate the total disturbance.

Step 2: Design a "$\widehat{d}(x, t)$" system from (2.3) to estimate the total disturbance. For this purpose, design

$$
\begin{cases}
\widehat{d}_{tt}(x, t) = \Delta \widehat{d}(x, t) - a(x)\widehat{d}_t(x, t), \\
\widehat{d}(x, t)|_{\Gamma_0} = 0, \\
\widehat{d}(x, t)|_{\Gamma_1} = -[z(x, t) - w(x, t)], \\
\widehat{d}(x, 0) = \widehat{d}_0(x), \; \widehat{d}_t(x, 0) = \widehat{d}_1(x),
\end{cases}
\tag{2.4}
$$

which is determined by $z(x, t)$ and output $w(x, t)|_{\Gamma_1}$ and hence (2.4) is determined by input and output only. Let $\widetilde{d}(x, t) = \widehat{z}(x, t) + \widehat{d}(x, t)$. Then $\widetilde{d}(x, t)$ satisfies

$$
\begin{cases}
\widetilde{d}_{tt}(x, t) = \Delta \widetilde{d}(x, t) - a(x)\widetilde{d}_t(x, t), \\
\widetilde{d}(x, t)|_{\Gamma = \Gamma_0 \cup \Gamma_1} = 0,
\end{cases}
\tag{2.5}
$$

which is exponentially stable ([14]). In other words, system (2.4) is an unknown input observer for system (2.3) that $-\widehat{d}(x, t)$ gives an estimate of $\widehat{z}(x, t)$. Now,

$$\frac{\partial \widetilde{d}(x,t)}{\partial v} = \frac{\partial \widehat{d}(x,t)}{\partial v} + k\widehat{d_t}(x,t) - F(w(\cdot,t)). \tag{2.6}$$

Since we can also show that

$$\frac{\partial \widetilde{d}}{\partial v} \in L^2(0, \infty; L^2(\Gamma_1)), \tag{2.7}$$

which means that the error between $\frac{\partial \widehat{d}(x,t)}{\partial v} + k\widehat{d_t}(x,t)$ and $F(w(\cdot,t))$ is at most of an error of $L^2(0, \infty; L^2(\Gamma_1))$, we can therefore consider

$$\frac{\partial \widehat{d}(x,t)}{\partial v} + k\widehat{d_t}(x,t) \approx F(w(\cdot,t)). \tag{2.8}$$

The error system (2.5) is independent of total disturbance ($\widetilde{d}(x,0)$ depends actually on the initial value $w(x,0)$ only because $z(x,0)$ and $\widehat{d}(x,0)$ can be assigned arbitrarily). In other words, no matter what the total disturbance is, the convergence rate is always the same or equivalently, the total disturbance is sufficiently estimated by (2.4). This is a remarkable merit of this design.

Since $\widetilde{d}(x,t) = \widehat{z}(x,t) + \widehat{d}(x,t) = z(x,t) + \widehat{d}(x,t) - w(x,t)$, it follows from the exponential stability of (2.5) that $z(x,t) + \widehat{d}(x,t)$ gives an estimate of $w(x,t)$. In other words, the systems (2.1) and (2.4) together give an unknown input observer for original system (1.1). This is the first time we have obtained an unknown input observer for system (1.1). However, since control appears explicitly in (2.1) only, it is a little bit not convenient to take care of (2.5) for design control. To round this obstacle, we present Step 3.

Step 3: Compensate the total disturbance to obtain a state observer. Actually let

$$\begin{cases} \widehat{w}_{tt}(x,t) = \Delta\widehat{w}(x,t) - a(x)[\widehat{w}_t(x,t) - w_t(x,t)], \\ \widehat{w}(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial \widehat{w}(x,t)}{\partial v}\Big|_{\Gamma_1} = -k\widehat{w}_t(x,t) + \dfrac{\partial \widehat{d}(x,t)}{\partial v} + k\widehat{d_t}(x,t) + u(x,t), \\ \widehat{w}(x,0) = \widehat{w}_0(x), \ \widehat{w}_t(x,0) = \widehat{w}_1(x), \end{cases} \tag{2.9}$$

which is determined by input and output. Then $\widetilde{w}(x,t) = \widehat{w}(x,t) - w(x,t)$ where $w(x,t)$ is the solution of (1.10), satisfies

$$\begin{cases} \widetilde{w}_{tt}(x,t) = \Delta\widetilde{w}(x,t) - a(x)\widetilde{w}_t(x,t), \\ \widetilde{w}(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial \widetilde{w}(x,t)}{\partial v}\Big|_{\Gamma_1} = -k\widetilde{w}_t(x,t) + \dfrac{\partial \widetilde{d}(x,t)}{\partial v}, \end{cases} \tag{2.10}$$

which could be shown to be asymptotically stable due to "smallness" of inhomogeneous term $\frac{\partial \widetilde{d}}{\partial v} \in L^2(0, \infty; L^2(\Gamma_1))$. Therefore $\widehat{w}(x,t)$ can be regarded as an estimate of $w(x,t)$ as $t \to +\infty$. In other words, (2.9) is served as another unknown input observer for original system (1.1). This is the second unknown input observer we designed for system (1.1). System (2.10) is more convenient for control design.

Finally, putting all these systems (2.1), (2.4), and (2.9) together, we obtain an extended state observer for system (1.10) as follows:

$$
\begin{cases}
\widehat{w}_{tt}(x,t) = \Delta \widehat{w}(x,t) - a(x)[\widehat{w}_t(x,t) - w_t(x,t)], \\
\widehat{w}(x,t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widehat{w}(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -k\widehat{w}_t(x,t) + \dfrac{\partial \widehat{d}(x,t)}{\partial \nu} + k\widehat{d}_t(x,t) + u(x,t), \\
\widehat{d}_{tt}(x,t) = \Delta \widehat{d}(x,t) - a(x)\widehat{d}_t(x,t), \\
\widehat{d}(x,t)|_{\Gamma_0} = 0, \quad \widehat{d}(x,t)|_{\Gamma_1} = -[z(x,t) - w(x,t)], \\
z_{tt}(x,t) = \Delta z(x,t) - a(x)[z_t(x,t) - w_t(x,t)], \\
z(x,t)|_{\Gamma_0} = 0, \quad \dfrac{\partial z(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -kz_t(x,t) + u(x,t), \\
\widehat{w}(x,0) = \widehat{w}_0(x), \quad \widehat{w}_t(x,0) = \widehat{w}_1(x), \\
\widehat{d}(x,0) = \widehat{d}_0(x), \quad \widehat{d}_t(x,0) = \widehat{d}_1(x), \\
z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x),
\end{cases}
\tag{2.11}
$$

which estimates not only the state but also the total disturbance.

2.2. Well-posedness of extended state observer

In this subsection, we explain the well-posedness of extended state observer (2.11). Lemma 2.1 is on system (2.3).

Lemma 2.1. *For any* $u \in L^2_{loc}(0,\infty; L^2(\Gamma_1))$, $d \in L^2_{loc}(0,\infty; L^2(\Gamma_1))$, *and* $(w_0, w_1) \in \mathcal{H}$, *suppose that* $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ *is continuous with* $f(0) = 0$ *and that (1.10) admits a unique solution* $(w, w_t)^\top \in C(0,\infty; \mathcal{H})$. *Then, for any initial state* $(\widehat{z}_0, \widehat{z}_1) \in \mathcal{H}$, *system (2.3) admits a unique solution* $(\widehat{z}, \widehat{z}_t) \in C(0,\infty; \mathcal{H})$. *Moreover, if* $d \in L^\infty(0,\infty; L^2(\Gamma_1))$ *and the solution* $(w(x,t), w_t(x,t))^\top$ *of (1.10) is bounded, then* $(\widehat{z}(x,t), \widehat{z}_t(x,t))^\top$ *is also bounded, i.e.,*

$$
\sup_{t \geq 0} \|(\widehat{z}(\cdot,t), \widehat{z}_t(\cdot,t))\|_{\mathcal{H}} < +\infty.
$$

Next, we consider (2.5) and (2.10) together, namely, the following system:

$$
\begin{cases}
\widetilde{w}_{tt}(x,t) = \Delta \widetilde{w}(x,t) - a(x)\widetilde{w}_t(x,t), \\
\widetilde{w}(x,t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widetilde{w}(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -k\widetilde{w}_t(x,t) + \dfrac{\partial \widetilde{d}(x,t)}{\partial \nu}, \\
\widetilde{d}_{tt}(x,t) = \Delta \widetilde{d}(x,t) - a(x)\widetilde{d}_t(x,t), \\
\widetilde{d}(x,t)|_{\Gamma = \Gamma_0 \cup \Gamma_1} = 0, \\
\widetilde{w}(x,0) = \widetilde{w}_0(x), \quad \widetilde{w}_t(x,0) = \widetilde{w}_1(x), \\
\widetilde{d}(x,0) = \widetilde{d}_0(x), \quad \widetilde{d}_t(x,0) = \widetilde{d}_1(x).
\end{cases}
\tag{2.12}
$$

Let $\mathbb{H} = H_0^1(\Omega) \times L^2(\Omega)$. It is well known ([14]) that the "$\widetilde{d}$-part" of system (2.12) admits a unique solution $(\widetilde{d}, \widetilde{d}_t) \in C(0, \infty; \mathbb{H})$ such that

$$E(t) := \frac{1}{2} \int_\Omega \left[|\nabla \widetilde{d}(x,t)|^2 + |\widetilde{d}_t(x,t)|^2 \right] dx \le M_d e^{-\omega_d t} \|(\widetilde{d}(\cdot, 0), \widetilde{d}_t(\cdot, 0))\|_{\mathbb{H}}^2, \tag{2.13}$$

for some constants $M_d, \omega_d > 0$. It is worth mentioning that the decay rate ω_d depends on the damping coefficient $a(x)$, and the measure of the interior observation domain ω, but independent of total disturbance. The last property is one of the remarkable merits of this design.

