



# Singular stochastic Allen–Cahn equations with dynamic boundary conditions

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## Abstract

We prove a well-posedness result for stochastic Allen–Cahn type equations in a bounded domain coupled with generic boundary conditions. The (nonlinear) flux at the boundary aims at describing the interactions with the hard walls and is motivated by some recent literature in physics. The singular character of the drift part allows for a large class of maximal monotone operators, generalizing the usual double-well potentials. One of the main novelties of the paper is the absence of any growth condition on the drift term of the evolution, neither on the domain nor on the boundary. A well-posedness result for variational solutions of the system is presented using *a priori* estimates as well as monotonicity and compactness techniques. A vanishing viscosity argument for the dynamic on the boundary is also presented.

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## 1. Introduction

Allen–Cahn type equations were introduced within the Van der Waals theory of phase transitions as a basic model to describe the evolution of a two-phase fluid. Generally, the unknown

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process is called (non-conserved) *order parameter* and represents the normalized density of one of the two involved phases. The starting point of the theory is the definition of a free energy functional associated to the order parameter  $x(\cdot)$ , which is given by

$$\mathcal{E}(x) := \int_D \left[ \frac{1}{2} |\nabla x|^2 + F(x) \right] dz. \quad (1.1)$$

In the classical setting  $F : \mathbb{R} \rightarrow [0, +\infty)$  is a smooth double well potential representing the energy density (e.g.  $F(x) = 1/4(x^2 - 1)^2$ ) which favours the pure states. The gradient part instead, taking into account the interactions at small scales, prevents instantaneous jumps between the two pure phases, penalizing the variation of  $x$ . Then, the Allen–Cahn equation can be viewed as the  $L^2$ -gradient flow of the Ginzburg–Landau free energy (1.1), i.e. the semilinear parabolic PDE of the form

$$\partial_t x - \Delta x + F'(x) = 0. \quad (1.2)$$

In order to model the thermal fluctuation of the system it is quite natural to perturb the equation with a random force and, from a phenomenological point of view, the choice of space time white noise seems very natural. Unfortunately, nonlinear equations such as (1.2) become ill-posed in space dimensions  $N \geq 2$  as soon as a white noise term is added. A classical way to bypass the problem is to smooth out the noise via a suitable covariance operator. Given  $D \subseteq \mathbb{R}^N$  ( $N \geq 2$ ) a smooth bounded domain with smooth boundary  $\Gamma$ , what we end up with is the so called stochastic Allen–Cahn equation:

$$dx_t - \Delta x_t dt - F'(x_t) dt = B(t, x_t) dW_t \quad \text{in } (0, T) \times D \quad (1.3)$$

with a given initial datum

$$x(0) = x_0 \quad \text{in } D,$$

where  $T > 0$  is a fixed final time,  $W$  is a cylindrical Wiener process on a separable Hilbert space  $U$  and  $B$  is a Hilbert–Schmidt operator from  $U$  to  $L^2(D)$  depending on  $x$  as well.

In the classical literature on the deterministic and stochastic Allen–Cahn equation, the order parameter is usually assumed to satisfy homogeneous Neumann conditions on the boundary  $\Gamma$ , which may be interpreted as a null interaction of the phase-transition with the hard walls. In this setting, well-posedness results for the stochastic Allen–Cahn equation can be found e.g. in the monograph [14], within the framework of dissipative SPDEs.

More recently, the class of energy functionals has been extended in several models to take into account also a possible interaction of the phase transition phenomenon with the hard walls. The main idea is to require that the free energy functional could (possibly) penalize the variation of  $x$  and the pure states also on the boundary  $\Gamma$ : to this aim, if we denote with  $\nabla_\Gamma$  the surface gradient on the boundary, the form of the functional becomes

$$\mathcal{E}(x) := \int_D \left[ \frac{1}{2} |\nabla x|^2 + F(x) \right] dz + \int_\Gamma \left[ \frac{\varepsilon}{2} |\nabla_\Gamma y|^2 + F_\Gamma \right] d\zeta, \quad (1.4)$$

where  $\varepsilon \geq 0$  is fixed and  $F_\Gamma : \mathbb{R} \rightarrow [0, +\infty)$  is another smooth double-well potential acting on the boundary. Arguing as before, the  $L^2$ -gradient flow of this generalized free energy describes the following system

$$\begin{aligned} \partial_t x - \Delta x + F'(x) &= 0 & \text{in } (0, T) \times D \\ \partial_t x + \partial_{\mathbf{n}} x - \varepsilon \Delta_\Gamma x + F'_\Gamma(x) &= 0 & \text{in } (0, T) \times \Gamma \\ x(0) &= x_0 & \text{in } D \\ x(0) &= x_0|_\Gamma & \text{in } \Gamma, \end{aligned}$$

where the symbol  $\partial_{\mathbf{n}}$  denotes the outward normal derivative on  $\Gamma$  and  $\Delta_\Gamma$  is the usual Laplace–Beltrami operator. Note that the presence of a free energy term on the boundary leads to non-standard dynamic conditions on the boundary (i.e. that involve the time derivative of  $x$  on  $\Gamma$ ), in contrast with the classical homogeneous Neumann conditions for  $x$ . Let us point out that in the last few years there has been a lot of interest in describing phase separation phenomena in confined systems under general boundary conditions, see e.g. [2], [16].

Finally, since the physical system may be subject also to a thermal fluctuation in the boundary (due both to the diffusion from the interior of  $D$  and to a boundary noise) the natural idea is to perturb the equation satisfied by  $x$  on  $\Gamma$  the same way that we have done for the one in the interior of  $D$ . These considerations lead to consider stochastic boundary conditions of the type

$$dx_t + \partial_{\mathbf{n}} x_t dt - \varepsilon \Delta_\Gamma x_t dt + F'_\Gamma(x_t) dt = B_\Gamma(t, x_t) dW_t^\Gamma \quad \text{in } (0, T) \times \Gamma, \quad (1.5)$$

where here  $W^\Gamma$  is a cylindrical Wiener process on another separable Hilbert space  $U_\Gamma$ , independent of  $W$ , and  $B_\Gamma$  is a random time-dependent Hilbert–Schmidt operator from  $U_\Gamma$  to  $L^2(\Gamma)$  of multiplicative type.

