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Positivity results for indefinite sublinear elliptic problems via a continuity argument

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Abstract

We establish a positivity property for a class of semilinear elliptic problems involving indefinite sublinear nonlinearities. Namely, we show that any nontrivial nonnegative solution is positive for a class of problems the strong maximum principle does not apply to. Our approach is based on a continuity argument combined with variational techniques, the sub and supersolutions method and some *a priori* bounds. Both Dirichlet and Neumann homogeneous boundary conditions are considered. As a byproduct, we deduce some existence and uniqueness results. Finally, as an application, we derive some positivity results for indefinite concave-convex type problems.

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1. Introduction

Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ with $N \geq 1$. The purpose of this article is to discuss the existence of positive solutions for the problems

$$\begin{cases} -\Delta u = a(x) f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P}$$

and

$$\begin{cases} -\Delta u = a(x) f(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P'}$$

where ν is the outward unit normal to $\partial\Omega$.

Here $a \in L^r(\Omega)$, $r > N$, is a function that **changes sign**, and $f : [0, \infty) \to [0, \infty)$ is continuous and sublinear in the following sense:

$$\lim_{s \to 0^+} \frac{f(s)}{s} = \infty \quad \text{and} \quad \lim_{s \to \infty} \frac{f(s)}{s} = 0. \tag{H_1}$$

In addition, we assume that $f(s) > 0$ for $s > 0$. The model for such f is $f(s) = s^q$ with $0 \leq q < 1$.

By a *nonnegative solution* of (P) we mean a function $u \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ (and thus $u \in \mathcal{C}^1(\overline{\Omega})$) that satisfies the equation for the weak derivatives and $u \geq 0$ in Ω. If, in addition, $u > 0$ in Ω, then we call it a *positive solution* of (P). Similarly, by a *nonnegative solution* of (P') we mean a function $u \in W^{2,r}(\Omega)$ that satisfies the equation for the weak derivatives and the boundary condition in the usual sense, and such that $u \geq 0$ in Ω. If, in addition, $u > 0$ in Ω, then we call it a *positive solution* of (P').

Under a stronger regularity condition on a, the existence of a nontrivial nonnegative solution of (P) has been proved in [4,18]. In addition, the existence of a nontrivial nonnegative solution of (P') has been obtained in [5] (see also [1]), under the additional assumption that $\int_\Omega a < 0$. Furthermore, the authors in [5] also proved that the latter condition is necessary for the existence of positive solutions for (P'), if $f \in C^1(0, \infty)$ and $f'(s) > 0$ for $s > 0$. However, due to the non-Lipschitzian character of f at $s = 0$ and the change of sign in a, neither the strong maximum principle nor Hopf's Lemma applies to (P) and (P'). As a consequence, one can't deduce the positivity of nontrivial nonnegative solutions of (P) or (P'). Let us also point out that nontrivial nonnegative solutions of (P) and (P') are not necessarily unique, see [4,5].

In fact, the existence of a positive solution for these indefinite sublinear problems is a delicate issue and very few papers in the literature have addressed this question. Regarding (P), when $f(s) = s^q$, it was first proved in [14] that if the unique solution φ of the linear problem

$$\begin{cases} -\Delta \varphi = a(x) & \text{in} \quad \Omega, \\ \varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

is such that $\varphi > 0$ in Ω, then (P) has a positive solution (which may *not* belong to the interior of the positive cone) for any $0 < q < 1$. This condition, however, is not sharp, since one can find a function a such that (P) possesses a positive solution for some $0 < q < 1$, but the corresponding φ satisfies that $\varphi < 0$ in Ω (see [12, Section 1]). Later on, in the aforementioned article [12], the authors considered the same problem in the one-dimensional and radial cases, providing several sufficient conditions on a (as well as some necessary conditions) for the existence of a positive solution of (P). Some of these results were then extended to the case of a general bounded domain in [13]. We point out that in all these papers the only tool used was essentially the well known sub and supersolutions method in the presence of weak and well-ordered sub and supersolutions (see e.g. [11]).

On the other hand, for the Neumann problem (P'), even with $f(s) = s^q$, to the best of our knowledge, no **sufficient** conditions for the existence of positive solutions are known.

In this article, we shall not only prove that in some cases (P') and (P) admit positive solutions, but even more, that **every** nontrivial nonnegative solution of (P) and (P') is a positive solution. This will be done using a continuity argument inspired by [15] (see also [16]), where the author proves the existence of a positive solution for the problem

$$\begin{cases} -\Delta u = u^p + f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < p < \frac{N+2}{N-2}$, $N \geq 3$, and $f \in L^s(\Omega)$, with $s > \frac{N}{2}$. Under a smallness condition on f (which may change sign), the author shows that this problem has a mountain-pass solution u_f which depends continuously on f, in the sense that, up to a subsequence, $u_f \to u_0$ in $\mathcal{C}^1(\overline{\Omega})$ as $f \to 0$ in $L^s(\Omega)$, where u_0 is a nontrivial nonnegative solution of (1.2) with $f \equiv 0$. Furthermore, by the strong maximum principle and Hopf's Lemma, u_0 lies in the interior of the positive cone of $\mathcal{C}^1(\overline{\Omega})$, and consequently so does u_f if f is close enough to zero. We shall exploit this idea, dealing now with a class of sublinear problems and deducing the positivity of not only one solution, but every nontrivial nonnegative solution.

Roughly speaking, we shall see that the positivity of nontrivial nonnegative solutions can be recovered if (P) or (P') are somehow sufficiently close to a problem the strong maximum principle applies to. This situation occurs, for instance, if the negative part of a is small enough (for (P)) or if $f(s) = s^q$ with q close enough to 1 (for both (P) and (P')). We rely here on the fact that the strong maximum principle applies to (P) and (P') if either $a \geq 0$ or $f(s) = s$.

We set $a^{\pm} := \max(\pm a, 0)$. Observe that the assumption that a changes sign means that $|\text{supp } a^{\pm}| > 0$, where $|A|$ stands for the Lebesgue measure of $A \subset \mathbb{R}^N$. We denote by Ω_+ the largest open subset of Ω where $a > 0$ *a.e.*, and assume that

$$\Omega_+ \text{ has finitely many connected components and } |(\text{supp } a^+) \setminus \Omega_+| = 0. \tag{H_2}$$

In particular, we see that Ω_+ is nonempty.

The above condition will be used to deduce that nontrivial nonnegative solutions of (P) and (P') are positive in a subdomain of Ω_+, and consequently uniformly bounded away from zero therein (see Lemma 2.2). To this end, we shall also assume the following technical condition, which is related to the use of the strong maximum principle:

$$K_{s_0} := \inf_{0 \leq t < s \leq s_0} \frac{f(s) - f(t)}{s - t} > -\infty, \quad \text{for all } s_0 > 0. \tag{H_3}$$

Note in particular that this condition is satisfied, for instance, if f is nondecreasing (in which case $K_{s_0} \geq 0$), and in particular, $f(s) = s^q$ with $0 \leq q < 1$.

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Our positivity results for (P) shall provide us with solutions that lie in the interior of the positive cone of $\mathcal{C}_0^1(\overline{\Omega}) := \{u \in \mathcal{C}^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$, which is denoted by

$$\mathcal{P}_D^\circ := \left\{ u \in \mathcal{C}_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega \right\}.$$

Regarding (P'), we shall obtain solutions that belong to

$$\mathcal{P}_N^\circ := \left\{ u \in \mathcal{C}^1(\overline{\Omega}) : u > 0 \text{ on } \overline{\Omega} \right\}.$$

Note that a positive solution of (P) (respect. (P')) *need not* belong to $\mathcal{P}_D^\circ$ (respect. $\mathcal{P}_N^\circ$), as shown in Proposition 2.9 below.

We state now our main results.

Theorem 1.1. *Assume (H_1), (H_2), and (H_3). Then there exists $\delta > 0$ (possibly depending on f and a^+) such that every nontrivial nonnegative solution of (P) belongs to $\mathcal{P}_D^\circ$ if $\|a^-\|_{L^r(\Omega)} < \delta$.*

Remark 1.2. As already mentioned, if $f \in C^1(0,\infty)$ and $f' > 0$ in $(0,\infty)$, then the condition $\int_\Omega a < 0$ is necessary for the existence of positive solutions of (P'), cf. [5, Lemma 2.1]. In view of this fact, we can't expect an analogue of Theorem 1.1 for (P').