We consider system (2.12) in the energy Hilbert state space $\mathcal{X} = \mathcal{H} \times \mathbb{H}$ with the usual inner product given by

$$\langle (\phi_1, \psi_1, p_1, q_1)^\top, (\phi_2, \psi_2, p_2, q_2)^\top \rangle_{\mathcal{X}} = \int_\Omega [\nabla \phi_1(x) \overline{\nabla \phi_2(x)} + \psi_1(x) \overline{\psi_2(x)}] dx$$
$$+ \int_\Omega [\nabla p_1(x) \overline{\nabla p_2(x)} + q_1(x) \overline{q_2(x)}] dx, \ \forall \ (\phi_i, \psi_i, p_i, q_i)^\top \in \mathcal{X}, i = 1, 2. \tag{2.14}$$

Lemma 2.2. *For any initial value* $(\widetilde{w}_0, \widetilde{w}_1, \widetilde{d}_0, \widetilde{d}_1)^\top \in \mathcal{X}$, *system (2.12) admits a unique solution* $(\widetilde{w}, \widetilde{w}_t, \widetilde{d}, \widetilde{d}_t)^\top \in C(0, \infty; \mathcal{X})$ *satisfying*

$$\lim_{t \to \infty} \|(\widetilde{w}(\cdot, t), \widetilde{w}_t(\cdot, t), \widetilde{d}(\cdot, t), \widetilde{d}_t(\cdot, t))\|_{\mathcal{X}} = 0.$$

Moreover,

$$\frac{\partial \widetilde{d}}{\partial \nu} \in L^2(0, \infty; L^2(\Gamma_1)), \tag{2.15}$$

which is said to be the hidden regularity of the PDE satisfied by $\widetilde{d}(x, t)$.

Theorem 2.1 is about the well-posedness of the extended state observer (2.11). A first main result of present paper.

Theorem 2.1. *For any* $u \in L_{loc}^2(0, \infty; L^2(\Gamma_1))$, $d \in L_{loc}^2(0, \infty; L^2(\Gamma_1))$, *and* $(w_0, w_1) \in \mathcal{H}$, *suppose that* $f : H_{\Gamma_0}^1(\Omega) \to L^2(\Gamma_1)$ *is continuous with* $f(0) = 0$ *and that (1.10) admits a unique solution* $(w, \dot{w})^\top \in C(0, \infty; \mathcal{H})$. *Then, the extended state observer (2.11) is well-posed and for any* $(\widehat{w}_0, \widehat{w}_1, \widehat{d}_0, \widehat{d}_1, z_0, z_1) \in \mathcal{H}^3$ *with compatible condition*

$$\widehat{d}_0(x) + z_0(x) - w_0(x) = 0 \ on \ \Gamma_1,$$

(2.11) admits a unique solution $(\widehat{w}, \widehat{w}_t, \widehat{d}, \widehat{d}_t, z, z_t) \in C(0, \infty; \mathcal{H}^3)$. *Moreover,* $(\widehat{w}(x,t), \widehat{w}_t(x,t))$ *satisfies*

$$\lim_{t \to \infty} \|(\widehat{w}(\cdot, t) - w(\cdot, t), \widehat{w}_t(\cdot, t) - w_t(\cdot, t))\|_{\mathcal{H}} = 0, \tag{2.16}$$

and $\widehat{d}(x,t)$ satisfies

$$\frac{\partial \widehat{d}}{\partial v} + k\widehat{d}_t - F(w) \in L^2(0, \infty; L^2(\Gamma_1)). \tag{2.17}$$

Notice that (2.17) is just (2.7), in other words, (2.17) confirms the approximation (2.8). Now we want to show much stronger approximation than (2.17). To do this, we consider system (2.5) further. It is seen that system (2.5) is an independent system with the system operator given by

$$\begin{cases} \mathbb{A}_1(\phi, \psi)^\top = (\psi, \Delta\phi - a\psi)^\top, \ \forall \, (\phi, \psi)^\top \in D(\mathbb{A}_1), \\ D(\mathbb{A}_1) = \{(\phi, \psi)^\top \in \mathbb{H} \cap (H^2(\Omega) \times H^1(\Omega)) | \ \phi|_\Gamma = \psi|_\Gamma = 0\}. \end{cases} \tag{2.18}$$

Lemma 2.3. *For any given $(\widetilde{d}(\cdot, 0), \widetilde{d}_t(\cdot, 0))^\top \in D(\mathbb{A}_1)$, the solution of (2.5) satisfies*

$$\left\| \frac{\partial \widetilde{d}(\cdot, t)}{\partial v} \right\|_{L^2(\Gamma_1)} \leq Ce^{-\omega_{d*}t},$$

with some constants $C, \omega_{d*} > 0$, where C depends on $(\widetilde{d}(\cdot, 0), \widetilde{d}_t(\cdot, 0))$ only.

Corollary 2.1 is a direct consequence of Theorem 2.1 and Lemma 2.3.

Corollary 2.1. *Suppose that $(\widehat{w}_0, \widehat{w}_1, \widehat{d}_0, \widehat{d}_1, z_0, z_1) \in \mathcal{H}^3 \cap (H^2(\Omega) \times H^1(\Omega))^3$ with compatible condition*

$$\widehat{d}_0(x) + z_0(x) - w_0(x) = 0 \text{ on } \Gamma_1.$$

Then the "$(\widehat{d}, z)$-part" of (2.11) admits a unique solution $(\widehat{d}, z)$ such that

$$\left\| \frac{\partial \widehat{d}(\cdot, t)}{\partial v} + k\widehat{d}_t(\cdot, t) - F(w(\cdot, t)) \right\|_{L^2(\Gamma_1)} \leq Ce^{-\omega_d t}. \tag{2.19}$$

Theorem 2.2. *For any $u \in L^2_{loc}(0, \infty; L^2(\Gamma_1))$, $d \in L^2_{loc}(0, \infty; L^2(\Gamma_1))$, and $(w_0, w_1) \in \mathcal{H}$, suppose that $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ is continuous with $f(0) = 0$ and that (1.10) admits a unique solution $(w, \dot{w})^\top \in C(0, \infty; \mathcal{H})$. Then, the extended state observer (2.11) is well-posed and for any $(\widehat{w}_0, \widehat{w}_1, \widehat{d}_0, \widehat{d}_1, z_0, z_1) \in \mathcal{H}^3 \cap (H^2(\Omega) \times H^1(\Omega))^3$, (2.11) admits a unique solution $(\widehat{w}, \widehat{w}_t, \widehat{d}, \widehat{d}_t, z, z_t) \in C(0, \infty; \mathcal{H}^3)$. Moreover, $(\widehat{w}(x, t), \widehat{w}_t(x, t))$ satisfies*

$$\|(\widehat{w}(\cdot, t) - w(\cdot, t), \widehat{w}_t(\cdot, t) - w_t(\cdot, t)\|_\mathcal{H} \leq Me^{-\mu t}, \tag{2.20}$$

where $M, \mu > 0$.

2.3. Extended state observer based output feedback

In this subsection, we consider the second main problem of this paper: extended state observer based output feedback stabilization. By Lemmas 2.2 and 2.3, $\frac{\partial \widehat{d}(x,t)}{\partial \nu} + k\widehat{d}_t(x,t)$ can be regarded as an estimate of the total disturbance $F(w(\cdot,t))$. Thus, we can cancel the total disturbance by designing output feedback control to (1.10) as follows:

$$u(x,t) = -\frac{\partial \widehat{d}(x,t)}{\partial \nu} - k\widehat{d}_t(x,t). \tag{2.21}$$

Under feedbacks (1.9) and (2.21), the closed-loop system of (1.1) becomes

$$\begin{cases}
w_{tt}(x,t) = \Delta w(x,t), \\
w(x,t)|_{\Gamma_0} = 0, \\
\frac{\partial w}{\partial \nu}\Big|_{\Gamma_1} = -kw_t(x,t) + f(w(x,t)) + d(x,t) - \frac{\partial \widehat{d}(x,t)}{\partial \nu} - k\widehat{d}_t(x,t), \\
\widehat{w}_{tt}(x,t) = \Delta \widehat{w}(x,t) - a(x)[\widehat{w}_t(x,t) - w_t(x,t)], \\
\widehat{w}(x,t)|_{\Gamma_0} = 0, \quad \frac{\partial \widehat{w}}{\partial \nu}\Big|_{\Gamma_1} = -k\widehat{w}_t(x,t), \\
\widehat{d}_{tt}(x,t) = \Delta \widehat{d}(x,t) - a(x)\widehat{d}_t(x,t), \\
\widehat{d}(x,t)|_{\Gamma_0} = 0, \quad \widehat{d}(x,t)|_{\Gamma_1} = -[z(x,t) - w(x,t)], \\
z_{tt}(x,t) = \Delta z(x,t) - a(x)[z_t(x,t) - w_t(x,t)], \\
z(x,t)|_{\Gamma_0} = 0, \\
\frac{\partial z(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -kz_t(x,t) - \frac{\partial \widehat{d}(x,t)}{\partial \nu} - k\widehat{d}_t(x,t).
\end{cases} \tag{2.22}$$

Theorem 2.3 is about convergence of system (2.22), the second main result of this paper.

Theorem 2.3. *Suppose that $d \in L^\infty(0,+\infty, L^2(\Gamma_1))$ and $f \in C(H^1_{\Gamma_0}(\Omega), L^2(\Gamma_1))$ satisfies $f(0) = 0$. Then, for any initial state $(w_0, w_1, \widehat{w}_0, \widehat{w}_1, \widehat{d}_0, \widehat{d}_1, z_0, z_1)^\top \in \mathcal{H}^4$ with compatible condition*

$$\widehat{d}_0(x) + z_0(x) - w_0(x) = 0 \text{ on } \Gamma_1, \tag{2.23}$$

the closed-loop system (2.22) admits a unique solution $(w, w_t, \widehat{w}, \widehat{w}_t, \widehat{d}, \widehat{d}_t, z, z_t) \in C(0, \infty; \mathcal{H}^4)$ which satisfies

$$\lim_{t\to\infty} \|(w(\cdot,t), w_t(\cdot,t), \widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t))\|_{\mathcal{H}^2} = 0, \tag{2.24}$$

and

$$\sup_{t\geq 0} \|(\widehat{d}(\cdot,t), \widehat{d}_t(\cdot,t), z(\cdot,t), z_t(\cdot,t))\|_{\mathcal{H}^2} < +\infty. \tag{2.25}$$

Moreover, if we assume further that

$$(z_0 - w_0 + \widehat{d}_0, z_1 - w_1 + \widehat{d}_1)^\top \in D(\mathbb{A}_1)$$

where $D(\mathbb{A}_1)$ is defined in (2.18), then

$$\left\| \frac{\partial \widehat{d}(\cdot, t)}{\partial \nu} + k\widehat{d}_t(\cdot, t) - F(w(\cdot, t)) \right\|_{L^2(\Gamma_1)} \le Ce^{-\omega_d t}, \tag{2.26}$$

and there exist constants $M, \mu > 0$ such that

$$\|(w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t))\|_{\mathcal{H}^2} \le Me^{-\mu t}. \tag{2.27}$$