In the present paper we are interested in studying the stochastic system arising from the equations (1.3) and (1.5) from a more general mathematical perspective. The main extension that we carry out concerns the form of the double-well potentials: more precisely, instead of assuming that  $F$  and  $F_\Gamma$  are smooth functions on  $\mathbb{R}$ , we simply require that  $F = j + G$  and  $F_\Gamma = j_\Gamma + G_\Gamma$ , where  $j, j_\Gamma : \mathbb{R} \rightarrow [0, +\infty)$  are given convex functions with subdifferentials  $\beta = \partial j$  and  $\beta_\Gamma = \partial j_\Gamma$  everywhere defined, respectively, and  $G, G_\Gamma$  are smooth functions with Lipschitz differentials  $\pi, \pi_\Gamma$ , respectively. This means essentially that we are looking at the double-well potentials  $F$  and  $F_\Gamma$  as sufficiently smooth concave perturbations of convex potentials. Bearing in mind these considerations, in the present paper we are concerned with the following system:

$$dx_t - \Delta x_t dt + \beta(x_t) dt + \pi(x_t) dt \ni B(t, x_t) dW_t \quad \text{in } (0, T) \times D \quad (1.6)$$

$$x = y \quad \text{in } (0, T) \times \Gamma \quad (1.7)$$

$$dy_t + \partial_{\mathbf{n}} x_t dt - \varepsilon \Delta_\Gamma y_t dt + \beta_\Gamma(y_t) dt + \pi_\Gamma(y_t) dt \ni B_\Gamma(t, y_t) dW_t^\Gamma \quad \text{in } (0, T) \times \Gamma \quad (1.8)$$

$$x(0) = x_0 \quad \text{in } D \quad (1.9)$$

$$y(0) = x_0|_\Gamma \quad \text{in } \Gamma. \quad (1.10)$$

More specifically, we aim at proving well-posedness for problem (1.6)–(1.10) both in the case  $\varepsilon > 0$  and  $\varepsilon = 0$ , as well as a suitable continuity of the solutions with respect to the parameter  $\varepsilon \geq 0$ , under no restrictive growth assumptions on the potentials.

Mathematical results concerning the Allen–Cahn (and similarly Cahn–Hilliard) equation have been obtained recently in the framework of generalized (deterministic) boundary dynamics. Let us mention e.g. [9,11,12,17,23] and the references therein. On the contrary, not much is known on the stochastic counterpart, where one is interested in the dynamical impact of a noise term on the boundary. In this direction we have to mention [5,7], where the authors study (nonlinear) diffusion problems with stochastic boundary conditions in a variational framework, and [37] where long-time properties of the Cahn–Hilliard equation are investigated. We also mention the contributions [13,32] dealing with the stochastic Cahn–Hilliard equation.

Concerning general well-posedness results for stochastic PDEs, let us mention [19] and the references therein, where unique existence of analytical strong solutions for a large class of SPDEs of gradient type is exhibited. In that case, a crucial hypothesis used by the author is the sub-homogeneous character of the potential, which unfortunately forbids e.g. exponential growth. In order to avoid any growth condition, in the present paper we only ask that  $D(\beta) = \mathbb{R}$ . Still, this hypothesis does not seem to be optimal: it is not needed for the well-posedness of deterministic systems (see e.g. [9]), whereas in the stochastic formulation seems to be essential, at least in the approach we develop. Actually, in our case of interest  $\beta = \partial j$  with  $j$  being a convex potential, this restriction on the domain is the most general assumption in literature: it was considered for the first time in [3] in a problem related to existence of semilinear Laplace-driven stochastic equations and then in [26] when studying well-posedness for a class of abstract semilinear SPDEs with singular drift, avoiding any conditions on the growth of  $\beta$ . Again, let us remark that we are not able to consider a graph of the form  $\tilde{\beta} = \partial I_{[-1,1]}$ , but only an approximation of it defined everywhere in  $\mathbb{R}$ . With respect to the result obtained in [26], tailored for a large class of singular dissipative SPDEs, here we focus on a precise choice of diffusion operator, the Laplacian. This is motivated by the physical description of the model, but greater generality can be achieved without any substantial change in the proof. Within this framework, the key idea is to get good estimates in expectation and produce the pathwise counterpart in a set  $\Omega'$  of probability 1 using a suitable regularization on the noise.

The strategy of the proof is as follows. We start by rewriting the system with additive noise as an equation for the pair  $(x, y)$  in the product space  $L^2(D) \times L^2(\Gamma)$ . Here we develop a variational approach à la Krylov, Rozovskiĭ and Pardoux. In particular we define a suitable Gelfand triple and we smooth out the equation via Yosida approximations of the singular part. In this way, the approximated system satisfies the usual assumptions and a version of the Itô formula can be applied. We derive estimates in expectation of the solution as well as of the monotone maps. By compactness we pass to the limit pathwise to get a candidate limit equation and we identify the drift part as an element of the maximal monotone graphs  $\beta$  and  $\beta_\Gamma$ . Then we recover uniqueness of the solution which is essential to infer measurability in  $\omega$  of the limit. At last we generalize the well-posedness result also to noises of multiplicative type using a standard fixed-point argument.