In the case that f is a power, we write (P) and (P') as

$$\begin{cases} -\Delta u = a(x)u^q & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_D}$$

and

$$\begin{cases} -\Delta u = a(x)u^q & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P_N}$$

Theorem 1.3. *Assume (H_2). Then, given $a \in L^r(\Omega)$, there exists $q_0 = q_0(a) \in (0,1)$ such that every nontrivial nonnegative solution of (P_D) belongs to $\mathcal{P}_D^\circ$ if $q_0 < q < 1$.*

As a consequence of Theorems 1.1 and 1.3, we derive the following existence and uniqueness results:

Corollary 1.4. *Under the conditions of Theorem 1.1, let $\|a^-\|_{L^r(\Omega)} < \delta$. Assume in addition that $f \in \mathcal{C}^1(0,\infty)$, f' is nonincreasing in $(0,\infty)$ and $\int_0^t \frac{1}{f(s)} ds < \infty$ for $t > 0$. Then (P) has a solution in $\mathcal{P}_D^\circ$ and has no other nontrivial nonnegative solutions.*

Corollary 1.5. *Under the assumptions of Theorem 1.3, let $q_0 < q < 1$. Then (P_D) has a solution in $\mathcal{P}_D^\circ$ and has no other nontrivial nonnegative solutions.*

Remark 1.6. Let us mention that if f is nondecreasing and $k_1 s^q \le f(s) \le k_2 s^q$ for some $k_1, k_2 > 0$ and all $s \ge 0$, as a consequence of [13, Theorem 3.1], one can deduce the existence of

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a solution $u \in \mathcal{P}_D^\circ$ for (P) if a^- is sufficiently small with respect to a^+. On the other side, in the one-dimensional and radial cases one can derive the existence of a positive solution of (P_D) (but not necessarily belonging to $\mathcal{P}_D^\circ$) provided that q is close enough to 1 (cf. [12]). In this sense, Corollaries 1.4 and 1.5 are consistent with the existence results from [12,13].

For the Neumann problem (P_N), we establish the following analogue of Theorem 1.3:

Theorem 1.7. *Assume (H_2). Then, given $a \in L^r(\Omega)$, there exists $q_0 = q_0(a) \in (0,1)$ such that every nontrivial nonnegative solution of (P_N) belongs to $\mathcal{P}_N^\circ$ if $q_0 < q < 1$. If, in addition, $\int_\Omega a \geq 0$, then (P_N) has no nontrivial nonnegative solution for $q_0 < q < 1$.*

Corollary 1.8. *Under the assumptions of Theorem 1.7, let $q_0 < q < 1$. Assume in addition that $\int_\Omega a < 0$. Then (P_N) has a solution in $\mathcal{P}_N^\circ$ and has no other nontrivial nonnegative solutions.*

Our next result concerns the sets

$$\mathcal{A}_D := \{q \in (0,1) : \text{any nontrivial nonnegative solution of } (P_D) \text{ lies in } \mathcal{P}_D^\circ\},$$

$$\mathcal{A}_N := \{q \in (0,1) : \text{any nontrivial nonnegative solution of } (P_N) \text{ lies in } \mathcal{P}_N^\circ\}.$$

Theorem 1.9. *Assume (H_2) and fix $a \in L^r(\Omega)$. Then:*

(i) *$\mathcal{A}_D$ is a nonempty open interval.*
(ii) *$\mathcal{A}_N$ is a nonempty open interval under the condition $\int_\Omega a < 0$.*

In particular, there exists $q_0 \in [0,1)$ such that $\mathcal{A}_D = (q_0, 1)$, and a similar characterization holds for $\mathcal{A}_N$.

Remark 1.10. Although Theorem 1.9 states that (under (H_2)) the sets $\mathcal{A}_D$ and $\mathcal{A}_N$ (assuming $\int_\Omega a < 0$) are always nonempty, as a consequence of Proposition 2.9 below, we shall see that given any $q \in (0,1)$, we may find a in such a way that (P_D) (respect. (P_N)) has nontrivial nonnegative solutions that do not belong to $\mathcal{P}_D^\circ$ (respect. $\mathcal{P}_N^\circ$). This fact shows that $\mathcal{A}_D$ and $\mathcal{A}_N$ can be made arbitrarily small by choosing a in a suitable way.

The rest of the paper is organized as follows. In the next section we prove some auxiliary results concerning (P) and (P'), whereas in Section 3 we supply the proofs of our main results. Finally, in Section 4 we apply some of our theorems to derive positivity results (as well as existence and multiplicity results of positive solutions) for indefinite concave-convex type problems.

2. Preliminary results

Let us fix the notation to be used in the sequel.

Given $m \in L^r(\Omega)$, $r > N$, and a subdomain $B \subseteq \Omega$ such that $m^+ \not\equiv 0$ in B, we denote by $\lambda_1(m, B)$ the first positive eigenvalue of the problem

$$\begin{cases} -\Delta\phi = \lambda m(x)\phi & \text{in} \quad B, \\ \phi = 0 & \text{on} \quad \partial B. \end{cases}$$

We shall deal with several norms, which will be denoted as follows: $\|u\|_{L^r(\Omega)} := (\int_\Omega |u|^r)^{\frac{1}{r}}$, $\|u\|_{H_0^1(\Omega)} := (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$ and $\|u\|_{H^1(\Omega)} := (\int_\Omega (|\nabla u|^2 + u^2))^{\frac{1}{2}}$.

To begin with, we provide several useful lower bounds for nontrivial nonnegative solutions of (P) and (P').

Lemma 2.1. *Assume (H_2) and let u be a nontrivial nonnegative solution of (P) or (P'). Then there exists a subdomain $\Omega' \subset \Omega_+$ such that $u > 0$ in Ω'.*

Proof. If u is a nontrivial nonnegative solution of (P) then it satisfies

$$0 < \int_\Omega |\nabla u|^2 = \int_\Omega a(x)f(u)u \leq \int_{\operatorname{supp} a^+} a^+(x)f(u)u = \int_{\Omega_+} a^+(x)f(u)u,$$

where we used the assumption that $|(\operatorname{supp} a^+) \setminus \Omega_+| = 0$. It follows that $u \not\equiv 0$ in Ω_+, and consequently $u > 0$ in some subdomain of Ω_+. The same argument applies if u is a nontrivial nonnegative solution of (P'), since u can't be a constant. $\square$

Lemma 2.2. *Assume that (H_3) holds, $\lim_{s\to 0^+} s^{-1}f(s) = \infty$ and $\Omega' \neq \emptyset$ is a subdomain of Ω_+. Then, for any open ball B such that $\overline{B} \subset \Omega'$ there exists a function $\psi \in W^{2,r}(B)$ such that $u \geq \psi > 0$ in B for every nontrivial nonnegative supersolution of*

$$-\Delta u = a(x)f(u) \quad \text{in } \Omega'. \tag{2.1}$$

Proof. Let u be a nontrivial nonnegative supersolution of (2.1) and B be an open ball such that $\overline{B} \subset \Omega'$. Then $a \geq 0$ and $a \not\equiv 0$ in B. Let $\phi \in W^{2,r}(B) \cap W_0^{1,r}(B)$ be a positive eigenfunction associated to $\lambda_1(a, B)$, with $\|\phi\|_\infty = 1$. We observe that for all $\varepsilon > 0$ sufficiently small it holds that

$$-\Delta(\varepsilon\phi) \leq a(x)f(\varepsilon\phi) \quad \text{in } B.$$

Indeed, note that $-\Delta(\varepsilon\phi) = \varepsilon\lambda_1(a, B)a(x)\phi$ in B. Hence, it is enough to check that $\varepsilon\lambda_1(a, B)a(x)\phi \leq a(x)f(\varepsilon\phi)$, i.e.

$$\lambda_1(a, B) \leq \frac{f(\varepsilon\phi)}{\varepsilon\phi} \quad \text{in } B.$$

Since $\frac{f(s)}{s} \to \infty$ as $s \to 0^+$ and ϕ is bounded, we see that there exists $\varepsilon_0 > 0$ such that the above inequality holds for all $0 < \varepsilon \leq \varepsilon_0$.

To conclude the proof we show that $u \geq \varepsilon_0\phi$ in B (note that ε_0 does not depend on u). Indeed, suppose this is not true. Since Ω' is connected, it follows from the strong maximum principle that $u > 0$ in Ω', so that $u > 0$ on $\overline{B}$. Moreover, $\phi = 0$ on ∂B, so there exists $s \in (0, 1)$ such that $u \geq s\varepsilon_0\phi$ in B and $u(x_0) = s\varepsilon_0\phi(x_0)$ for some $x_0 \in B$. Setting $s_0 := \|u\|_{L^\infty(\Omega)}$ and $M(x) := |K_{s_0}|a(x)$, where K_{s_0} is given by (H_3), one can see that the map $s \to M(x)s + a(x)f(s)$ is nondecreasing for all $s \in (0, \|u\|_\infty)$ and *a.e.* $x \in B$. Then,

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$$
\begin{aligned}
&-\Delta(u-s\varepsilon_0\phi)+M(x)(u-s\varepsilon_0\phi)\\
&\quad\geq M(x)(u-s\varepsilon_0\phi)+a(x)(f(u)-f(s\varepsilon_0\phi))\geq 0 \quad \text{in } B,
\end{aligned}
$$

and $u > s\varepsilon_0\phi$ on ∂B. Therefore, the strong maximum principle (e.g. [20]) says that $u > s\varepsilon_0\phi$ in B, which is a contradiction. The proof is complete. $\square$

Remark 2.3. Let us point out that the proof of Lemma 2.2 is the only instance where (H_3) is employed.