Remark 2.1. For the closed-loop system (2.22) in Theorem 2.3, different to Lemma 2.1, Theorems 2.1 and 2.2, we remove the assumption that the "w-part" of closed-loop system (2.22) admits a unique solution $(w, \dot{w})^\top \in C(0, \infty; \mathcal{H})$. This sounds extraordinary because the closed-loop system (2.22) is a nonlinear system. However, the closed system (2.22) is equivalent to system (2.28):

$$
\begin{cases}
w_{tt}(x, t) = \Delta w(x, t), \\
w(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial w(x, t)}{\partial \nu}\Big|_{\Gamma_1} = -kw_t(x, t) - \dfrac{\partial \widetilde{d}(x, t)}{\partial \nu}, \\
\widetilde{w}_{tt}(x, t) = \Delta \widetilde{w}(x, t) - a(x)\widetilde{w}_t(x, t), \\
\widetilde{w}(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widetilde{w}(x, t)}{\partial \nu}\Big|_{\Gamma_1} = -k\widetilde{w}_t(x, t) + \dfrac{\partial \widetilde{d}(x, t)}{\partial \nu}, \\
\widetilde{d}_{tt}(x, t) = \Delta \widetilde{d}(x, t) - a(x)\widetilde{d}_t(x, t), \\
\widetilde{d}(x, t)|_{\Gamma = \Gamma_0 \cup \Gamma_1} = 0, \\
\widehat{z}_{tt}(x, t) = \Delta \widehat{z}(x, t) - a(x)\widehat{z}_t(x, t), \\
\widehat{z}(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widehat{z}(x, t)}{\partial \nu}\Big|_{\Gamma_1} = -k\widehat{z}_t(x, t) - f(w(x, t)) - d(x, t),
\end{cases} \tag{2.28}
$$

which is (3.35) is next section, where we can see that the "$(w, \widetilde{w}, \widetilde{d})$-subsystem" of (2.28) admits a unique solution because it is linear and independent of "$\widehat{z}$-subsystem". The principal part of "$\widehat{z}$-subsystem" is linear and the nonlinear term $f(w(x, t))$ is its inhomogeneous term and is obtained from the linear "$(w, \widetilde{w}, \widetilde{d})$-subsystem". This interesting trick solves well-posedness of actual nonlinear system (2.22).

Remark 2.2. From Remark 1.1, the signal $w_t(x, t)|_{\Gamma_1}$ is used to make system (1.10) (Proposition 1.2) as well as extended state observer (2.11) well-posed (Theorem 2.1). In other words, if we consider stabilization of closed-loop system only without consideration of well-posedness

for open loop system and extended state observer, the output $y(x, t) = (w(x, t)|_{\Gamma_1}, w_t(x, t)|_\omega)$ is sufficient to be used to stabilize system (1.1). Indeed, setting $k = 0$ in (2.11) and replacing the control (2.21) by $u(x, t) = -\widehat{w}_t(x, t) - \frac{\partial \widehat{d}(x,t)}{\partial v}$, we have the closed-system as follows:

$$
\begin{cases}
w_{tt}(x, t) = \Delta w(x, t), \\
w(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial w}{\partial v}\Big|_{\Gamma_1} = -\widehat{w}_t(x, t) + f(w(x, t)) + d(x, t) - \dfrac{\partial \widehat{d}(x, t)}{\partial v}, \\
\widehat{w}_{tt}(x, t) = \Delta \widehat{w}(x, t) - a(x)[\widehat{w}_t(x, t) - w_t(x, t)], \\
\widehat{w}(x, t)|_{\Gamma_0} = 0, \quad \dfrac{\partial \widehat{w}}{\partial v}\Big|_{\Gamma_1} = -\widehat{w}_t(x, t), \\
\widehat{d}_{tt}(x, t) = \Delta \widehat{d}(x, t) - a(x)\widehat{d}_t(x, t), \\
\widehat{d}(x, t)|_{\Gamma_0} = 0, \quad \widehat{d}(x, t)|_{\Gamma_1} = -[z(x, t) - w(x, t)], \\
z_{tt}(x, t) = \Delta z(x, t) - a(x)[z_t(x, t) - w_t(x, t)], \\
z(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial z(x, t)}{\partial v}\Big|_{\Gamma_1} = -\dfrac{\partial \widehat{d}(x, t)}{\partial v} - \widehat{w}_t(x, t).
\end{cases}
\tag{2.29}
$$

Following the proofs of Lemmas 2.1 and 2.2, and Theorem 2.3 in next section with slight modification, we can prove that for any initial state $(w_0, w_1, \widehat{w}_0, \widehat{w}_1, \widehat{d}_0, \widehat{d}_1, z_0, z_1)^\top \in \mathcal{H}^4$ with compatible condition $\widehat{d}_0(x) + z_0(x) - w_0(x) = 0$ on Γ_1, the closed-loop system (2.29) admits a unique solution $(w, w_t, \widehat{w}, \widehat{w}_t, \widehat{d}, \widehat{d}_t, z, z_t) \in C(0, \infty; \mathcal{H}^4)$ satisfying $\lim_{t \to \infty} \|(w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t))\|_{\mathcal{H}^2} = 0$ and $\sup_{t \geq 0} \|(\widehat{d}(\cdot, t), \widehat{d}_t(\cdot, t), z(\cdot, t), z_t(\cdot, t))\|_{\mathcal{H}^2} < +\infty$. As pointed in [15, p. 669], the system (1.1) is not well-posed even if $f \equiv 0$ and $d \equiv 0$ when the dimension $n \geq 2$. In this case, the extended state observer (2.11) with $k = 0$ is not well-posed. Since the solvability of observer is important as indicated in [7], we include signal $w_t(x, t)|_{\Gamma_1}$ in the output to make (1.10) well-posed.

3. Proof of the main results

Proof of Lemma 2.1. We first notice that system (2.3) can be rewritten as an evolution equation in $\mathcal{H}$:

$$
\frac{d}{dt}(\widehat{z}(\cdot, t), \widehat{z}_t(\cdot, t))^\top = \mathbb{A}_0(\widehat{z}(\cdot, t), \widehat{z}_t(\cdot, t))^\top + \mathbb{B}(f(w(\cdot, t)) + d(\cdot, t)),
\tag{3.1}
$$

where $\mathbb{B}$ is the same as that in (1.12) and the operator $\mathbb{A}_0$ is given by

$$
\begin{cases}
\mathbb{A}_0(\phi, \psi)^\top = (\psi, \Delta\phi - a\psi)^\top, \ \forall (\phi, \psi)^\top \in D(\mathbb{A}_0), \\
D(\mathbb{A}_0) = \left\{(\phi, \psi)^\top \in \mathcal{H} \cap (H^2(\Omega) \times H^1_{\Gamma_0}(\Omega)) : \dfrac{\partial\phi}{\partial v}\Big|_{\Gamma_1} = -k\psi|_{\Gamma_1}\right\}.
\end{cases}
\tag{3.2}
$$

We will show that system (3.1) has a unique solution. Since $(w, w_t)^\top \in C(0, \infty; \mathcal{H})$ and $f : H^1_{\Gamma_0}(\Omega) \to L^2(\Gamma_1)$ is continuous, we know that $f(w(\cdot, \cdot)) \in C(0, \infty; L^2(\Gamma_1)))$, and thus

it suffices to prove that $\mathbb{A}_0$ generates a C_0-semigroup $e^{\mathbb{A}_0 t}$ and $\mathbb{B}$ is admissible for the semigroup $e^{\mathbb{A}_0 t}$ ([23]). Actually, for any $(\phi, \psi)^\top \in D(\mathbb{A}_0)$,

$$\mathrm{Re}\langle \mathbb{A}_0(\phi, \psi)^\top, (\phi, \psi)^\top \rangle_{\mathcal{H}} = \mathrm{Re} \int_\Omega \left[\nabla \psi(x) \cdot \overline{\nabla \phi(x)} + (\Delta \phi(x) - a\psi(x)) \overline{\psi(x)} \right] dx$$

$$= -k \int_{\Gamma_1} |\psi(x)|^2 dx - \int_\Omega a(x) |\psi(x)|^2 dx \leq 0. \tag{3.3}$$

This shows that $\mathbb{A}_0$ is dissipative. Now we show that $\mathbb{A}_0^{-1} \in \mathcal{L}(\mathcal{H})$. Solve the equation:

$$\mathbb{A}_0 \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \phi'' - a\psi \end{pmatrix} = \begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} \in \mathcal{H}$$

to obtain $\psi = \widehat{\phi} \in H^1_{\Gamma_0}(\Omega)$, and

$$\begin{cases} \Delta \phi(x) - a(x)\psi(x) = \widehat{\psi}(x), \\ \phi|_{\Gamma_0} = 0, \quad \dfrac{\partial \phi}{\partial \nu}\Big|_{\Gamma_1} = -k\psi|_{\Gamma_1}. \end{cases} \tag{3.4}$$

By the trace theorem and $\psi = \widehat{\phi} \in H^1_{\Gamma_0}(\Omega)$, (3.4) can be rewritten as

$$\begin{cases} \Delta \phi(x) = \widehat{\psi}(x) + a(x)\widehat{\phi}(x) \in L^2(\Omega), \ x \in \Omega, \\ \phi|_{\Gamma_0} = 0, \quad \dfrac{\partial \phi}{\partial \nu}\Big|_{\Gamma_1} = -k\widehat{\phi}|_{\Gamma_1} \in H^{1/2}(\Gamma_1). \end{cases} \tag{3.5}$$

By the elliptic partial differential equation theory, we know that (3.5) admits a unique solution $\phi \in H^2(\Omega)$ satisfying

$$\|\phi\|_{H^2(\Omega)} \leq C \left[\|\widehat{\psi} + a\widehat{\phi}\|_{L^2(\Omega)} + \|\widehat{\phi}\|_{H^{1/2}(\Gamma_1)} \right].$$

Hence $\mathbb{A}_0^{-1}(\widehat{\phi}, \widehat{\psi})^\top = (\phi, \widehat{\phi})$, where ϕ is uniquely defined by (3.5). It follows from the Lumer–Phillips theorem [17, Theorem 1.4.3] that $\mathbb{A}_0$ generates a C_0-semigroup of contractions $e^{\mathbb{A}_0 t}$ on $\mathcal{H}$. Moreover, we can show that $e^{\mathbb{A}_0 t}$ is exponentially stable on $\mathcal{H}$. Actually, the operator $\mathbb{A}_0$ can be decomposed into the sum of two operators, i.e., $\mathbb{A}_0 = \mathbb{A} + \mathbb{A}_a$, where $\mathbb{A}$ is given by (1.12) and $\mathbb{A}_a$ is defined by $\mathbb{A}_a(\phi, \psi)^\top = (0, -a\psi)^\top$ with $D(\mathbb{A}_a) = \mathcal{H}$. It is clear that for any $(\phi, \psi)^\top \in D(\mathbb{A}_a)$,

$$\mathrm{Re}\langle \mathbb{A}_a(\phi, \psi)^\top, (\phi, \psi)^\top \rangle_{\mathcal{H}} = -\int_\Omega a(x)|\psi(x)|^2 dx \leq 0,$$

which shows that $\mathbb{A}_a$ is dissipative. Since $\mathbb{A}_a$ is bounded on $\mathcal{H}$, $\mathbb{A}_a$ generates a C_0-semigroup of contractions $e^{\mathbb{A}_a t}$, i.e., $\|e^{\mathbb{A}_a t}\| \leq 1$ for all $t \geq 0$. Since we have proved that all $e^{(\mathbb{A}+\mathbb{A}_a)t}$, $e^{\mathbb{A} t}$ and