A crucial step to get uniqueness of the solution is the application of the Itô formula, for which a suitable smoothing of the equation is required. A classical way of proceeding is to apply the resolvent operator of the diffusion to the equation itself. In our case of interest, rewriting the equation in the product space  $L^2(D) \times L^2(\Gamma)$ , the diffusion operator is not “standard” and has the form

$$\mathcal{C}_\varepsilon(x, y) := (-\Delta x, \partial_{\mathbf{n}}x - \varepsilon \Delta_\Gamma y), \quad (1.11)$$

where  $y = \tau x$  is the trace part. Hence, given a couple of functions  $(f, g)$ , one needs to study the smoothing effect of the resolvent  $(I + \delta \mathcal{C}_\varepsilon)^{-1}$  through an *ad hoc* regularity analysis of the associated elliptic system

$$\begin{cases} u - \Delta u = f & \text{in } D, \\ u = v, \quad u + \partial_{\mathbf{n}}u - \varepsilon \Delta_\Gamma v = g & \text{in } \Gamma. \end{cases}$$

Precisely, what we show is the ultracontractivity of  $(I + \delta \mathcal{C}_\varepsilon)^{-1}$  from  $L^1(D) \times L^1(\Gamma)$  to  $L^\infty(D) \times L^\infty(\Gamma)$ . To this aim, we generalize a classical regularity result by Stampacchia for elliptic equations with homogeneous boundary conditions contained in [35] and subsequently prove a version of the maximum principle with data  $(f, g) \in L^1(D) \times L^1(\Gamma)$ . Let us note that the study of this operator forces us to impose some additional constraints on the relative growth of  $\beta$  and  $\beta_\Gamma$ , which are indeed quite natural from the point of view of the physical applications. All the results mentioned above are collected in the Appendix.

Throughout the paper we use the parameter  $\varepsilon$  to indicate the presence of the diffusion operator  $\varepsilon \Delta_\Gamma$  on the boundary. The presence/absence of this term creates a gap between the effective domain of  $\mathcal{C}_\varepsilon$ , when  $\varepsilon > 0$  and  $\varepsilon = 0$ . From the physical point of view, a vanishing viscosity argument on the boundary becomes interesting as it is related to the formation of sharp interfaces between the two phases. What we show is the well-posedness of the problem in the singular case  $\varepsilon = 0$  as well as the continuous dependence of the solutions to the system (1.6), (1.8) with respect to the variation of  $\varepsilon \geq 0$ .

The paper is organised as follows: in Section 2 we introduce the notations, assumptions and we present the main results. Section 3 is devoted to proving the well-posedness of the system: here, we study the approximated equation with additive noise and we pass to the limit using compactness arguments. Then we extend the previous result to the Allen–Cahn equation with multiplicative noise using a fixed point argument. In Section 4 we study the asymptotic behaviour of the system as  $\varepsilon \rightarrow 0$ . Finally, in the Appendix, we derive the smoothing properties of the diffusion operator  $\mathcal{C}_\varepsilon$ .

## 2. Notation, setting and main results

In the section we state the notation that we use and the precise assumptions of the work; moreover, the concept of solution and the main results are presented.

### 2.1. Notation

Throughout the paper,  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  is a filtered probability space, with the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  satisfying the so-called “usual conditions” (i.e. it is saturated and right continuous). As we have anticipated,  $D \subseteq \mathbb{R}^N$  is a smooth bounded domain with smooth boundary  $\Gamma$  and  $T > 0$  is a fixed time.

If  $U$  is a Banach space, for any  $t \in [0, T]$  we use the classical notations  $L^p(\Omega, \mathcal{F}_t, \mathbb{P}; U)$  and  $L^p(0, T; U)$  for the classes of  $U$ -valued  $p$ -Bochner-integrable functions on  $\Omega$  and  $(0, T)$ , respectively (without specifying the  $\sigma$ -algebra and the probability measure if  $t = T$ ). The symbol  $C_w^0([0, T]; U)$  denotes the space of continuous functions from  $[0, T]$  to the space  $U$  endowed

with the weak topology. Moreover, if  $U_1$  and  $U_2$  are separable Hilbert Spaces, we may write  $\mathcal{L}(U_1, U_2)$  and  $\mathcal{L}_2(U_1, U_2)$  to indicate the spaces of the linear continuous operators and Hilbert–Schmidt operators from  $U_1$  to  $U_2$ , respectively.

Let  $\tau$  be the trace operator  $\tau : H^1(D) \rightarrow L^2(\Gamma)$ . Recall that the rank of  $\tau$  coincides with the boundary Sobolev space  $H^{1/2}(\Gamma)$  and there is a constant  $M > 0$  such that  $\|\tau u\|_{H^{1/2}(\Gamma)} \leq M \|u\|_{H^1(D)}$  for any  $u \in H^1(D)$  (see [1, Thm. 7.39] and [4, p. 14]): hence the operator  $\tau : H^1(D) \rightarrow H^{1/2}(\Gamma)$  is well-defined, linear and continuous. Moreover, it is worth recalling that for any  $k \geq 1$ , we have that  $\tau \in \mathcal{L}(H^k(D), H^{k-1/2}(\Gamma))$ .

The symbol  $\Delta_\Gamma$  denotes the usual Laplace–Beltrami operator on  $L^2(\Gamma)$ , i.e.

$$\Delta_\Gamma : L^2(\Gamma) \rightarrow L^2(\Gamma), \quad \Delta_\Gamma v := \operatorname{div}_\Gamma \nabla_\Gamma v, \quad v \in D(\Delta_\Gamma) := H^2(\Gamma),$$

where  $\nabla_\Gamma := (\partial_{\tau_1}, \dots, \partial_{\tau_{N-1}})$  is the Riemannian gradient on  $\Gamma$  and  $\partial_{\tau_i}$  is the derivative along the  $i$ -th tangential direction  $\tau_i$  for every  $i \in \{1, \dots, N-1\}$ . Let us recall that the operator  $-\Delta_\Gamma$  is maximal monotone on  $L^2(\Gamma)$ ; moreover, for any  $\delta > 0$  and  $k \in \mathbb{N}$ , its resolvent  $(I - \delta \Delta_\Gamma)^{-1}$  belongs to  $\mathcal{L}(H^{k-1}(\Gamma), H^{k+1}(\Gamma))$ .