We prove now that $\|u\|_{H_0^1(\Omega)} \geq C$ for any nontrivial nonnegative solution of (P), for some constant $C > 0$ independent of a^-.

Lemma 2.4. *Assume that (H_2) and (H_3) hold, and $\lim_{s\to 0^+} s^{-1}f(s) = \infty$. Then there exists a constant $C > 0$ such that $\|u\|_{H_0^1(\Omega)} \geq C$ for every nontrivial nonnegative solution u of (P). Moreover, C does not depend on a^-.*

Proof. Assume by contradiction that there exists a sequence $\{u_n\}$ of solutions of (P) with $u_n \to 0$ in $H_0^1(\Omega)$. Then $u_n \to 0$ in $L^2(\Omega)$, and, up to a subsequence, we have $u_n \to 0$ *a.e.* in Ω. By Lemma 2.1, we know that any nontrivial nonnegative solution u of (P) is positive in some subdomain of Ω_+. Thus, since Ω_+ has finitely many connected components, we may assume that, for all $n \in \mathbb{N}$, $u_n > 0$ in some fixed subdomain $\widetilde{\Omega} \subset \Omega_+$. However, by Lemma 2.2, we have $u_n \geq \psi > 0$ in some open ball $B \subset \widetilde{\Omega}$, so we reach a contradiction. $\square$

Next we get an *a priori* bound from below for nontrivial nonnegative solutions of either (P_D) or (P_N). We remark that this estimate does not depend on q.

Lemma 2.5. *Assume that $\Omega' \neq \emptyset$ is a subdomain of Ω_+ such that $\lambda_1(a,\Omega') < 1$. Then there exist a domain B such that $\overline{B} \subset \Omega'$ and a function $\phi \in W^{2,r}(B)$ such that $u \geq \phi > 0$ in B, for every nontrivial nonnegative supersolution of*

$$
-\Delta u = a(x)u^q \quad \text{in } \Omega', \tag{2.2}
$$

and for every $q \in (0,1)$.

Proof. Let $\Omega'_\delta := \{x \in \Omega' : \text{dist}(x,\partial\Omega') > \delta\}$ for $\delta > 0$. From the variational characterization of $\lambda_1(a,B)$, we know that $\lambda_1(a,\Omega'_\delta) \to \lambda_1(a,\Omega')$ as $\delta \to 0^+$ (see e.g. Lemma 2.5 in [7]). We fix $\delta_0 > 0$ such that $\lambda_1(a,\Omega'_{\delta_0}) < 1$ and set $B := \Omega'_{\delta_0}$. Let $\phi > 0$ with $\|\phi\|_\infty = 1$ be as in Lemma 2.2, i.e. a solution of

$$
\begin{cases}
-\Delta\phi = \lambda_1(a,B)\,a(x)\,\phi & \text{in} \quad B,\\
\phi = 0 & \text{on} \quad \partial B,
\end{cases}
$$

and let

$$
0 < \varepsilon \leq \varepsilon_q := \frac{1}{\lambda_1(a,B)^{1/(1-q)}}.
$$

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Observe that since $\lambda_1(a, B) < 1$ we have $\varepsilon_q \geq 1$ for all q. Then, taking into account that $0 < \phi \leq 1$ and the definition of ε_q, we derive that

$$-\Delta(\varepsilon\phi) = \lambda_1(a, B)\, a(x)\, \varepsilon\phi \leq a(x)(\varepsilon\phi)^q \quad \text{in } B$$

for all $0 < \varepsilon \leq \varepsilon_q$.

Now, given any nontrivial nonnegative supersolution u of (2.2), we can argue as in the last paragraph of the proof of Lemma 2.2 to infer that $u \geq \phi$ in B for any $q \in (0, 1)$. This concludes the proof of the lemma. $\square$

The proof of the next estimates are similar to the one of Lemma 2.4 (we use now Lemma 2.5), so we omit it.

Lemma 2.6. *Assume that (H_2) holds, $0 < q < 1$, and $\lambda_1(a, \Omega') < 1$ for any open connected component $\Omega' \subset \Omega_+$. Then there exists a constant $C > 0$ such that:*

(i) $\|u\|_{H_0^1(\Omega)} \geq C$ *for every nontrivial nonnegative solution u of (P_D).*
(ii) $\|u\|_{H^1(\Omega)} \geq C$ *for every nontrivial nonnegative solution u of (P_N).*

Moreover, C does not depend on q.

To end this section, we prove some results on the sets $\mathcal{A}_D$ and $\mathcal{A}_N$.

Lemma 2.7. *Assume (H_2). If $q_0 \in \mathcal{A}_D$, then $\left(q_0, \frac{1}{2-q_0}\right) \subset \mathcal{A}_D$.*

Proof. Assume to the contrary that $q_0 \in \mathcal{A}_D$, $q_0 < q < \frac{1}{2-q_0}$, but $q \notin \mathcal{A}_D$. It follows that there exists a nontrivial nonnegative solution u of (P_D) such that $u \notin \mathcal{P}_D^{\circ}$. Let

$$\beta := \frac{1-q}{1-q_0} \in (0, 1), \qquad \gamma := \frac{q-q_0}{1-q} > 0,$$

and consider the auxiliary problem

$$\begin{cases} -\Delta w = \beta a(x) w^{-\gamma}(w^{1/\beta} - \varepsilon)^q & \text{in } \Omega, \\ w \geq \varepsilon^\beta & \text{in } \Omega, \\ w = \varepsilon^\beta & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

with $0 < \varepsilon \leq 1$. Equivalently, putting $\hat{w} := w - \varepsilon^\beta$, we consider

$$\begin{cases} -\Delta \hat{w} = \beta a(x)(\hat{w} + \varepsilon^\beta)^{-\gamma}\{(\hat{w} + \varepsilon^\beta)^{1/\beta} - \varepsilon\}^q & \text{in } \Omega, \\ w \geq 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

The limiting problem as $\varepsilon \to 0^+$ is understood as

$$\begin{cases} -\Delta w = \beta a(x) w^{q_0} & \text{in } \Omega, \\ w \geq 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Since $q_0 \in \mathcal{A}_D$, any nontrivial nonnegative solution of (2.5) belongs to $\mathcal{P}_D^\circ$.

In the sequel, we shall obtain a solution w_ε of (2.3) and show that, as $\varepsilon \to 0$, w_ε converges (up to a subsequence) to a nontrivial nonnegative solution of (2.5) that does not belong to $\mathcal{P}_D^\circ$. This will provide us with a contradiction. We divide the rest of the proof in several steps:

Step 1: Construction of a weak supersolution of (2.3).

We note that $\psi = \psi_\varepsilon := (u+\varepsilon)^\beta$ is a supersolution of (2.3). Indeed, since $1-\beta = \gamma\beta$, by direct computations we have that

$$-\Delta\psi \geq \beta a(x) \psi^{-\gamma} (\psi^{1/\beta} - \varepsilon)^q \quad \text{in } \Omega,$$

and $\psi = \varepsilon^\beta$ on $\partial\Omega$, as desired.

Step 2: Construction of a weak subsolution of (2.3).

By Lemma 2.1, there exists a ball B such that $a \geq 0$, $a \not\equiv 0$ *a.e.* in B and $u > 0$ on $\overline{B}$. Let $\phi \in W^{2,r}(B) \cap W_0^{1,r}(B)$ be a positive eigenfunction associated to $\lambda_1(a, B)$, with $\|\phi\|_\infty = 1$, and extend ϕ to $\overline{\Omega}$ by setting $\phi = 0$ in $\overline{\Omega} \setminus \overline{B}$. Given $0 < \delta \leq 1$, we set

$$\varphi_{\delta,\varepsilon} := \begin{cases} \delta\phi + \varepsilon^\beta & \text{in } B, \\ \varepsilon^\beta & \text{in } \overline{\Omega} \setminus \overline{B}. \end{cases} \tag{2.6}$$

We observe that

$$\begin{aligned} & -\Delta\varphi_{\delta,\varepsilon} - \beta a(x) \varphi_{\delta,\varepsilon}^{-\gamma} (\varphi_{\delta,\varepsilon}^{1/\beta} - \varepsilon)^q \\ & \leq a(x) \left(\lambda_1(a, B)\, \delta\phi - \beta(\delta\phi + \varepsilon^\beta)^{-\gamma} \{(\delta\phi + \varepsilon^\beta)^{1/\beta} - \varepsilon\}^q \right) \quad \text{in } B. \end{aligned}$$

We claim that there exists $c_0 > 0$, independent of $x \in B$, such that for $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$ we have

$$(\delta\phi + \varepsilon^\beta)^{-\gamma} \{(\delta\phi + \varepsilon^\beta)^{1/\beta} - \varepsilon\}^q \geq c_0 (\delta\phi)^{q_0+\gamma} \quad \text{in } B.$$