$e^{\mathbb{A}_a t}$ are C_0-semigroups of contractions, it follows from the Trotter product formula [21] that

$$e^{\mathbb{A}_0 t} = \lim_{n \to \infty} [e^{\mathbb{A}t/n} e^{\mathbb{A}_a t/n}]^n, \quad \forall t \geq 0. \tag{3.6}$$

Since $e^{\mathbb{A}t}$ is exponentially stable, there exist $M_{\omega_0} \geq 1, \omega_0 > 0$ such that $\|e^{\mathbb{A}t}\| \leq M_{\omega_0} e^{-\omega_0 t}$. Take $\tau_0 > 0$ such that $M_{\omega_0} e^{-\omega_0 \tau_0/2} = 1$. It is seen that $\|e^{\mathbb{A}t}\| \leq M_{\omega_0} e^{-\omega_0 t} = M_{\omega_0} e^{-\omega_0/2t} e^{-\omega_0/2t} \leq e^{-\omega_0 t/2}$ for $t \geq \tau_0$. By (3.6), we have

$$
\begin{aligned}
\|e^{\mathbb{A}_0 t}\| &= \|\lim_{n \to \infty} [e^{\mathbb{A}t/n} e^{\mathbb{A}_a t/n}]^n\| \leq \lim_{n \to \infty} \|[e^{\mathbb{A}t/n} e^{\mathbb{A}_a t/n}]^n\| \\
&\leq \lim_{n \to \infty} [\|e^{\mathbb{A}t/n}\| \|e^{\mathbb{A}_a t/n}\|]^n \leq \lim_{n \to \infty} [e^{-\omega_0 t/(2n)} \cdot 1]^n = e^{-\omega_0 t/2}, \quad \forall t \geq \tau_0.
\end{aligned} \tag{3.7}
$$

So, $e^{\mathbb{A}_0 t}$ is exponentially stable on $\mathcal{H}$.

Now, we show that $\mathbb{B}$ is admissible for the semigroup $e^{\mathbb{A}_0 t}$. For this purpose, we consider the following system:

$$\frac{d}{dt}\begin{pmatrix} p(\cdot, t) \\ p_t(\cdot, t) \end{pmatrix} = \mathbb{A}_0 \begin{pmatrix} p(\cdot, t) \\ p_t(\cdot, t) \end{pmatrix}. \tag{3.8}$$

Since $\mathbb{A}_0$ generates a C_0-semigroup on $\mathcal{H}$, which has been justified, for any $(p(\cdot, 0), \dot{p}(\cdot, 0))^\top \in D(\mathbb{A}_0)$, the solution to (3.8) satisfies $(p(\cdot, t), p_t(\cdot, t))^\top \in D(\mathbb{A}_0)$. Take the inner product on both sides of (3.8) with $(p(\cdot, t), p_t(\cdot, t))^\top$ and take (3.3) into account to obtain

$$\text{Re}\langle \ddot{p}, \dot{p} \rangle + \text{Re}\langle \nabla p, \nabla \dot{p} \rangle = -k\|B^* \dot{p}\|^2 - \|\sqrt{a}\dot{p}\|^2,$$

that is,

$$\dot{F}(t) = -k \int_{\Gamma_1} |B^* p_t(x, t)|^2 dx - \int_{\Omega} a(x)|p_t(x, t)|^2 dx, \quad F(t) = \frac{1}{2} \int_{\Omega} \big[\|\nabla p(x)\| + \|p_t(x)\|^2\big]dx.$$

Therefore,

$$k \int_0^T \int_{\Gamma_1} |B^* p_t(x, t)|^2 dx dt \leq F(0) - F(T) \leq F(0).$$

This shows that the operator $\mathbb{B}$ is admissible for the semigroup $e^{\mathbb{A}_0 t}$ ([23]). Therefore, system (2.3) has a unique solution given by

$$\begin{pmatrix} \widehat{z}(\cdot, t) \\ \widehat{z}_t(\cdot, t) \end{pmatrix} = e^{\mathbb{A}_0 t} \begin{pmatrix} \widehat{z}_0(\cdot) \\ \widehat{z}_1(\cdot) \end{pmatrix} + \int_0^t e^{\mathbb{A}_0(t-s)} \mathbb{B}(f(w(\cdot, t)) + d(\cdot, t))ds. \tag{3.9}$$

Since $e^{\mathbb{A}_0 t}$ is exponentially stable on $\mathcal{H}$, which has been justified, and since $\mathbb{B}$ is ∞-admissible for $e^{\mathbb{A}_0 t}$ by virtue of [23, Remark 4.7], it follows from [23, Remark 2.6] that there exists a constant

$M > 0$ independent of t such that

$$\left\| \int_0^t e^{\mathbb{A}_0(t-s)} \mathbb{B}(f(w(\cdot,t)) + d(\cdot,t))ds \right\|_{\mathcal{H}} \le M \|f(w) + d\|_{L^\infty(0,t;L^2(\Gamma_1))}. \tag{3.10}$$

Since $f : H_{\Gamma_0}^1(\Omega) \to L^2(\Gamma_1)$ is continuous and $(w(\cdot,t), \dot{w}(\cdot,t))^\top$ is bounded, there exists $M_1 > 0$ such that $\|f(w(\cdot,t))\|_{L^2(\Gamma_1)} \le M_1$ for all $t \ge 0$. It follows from (3.9) and (3.10) that $\sup_{t \ge 0} \|\widehat{z}(\cdot,t), \widehat{z}_t(\cdot,t))\|_{\mathcal{H}} < +\infty$. □

Proof of Lemma 2.2. It suffices to prove that the "$\widetilde{w}$-part" of system (2.12) admits a unique solution $(\widetilde{w}, \widetilde{w}_t) \in C(0,\infty; \mathcal{H})$ and is asymptotically stable. For this purpose, we first prove that (2.15) holds. Since the boundary Γ is of class C^2, it follows from Lemma 2.1 of [12, p. 18] that there exists vector function $\hat{h} \in C^1(\overline{\Omega}, \mathbb{R}^n)$ (different from h in (1.4)) such that an $\hat{h}(x) = \nu(x)$ on $x \in \Gamma$. Set

$$\rho(t) = \text{Re} \int_\Omega \widetilde{d}_t(x,t)\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)}dx. \tag{3.11}$$

It follows from Cauchy's inequality that

$$|\rho(t)| \le \max\left\{1, \|\hat{h}\|_{L^\infty(\Omega)}^2\right\} E(t), \ \forall\, t \ge 0, \tag{3.12}$$

where $E(t)$ is given by (2.13). Denote $\hat{H}(x) = \{\partial\hat{h}_i/\partial x_j\}_{i,j=1}^n$. Since

$$\nabla\widetilde{d}(x,t)|_\Gamma = \frac{\partial\widetilde{d}(x,t)}{\partial\nu}\nu(x) \text{ owing to } \widetilde{d}(x,t)|_\Gamma = 0, \tag{3.13}$$

it follows from the divergence theorem and the fact $\hat{h}(x) \cdot \nu(x) = 1$ on Γ that

$$\begin{aligned}
\text{Re} \int_\Omega \nabla\widetilde{d}(x,t) \cdot \nabla(\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)})dx &= \frac{1}{2}\text{Re} \int_\Omega \nabla\widetilde{d}(x,t)(\hat{H}(x) + \hat{H}^\top(x))\overline{\nabla\widetilde{d}(x,t)}dx \\
&+ \frac{1}{2} \int_\Omega \text{div}(|\nabla\widetilde{d}(x,t)|^2\hat{h}(x))dx - \frac{1}{2} \int_\Omega |\nabla\widetilde{d}(x,t)|^2\text{div}(\hat{h}(x))dx \\
&= \frac{1}{2} \int_\Gamma \left|\frac{\partial\widetilde{d}(x,t)}{\partial\nu}\right|^2 dx + \frac{1}{2}\text{Re} \int_\Omega \nabla\widetilde{d}(x,t)(\hat{H}(x) + \hat{H}^\top(x))\overline{\nabla\widetilde{d}(x,t)}dx \\
&- \frac{1}{2} \int_\Omega |\nabla\widetilde{d}(x,t)|^2\text{div}(\hat{h}(x))dx,
\end{aligned} \tag{3.14}$$

and

$$\int_\Omega \hat{h}(x) \cdot \nabla(|\widetilde{d}_t(x,t)|^2)dx \quad = \int_\Omega [\mathrm{div}(|\widetilde{d}_t(x,t)|^2\hat{h}(x)) - |\widetilde{d}_t(x,t)|^2\mathrm{div}(\hat{h}(x))]dx$$

$$= \int_\Gamma |\widetilde{d}_t(x,t)|^2\hat{h}(x) \cdot v(x)dx - \int_\Omega |\widetilde{d}_t(x,t)|^2\mathrm{div}(\hat{h}(x))dx \quad (3.15)$$

$$= -\int_\Omega |\widetilde{d}_t(x,t)|^2\mathrm{div}(\hat{h}(x))dx.$$

By (3.13), (3.14), and (3.15), differentiating $\rho(t)$ yields

$$\dot{\rho}(t) = \mathrm{Re}\int_\Omega \Delta\widetilde{d}(x,t)\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)}dx - \mathrm{Re}\int_\Omega a(x)\widetilde{d}_t(x,t)\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)}dx$$

$$+ \frac{1}{2}\int_\Omega h(x) \cdot \nabla(|\widetilde{d}_t(x,t)|^2)dx$$

$$= \mathrm{Re}\int_\Gamma \frac{\partial\widetilde{d}(x,t)}{\partial v}\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)}dx - \frac{1}{2}\int_\Gamma \left|\frac{\partial\widetilde{d}(x,t)}{\partial v}\right|^2 dx$$

$$- \frac{1}{2}\mathrm{Re}\int_\Omega \nabla\widetilde{d}(x,t)(\hat{H}(x) + \hat{H}^\top(x))\overline{\nabla\widetilde{d}(x,t)}dx + \frac{1}{2}\int_\Omega |\nabla\widetilde{d}(x,t)|^2\mathrm{div}(h(x))dx$$

$$- \frac{1}{2}\int_\Omega |\widetilde{d}_t(x,t)|^2\mathrm{div}(\hat{h}(x))dx - \mathrm{Re}\int_\Omega a(x)\widetilde{d}_t(x,t)\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)}dx$$

$$= \frac{1}{2}\int_\Gamma \left|\frac{\partial\widetilde{d}(x,t)}{\partial v}\right|^2 dx - \frac{1}{2}\mathrm{Re}\int_\Omega \nabla\widetilde{d}(x,t)(\hat{H}(x) + \hat{H}^\top(x))\overline{\nabla\widetilde{d}(x,t)}dx$$

$$+ \frac{1}{2}\int_\Omega |\nabla\widetilde{d}(x,t)|^2\mathrm{div}(\hat{h}(x))dx$$

$$- \frac{1}{2}\int_\Omega |\widetilde{d}_t(x,t)|^2\mathrm{div}(\hat{h}(x))dx - \mathrm{Re}\int_\Omega a(x)\widetilde{d}_t(x,t)\hat{h}(x) \cdot \overline{\nabla\widetilde{d}(x,t)}dx.$$