For every  $a, b \geq 0$ , we use the classical notation  $a \lesssim b$  to mean that there exists a positive constant  $C$  such that  $a \leq Cb$ .

## 2.2. Assumptions

We precise here the assumptions that are in order throughout the work.

**Assumptions on the double-well potentials.** We assume

$$\beta, \beta_\Gamma : \mathbb{R} \rightarrow 2^{\mathbb{R}} \quad \text{maximal monotone,} \quad 0 \in \beta(0) \cap \beta_\Gamma(0), \quad D(\beta) = D(\beta_\Gamma) = \mathbb{R},$$

$$\pi, \pi_\Gamma : \mathbb{R} \rightarrow \mathbb{R} \quad \text{Lipschitz continuous with Lipschitz constants } C_\pi, C_{\pi_\Gamma}.$$

In this setting, the following proper, convex and lower semicontinuous functions are well-defined:

$$j, j_\Gamma : \mathbb{R} \rightarrow [0, +\infty), \quad \text{such that} \quad \partial j = \beta, \quad \partial j_\Gamma = \beta_\Gamma, \quad j(0) = j_\Gamma(0) = 0.$$

Since  $\beta$  and  $\beta_\Gamma$  are everywhere defined,  $j$  and  $j_\Gamma$  are actually continuous. Moreover, the convex conjugates of  $j$  and  $j_\Gamma$ , i.e.

$$j^*, j_\Gamma^* : \mathbb{R} \rightarrow [0, +\infty], \quad j^*(s) := \sup_{r \in \mathbb{R}} \{rs - j(r)\}, \quad j_\Gamma^*(s) := \sup_{r \in \mathbb{R}} \{rs - j_\Gamma(r)\},$$

are superlinear at infinity (see [4, Prop. 1.8]). Precisely, we have

$$\lim_{|s| \rightarrow \infty} \frac{j^*(s)}{|s|} = \lim_{|s| \rightarrow \infty} \frac{j_\Gamma^*(s)}{|s|} = +\infty.$$

We need to make some further hypotheses on  $j$  and  $j_\Gamma$ . Firstly, we require a symmetry property for the growth of the two potentials at infinity, namely

$$\limsup_{|r| \rightarrow \infty} \frac{j(r)}{j(-r)} < +\infty \quad \text{and} \quad \limsup_{|r| \rightarrow \infty} \frac{j_\Gamma(r)}{j_\Gamma(-r)} < +\infty,$$

which is very common in literature (see [3,6,25–28,31]). Secondly, a natural assumption to make is that

$$j(r) \lesssim 1 + j_{\Gamma}(r), \quad j_{\Gamma}(r) \lesssim 1 + j(r) \quad \forall r \in \mathbb{R}, \quad (\text{H1})$$

which means essentially that  $j$  and  $j_{\Gamma}$  control each other at  $+\infty$ . If we keep in mind the physical interpretation of the problem, (H1) is very reasonable and can be reinterpreted as the requirement that the nonlinear flux on the boundary is of the same type as the one in the interior of the domain.

However, note that condition (H1) is much stronger than the corresponding one in the deterministic case, in which it is sufficient to assume just one of the two inequalities (see [9]). Consequently, for sake of completeness, it is worth introducing two other possible hypotheses, in which the potentials are allowed to have different growth at infinity, provided that they are bounded by specific polynomial functions:

$$j(r) \lesssim 1 + j_{\Gamma}(r), \quad \begin{cases} j_{\Gamma}(r) \lesssim 1 + |r|^{\frac{2N}{N-2}} & \text{if } N > 2 \\ \exists p \geq 1 : j_{\Gamma}(r) \lesssim 1 + |r|^p & \text{if } N = 2 \end{cases} \quad \forall r \in \mathbb{R}, \quad (\text{H2})$$

$$j_{\Gamma}(r) \lesssim 1 + j(r), \quad \begin{cases} j(r) \lesssim 1 + |r|^{\frac{2(N-1)}{N-3}} & \text{if } N > 3 \\ \exists p \geq 1 : j(r) \lesssim 1 + |r|^p & \text{if } N = 3 \\ \text{no restrictions on } j & \text{if } N = 2 \end{cases} \quad \forall r \in \mathbb{R}, \quad (\text{H3}_{\varepsilon>0})$$

$$j_{\Gamma}(r) \lesssim 1 + j(r), \quad \begin{cases} j(r) \lesssim 1 + |r|^{\frac{2(N-1)}{N-2}} & \text{if } N > 2 \\ \exists p \geq 1 : j(r) \lesssim 1 + |r|^p & \text{if } N = 2 \end{cases} \quad \forall r \in \mathbb{R}. \quad (\text{H3}_{\varepsilon=0})$$

Let us comment on these conditions, focusing in particular on the cases  $N = 2, 3$ , which are the most interesting in terms of applications. In (H2) we are requiring that  $j$  is controlled by  $j_{\Gamma}$  and that  $j_{\Gamma}$  is bounded by a polynomial of degree six if  $N = 3$ , or by any generic polynomial if  $N = 2$ . The second hypothesis requires instead that  $j_{\Gamma}$  is controlled by  $j$ , and it depends on whether we are working with  $\varepsilon > 0$  or  $\varepsilon = 0$ . In the case  $\varepsilon > 0$ , we can assume that  $j$  has any polynomial growth if  $N = 3$ , or any arbitrary growth if  $N = 2$ . In the case  $\varepsilon = 0$ ,  $j$  has to be controlled by a polynomial of degree four if  $N = 3$ , or by any generic polynomial if  $N = 2$ . In particular, note that the classical double-well potentials of degree four are included in the interesting cases  $N = 2, 3$ .