Indeed, since $q/\beta = q_0 + \gamma$, we note that for $x \in B$,

$$\frac{(\delta\phi + \varepsilon^\beta)^{-\gamma} \{(\delta\phi + \varepsilon^\beta)^{1/\beta} - \varepsilon\}^q}{(\delta\phi)^{q_0+\gamma}} \geq \frac{(\delta\phi)^{q/\beta}}{(\delta\phi + \varepsilon^\beta)^\gamma (\delta\phi)^{q_0+\gamma}} = \frac{1}{(\delta\phi + \varepsilon^\beta)^\gamma} \geq \frac{1}{2^\gamma} =: c_0,$$

as desired. Here, we have used the fact that if $\alpha > 1$ then $(s+t)^\alpha \geq s^\alpha + t^\alpha$ for $t, s \geq 0$. Thus the claim is proved. It follows that for $\delta > 0$ small enough, we have that

$$-\Delta\varphi_{\delta,\varepsilon} - \beta a(x) \varphi_{\delta,\varepsilon}^{-\gamma} \left(\varphi_{\delta,\varepsilon}^{1/\beta} - \varepsilon \right)^q \leq a(x) \left(\lambda_1(a, B)\, \delta\phi - c_0 \beta (\delta\phi)^{q_0+\gamma} \right) \leq 0 \quad \text{in } B,$$

since the assumption $q < \frac{1}{2-q_0}$ implies $q_0 + \gamma < 1$. Note that δ is determined uniformly in $\varepsilon \in (0, 1]$. Thus, employing the divergence theorem as stated e.g. in [6], p. 742, we deduce that $\varphi_{\delta,\varepsilon}$ is a weak subsolution of (2.3).

Step 3: The subsolution and the supersolution of (2.3) are well-ordered.

We shall see that, choosing δ and ε adequately, $(u+\varepsilon)^\beta$ and $\varphi_{\delta,\varepsilon}$ are well-ordered, i.e., $(u+\varepsilon)^\beta \geq \varphi_{\delta,\varepsilon}$ in Ω. We assert that there exist $\delta_1, \varepsilon_1 > 0$ such that if $\varepsilon \in (0, \varepsilon_1)$, then

$$(u+\varepsilon)^\beta \geq \delta_1 \phi + \varepsilon^\beta \quad \text{in } B.$$

Indeed, if we fix ε_1 and δ_1 such that

$$\varepsilon_1^\beta \leq \frac{1}{2}(\min_{\overline{B}} u)^\beta, \qquad \delta_1 \leq \frac{1}{4}(\min_{\overline{B}} u)^\beta,$$

then it is clear that

$$(u+\varepsilon)^\beta \geq \frac{1}{2}\left(u^\beta + \varepsilon^\beta\right) \geq \frac{1}{2}\{(\min_{\overline{B}} u)^\beta - \varepsilon^\beta\} + \varepsilon^\beta \geq \frac{1}{4}(\min_{\overline{B}} u)^\beta + \varepsilon^\beta \geq \delta_1 \phi + \varepsilon^\beta \quad \text{in } B.$$

Hence, for every $\varepsilon \in (0, \varepsilon_1)$, the method of weak sub and supersolutions (see e.g. [11, Theorem 4.9]) gives us some $w_\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution of (2.3), with

$$\varphi_{\delta_1,\varepsilon} \leq w_\varepsilon \leq (u+\varepsilon)^\beta \quad \text{in } \Omega. \tag{2.7}$$

Furthermore, by standard regularity arguments, $w_\varepsilon \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$.

Step 4: The limiting behavior of w_ε as $\varepsilon \to 0^+$.

We convert w_ε to (2.4) by $\hat{w}_\varepsilon = w_\varepsilon - \varepsilon^\beta$, so that $\hat{w}_\varepsilon = 0$ on $\partial\Omega$. Thus, we deduce that

$$\begin{aligned}
\int_\Omega |\nabla \hat{w}_\varepsilon|^2 &= \beta \int_\Omega a(x)\hat{w}_\varepsilon(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}((\hat{w}_\varepsilon + \varepsilon^\beta)^{1/\beta} - \varepsilon)^q \\
&\leq C \int_\Omega \hat{w}_\varepsilon(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}(\hat{w}_\varepsilon + \varepsilon^\beta)^{q/\beta} \\
&= C \int_\Omega \hat{w}_\varepsilon(\hat{w}_\varepsilon + \varepsilon^\beta)^{q_0}.
\end{aligned}$$

Since we see from (2.7) that $\|\hat{w}_\varepsilon\|_{L^\infty(\Omega)} \leq C$ as $\varepsilon \to 0^+$, we infer that $\|\hat{w}_\varepsilon\|_{H_0^1(\Omega)}$ is bounded as $\varepsilon \to 0^+$. It follows that, up to a subsequence, $\hat{w}_\varepsilon \rightharpoonup \hat{w}_0$ in $H_0^1(\Omega)$, and $\hat{w}_\varepsilon \to \hat{w}_0$ *a.e.* in Ω for some $\hat{w}_0 \in H_0^1(\Omega)$. Also, since $\hat{w}_\varepsilon$ is a weak solution of (2.4), we note that

$$\int_\Omega \nabla \hat{w}_\varepsilon \nabla v = \beta \int_\Omega a(x)(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}\{\left(\hat{w}_\varepsilon + \varepsilon^\beta\right)^{1/\beta} - \varepsilon\}^q v, \quad \forall v \in \mathcal{C}_0^1(\Omega).$$

So, from the fact that $\hat{w}_\varepsilon \rightharpoonup \hat{w}_0$ in $H_0^1(\Omega)$, we get that

$$\int_\Omega \nabla \hat{w}_\varepsilon \nabla v \to \int_\Omega \nabla \hat{w}_0 \nabla v, \quad \forall v \in \mathcal{C}_0^1(\Omega).$$

On the other hand, recalling (2.7) and that $-\gamma + q/\beta = q_0$, we see that

$$\begin{aligned}\left|a(x)(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}\{\left(\hat{w}_\varepsilon + \varepsilon^\beta\right)^{1/\beta} - \varepsilon\}^q v\right| &\leq C\,|a(x)|\,(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}(\hat{w}_\varepsilon + \varepsilon^\beta)^{q/\beta} \\ &= C\,|a(x)|\,(\hat{w}_\varepsilon + \varepsilon^\beta)^{q_0} \\ &\leq C'\,|a(x)| \in L^r(\Omega).\end{aligned}$$

Therefore, the Lebesgue convergence theorem yields that

$$\int_\Omega a(x)(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}\{\left(\hat{w}_\varepsilon + \varepsilon^\beta\right)^{1/\beta} - \varepsilon\}^q v \to \int_\Omega a(x)\hat{w}_0^{q_0} v, \quad \forall v \in \mathcal{C}_0^1(\Omega).$$

Indeed, if $w_0 > 0$, then

$$(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}\{\left(\hat{w}_\varepsilon + \varepsilon^\beta\right)^{1/\beta} - \varepsilon\}^q = w_\varepsilon^{-\gamma}(w_\varepsilon^{1/\beta} - \varepsilon)^q \to w_0^{-\gamma}(w_0^{1/\beta})^q = w_0^{q_0},$$

whereas if $w_0 = 0$, then

$$(\hat{w}_\varepsilon + \varepsilon^\beta)^{-\gamma}\{\left(\hat{w}_\varepsilon + \varepsilon^\beta\right)^{1/\beta} - \varepsilon\}^q = w_\varepsilon^{-\gamma}(w_\varepsilon^{1/\beta} - \varepsilon)^q \leq w_\varepsilon^{-\gamma} w_\varepsilon^{q/\beta} = w_\varepsilon^{q_0} \to 0.$$

Summing up, we have obtained that

$$\int_\Omega \nabla\hat{w}_0 \nabla v = \beta \int_\Omega a(x)\hat{w}_0^{q_0} v = 0, \quad \forall v \in \mathcal{C}_0^1(\Omega).$$

This implies that $\hat{w}_0$ is a weak solution of (2.5). Now, from (2.7), we recall that for $\varepsilon \in (0, \varepsilon_1)$,

$$\varphi_{\delta_1,\varepsilon} - \varepsilon^\beta \leq \hat{w}_\varepsilon \leq (u+\varepsilon)^\beta - \varepsilon^\beta \quad \text{in } \Omega.$$

Therefore, passing to the limit as $\varepsilon \to 0^+$, this inequality provides

$$\varphi_{\delta_1,0} \leq \hat{w}_0 \leq u^\beta \quad \text{in } \Omega.$$

This means that $\hat{w}_0$ is a nontrivial nonnegative solution of (2.5), but $\hat{w}_0 \notin \mathcal{P}_D^\circ$, since $u \notin \mathcal{P}_D^\circ$ by assumption. Hence we reach a contradiction, and the proof is complete. □

Remark 2.8. Lemma 2.7 also holds for $\mathcal{A}_N$. Indeed, we can prove it with some minor modifications: assume $q_0 \in \mathcal{A}_N$, $q_0 < q < \frac{1}{2-q_0}$, but $q \notin \mathcal{A}_N$. It follows that there exists a nontrivial nonnegative solution u of (P_N) such that u does not belong to $\mathcal{P}_N^\circ$. The rest of the proof proceeds with the following changes:

- $w = \varepsilon^\beta$ on $\partial\Omega$ replaced by $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ in (2.3);
- $w = 0$ on $\partial\Omega$ replaced by $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ in (2.5);
- no consideration of (2.4). Note that $\frac{\partial w_\varepsilon}{\partial \nu} = 0$ on $\partial\Omega$;
- in Step 4, the test functions are now taken in $\mathcal{C}^1(\overline{\Omega})$.