Hence,

$$\frac{1}{2}\int_0^t\int_\Gamma \left|\frac{\partial\widetilde{d}(x,t)}{\partial v}\right|^2 dxds = \frac{1}{2}\mathrm{Re}\int_0^t\int_\Omega \nabla\widetilde{d}(x,t)(\hat{H}(x) + \hat{H}^\top(x))\overline{\nabla\widetilde{d}(x,t)}dxds$$

$$- \frac{1}{2}\int_0^t\int_\Omega |\nabla\widetilde{d}(x,s)|^2\mathrm{div}(\hat{h}(x))dxds + \frac{1}{2}\int_0^t\int_\Omega |\widetilde{d}_t(x,t)|^2\mathrm{div}(\hat{h}(x))dxds \quad (3.16)$$

$$+ \operatorname{Re} \int_0^t \int_\Omega a(x) \widetilde{d}_t(x,t) h(x) \cdot \overline{\nabla \widetilde{d}(x,s)} dx ds + \rho(t) - \rho(0)$$

$$\leq C \int_0^t E(s) ds + \rho(t) - \rho(0),$$

where C is a positive constant depending only on $\hat{h}(x)$. By (2.13), (3.12) and (3.16), we have

$$\int_0^\infty \int_\Gamma \left| \frac{\partial \widetilde{d}(x,t)}{\partial \nu} \right|^2 dx ds < +\infty, \tag{3.17}$$

which implies (2.15). Next, we consider the "$\widetilde{w}$-part" of system (2.12). Notice that the "$\widetilde{w}$-part" of system (2.12) can be written as

$$\frac{d}{dt} (\widetilde{w}(\cdot,t), \widetilde{w}_t(\cdot,t))^\top = \mathbb{A}_0 (\widetilde{w}(\cdot,t), \widetilde{w}_t(\cdot,t))^\top + \mathbb{B} \frac{\partial \widetilde{d}(\cdot,t)}{\partial \nu}, \tag{3.18}$$

where the operators $\mathbb{A}_0$ and $\mathbb{B}$ are given by (3.2). By Lemma 2.1, we know that $\mathbb{B}$ is admissible for $e^{\mathbb{A}_0 t}$. Therefore, the "$\widetilde{w}$-part" of system (2.12) has a unique solution given by

$$(\widetilde{w}(\cdot,t), \widetilde{w}_t(\cdot,t))^\top = e^{\mathbb{A}_0 t} (\widetilde{w}_0, \widetilde{w}_1)^\top + \int_0^t e^{\mathbb{A}_0 (t-s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu} ds. \tag{3.19}$$

By (3.17), for any given $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$\int_{t_0}^\infty \int_{\Gamma_1} \left| \frac{\partial \widetilde{d}(x,s)}{\partial \nu} \right|^2 dx ds \leq \varepsilon^2. \tag{3.20}$$

We rewrite (3.19) as

$$\begin{aligned} (\widetilde{w}(\cdot,t), \widetilde{w}_t(\cdot,t))^\top &= e^{\mathbb{A}_0 t} (\widetilde{w}_0, \widetilde{w}_1)^\top + e^{\mathbb{A}_0 (t-t_0)} \int_0^{t_0} e^{\mathbb{A}_0 (t_0 - s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu} ds \\ &\quad + \int_{t_0}^t e^{\mathbb{A}_0 (t-s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu} ds. \end{aligned} \tag{3.21}$$

It follows from the admissibility of $\mathbb{B}$, the exponential stability of the semigroup $e^{\mathbb{A}_0 t}$ and [23, Remark 2.6] that

$$\left\| \int_{t_0}^{t} e^{\mathbb{A}_0(t-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial v}ds \right\|_{\mathcal{H}} \leq \left\| \int_{0}^{t} e^{\mathbb{A}_0(t-s)}\mathbb{B}\left(0\underset{t_0}{\square}\frac{\partial \widetilde{d}(\cdot,s)}{\partial v}\right)ds \right\|_{\mathcal{H}} \tag{3.22}$$

$$\leq L\left\| \frac{\partial \widetilde{d}(\cdot,s)}{\partial v} \right\|_{L^2(t_0,\infty;L^2(\Gamma_1))} \leq L\varepsilon,$$

where L is a constant that is independent of $\widetilde{d}(x,t)$, and

$$(u\underset{\tau}{\square}v)(t) = \begin{cases} u(t), & 0 \leq t \leq \tau, \\ v(t), & t > \tau. \end{cases}$$

Since $e^{\mathbb{A}_0 t}$ is exponentially stable claimed by Lemma 2.1, there exist constants $L_0, \omega_0 > 0$ such that $\|e^{\mathbb{A}_0 t}\| \leq L_0 e^{-\omega_0 t}$. By (3.21) and (3.22), it follows that

$$\left\| \begin{pmatrix} \widetilde{w}(\cdot,t) \\ \widetilde{w}_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L_0 e^{-\omega_0 t}\left\| \begin{pmatrix} \widetilde{w}_0 \\ \widetilde{w}_1 \end{pmatrix} \right\|_{\mathcal{H}} + L_0 e^{-\omega_0(t-t_0)}\left\| \int_{0}^{t_0} e^{\mathbb{A}_0(t_0-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial v}ds \right\|_{\mathcal{H}} + L\varepsilon. \tag{3.23}$$

Passing to the limit as $t \to \infty$, we finally obtain

$$\lim_{t \to \infty}\left\| \begin{pmatrix} \widetilde{w}(\cdot,t) \\ \widetilde{w}_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L\varepsilon. \tag{3.24}$$

This proves that $\lim_{t \to \infty}\|(\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t)\|_{\mathcal{H}} = 0$. $\quad\square$

Proof of Theorem 2.1. Let $\widehat{z}(x,t)$ be given by (2.2) and let $(\widetilde{w}(x,t),\widetilde{d}(x,t)) = (\widehat{w}(x,t) - w(x,t),\widehat{z}(x,t) + \widehat{d}(x,t))$. It is obvious that

$$\begin{pmatrix} \widehat{w}(x,t) \\ \widehat{w}_t(x,t) \\ \widehat{d}(x,t) \\ \widehat{d}_t(x,t) \\ z(x,t) \\ z_t(x,t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & -I & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & I \end{pmatrix}\begin{pmatrix} \widetilde{d}(x,t) \\ \widetilde{d}_t(x,t) \\ \widehat{z}(x,t) \\ \widehat{z}_t(x,t) \\ w(x,t) \\ w_t(x,t) \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{w}(x,t) \\ \widetilde{w}_t(x,t) \end{pmatrix}.$$

It follows from Lemmas 2.1 and 2.2 that (2.11) admits a unique solution $(\widehat{w},\widehat{w}_t,\widehat{d},\widehat{d}_t,z,z_t) \in C(0,\infty;\mathcal{H}^3)$. By Lemma 2.2, we know that (2.16) and (2.17) are valid. $\quad\square$

Proof of Lemma 2.3. For any given $(\widetilde{d}(\cdot,0),\widetilde{d}_t(\cdot,0))^{\top} \in D(\mathbb{A}_1)$, $(\widetilde{d},\widetilde{d}_t)^{\top} \in C(0,\infty;D(\mathbb{A}_1))$ is the classical solution to (2.5). Denote $d^*(x,t) = \widetilde{d}_t(x,t)$. Then $d^*(x,t)$ satisfies

$$\begin{cases} d_{tt}^*(x,t) = \Delta d^*(x,t) - a(x)d_t^*(x,t), \\ d^*(x,t)|_{\Gamma_0} = 0, \\ d^*(x,t)|_{\Gamma_1} = 0, \end{cases} \tag{3.25}$$

and $(d^*(\cdot, 0), d_t^*(\cdot, 0)) \in \mathbb{H}$. Hence, system (3.25) admits a unique solution $(d^*, d_t^*) \in C(0, \infty; \mathbb{H})$ which is exponentially stable, i.e., there exist constants $M_{d^*}, \omega_{d^*} > 0$ such that

$$\|(d^*(\cdot, t), d_t^*(\cdot, t))\|_{\mathbb{H}} \le M_{d^*} e^{-\omega_{d^*} t} \|(d^*(\cdot, 0), d_t^*(\cdot, 0))\|_{\mathbb{H}}, \tag{3.26}$$

which implies that

$$\|\nabla \widetilde{d}_t(\cdot, t)\|_{L^2(\Omega)} = \|\nabla d^*(\cdot, t)\|_{L^2(\Omega)} \le M_{d^*} e^{-\omega_{d^*} t} \|(d^*(\cdot, 0), d_t^*(\cdot, 0))\|_{\mathbb{H}}, \tag{3.27}$$

and

$$\|\widetilde{d}_{tt}(\cdot, t)\|_{L^2(\Omega)} = \|d_t^*(\cdot, t)\|_{L^2(\Omega)} \le M_{d^*} e^{-\omega_{d^*} t} \|(d^*(\cdot, 0), d_t^*(\cdot, 0))\|_{\mathbb{H}}. \tag{3.28}$$