Polynomial growths of this type for the potentials have been widely used in the deterministic setting, also in the framework of Cahn–Hilliard and quasilinear equations. Among the great literature, we can mention the works [10,15,17,18,34] and the references therein.

Throughout the paper, we will assume either hypothesis (H1) or (H2) or (H3<sub>ε>0</sub>)–(H3<sub>ε=0</sub>).

We introduce also the multivalued operator

$$\gamma : \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}, \quad \gamma(u, v) := \{(r, w) \in \mathbb{R}^2 : r \in \beta(u), w \in \beta_{\Gamma}(v)\}, \quad (u, v) \in \mathbb{R}^2$$

and the proper, convex and lower semicontinuous function

$$k : \mathbb{R}^2 \rightarrow [0, +\infty), \quad k(u, v) := j(u) + j_{\Gamma}(v), \quad (u, v) \in \mathbb{R}^2 :$$

it is not difficult to check that  $\gamma$  is maximal monotone on  $\mathbb{R}^2$  and

$$0 \in \gamma(0), \quad D(\gamma) = \mathbb{R}^2, \quad \gamma = \partial k, \quad k(0) = 0.$$

Finally, we define

$$\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \mathcal{P}(u, v) := (\pi(u), \pi_\Gamma(v)), \quad (u, v) \in \mathbb{R}^2,$$

which is Lipschitz continuous with Lipschitz constant  $C_{\mathcal{P}} := \max\{C_\pi, C_{\pi_\Gamma}\}$ .

**Assumptions on the variational setting.** For any  $\varepsilon \geq 0$ , we define the spaces

$$H := L^2(D), \quad H_\Gamma := L^2(\Gamma), \quad V := H^1(D), \quad V_\Gamma^\varepsilon := \begin{cases} H^1(\Gamma) & \text{if } \varepsilon > 0, \\ H^{1/2}(\Gamma) & \text{if } \varepsilon = 0, \end{cases}$$

endowed with their natural norms  $\|\cdot\|_H$ ,  $\|\cdot\|_{H_\Gamma}$ ,  $\|\cdot\|_V$  and  $\|\cdot\|_{V_\Gamma^\varepsilon}$ , respectively. We also use the notations  $(\cdot, \cdot)_H$ ,  $(\cdot, \cdot)_{H_\Gamma}$ ,  $\langle \cdot, \cdot \rangle_V$  and  $\langle \cdot, \cdot \rangle_{V_\Gamma^\varepsilon}$  for the standard scalar products of  $H$  and  $H_\Gamma$  and the duality pairings between  $V^*$  and  $V$ ,  $(V_\Gamma^\varepsilon)^*$  and  $V_\Gamma^\varepsilon$ , respectively. In this way, for every  $\varepsilon \geq 0$ ,  $(V, H, V^*)$  and  $(V_\Gamma^\varepsilon, H_\Gamma, (V_\Gamma^\varepsilon)^*)$  are Hilbert triplets with compact inclusions  $V \xhookrightarrow{c} H$  and  $V_\Gamma^\varepsilon \xhookrightarrow{c} H_\Gamma$ . The variational setting is obtained considering the product spaces

$$\mathcal{H} := H \times H_\Gamma, \quad \mathcal{V}_\varepsilon := \{(u, v) \in V \times V_\Gamma^\varepsilon : v = \tau u\}$$

endowed with the norms

$$\begin{aligned} \|(u, v)\|_{\mathcal{H}} &:= \sqrt{\|u\|_H^2 + \|v\|_{H_\Gamma}^2} & \forall (u, v) \in \mathcal{H}, \\ \|(u, v)\|_{\mathcal{V}_\varepsilon} &:= \sqrt{\|u\|_V^2 + \|v\|_{V_\Gamma^\varepsilon}^2} & \forall (u, v) \in \mathcal{V}_\varepsilon. \end{aligned}$$

Let us recall that, using the continuity of  $\tau : H^1(D) \rightarrow H^{1/2}(\Gamma)$ , an equivalent norm on  $\mathcal{V}_\varepsilon$  is given by

$$|(u, v)|_{\mathcal{V}_\varepsilon} := \sqrt{\|u\|_V^2 + \varepsilon \|\nabla v\|_{H_\Gamma}^2} \quad \forall (u, v) \in \mathcal{V}_\varepsilon.$$

Notice that the above equivalence is not uniform in  $\varepsilon$ . It is clear that  $\mathcal{H}$  is a Hilbert space with respect to the scalar product

$$((u_1, v_1), (u_2, v_2))_{\mathcal{H}} := (u_1, u_2)_H + (v_1, v_2)_{H_\Gamma} \quad \forall (u_1, v_1), (u_2, v_2) \in \mathcal{H},$$

and that, for every  $\varepsilon \geq 0$ ,  $\mathcal{V}_\varepsilon$  is included in  $\mathcal{H}$  continuously and densely, so that  $(\mathcal{V}_\varepsilon, \mathcal{H}, \mathcal{V}_\varepsilon^*)$  is a Hilbert triplet. Furthermore, we introduce the space

$$\mathcal{Z} := \{(u, v) \in H^{N+1}(D) \times H^N(\Gamma) : v = \tau u\}$$

and note that  $\mathcal{Z} \hookrightarrow \mathcal{V}_\varepsilon \cap (L^\infty(D) \times L^\infty(\Gamma))$ , with  $\mathcal{Z} \hookrightarrow \mathcal{V}_\varepsilon$  densely. Now, it is natural to introduce the operator



$$\begin{aligned}\tilde{\mathcal{A}}_\varepsilon : C^\infty(\overline{D}) \times \tau(C^\infty(\overline{D})) &\rightarrow C^\infty(\overline{D}) \times C^\infty(\Gamma), \\ \tilde{\mathcal{A}}_\varepsilon(u, v) &:= (-\Delta u + \pi(u), \partial_{\mathbf{n}} u - \varepsilon \Delta_\Gamma v + \pi_\Gamma(v)).\end{aligned}$$