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The following proposition shows that in general it is hard to give a lower estimate for q_0 in Theorems 1.3 and 1.7. It also shows that the sets $\mathcal{A}_D$ and $\mathcal{A}_N$ can be made arbitrary small by choosing a suitable a. On the other hand, it is an interesting open question whether these sets can be equal to the whole interval $(0, 1)$.

Proposition 2.9. *Let $q \in (0, 1)$ and $\Omega = (x_0, x_1) \subset \mathbb{R}$. Then there exists $a \in \mathcal{C}^2(\overline{\Omega})$ such that $q \notin \mathcal{A}_D$ and $q \notin \mathcal{A}_N$.*

Proof. After a translation and a dilation, we can assume that $\Omega = (0, \pi)$. We set

$$r := \frac{2}{1-q} \in (2, \infty), \quad \text{and} \quad a(x) := r^{1-\frac{2}{r}} \left(1 - r\cos^2 x\right), \quad \text{for } x \in \overline{\Omega}.$$

Clearly a changes sign in Ω. We now set

$$u(x) := \frac{\sin^r x}{r} \in \mathcal{C}^2(\overline{\Omega}).$$

Note that $u > 0$ in Ω. We claim that

$$\begin{cases} -u'' = a(x)u^q & \text{in} \quad \Omega, \\ u = u' = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Indeed, it is immediate to see that the boundary conditions are satisfied. Also, taking into account that $rq = r - 2$ (and so, $q = 1 - 2/r$), a few computations show that

$$\begin{aligned} -u'' &= -\left((r-1)\sin^{r-2} x\cos^2 x - \sin^r x\right) = \sin^{r-2} x\left(1 - r\cos^2 x\right) \\ &= \left(1 - r\cos^2 x\right)\sin^{rq} x = r^{1-\frac{2}{r}}\left(1 - r\cos^2 x\right)\left(\frac{\sin^r x}{r}\right)^q = a(x)u^q \end{aligned}$$

and therefore the claim follows.

To conclude the proof we note that, since $u > 0$ in Ω and $u = u' = 0$ on $\partial\Omega$, we have that $q \notin \mathcal{A}_D$ and $q \notin \mathcal{A}_N$. □

Remark 2.10. Let q, Ω, a and u be as in the above proposition. Consider any bounded open interval Ω' with $\Omega' \supset \overline{\Omega}$, and extend (to Ω') u by zero and a in any way. Then we clearly see that u is a nontrivial nonnegative solution having a *dead core* in Ω' (i.e. an open subset with compact closure in Ω' where u vanishes) of both (P_D) and (P_N), with Ω' instead of Ω.

3. Proofs of main results

Remark 3.1. The following fact shall be repeatedly used in the sequel. Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded sequence such that

$$-\Delta u_n = h_n(x, u_n) \quad \text{in } \Omega.$$

Here $h_n : \Omega \times [0, \infty) \to \mathbb{R}$ are Carathéodory functions satisfying

$$|h_n(x,s)| \le b(x)(1+s) \quad \text{for } x \in \Omega, \text{ and } s \ge 0,$$

where $b \in L^r(\Omega)$, $r > N$. Then $\{u_n\}$ has a convergent subsequence in $\mathcal{C}^1(\overline{\Omega})$. Indeed, by using the above inequality on h_n, Hölder's inequality and the Sobolev embedding theorem, we can derive that $\|u_n\|_{W^{2,\sigma_k}(\Omega)}$ is bounded for each $\sigma_k = \frac{2N}{N-2k}$, $k = 1, 2, 3, \dots$. Hence, employing Sobolev's embedding theorem again, we obtain that $\|u_n\|_{\mathcal{C}^{1+\theta}(\overline{\Omega})}$ is bounded for some $\theta \in (0,1)$. The desired conclusion follows by the Ascoli–Arzelà theorem. We also note that a similar argument applies to the analogous Neumann problem.

Proof of Theorem 1.1. Assume by contradiction that $\{a_n\}$ is a sequence such that $a_n^- \to 0$ in $L^r(\Omega)$ and u_n are nontrivial nonnegative solutions of (P) with $a = a_n$, satisfying that $u_n \notin \mathcal{P}_D^\circ$. Let us stress the fact that $a_n^+ = a^+$ does not depend on n. We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, by our assumptions on f, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$0 \le f(s) \le C_\varepsilon + \varepsilon s, \quad \forall s \ge 0. \tag{3.1}$$

Hence, for some $C, \tilde{C}_\varepsilon > 0$, we have

$$\|u_n\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla u_n|^2 \le \int_\Omega a^+(x)\,(C_\varepsilon + \varepsilon u_n)\,u_n \le \tilde{C}_\varepsilon \|u_n\|_{H_0^1(\Omega)} + C\varepsilon \|u_n\|_{H_0^1(\Omega)}^2,$$

where we have used Poincaré´s inequality. Taking $\varepsilon > 0$ small enough, we deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. We can then assume that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \to u_0$ in $L^p(\Omega)$, with $p \in (1, 2^*)$, and $u_n \to u_0$ *a.e.* in Ω, for some u_0. We claim that $u_0 \not\equiv 0$. Indeed, if $u_0 \equiv 0$ then, since $u_n \to 0$ in $L^p(\Omega)$, a_n is bounded in $L^r(\Omega)$, and

$$\int_\Omega \nabla u_n \nabla \phi = \int_\Omega a_n(x) f(u_n)\phi, \quad \forall \phi \in H_0^1(\Omega), \tag{3.2}$$

taking $\phi = u_n$ we see that $u_n \to 0$ in $H_0^1(\Omega)$, which contradicts Lemma 2.4. Therefore $u_0 \not\equiv 0$, as claimed. In addition, since $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, recalling (3.1) and choosing $\phi = u_n - u_0$ in (3.2), we obtain $u_n \to u_0$ in $H_0^1(\Omega)$. Moreover, since $a_n^- \to 0$ in $L^r(\Omega)$, u_0 is a nontrivial nonnegative solution of

$$\begin{cases} -\Delta u_0 = a^+(x) f(u_0) & \text{in} \quad \Omega, \\ u_0 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By the strong maximum principle and Hopf's Lemma, we have $u_0 \in \mathcal{P}_D^\circ$. Furthermore, standard elliptic regularity yields, up to a subsequence, that $u_n \to u_0$ in $\mathcal{C}^1(\overline{\Omega})$ (see Remark 3.1, with $h_n(x,s) = a_n(x) f(s)$). Thus we must have $u_n \in \mathcal{P}_D^\circ$ for n large enough, which contradicts the assumption that $u_n \notin \mathcal{P}_D^\circ$. $\square$

Proof of Theorem 1.3. First we note that, for every $c > 0$, u is a nonnegative solution of (P_D) if and only if $v := c^{1/(1-q)} u$ is a nonnegative solution of (P_D) with a replaced by ca. Let $\Omega' \neq \emptyset$ be an open connected component of Ω_+. Since $\lambda_1(ca, \Omega') = c^{-1}\lambda_1(a, \Omega') \to 0$ as $c \to \infty$, we can then assume without loss of generality that $\lambda_1(a, \Omega') < 1$.

Assume by contradiction that $q_n \to 1^-$ and u_n are nontrivial nonnegative solutions of (P_D) with $q = q_n$ and $u_n \notin \mathcal{P}_D^\circ$.

First we suppose that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. We can assume that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \to u_0$ in $L^p(\Omega)$ with $p \in (1, 2^*)$, and $u_n \to u_0$ *a.e.* in Ω, for some u_0. From

$$\int_\Omega \nabla u_n \nabla (u_n - u_0) = \int_\Omega a(x) u_n^{q_n} (u_n - u_0) \to 0,$$

we infer that $u_n \to u_0$ in $H_0^1(\Omega)$. By Lemma 2.6 (i), we have that $u_0 \not\equiv 0$. Moreover, u_0 satisfies

$$-\Delta u_0 = a(x) u_0, \quad u_0 \geq 0, \quad u_0 \in H_0^1(\Omega).$$

We deduce then, by Remark 3.1 (with $h_n(x, s) = a(x)s^{q_n}$), that $u_n \to u_0$ in $\mathcal{C}^1(\overline{\Omega})$, up to a subsequence. Moreover, by the strong maximum principle and Hopf's Lemma we get that $u_0 \in \mathcal{P}_D^\circ$ and consequently $u_n \in \mathcal{P}_D^\circ$ for n large enough, which yields a contradiction.