By the first equation of (2.5), the Poincare's inequality, (3.27) and (3.28), we obtain

$$
\begin{aligned}
\|\Delta \widetilde{d}(\cdot, t)\|_{L^2(\Omega)} &\le \|\widetilde{d}_{tt}(\cdot, t)\|_{L^2(\Omega)} + \|a(\cdot)\|_{L^\infty(\omega)} \|\widetilde{d}_t(\cdot, t)\|_{L^2(\Omega)} \\
&\le \|\widetilde{d}_{tt}(\cdot, t)\|_{L^2(\Omega)} + C_0 \|a(\cdot)\|_{L^\infty(\omega)} \|\nabla \widetilde{d}_t(\cdot, t)\|_{L^2(\Omega)} \\
&\le (1 + C_0 \|a(\cdot)\|_{L^\infty(\omega)}) M_{d^*} e^{-\omega_{d^*} t} \|(d^*(\cdot, 0), d_t^*(\cdot, 0))\|_{\mathbb{H}},
\end{aligned}
\tag{3.29}
$$

for some constant C_0. It follows from the Sobolev embedding theorem, the trace theorem, (2.13) and (3.29) that there are constants $C_1, C_2 > 0$ such that

$$
\begin{aligned}
\left\| \frac{\partial \widetilde{d}(\cdot, t)}{\partial \nu} \right\|_{L^2(\Gamma_1)} &= \|\nabla \widetilde{d}(\cdot, t) \cdot \nu\|_{L^2(\Gamma_1)} \le \|\nabla \widetilde{d}(\cdot, t)\|_{L^2(\Gamma_1)} \\
&\le C_1 \|\nabla \widetilde{d}(\cdot, t)\|_{H^{1/2}(\Gamma_1)} \le C_1 C_2 \|\nabla \widetilde{d}(\cdot, t)\|_{H^1(\Omega)} \\
&\le C_1 C_2 [\|\nabla \widetilde{d}(\cdot, t)\|_{L^2(\Omega)} + \|\Delta \widetilde{d}(\cdot, t)\|_{L^2(\Omega)}] \\
&\le C_1 C_2 M_1 e^{-\mu_1 t} [\|(\widetilde{d}(\cdot, 0), \widetilde{d}_t(\cdot, 0))\|_{\mathbb{H}} + \|(d^*(\cdot, 0), d_t^*(\cdot, 0))\|_{\mathbb{H}}],
\end{aligned}
\tag{3.30}
$$

with $M_1 = [(1 + C_0 \|a(\cdot)\|_{L^\infty(\omega)}) M_{d^*} + 2 M_d]$, $\mu_1 = \min\{\omega_{d^*}, \omega_d / 2\}$. $\quad \square$

Proof of Theorem 2.2. By Theorem 2.1, it suffices to prove (2.20). Let $(\widetilde{w}(x, t), \widetilde{d}(x, t)) = (\widehat{w}(x, t) - w(x, t), \widehat{z}(x, t) + \widehat{d}(x, t))$. Then $\widetilde{w}(x, t)$ satisfies

$$
\begin{cases}
\widetilde{w}_{tt}(x, t) = \Delta \widetilde{w}(x, t) - a(x) \widetilde{w}_t(x, t), \\
\widetilde{w}(x, t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widetilde{w}(x, t)}{\partial \nu}\Big|_{\Gamma_1} = -k \widetilde{w}_t(x, t) + \dfrac{\partial \widetilde{d}(x, t)}{\partial \nu}, \\
\widetilde{w}(x, 0) = \widehat{w}(x) - w_0(x), \quad \widetilde{w}_t(x, 0) = \widehat{w}_1(x) - w_1(x),
\end{cases}
\tag{3.31}
$$

which gives

$$(\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t))^\top = e^{\mathbb{A}_0 t}(\widetilde{w}_0,\widetilde{w}_1)^\top + \int_0^t e^{\mathbb{A}_0(t-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds$$

$$= e^{\mathbb{A}_0 t}(\widetilde{w}_0,\widetilde{w}_1)^\top + e^{\mathbb{A}_0(t-t/2)}\int_0^{t/2} e^{\mathbb{A}_0(t/2-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds + \int_{t/2}^t e^{\mathbb{A}_0(t-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds. \tag{3.32}$$

By Lemma 2.1, $e^{\mathbb{A}_0 t}$ is exponentially stable, and hence there exist $M_0, \omega_0 > 0$ such that $\|e^{\mathbb{A}_0 t}\| \le M_0 e^{-\omega_0 t}$. It follows from (2.15), the admissibility of $\mathbb{B}$ and [23, Remark 2.6] that there exists a constant $L > 0$ such that

$$\left\| e^{\mathbb{A}_0(t-t/2)}\int_0^{t/2} e^{\mathbb{A}_0(t/2-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds \right\|_{\mathcal{H}} \le \|e^{\mathbb{A}_0(t-t/2)}\| \left\| \int_0^{t/2} e^{\mathbb{A}_0(t/2-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds \right\|_{\mathcal{H}}$$

$$\le L\|e^{\mathbb{A}_0(t-t/2)}\| \left\| \frac{\partial \widetilde{d}}{\partial \nu}\right\|_{L^2(0,t/2;L^2(\Gamma_1))} = LM_0 e^{-\omega_0 t/2}\left\|\frac{\partial \widetilde{d}}{\partial \nu}\right\|_{L^2(0,\infty;L^2(\Gamma_1))}, \tag{3.33}$$

and

$$\left\| \int_{t/2}^t e^{\mathbb{A}_0(t-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds \right\|_{\mathcal{H}} \le \left\| \int_0^t e^{\mathbb{A}_0(t-s)}\mathbb{B}\left(0\underset{t/2}{\square}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}\right)ds \right\|_{\mathcal{H}}$$

$$\le L\left\| 0\underset{t/2}{\square}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}\right\|_{L^2(0,t;L^2(\Gamma_1))} = L\left\|\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}\right\|_{L^2(t/2,t;L^2(\Gamma_1))} \tag{3.34}$$

where L is a constant that is independent of $\widetilde{d}(x,t)$. By Lemma 2.3, (3.32), (3.33) and (3.34), we have

$$\|(\widetilde{w}(\cdot,t),\widetilde{w}_t(\cdot,t))^\top\|_{\mathcal{H}} \le M_0 e^{-\omega_0 t}\|(\widetilde{w}_0,\widetilde{w}_1)^\top\|_{\mathcal{H}} + LM_0 e^{-\omega_0 t/2}\left\|\frac{\partial \widetilde{d}}{\partial \nu}\right\|_{L^2(0,\infty;L^2(\Gamma_1))}$$

$$+ L\left\|\frac{\partial \widetilde{d}}{\partial \nu}\right\|_{L^2(t/2,t;L^2(\Gamma_1))} = M_0 e^{-\omega_0 t}\|(\widetilde{w}_0,\widetilde{w}_1)^\top\|_{\mathcal{H}}$$

$$+ \frac{LCM_0}{2\omega_{d*}}e^{-\omega_0 t/2} + \frac{LC}{\omega_{d*}}(e^{-\omega_{d*} t/2} - e^{-\omega_{d*} t}),$$

which implies that $(\widetilde{w}(x,t),\widetilde{w}_t(x,t))^\top$ is exponentially stable on $\mathcal{H}$. This proves (2.20). $\quad\square$

Proof of Theorem 2.3. Using the error variables $(\widetilde{w}(x,t),\widetilde{d}(x,t))$ defined by (2.12), and $\widehat{z}(x,t)$ defined by (2.3), and the invertible transformation:

$$\begin{pmatrix} w \\ \widetilde{w} \\ \widetilde{d} \\ \widehat{z} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ -I & I & 0 & 0 \\ -I & 0 & I & I \\ -I & 0 & 0 & I \end{pmatrix}\begin{pmatrix} w \\ \widehat{w} \\ \widehat{d} \\ z \end{pmatrix},$$

we can convert system (2.22) into system (2.28) in section 2, which is rewritten here as follows:

$$
\begin{cases}
w_{tt}(x,t) = \Delta w(x,t), \\
w(x,t)|_{\Gamma_0} = 0, \\
\dfrac{\partial w(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -kw_t(x,t) - \dfrac{\partial \widetilde{d}(x,t)}{\partial \nu}, \\
\widetilde{w}_{tt}(x,t) = \Delta \widetilde{w}(x,t) - a(x)\widetilde{w}_t(x,t), \\
\widetilde{w}(x,t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widetilde{w}(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -k\widetilde{w}_t(x,t) + \dfrac{\partial \widetilde{d}(x,t)}{\partial \nu}, \\
\widetilde{d}_{tt}(x,t) = \Delta \widetilde{d}(x,t) - a(x)\widetilde{d}_t(x,t), \\
\widetilde{d}(x,t)|_{\Gamma = \Gamma_0 \cup \Gamma_1} = 0, \\
\widehat{z}_{tt}(x,t) = \Delta \widehat{z}(x,t) - a(x)\widehat{z}_t(x,t), \\
\widehat{z}(x,t)|_{\Gamma_0} = 0, \\
\dfrac{\partial \widehat{z}(x,t)}{\partial \nu}\Big|_{\Gamma_1} = -k\widehat{z}_t(x,t) - f(w(x,t)) - d(x,t).
\end{cases}
\tag{3.35}
$$

Obviously, it suffices to prove that the "$(w, \widetilde{w}, \widetilde{d})$-part" in (3.35) is convergent as $t \to \infty$ and the "$\widehat{z}$-part" is bounded. However, the "$(\widetilde{w}, \widetilde{d})$-part" in (3.35) has been shown in Lemma 2.2. Now we only need to consider the "$(w, \widehat{z})$-part" of system (3.35). Actually, the "w-part" in (3.35) can be rewritten as

$$
\begin{cases}
w_{tt}(x,t) = \Delta w(x,t), \\
w(x,t)|_{\Gamma_0} = 0, \\
\dfrac{\partial w}{\partial \nu}\Big|_{\Gamma_1} = -kw_t(x,t) - \dfrac{\partial \widetilde{d}(x,t)}{\partial \nu}.
\end{cases}
\tag{3.36}
$$

By Proposition 1.2, we can write the solution of (3.36) as

$$
(w(\cdot,t), w_t(\cdot,t))^{\top} = e^{\mathbb{A}t}(w_0, w_1)^{\top} + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{B}\frac{\partial \widetilde{d}(\cdot,s)}{\partial \nu}ds,
\tag{3.37}
$$

where the operators $\mathbb{A}$ and $\mathbb{B}$ are given by (1.12). By (2.15) in Lemma 2.2, for any given $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$
\int_{t_0}^{\infty} \int_{\Gamma_1} \left| \frac{\partial \widetilde{d}(x,s)}{\partial \nu} \right|^2 dxds \leq \varepsilon^2.
\tag{3.38}
$$

We rewrite (3.37) as

$$(w(\cdot, t), w_t(\cdot, t))^\top = e^{\mathbb{A}t}(w_0, w_1)^\top + e^{\mathbb{A}(t-t_0)} \int_0^{t_0} e^{\mathbb{A}(t_0-s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot, s)}{\partial \nu} ds + \int_{t_0}^{t} e^{\mathbb{A}(t-s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot, s)}{\partial \nu} ds.$$

(3.39)

Since $e^{\mathbb{A}t}$ is exponentially stable, it follows from the admissibility of $\mathbb{B}$ and [23, Remark 2.6] that

$$\left\| \int_{t_0}^{t} e^{\mathbb{A}(t-s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot, s)}{\partial \nu} ds \right\|_{\mathcal{H}} \leq \left\| \int_{t_0}^{t} e^{\mathbb{A}(t-s)} \mathbb{B} \left(0 \underset{t_0}{\square} \frac{\partial \widetilde{d}(\cdot, s)}{\partial \nu} \right) ds \right\|_{\mathcal{H}}$$

$$\leq L \left\| \frac{\partial \widetilde{d}(\cdot, s)}{\partial \nu} \right\|_{L^2(t_0, \infty; L^2(\Gamma_1))} \leq L\varepsilon,$$

(3.40)

where L is a constant that is independent of $\widetilde{d}(x, t)$.