Then it is immediate to check that for every  $(\varphi, \psi) \in \mathcal{V}_\varepsilon$  we have

$$(\tilde{\mathcal{A}}_\varepsilon(u, v), (\varphi, \psi))_{\mathcal{H}} = \int_D \nabla u \cdot \nabla \varphi + \int_D \pi(u) \varphi + \varepsilon \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \psi + \int_\Gamma \pi_\Gamma(v) \psi.$$

Analysing separately the cases  $\varepsilon > 0$  and  $\varepsilon = 0$  and using the fact that  $\tau : H^1(D) \rightarrow L^2(\Gamma)$  is continuous, it is not difficult to check that the previous expression defines a linear continuous functional on  $\mathcal{V}_\varepsilon$  for every  $\varepsilon \geq 0$ . Hence, the operator  $\tilde{\mathcal{A}}_\varepsilon$  can be extended to

$$\begin{aligned}\mathcal{A}_\varepsilon : \mathcal{V}_\varepsilon &\rightarrow \mathcal{V}_\varepsilon^*, \\ \langle \mathcal{A}_\varepsilon(u, v), (\varphi, \psi) \rangle_{\mathcal{V}_\varepsilon} &:= \int_D \nabla u \cdot \nabla \varphi + \int_D \pi(u) \varphi + \varepsilon \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \psi + \int_\Gamma \pi_\Gamma(v) \psi \\ \forall (u, v), (\varphi, \psi) &\in \mathcal{V}_\varepsilon.\end{aligned}$$

In the sequel, we will denote by  $\mathcal{C}_\varepsilon : \mathcal{V}_\varepsilon \rightarrow \mathcal{V}_\varepsilon^*$  the linear component of  $\mathcal{A}_\varepsilon$ , i.e.

$$\langle \mathcal{C}_\varepsilon(u, v), (\varphi, \psi) \rangle_{\mathcal{V}_\varepsilon} := \int_D \nabla u \cdot \nabla \varphi + \varepsilon \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \psi, \quad (u, v), (\varphi, \psi) \in \mathcal{V}_\varepsilon,$$

so that we have the representation  $\mathcal{A}_\varepsilon = \mathcal{C}_\varepsilon + \mathcal{P}$ , where we have used the same symbol for  $\mathcal{P}$  and its corresponding Lipschitz operator induced on  $\mathcal{H}$ .

**Assumptions on the noises.** Let  $W$  and  $W_\Gamma$  be two independent cylindrical Wiener processes on two separable Hilbert spaces  $U$  and  $U_\Gamma$ , respectively. We introduce

$$\begin{aligned}B : \Omega \times [0, T] \times H &\rightarrow \mathcal{L}_2(U, H) \quad \text{progressively measurable,} \\ B_\Gamma : \Omega \times [0, T] \times H_\Gamma &\rightarrow \mathcal{L}_2(U_\Gamma, H_\Gamma) \quad \text{progressively measurable.}\end{aligned}$$

Then, setting

$$\begin{aligned}\mathcal{U} &:= U \times U_\Gamma, \quad \mathcal{W} := \begin{pmatrix} W \\ W_\Gamma \end{pmatrix}, \\ \mathcal{B} : \Omega \times [0, T] \times \mathcal{H} &\rightarrow \mathcal{L}_2(\mathcal{U}, \mathcal{H}), \quad \mathcal{B} := \begin{bmatrix} B & 0 \\ 0 & B_\Gamma \end{bmatrix},\end{aligned}$$

we have that  $\mathcal{W}$  is a cylindrical Wiener process on  $\mathcal{U}$  and  $\mathcal{B}$  is progressively measurable. Moreover, we assume that  $B$  and  $B_\Gamma$  are Lipschitz-continuous and at most with linear growth in their third arguments, uniformly on  $\Omega \times [0, T]$ , i.e. that there exists a positive constant  $C$  such that

$$\begin{aligned} \|B(\cdot, \cdot, u_1) - B(\cdot, \cdot, u_2)\|_{\mathcal{L}_2(U, H)} &\leq C \|u_1 - u_2\|_H \quad \forall u_1, u_2 \in H, \\ \|B_\Gamma(\cdot, \cdot, v_1) - B_\Gamma(\cdot, \cdot, v_2)\|_{\mathcal{L}_2(U_\Gamma, H_\Gamma)} &\leq C \|v_1 - v_2\|_{H_\Gamma} \quad \forall v_1, v_2 \in H_\Gamma, \\ \|B(\cdot, \cdot, u)\|_{\mathcal{L}_2(U, H)} &\leq C (1 + \|u\|_H) \quad \forall u \in H, \\ \|B(\cdot, \cdot, v)\|_{\mathcal{L}_2(U_\Gamma, H_\Gamma)} &\leq C (1 + \|v\|_{H_\Gamma}) \quad \forall v \in H_\Gamma. \end{aligned}$$

Then, it is clear that the same hypotheses hold also for  $\mathcal{B}$  in its corresponding spaces for a certain positive constant  $C_{\mathcal{B}}$ .