We suppose now that $\{u_n\}$ is unbounded in $H_0^1(\Omega)$. Then we can assume that

$$\|u_n\| := \|u_n\|_{H_0^1(\Omega)} \to \infty, \quad v_n := \frac{u_n}{\|u_n\|} \rightharpoonup v_0 \text{ in } H_0^1(\Omega),$$

$$v_n \to v_0 \text{ in } L^p(\Omega), \text{with } p \in (1, 2^*), \text{ and } v_n \to v_0 \text{ } a.e. \text{ in } \Omega,$$

for some v_0. Note that v_n satisfies

$$-\Delta v_n = a(x) \frac{v_n^{q_n}}{\|u_n\|^{1-q_n}}, \quad v_n \geq 0, \quad v_n \in H_0^1(\Omega). \tag{3.3}$$

Since $\|u_n\| \geq 1$ for n large enough, we have either $\|u_n\|^{1-q_n} \to \infty$ or $\|u_n\|^{1-q_n}$ is bounded. In the first case, from (3.3) we have

$$1 = \|v_n\|^2 = \frac{\int_\Omega a(x) v_n^{q_n+1}}{\|u_n\|^{1-q_n}} \to 0,$$

which is a contradiction. Now, if $\|u_n\|^{1-q_n}$ is bounded then we can assume that $\|u_n\|^{1-q_n} \to d \geq 1$. From (3.3), we obtain

$$\int_\Omega \nabla v_0 \nabla \phi = \frac{1}{d} \int_\Omega a(x) v_0 \phi, \quad \forall \phi \in H_0^1(\Omega),$$

i.e.

$$-\Delta v_0 = \frac{1}{d} a(x) v_0 \quad \text{in } \Omega, \quad v_0 \in H_0^1(\Omega).$$

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In addition, $v_n \to v_0$ in $H_0^1(\Omega)$, so that $v_0 \not\equiv 0$ and $v_0 \geq 0$. Once again, by the strong maximum principle and Hopf's Lemma, we deduce that $v_0 \in \mathcal{P}_D^\circ$. Furthermore, recalling Remark 3.1 (with $h_n(x,s) = a(x)\|u_n\|^{q_n-1}s^{q_n}$) we have that $v_n \to v_0$ in $\mathcal{C}^1(\overline{\Omega})$, up to a subsequence. Consequently $v_n \in \mathcal{P}_D^\circ$ for n large enough. Hence $u_n \in \mathcal{P}_D^\circ$, and we get another contradiction, which concludes the proof. □

Proof of Theorem 1.7. We proceed as in the proof of Theorem 1.3: we assume by contradiction that $q_n \to 1^-$, u_n are nontrivial nonnegative solutions of (P_N) with $q = q_n$, and $u_n \notin \mathcal{P}_N^\circ$.

First we suppose that $\{u_n\}$ is unbounded in $H^1(\Omega)$. Then we can assume that

$$\|u_n\| := \|u_n\|_{H^1(\Omega)} \to \infty, \quad v_n := \frac{u_n}{\|u_n\|} \rightharpoonup v_0 \text{ in } H^1(\Omega),$$

$$v_n \to v_0 \text{ in } L^p(\Omega), \text{ with } p \in (1, 2^*), \text{ and } v_n \to v_0 \ a.e. \text{ in } \Omega,$$

for some v_0. Note that v_n satisfies

$$-\Delta v_n = a(x) \frac{v_n^{q_n}}{\|u_n\|^{1-q_n}}, \quad v_n \geq 0, \quad v_n \in H^1(\Omega), \tag{3.4}$$

and so we have

$$\int_\Omega \nabla v_n \nabla \phi = \int_\Omega a(x) \frac{v_n^{q_n}}{\|u_n\|^{1-q_n}} \phi, \quad \forall \phi \in H^1(\Omega).$$

Now, if $\|u_n\|^{1-q_n} \to \infty$, taking $\phi = v_n$ we obtain that $\int_\Omega |\nabla v_n|^2 \to 0$, which implies that $v_n \to v_0$ in $H^1(\Omega)$ and v_0 is a nonnegative constant. Since $\|v_n\| = 1$, we infer that v_0 is a positive constant. By Remark 3.1, we have $v_n \to v_0$ in $\mathcal{C}^1(\overline{\Omega})$. Consequently, $v_n \in \mathcal{P}_N^\circ$ for n large enough, which yields a contradiction.

On the other hand, if $\|u_n\|^{1-q_n}$ is bounded, reasoning as in the proof of Theorem 1.3 we derive now that v_0 satisfies

$$-\Delta v_0 = \frac{1}{d} a(x) v_0 \text{ in } \Omega, \quad \text{with} \quad \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

In addition, $v_n \to v_0$ in $H^1(\Omega)$, so that $v_0 \not\equiv 0$ and $v_0 \geq 0$. By the strong maximum principle, we deduce that $v_0 \in \mathcal{P}_N^\circ$. Once again, by Remark 3.1, we have $v_n \to v_0$ in $\mathcal{C}^1(\overline{\Omega})$, so that again we reach a contradiction.

Now, if $\{u_n\}$ is bounded in $H^1(\Omega)$ then we can argue again as in the proof of Theorem 1.3. Indeed, by Lemma 2.6 (ii), we have that $v_0 \not\equiv 0$. The rest of the argument is similar, so we omit it.

Finally, the non-existence assertion follows as in the proof of Corollary 2.1 in [5]. □

Proof of Corollary 1.4. We first claim that (P) admits a nontrivial nonnegative solution u. Indeed, let $\underline{u}_\gamma := \varphi_{\gamma,0}$, where $\varphi_{\gamma,0}$ is given by (2.6) (with $\delta = \gamma$ and $\varepsilon = 0$). Using the condition that $\lim_{s\to 0^+} \frac{f(s)}{s} = \infty$ and arguing as in the first part of the proof of Lemma 2.2, it is easy to see that, for all $\gamma > 0$ sufficiently small, $\underline{u}_\gamma$ is a (nonnegative) subsolution of (P). On the other side, let $\psi > 0$ be the unique solution of the problem

$$\begin{cases} -\Delta\psi = a^+(x) & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $\lim_{s\to\infty} \frac{f(s)}{s} = 0$, we note that, for every $k > 0$ large enough, $\overline{u}_k := k\psi$ is a supersolution of (P). Indeed, there exists $C > 0$ such that

$$f(s) \le \frac{1}{2\|\psi\|_{L^\infty(\Omega)}} s + C, \quad s \ge 0.$$

It follows that

$$-\Delta(k\psi) - a(x)f(k\psi) \ge a^+(x)\left(\frac{k}{2} - C\right) \ge 0 \quad \text{in } \Omega, \quad \text{if} \quad k \ge 2C.$$

Moreover, since $\underline{u}_\gamma = 0$ in a neighborhood of $\partial\Omega$, making γ smaller and k larger if necessary, we have that $\underline{u}_\gamma \le \overline{u}_k$, and it follows that (P) has a nontrivial nonnegative solution u.

Since $\left\|a^-\right\|_{L^r(\Omega)} < \delta$, from Theorem 1.1, we know that any nontrivial nonnegative solution of (P) belongs to $\mathcal{P}_D^\circ$. In particular, $u \in \mathcal{P}_D^\circ$.

Finally, by the assumptions on f and Theorem 2.1 in [10], we also know that there is at most one positive solution of (P). Therefore there are no other nontrivial nonnegative solutions of (P). $\square$

Proof of Corollary 1.5. The proof is similar to the previous one. It suffices to note that $f(s) = s^q$ satisfies (H_1), so that the previous proof and Theorem 1.3 yield the existence assertion. In addition, $f(s) = s^q$ satisfies the conditions of Corollary 1.4, so that the nonexistence assertion is proved in the same way. $\square$

Proof of Corollary 1.8. We set now

$$I(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{q+1}a(x)|u|^{q+1}\right), \quad \text{for } u \in H^1(\Omega).$$

We claim that I is coercive on $H^1(\Omega)$. Indeed, assume by contradiction that

$$u_n \in H^1(\Omega), \quad \|u_n\| := \|u_n\|_{H^1(\Omega)} \to \infty, \quad \text{and} \quad I(u_n) \text{ is bounded from above.}$$

We may assume that

$$v_n := \frac{u_n}{\|u_n\|} \rightharpoonup v_0 \text{ in } H^1(\Omega), \quad \text{and} \quad v_n \to v_0 \text{ in } L^p(\Omega) \text{ for } p \in (1, 2^*),$$

for some v_0. Then

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 - \frac{1}{(q+1)\,\|u_n\|^{1-q}}\int_\Omega a(x)|v_n|^{q+1} = \frac{I(u_n)}{\|u_n\|^2},$$

so that $\int_\Omega |\nabla v_n|^2 \to 0$. It follows that $v_n \to v_0$ in $H^1(\Omega)$ and v_0 is a nonzero constant. Moreover, from

$$-\frac{1}{q+1}\int_\Omega a(x)|v_n|^{q+1} < \frac{I(u_n)}{\|u_n\|^{q+1}},$$

we have that

$$\int_\Omega a(x)|v_0|^{q+1} = \lim \int_\Omega a(x)|v_n|^{q+1} \geq 0,$$

and consequently $\int_\Omega a \geq 0$, which contradicts our assumption. Therefore I is coercive so that it has a global maximum. Taking u_0 such that $\int_\Omega a(x)|u_0|^{q+1} > 0$, we see that $I(tu_0) < 0$ if $t > 0$ is sufficiently small. This shows that I has a nontrivial global minimizer. Finally, since I is even, it has a nonnegative global minimizer, which is a nontrivial nonnegative solution of (P'). By Theorem 1.7, this solution (and any other nontrivial nonnegative solution) belongs to $\mathcal{P}_N^\circ$ for $q_0 < q < 1$.