Suppose that $\|e^{\mathbb{A}t}\| \leq L_0 e^{-\omega_0 t}$ for some $L_0, \omega_0 > 0$, it follows from (3.39) and (3.40) that

$$\left\| \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L_0 e^{-\omega_0 t} \left\| \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\|_{\mathcal{H}} + L_0 e^{-\omega_0(t-t_0)} \left\| \int_0^{t_0} e^{\mathbb{A}(t_0-s)} \mathbb{B} \frac{\partial \widetilde{d}(\cdot, s)}{\partial \nu} ds \right\|_{\mathcal{H}} + L\varepsilon.$$

(3.41)

Passing to the limit as $t \to \infty$, we finally obtain

$$\lim_{t \to \infty} \left\| \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} \right\|_{\mathcal{H}} \leq L\varepsilon.$$

(3.42)

This proves that $\lim_{t \to \infty} \|(w(\cdot, t), w_t(\cdot, t))\|_{\mathcal{H}} = 0$. The "$\widehat{z}$-part" of system (3.35) can be rewritten as
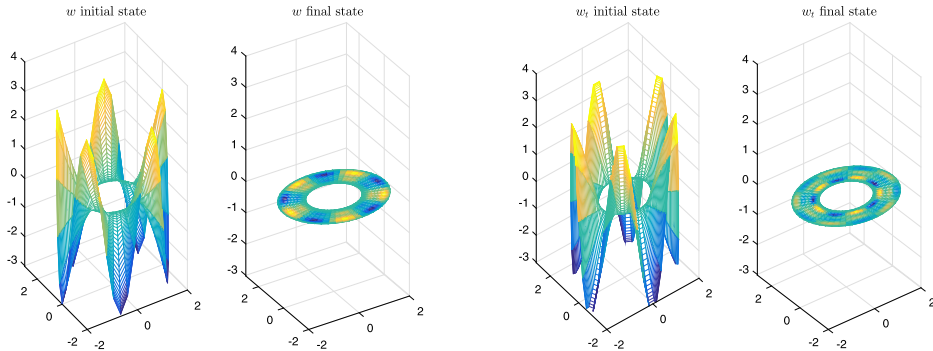
$$\begin{cases} \widehat{z}_{tt}(x, t) = \Delta \widehat{z}(x, t) - a(x) \widehat{z}_t(x, t), \\ \widehat{z}(x, t)|_{\Gamma_0} = 0, \\ \dfrac{\partial \widehat{z}(x, t)}{\partial \nu} \Big|_{\Gamma_1} = -k \widehat{z}_t(x, t) - f(w(x, t)) - d(x, t), \\ \widehat{z}(x, 0) = z_0(x) - w_0(x), \ \widehat{z}_t(x, 0) = z_1(x) - w_1(x). \end{cases}$$

(3.43)

Since $f \in C(H_{\Gamma_0}^1(\Omega), L^2(\Gamma_1))$ and $\lim_{t \to \infty} \|(w(\cdot, t), w_t(\cdot, t))\|_{\mathcal{H}} = 0$, we obtain $f(w) \in L^\infty(0, \infty; L^2(\Gamma_1))$, and thus $f(w) + d \in L^\infty(0, \infty; L^2(\Gamma_1))$. By Lemma 2.1, we know that $(\widehat{z}(x, t), \widehat{z}_t(x, t))$ is bounded. The inequalities (2.26) and (2.27) follow from Corollary 2.1 and Theorem 2.2. $\square$

4. Numerical simulation

In this section, we present some numerical simulations for system (2.22) for illustration. For simplicity, we just take dimension $n = 2$. We consider $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | 1 < x_1^2 + x_2^2 < 4\}$, $\Gamma_0 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}$, $\Gamma_1 = \Gamma \setminus \Gamma_0$. Take the interior observation domain $\omega = \{(x_1, x_2) \in \mathbb{R}^2 | 3.9 < x_1^2 + x_2^2 < 4\}$. Obviously, ω satisfies the condition (1.2). For numerical computations, we take parameter $k = 3$, internal uncertainty $f(w(x, t)) = w^2(x, t)|_{\Gamma_1}$, disturbance $d(x, t) = \sin(x_1 t) + \cos(x_2 t)$. Since the spatial domain consists of a two-dimensional annulus, we can more easily solve (2.22) in the polar coordinate (r, θ) and then convert back to the original coordinate for some figures if necessary. Under the polar coordinate, system (2.22) can be written as:

$$
\begin{cases}
\dfrac{\partial^2 w(r, \theta, t)}{\partial t^2} - \dfrac{\partial^2 w(r, \theta, t)}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial w(r, \theta, t)}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 w(r, \theta, t)}{\partial \theta^2} = 0, \\
\quad 1 < r < 2, \ 0 < \theta < 2\pi, \ t > 0, \\
w(1, \theta, t) = 0, \ 0 \le \theta \le 2\pi, \ t \ge 0, \\
\dfrac{\partial w(2, \theta, t)}{\partial r} = -k\dfrac{\partial w(2, \theta, t)}{\partial t} + f(w(2, \theta, t)) + d(2\cos(\theta), 2\sin(\theta), t) \\
\qquad\qquad - \dfrac{\partial \widehat{d}(2, \theta, t)}{\partial r} - k\widehat{d}_t(2, \theta, t), \ 0 < \theta < 2\pi, \ t > 0, \\
\dfrac{\partial^2 \widehat{w}(r, \theta, t)}{\partial t^2} - \dfrac{\partial^2 \widehat{w}(r, \theta, t)}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial w(r, \theta, t)}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 \widehat{w}(r, \theta, t)}{\partial \theta^2} \\
\qquad\qquad - a(r, \theta)\left[\dfrac{\partial \widehat{w}(r, \theta, t)}{\partial t} - \dfrac{\partial w(r, \theta, t)}{\partial t}\right] = 0, \ 1 < r < 2, \ 0 < \theta < 2\pi, \ t > 0, \\
\widehat{w}(1, \theta, t) = 0, \ 0 \le \theta \le 2\pi, \ t \ge 0, \\
\dfrac{\partial \widehat{w}(2, \theta, t)}{\partial r} = -k\dfrac{\partial \widehat{w}(2, \theta, t)}{\partial t}, \ t \ge 0, \\
\dfrac{\partial^2 \widehat{d}(r, \theta, t)}{\partial t^2} - \dfrac{\partial^2 \widehat{d}(r, \theta, t)}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial \widehat{d}(r, \theta, t)}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 \widehat{d}(r, \theta, t)}{\partial \theta^2} \\
\qquad\qquad + a(r, \theta)\dfrac{\partial \widehat{d}(r, \theta, t)}{\partial t} = 0, \ 1 < r < 2, \ 0 < \theta < 2\pi, \ t > 0, \\
\widehat{d}(1, \theta, t) = 0, \ 0 \le \theta \le 2\pi, \ t \ge 0, \\
\widehat{d}(2, \theta, t) = -[z(2, \theta, t) - w(2, \theta, t)], \ 0 \le \theta \le 2\pi, \ t \ge 0, \\
\dfrac{\partial^2 z(r, \theta, t)}{\partial t^2} - \dfrac{\partial^2 z(r, \theta, t)}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial z(r, \theta, t)}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 z(r, \theta, t)}{\partial \theta^2} \\
\qquad - a(r, \theta)\left[\dfrac{\partial z(r, \theta, t)}{\partial t} - \dfrac{\partial w(r, \theta, t)}{\partial t}\right] = 0, \ 1 < r < 2, \ 0 < \theta < 2\pi, \ t > 0, \\
z(1, \theta, t) = 0, \ 0 \le \theta \le 2\pi, \ t \ge 0, \\
\dfrac{\partial z(2, \theta, t)}{\partial r} = -k\dfrac{\partial z(2, \theta, t)}{\partial t} - \dfrac{\partial \widehat{d}(2, \theta, t)}{\partial r} - k\widehat{d}_t(2, \theta, t), \ 0 \le \theta \le 2\pi, \ t \ge 0,
\end{cases}
$$

(4.1)

(a) Displacement w at initial time $t = 0$ and time $t = 15$. (b) Velocity w_t at initial time $t = 0$ and final time $t = 15$.

Fig. 1. The initial state and state at $t = 15$ of system (4.1) with total disturbance $F(x, t) = w^2(x, t)|_{\Gamma_1} + \sin(x_1 t) + \cos(x_2 t)$ (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

where we still use $w(r, \theta, t)$, $\widehat{w}(r, \theta, t)$, $\widehat{d}(r, \theta, t)$ and $z(r, \theta, t)$ to denote the states under the polar coordinate for notation simplicity, which is clear from the context. The function $a(r, \theta)$ in (4.1) is defined by $a(r, \theta) = \max\{0, \min\{1, 1000(r - \sqrt{3.9} - 0.001), -1000(r - 1.999)\}\}$ with $r \in [1, 2]$, $\theta \in [0, 2\pi]$.

The initial value (4.1) is taken as

$$
\begin{cases}
w(r, \theta, 0) = (r^2 - 1)\cos(5\theta), \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \\
w_t(r, \theta, 0) = 4\sin(r - 1)\sin(5\theta), \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \\
\widehat{d}(r, \theta, 0) = -3(r - 1)\cos(5\theta), \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \\
\widehat{d}_t(r, \theta, 0) = 4\sin(r - 1)\sin(5\theta), \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \\
\widehat{w}(r, \theta, 0) = -(r^2 - 1)\cos(5\theta), \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \\
\widehat{w}_t(r, \theta, 0) = -4\sin(r - 1)\sin(5\theta), \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \\
z(r, \theta, 0) = z_t(r, \theta, 0) = 0, \ 1 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi.
\end{cases}
\tag{4.2}
$$

It is clear that the above initial value satisfies the compatible condition (2.23). The backward Euler method in time and the Chebyshev spectral method for polar variables are used to discretize system (4.1). Here, we take the grid size $r_N = 30$ for r, the grid size $\theta_N = 50$ for θ, and the time step $dt = 5 \times 10^{-4}$. The numerical algorithm is programmed by Matlab [22] and the numerical results are plotted in Figures 1–5.