**Assumption on the initial datum.** We assume that the initial datum satisfies

$$(x_0, y_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H}).$$

### 2.3. Formulation of the problem and main results

In this setting, we can write the SPDE of the joint process  $(x_t, y_t)$  as follows:

$$d(x_t, y_t) + \mathcal{A}_\varepsilon(x_t, y_t) dt + \gamma(x_t, y_t) dt \ni \mathcal{B}(t, x_t, y_t) d\mathcal{W}_t, \quad (x(0), y(0)) = (x_0, y_0). \quad (2.1)$$

**Definition 2.1.** A strong solution to problem (2.1) is a quadruplet  $(x, y, \xi, \xi_\Gamma)$  such that

$$(x, y) \in L^2\left(\Omega; C^0([0, T]; \mathcal{H})\right) \cap L^2(\Omega \times (0, T); \mathcal{V}_\varepsilon), \quad (2.2)$$

$$\xi \in L^1(\Omega \times (0, T) \times D), \quad \xi_\Gamma \in L^1(\Omega \times (0, T) \times \Gamma), \quad (2.3)$$

$$j(x) + j^*(\xi) \in L^1(\Omega \times (0, T) \times D), \quad j_\Gamma(y) + j_\Gamma^*(\xi_\Gamma) \in L^1(\Omega \times (0, T) \times \Gamma), \quad (2.4)$$

$$\xi \in \beta(x) \quad \text{a.e. in } \Omega \times (0, T) \times D, \quad \xi_\Gamma \in \beta_\Gamma(y) \quad \text{a.e. in } \Omega \times (0, T) \times \Gamma, \quad (2.5)$$

$$(\xi, \xi_\Gamma) \quad \text{is predictable in } L^1(D) \times L^1(\Gamma), \quad (2.6)$$

Moreover the following equality holds in  $\mathcal{V}_\varepsilon^* \cap (L^1(D) \times L^1(\Gamma))$

$$\begin{aligned} (x(t), y(t)) + \int_0^t \mathcal{A}_\varepsilon(x(s), y(s)) ds + \int_0^t (\xi(s), \xi_\Gamma(s)) ds \\ = (x_0, y_0) + \int_0^t \mathcal{B}(s, x(s), y(s)) d\mathcal{W}_s \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (2.7)$$

**Definition 2.2.** We say that problem (2.1) is well-posed for a given  $\varepsilon \geq 0$  if for any initial datum  $(x_0, y_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H})$  there exists a unique strong solution to (2.1) in the sense of Definition 2.1 and the following solution map is Lipschitz-continuous:

$$\Lambda_\varepsilon : L^2(\Omega; \mathcal{H}) \rightarrow L^2\left(\Omega; C^0([0, T]; \mathcal{H})\right) \cap L^2(\Omega \times (0, T); \mathcal{V}_\varepsilon), \quad (x_0, y_0) \mapsto (x, y).$$

**Theorem 2.3.** *The problem (2.1) is well-posed for any  $\varepsilon \geq 0$ .*

**Theorem 2.4.** *Let  $(x_\varepsilon, y_\varepsilon, \xi_\varepsilon, \xi_{\Gamma, \varepsilon})$  and  $(x, y, \xi, \xi_\Gamma)$  be the unique strong solutions to (2.1) given by Theorem 2.3 in the cases  $\varepsilon > 0$  and  $\varepsilon = 0$ , respectively. Then for every sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$ , with  $\varepsilon_k \searrow 0$  as  $k \rightarrow \infty$ , we have, as  $k \rightarrow \infty$ ,*

$$(x_{\varepsilon_k}, y_{\varepsilon_k}) \rightarrow (x, y) \quad \text{in } L^p \left( \Omega; L^2(0, T; \mathcal{H}) \right) \quad \forall p \in [1, 2), \quad (2.8)$$

$$(x_{\varepsilon_k}, y_{\varepsilon_k}) \xrightarrow{*} (x, y) \quad \text{in } L^\infty \left( 0, T; L^2(\Omega; \mathcal{H}) \right), \quad (2.9)$$

$$(x_{\varepsilon_k}, y_{\varepsilon_k}) \rightharpoonup (x, y) \quad \text{in } L^2(\Omega \times (0, T); \mathcal{V}_0), \quad (2.10)$$

$$\xi_{\varepsilon_k} \rightharpoonup \xi \quad \text{in } L^1(\Omega \times (0, T) \times D), \quad \xi_{\Gamma, \varepsilon_k} \rightharpoonup \xi_\Gamma \quad \text{in } L^1(\Omega \times (0, T) \times \Gamma), \quad (2.11)$$

$$\varepsilon_k y_{\varepsilon_k} \rightarrow 0 \quad \text{in } L^2 \left( \Omega \times (0, T); H^1(\Gamma) \right). \quad (2.12)$$

### 3. Well-posedness

First of all, we prove well-posedness for the problem (2.1) with additive and more regular noise. Namely, let  $\mathcal{Z}$  be the separable Hilbert space defined in the previous section and consider the problem

$$d(x_t, y_t) + \mathcal{A}_\varepsilon(x_t, y_t) dt + \gamma(x_t, y_t) dt \ni \mathcal{B}(t) d\mathcal{W}_t, \quad (x(0), y(0)) = (x_0, y_0), \quad (3.1)$$

where

$$\mathcal{B} \in L^2 \left( \Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{Z})) \right). \quad (3.2)$$

The hypothesis on  $\mathcal{B}$  will be removed at the end of the section. Existence of a solution is proved using a suitable approximation on the equation and then passing to the limit using compactness results and monotonicity arguments. Continuous dependence on the data is obtained using an appropriate version of Itô's formula.

Throughout the section,  $\varepsilon \geq 0$  is a fixed constant, so that the argument fits both to the case  $\varepsilon = 0$  and  $\varepsilon > 0$  at the same time; when two different approaches are needed, we will specify it explicitly.