Lastly, reasoning exactly as in Lemma 3.1 in [5], we infer that there are no other nontrivial nonnegative solutions of (P_N). □

Proof of Theorem 1.9. Note that Theorem 1.3 says that $\mathcal{A}_D$ is nonempty. Now, first we show, via the continuity argument used in the proofs of Theorems 1.3 and 1.7, that $\mathcal{A}_D$ is open. Indeed, assume to the contrary that there exist $q \in \mathcal{A}_D$ and $q_n \notin \mathcal{A}_D$ such that $q_n \to q$. We take nontrivial nonnegative solutions $u_n \notin \mathcal{P}_D^\circ$ of (P_D) with $q = q_n$. Using Lemma 2.6 and arguing as in the proof of Theorem 1.3, we may deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and consequently, up to a subsequence, $u_n \to u_0$ in $\mathcal{C}^1(\overline{\Omega})$, where u_0 is a nontrivial nonnegative solution of (P_D). Since $q \in \mathcal{A}_D$, we have $u_0 \in \mathcal{P}_D^\circ$, and so $u_n \in \mathcal{P}_D^\circ$ for n large enough, which is a contradiction. Thus $\mathcal{A}_D$ is open.

Next we prove that $\mathcal{A}_D$ is connected. To this end, we show that if $q_0 \in \mathcal{A}_D$ then $(q_0, 1) \subset \mathcal{A}_D$. Let $q_0 \in \mathcal{A}_D$. By Lemma 2.7, we know that

$$\left(q_0, \frac{1}{2-q_0} - \sigma_0\right] \subset \mathcal{A}_D$$

with $\sigma_0 = \frac{1}{10}\frac{(q_0-1)^2}{2-q_0}$, where $\frac{(q_0-1)^2}{2-q_0}$ is the length of the interval $(q_0, \frac{1}{2-q_0})$. By iteration, since $q_1 = \frac{1}{2-q_0} - \sigma_0 \in A$, we have, again by Lemma 2.7, that

$$\left(q_1, \frac{1}{2-q_1} - \sigma_1\right] \subset \mathcal{A}_D,$$

where $\sigma_1 = \frac{1}{10}\frac{(q_1-1)^2}{2-q_1}$. More generally, we have

$$\left(q_{n-1}, \frac{1}{2-q_{n-1}} - \sigma_{n-1}\right] \subset \mathcal{A}_D,$$

where $\sigma_n := \frac{1}{10}\frac{(q_n-1)^2}{2-q_n}$, and $q_n := \frac{1}{2-q_{n-1}} - \sigma_{n-1}$. Then, we obtain by induction that $\{q_n\}$ is nondecreasing and $q_n \leq 1$, so that $q_n \to q_*$ for some $q_* \leq 1$. Passing to the limit as $n \to \infty$, we have

$$q_* = \frac{1}{2-q_*} - \frac{1}{10}\frac{(q_*-1)^2}{2-q_*},$$

so that $\frac{9}{10}\frac{(q_*-1)^2}{2-q_*} = 0$, and thus, $q_* = 1$. Hence we have proved that $(q_0, 1) \subset \mathcal{A}_D$.

Finally, the proof that $\mathcal{A}_N$ is open and connected can be carried out in the same manner as for $\mathcal{A}_D$. In addition, we know by Corollary 1.8 that $\int_\Omega a < 0$ is sufficient for the existence of some $q \in \mathcal{A}_N$. The proof is now complete. □

4. Positivity results for concave-convex type problems

As an application of some of our previous results, we consider now the problem

$$\begin{cases} -\Delta u = \lambda a(x) u^q + g(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where now $a \in L^\infty(\Omega)$, $0 < q < 1$, $\lambda > 0$, and $N \geq 3$. In addition, we assume that $g : [0, \infty) \to [0, \infty)$ is continuous and superlinear in the following sense:

$$\lim_{s\to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{s^p} = 1, \quad \text{for some } 1 < p < \frac{N+2}{N-2}. \tag{H_4}$$

We shall also assume that $g(s) > 0$ for $s > 0$.

The problem above has been investigated in [3] for $a \equiv 1$, and $g(s) = s^p$. The authors proved that (P_λ) has two positive solutions for $\lambda > 0$ sufficiently small. This result was extended to a more general nonlinearity, with $a \geq 0$, in [9]. In addition, in [8], the authors allowed a to change sign and proved the existence of two nontrivial nonnegative solutions of (P_λ).

The growth condition at infinity in (H_4) ensures, in particular, an *a priori* bound for nonnegative solutions of (P_λ) (see [2], and also [17]), which will be used to prove the following positivity result:

Theorem 4.1. *Assume (H_2) and (H_4). In addition, assume that every nontrivial nonnegative solution of (P_D) belongs to $\mathcal{P}_D^\circ$. Then there exists $\lambda_0 > 0$ such that every nontrivial nonnegative solution of (P_λ) belongs to $\mathcal{P}_D^\circ$ for $0 < \lambda < \lambda_0$.*

Proof. Assume by contradiction that $\lambda_n \to 0^+$ and u_n are nontrivial nonnegative solutions of (P_λ) with $\lambda = \lambda_n$ and $u_n \notin \mathcal{P}_D^\circ$. By the *a priori* bounds in [2], there exists $K > 0$ such that $\|u_n\|_\infty \leq K$ for every n. It follows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thus, we can assume that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, $u_n \to u_0$ in $L^s(\Omega)$, $1 < s < 2^*$, and $u_n \to u_0$ *a.e.* in Ω, for some u_0. Taking as test function $u_n - u_0$ in (P_{λ_n}) we see that $u_n \to u_0$ in $H_0^1(\Omega)$ and u_0 is a solution of (P_λ) with $\lambda = 0$. Moreover, by elliptic regularity, we have, up to a subsequence, that $u_n \to u_0$ in $\mathcal{C}^1(\overline{\Omega})$,

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see Remark 4.2 below. If $u_0 \not\equiv 0$ then, by the strong maximum principle, we have that $u_0 \in \mathcal{P}_D^\circ$, and consequently $u_n \in \mathcal{P}_D^\circ$ for n large enough, which provides a contradiction. Now, if $u_0 \equiv 0$, then we consider $v_n := \lambda_n^{\frac{1}{q-1}} u_n$. We see that v_n are nontrivial nonnegative solutions of

$$\begin{cases} -\Delta v = a(x)v^q + \lambda^{\frac{1}{q-1}} g(\lambda^{\frac{1}{1-q}} v) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $\lambda = \lambda_n$. Hence, v_n are nontrivial nonnegative supersolutions of

$$\begin{cases} -\Delta v = a(x)v^q & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $\lambda = \lambda_n$.