Figs. 1(a) and 1(b) display the displacement $w(r, \theta, t)$ and the velocity $w_t(r, \theta, t)$ at the initial time $t = 0$ and the time $t = 15$, respectively. It is seen that the convergence for both $w(r, \theta, t)$ and $w_t(r, \theta, t)$ is very satisfactory.

We plot $(w(r, \pi, t), w_t(r, \pi, t))$ and $(\widehat{w}(r, \pi, t), \widehat{w}_t(r, \pi, t))$ (there is no speciality for π which can be any angle) in the polar coordinate for system (4.1) in Figs. 2 and 3, respectively. It is clearly seen that in both cases, the convergence is fast and satisfactory.

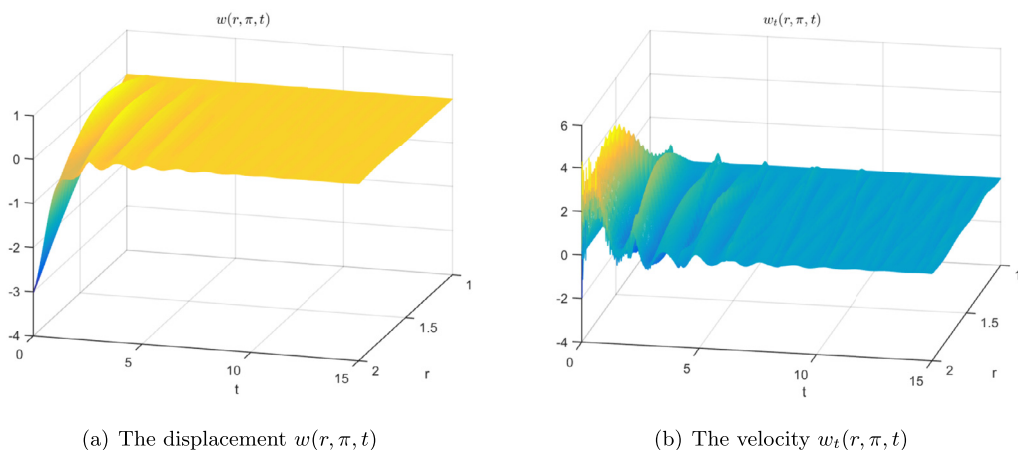(a) The displacement $w(r, \pi, t)$ (b) The velocity $w_t(r, \pi, t)$

Fig. 2. The evolution of $(w(r, \pi, t), w_t(r, \pi, t))$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).
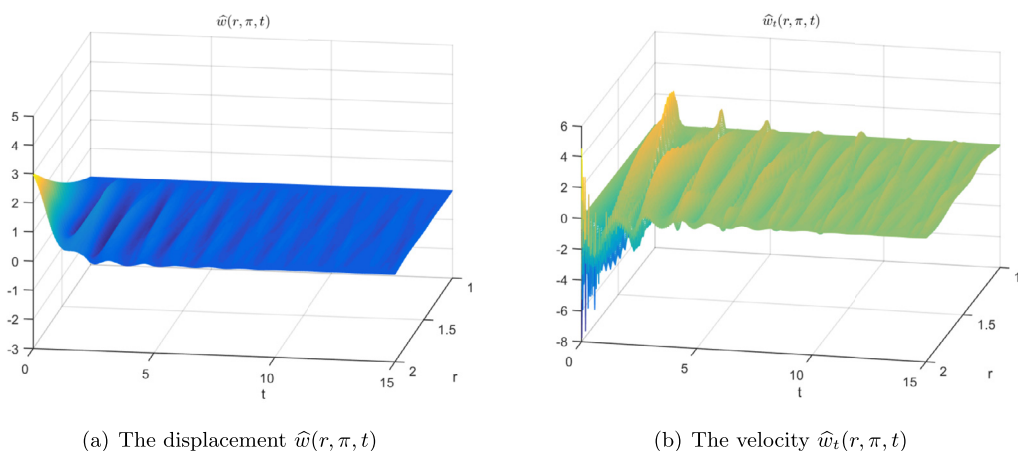

(a) The displacement $\widehat{w}(r, \pi, t)$ (b) The velocity $\widehat{w}_t(r, \pi, t)$

Fig. 3. The evolution of $(\widehat{w}(r, \pi, t), \widehat{w}_t(r, \pi, t))$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

We plot $(\widehat{d}(r, \pi, t), \widehat{d}_t(r, \pi, t))$ and $(z(r, \pi, t), z_t(r, \pi, t))$ in the polar coordinate for system (4.1) in Figs. 4 and 5, respectively. It is clearly seen that in both cases, the state $(\widehat{d}(r, \pi, t), \widehat{d}_t(r, \pi, t))$ and $(z(r, \pi, t), z_t(r, \pi, t))$ are bounded.

The total disturbance $f(w) + d$ and the error between total disturbance and its estimation $\frac{\partial \widehat{d}}{\partial \nu} + k\widehat{d}_t - F(w)$ are plotted in Fig. 6. The convergence is quite fast.

5. Concluding remarks

In this paper, we consider output feedback stabilization for a multi-dimensional wave equation with Neumann boundary control and control matched disturbance suffered from the boundary disturbance and the nonlinear internal uncertainty. We propose a new infinite-dimensional
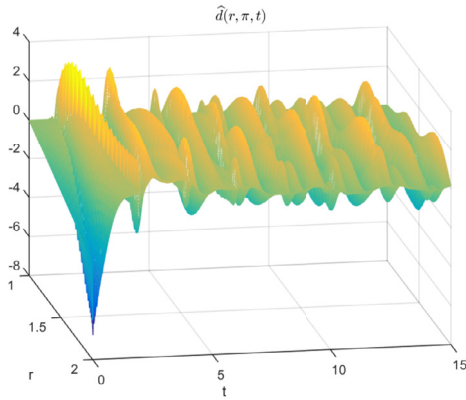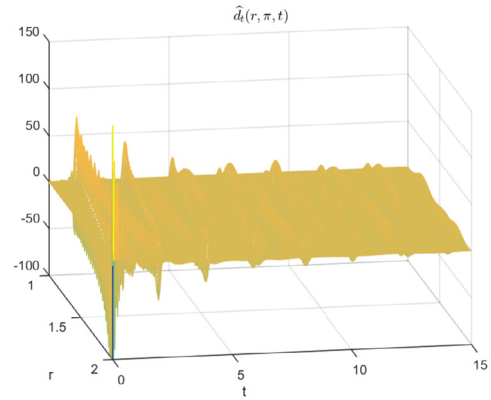
(a) The displacement $\widehat{d}(r, \pi, t)$ (b) The velocity $\widehat{d}_t(r, \pi, t)$

Fig. 4. The evolution of $(\widehat{d}(r, \pi, t), \widehat{d}_t(r, \pi, t))$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).
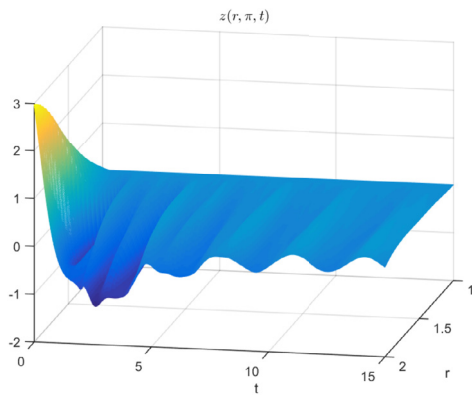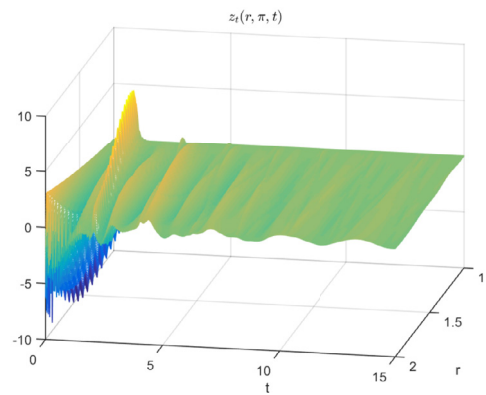

(a) The displacement $z(r, \pi, t)$ (b) The velocity $z_t(r, \pi, t)$

Fig. 5. The evolution of $(z(r, \pi, t), z_t(r, \pi, t))$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

extended state observer to estimate unknown total disturbance. The disturbance $d(x, t)$ is particularly supposed to satisfy $d \in L^\infty(0, \infty; L^2(\Gamma_1))$ only. This type of disturbance is much more general than the existing works by active disturbance rejection control where the derivative of disturbance with respect to time variable t is required to be bounded and the disturbance with respect to spatial variable x is supposed to be Hölder continuous [8]. Moreover, we remove the limitation of active disturbance rejection control of using high-gain to estimate the disturbance, which avoids consequently the peaking value problem. Mathematically, the proposed scheme has two additionally merits. First, the convergence rate of the extended state observer is independent of the total disturbance (2.5)–(2.6), and second, the actual nonlinear closed-loop system is solved by linear method (2.28).
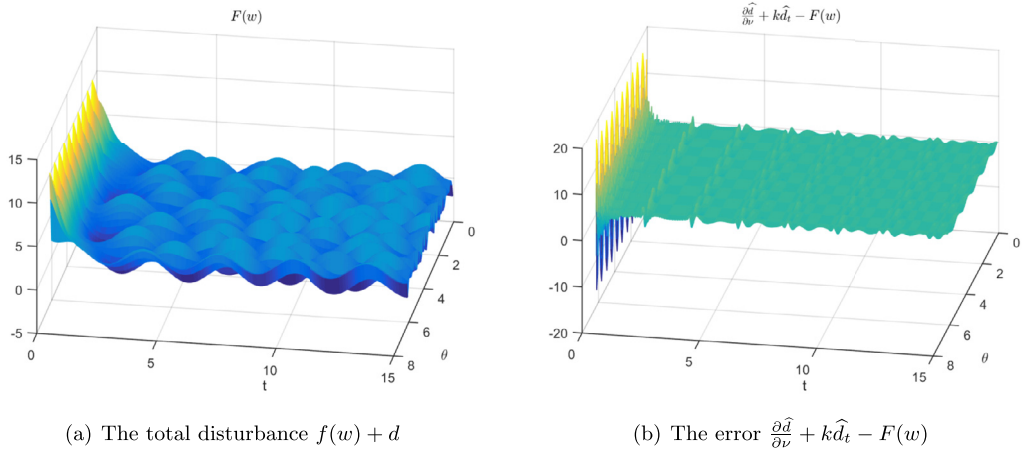
(a) The total disturbance $f(w) + d$

(b) The error $\frac{\partial \widehat{d}}{\partial \nu} + k\widehat{d}_t - F(w)$

Fig. 6. The evolution of $(f(w)+d, \frac{\partial \widehat{d}}{\partial \nu} + k\widehat{d}_t - F(w))$ under the polar coordinate (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

Acknowledgments

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