We claim that $v_n \not\equiv 0$ in Ω_+. Indeed, note that if $v_n \equiv 0$ in Ω_+ then $u_n \equiv 0$ in Ω_+, so that

$$\int_\Omega |\nabla u_n|^2 \le \int_\Omega g(u_n)u_n \le \varepsilon \|u_n\|^2_{H_0^1(\Omega)} + C_\varepsilon \|u_n\|^{p+1}_{H_0^1(\Omega)}.$$

Taking $\varepsilon > 0$ sufficiently small we see that $\|u_n\|_{H_0^1(\Omega)} \ge C > 0$, which contradicts $u_n \to 0$ in $H_0^1(\Omega)$. Therefore the claim is proved. Since Ω_+ has finitely many connected components, we can assume that $v_n \not\equiv 0$ in some fixed subdomain $\Omega' \subset \Omega_+$. Let ϕ be as in the proof of Lemma 2.2. Arguing as in that proof, we have that $\varepsilon\phi$ is a nonnegative subsolution of

$$-\Delta u = a(x)u^q \quad \text{in } B,$$

where B is an open ball such that $\overline{B} \subset \Omega'$. We extend ϕ by zero to $\overline{\Omega} \setminus \overline{B}$. For $\varepsilon > 0$ small enough, we have that $\varepsilon\phi \le v_n$ for every n. Thus, we find a nonnegative solution w_n of (P) such that $\varepsilon\phi \le w_n \le v_n$. But, by our assumption, we have that $w_n \in \mathcal{P}_D^\circ$, which contradicts the assumption that $v_n \notin \mathcal{P}_D^\circ$. The proof is now complete. □

Remark 4.2. In the same way as Remark 3.1, we give some further details on the regularity argument used in the previous proof. We set now

$$h_\lambda(x,s) := \lambda a(x)s^q + g(s)$$

and use the conditions

$$a \in L^\infty(\Omega), \quad 0 < q < 1, \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{s^p} = 1 \quad \text{for some} \quad 1 < p < \frac{N+2}{N-2},$$

to infer that, given $\overline{\lambda} > 0$, there exists $C > 0$ such that

$$|h_\lambda(x,s)| \le C(1+s^p) \quad \text{for } x \in \Omega,\ s \ge 0, \text{ and } |\lambda| \le \overline{\lambda}.$$

In the same manner as in Remark 3.1, we can deduce that $\|u_n\|_{W^{2,\sigma_k}(\Omega)}$ is bounded for each $\sigma_k = \frac{2N}{p_k}$, $k = 1, 2, 3, \dots$, where

$$p_k = \frac{4p}{p-1} - p^k \left(\frac{4}{p-1} - (N-2) \right).$$

Since $p > 1$ and $\frac{4}{p-1} - (N-2) > 0$, we can choose $\sigma_k > N$ such that $\|u_n\|_{W^{2,\sigma_k}(\Omega)}$ is bounded. Then, the argument proceeds in the same way as in Remark 3.1.

As a consequence of Theorem 4.1, we obtain two positive solutions of (P_λ) for $\lambda > 0$ small, if either a^- is small or q is close to 1:

Corollary 4.3. *Assume (H_2) and (H_4). Then there exist $\delta > 0$ and $q_0 \in (0, 1)$ such that, if either $\|a^-\|_{L^r(\Omega)} < \delta$ or $q_0 < q < 1$, then there exists $\lambda_0 > 0$ with the following properties:*

(i) *any nontrivial nonnegative solution of (P_λ) belongs to $\mathcal{P}_D^\circ$ for $0 < \lambda < \lambda_0$.*
(ii) *(P_λ) has two solutions in $\mathcal{P}_D^\circ$ for $0 < \lambda < \lambda_0$.*

Proof. (i) We apply Theorems 1.1 and 1.3 to (P_D) and obtain, respectively, $\delta > 0$ and $q_0 \in (0, 1)$ such that every nontrivial nonnegative solution of (P_D) belongs to $\mathcal{P}_D^\circ$ if either $\|a^-\|_{L^r(\Omega)} < \delta$ or $q_0 < q < 1$. Theorem 4.1 yields the conclusion.

(ii) We use Theorem 2.1 from [8]. One can easily show that assumptions (H_0)–(H_5) from [8] are satisfied under our conditions. Thus there exists $\lambda > 0$ such that for $0 < \lambda < \lambda_0$ there exist two nontrivial nonnegative solutions of (P_λ). Decreasing λ_0 if necessary, by the previous item, we infer that these solutions belong to $\mathcal{P}_D^\circ$ if either $\|a^-\|_{L^r(\Omega)} < \delta$ or $q_0 < q < 1$. □

Remark 4.4. Let us set

$$\mathcal{B}_D := \{\lambda > 0 : \text{any nontrivial nonnegative solution of } (P_\lambda) \text{ belongs to } \mathcal{P}_D^\circ\},$$

and assume (H_2), (H_4), and either $\|a^-\|_{L^r(\Omega)} < \delta$ or $q_0 < q < 1$, where δ and q_0 are provided by Corollary 4.3. Then $(0, \lambda_0) \subset \mathcal{B}_D$. Moreover, arguing as in the proof of Theorem 4.1 we can show that $\mathcal{B}_D$ is an open set. Indeed, one may easily see that the proof of Theorem 4.1 carries on taking now a sequence $\lambda_n \to \lambda_0 \in \mathcal{B}_D$.

We establish now a result analogue to Theorem 4.1 for the problem

$$\begin{cases} -\Delta u = \lambda a(x) u^q + g(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{Q_λ}$$

Instead of (H_4), we shall assume now

$$\lim_{s \to 0^+} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s \to \infty} \frac{g(s)}{s^p} = 1, \text{ for some } 1 < p < \frac{N+1}{N-1}. \tag{H_4'}$$

The above problem, with $g(s) = s^p$, has been recently investigated in [19]. The authors established existence and multiplicity results for nontrivial nonnegative solutions of (Q_λ), for $\lambda > 0$ sufficiently small. Furthermore, the asymptotic behavior of these solutions as $\lambda \to 0^+$ provides the positivity of some of these solutions, in certain cases. We shall now prove a general positivity result for (Q_λ):

Theorem 4.5. *Assume* (H_2) *and* (H_4'). *In addition, assume that every nontrivial nonnegative solution of* (P_N) *belongs to* $\mathcal{P}_N^\circ$. *Then there exists* $\lambda_0 > 0$ *such that every nontrivial nonnegative solution of* (Q_λ) *belongs to* $\mathcal{P}_N^\circ$ *for* $0 < \lambda < \lambda_0$.

Proof. The proof is similar to the one of Theorem 4.1. Again by the *a priori* bounds of [2] we get that $\{u_n\}$ is bounded in $H^1(\Omega)$. To show that if $u_n \to 0$ in $H^1(\Omega)$ then $u_n \not\equiv 0$ in Ω_+, we proceed in the following way: assume the contrary and set $w_n := \frac{u_n}{\|u_n\|}$, where $\|u_n\| := \|u_n\|_{H^1(\Omega)}$. So we can assume that $w_n \rightharpoonup w_0$ in $H^1(\Omega)$, $w_n \to w_0$ in $L^s(\Omega)$, with $1 < s < 2^*$, and $w_n \to w_0$ *a.e.* in Ω, for some w_0. Thus, from

$$\int_\Omega |\nabla u_n|^2 \leq \int_\Omega g(u_n)u_n,$$

we obtain that

$$\int_\Omega |\nabla w_n|^2 \leq \int_\Omega \frac{g(\|u_n\|w_n)}{\|u_n\|} w_n = \int_{\operatorname{supp} w_n} \frac{g(\|u_n\|w_n)}{\|u_n\|w_n} w_n^2$$

Since $\|u_n\| \to 0$ and $\frac{g(s)}{s} \to 0$ as $s \to 0^+$, we easily see that

$$\int_\Omega |\nabla w_n|^2 \to 0,$$

so that $w_n \to w_0$ in $H^1(\Omega)$ and w_0 is a positive constant. This contradicts the assumption that $w_n \equiv 0$ in Ω_+. The rest of the proof carries on in a similar way. □

Corollary 4.6. *Assume* (H_2) *and* (H_4'). *Let* $q_0 \in (0,1)$ *be given by Theorem 1.7. If* $q_0 < q < 1$ *then there exists* $\lambda_0 > 0$ *such that any nontrivial nonnegative solution of* (Q_λ) *belongs to* $\mathcal{P}_N^\circ$ *for* $0 < \lambda < \lambda_0$.

Corollary 4.7. *Let* $g(s) = s^p$, *with* $1 < p < \frac{N+1}{N-1}$, *and assume* $q_0 < q < 1$, *where* $q_0 \in (0,1)$ *is given by Theorem 1.7. If* $\int_\Omega a < 0$ *then there exists* $\lambda_0 > 0$ *such that* (Q_λ) *has two solutions in* $\mathcal{P}_N^\circ$ *for* $0 < \lambda < \lambda_0$.

Proof. We apply Corollary 1.3 (2) from [19] to obtain $\lambda_0 > 0$ such that (Q_λ) has two solutions $u_{1,\lambda}$, $u_{2,\lambda}$ such that $u_{2,\lambda} > u_{1,\lambda} \geq 0$ on $\overline{\Omega}$ for $0 < \lambda < \lambda_0$. By Corollary 4.6, decreasing λ_0 if necessary, we have that $u_{1,\lambda}$ and $u_{2,\lambda}$ belong to $\mathcal{P}_N^\circ$. □

Remark 4.8. Let us set

$$\mathcal{B}_N := \{\lambda > 0 : \text{any nontrivial nonnegative solution of } (Q_\lambda) \text{ belongs to } \mathcal{P}_N^\circ\}.$$

We assume that (H_2), (H_4') hold, and $q_0 < q < 1$, where q_0 is provided by Corollary 4.6. Then, by arguing in the same way as for (P_λ), we observe that $(0, \lambda_0) \subset \mathcal{B}_N$, and in addition, $\mathcal{B}_N$ is open.

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