#### 3.1. The approximated problem

For any  $\lambda \in (0, 1)$ , let

$$\begin{aligned} \beta_\lambda &:= \frac{1}{\lambda} (I - (I + \lambda \beta)^{-1}), & \beta_{\Gamma, \lambda} &:= \frac{1}{\lambda} (I - (I + \lambda \beta_\Gamma)^{-1}) \\ j_\lambda(r) &:= \inf_{s \in \mathbb{R}} \left( j(r) + \frac{|r - s|^2}{2\lambda} \right), & j_{\Gamma, \lambda}(r) &:= \inf_{s \in \mathbb{R}} \left( j_\Gamma(r) + \frac{|r - s|^2}{2\lambda} \right) \end{aligned}$$

denote the Yosida approximations of the graphs  $\beta$  and  $\beta_\Gamma$  and the Moreau regularizations of the functions  $j$  and  $j_\Gamma$ , respectively. With this notation, it is a standard matter to check that the Yosida approximation of  $\gamma$  and the Moreau regularization of  $k$  are given by

$$\gamma_\lambda = (\beta_\lambda, \beta_{\Gamma,\lambda}), \quad k_\lambda(u, v) = j_\lambda(u) + j_{\Gamma,\lambda}(v) \quad \forall (u, v) \in \mathbb{R}^2.$$

We consider the following approximated problem:

$$d(x_\lambda, y_\lambda) + \mathcal{A}_\varepsilon(x_\lambda, y_\lambda) dt + \gamma_\lambda(x_\lambda, y_\lambda) dt = \mathcal{B}(t) d\mathcal{W}_t, \quad (3.3)$$

$$(x_\lambda(0), y_\lambda(0)) = (x_0, y_0). \quad (3.4)$$

For sake of simplicity, let us use the notation

$$\mathcal{A}_{\varepsilon,\lambda} : \mathcal{V}_\varepsilon \rightarrow \mathcal{V}_\varepsilon^*, \quad \mathcal{A}_{\varepsilon,\lambda}(u, v) := \mathcal{A}_\varepsilon(u, v) + \gamma_\lambda(u, v) \quad \forall (u, v) \in \mathcal{V}_\varepsilon,$$

so that we can write the approximated problem as

$$d(x_\lambda, y_\lambda) + \mathcal{A}_{\varepsilon,\lambda}(x_\lambda, y_\lambda) dt = \mathcal{B}(t) d\mathcal{W}_t, \quad (x_\lambda(0), y_\lambda(0)) = (x_0, y_0).$$

We recall some properties of the operator  $\mathcal{A}_{\varepsilon,\lambda}$  in the following lemma.

**Lemma 3.1.** *The operator  $\mathcal{A}_{\varepsilon,\lambda} : \mathcal{V}_\varepsilon \rightarrow \mathcal{V}_\varepsilon^*$  is hemicontinuous, weakly monotone, weakly coercive and bounded. More specifically, there exist two positive constants  $C$  and  $C_{\varepsilon,\lambda}$ , with the first being independent of  $\varepsilon$  and  $\lambda$ , such that the following conditions hold for any  $(u_1, v_1), (u_2, v_2), (u, v) \in \mathcal{V}_\varepsilon$ :*

$$\begin{aligned} \langle \mathcal{A}_{\varepsilon,\lambda}(u_1, v_1) - \mathcal{A}_{\varepsilon,\lambda}(u_2, v_2), (u_1, v_1) - (u_2, v_2) \rangle_{\mathcal{V}_\varepsilon} &\geq -C \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{H}}^2, \\ \langle \mathcal{A}_{\varepsilon,\lambda}(u, v), (u, v) \rangle_{\mathcal{V}_\varepsilon} &\geq \|\nabla u\|_H^2 + \varepsilon \|\nabla_\Gamma v\|_{H_\Gamma}^2 - C \|(u, v)\|_{\mathcal{H}}^2 - C, \\ \|\mathcal{A}_{\varepsilon,\lambda}(u, v)\|_{\mathcal{V}_\varepsilon^*} &\leq C_{\varepsilon,\lambda} (1 + \|(u, v)\|_{\mathcal{V}_\varepsilon}). \end{aligned}$$

**Proof.** Firstly, let  $(r_1, s_1), (r_2, s_2), (r_3, s_3) \in \mathcal{V}_\varepsilon$ : for any  $t \in \mathbb{R}$  we have

$$\begin{aligned} &\langle \mathcal{A}_{\varepsilon,\lambda}((r_1, s_1) + t(r_2, s_2)), (r_3, s_3) \rangle_{\mathcal{V}_\varepsilon} \\ &= \int_D \nabla(r_1 + tr_2) \cdot \nabla r_3 + \int_D \pi(r_1 + tr_2)r_3 + \int_D \beta_\lambda(r_1 + tr_2)r_3 \\ &\quad + \varepsilon \int_\Gamma \nabla_\Gamma(s_1 + ts_2) \cdot \nabla_\Gamma s_3 + \int_\Gamma \pi_\Gamma(s_1 + ts_2)s_3 + \int_\Gamma \beta_{\Gamma,\lambda}(s_1 + ts_2)s_3, \end{aligned}$$

and by the Lipschitz continuity of  $\pi$ ,  $\pi_\Gamma$ ,  $\beta_\lambda$  and  $\beta_{\Gamma,\lambda}$ , the right-hand side is a continuous function of  $t$ . Hence,  $\mathcal{A}_{\varepsilon,\lambda}$  is hemicontinuous. Secondly, using the Lipschitz continuity of  $\pi$  and  $\pi_\Gamma$  and the monotonicity of  $\beta_\lambda$  and  $\beta_{\Gamma,\lambda}$ , we have

$$\begin{aligned}
 & \langle \mathcal{A}_{\varepsilon,\lambda}(u_1, v_1) - \mathcal{A}_{\varepsilon,\lambda}(u_2, v_2), (u_1, v_1) - (u_2, v_2) \rangle_{\mathcal{V}_\varepsilon} = \int_D |\nabla(u_1 - u_2)|^2 \\
 & + \int_D (\pi(u_1) - \pi(u_2))(u_1 - u_2) + \int_D (\beta_\lambda(u_1) - \beta_\lambda(u_2))(u_1 - u_2) + \varepsilon \int_\Gamma |\nabla_\Gamma(v_1 - v_2)|^2 \\
 & + \int_\Gamma (\pi_\Gamma(v_1) - \pi_\Gamma(v_2))(v_1 - v_2) + \int_\Gamma (\beta_{\Gamma,\lambda}(v_1) - \beta_{\Gamma,\lambda}(v_2))(v_1 - v_2) \\
 & \geq -C_\pi \|u_1 - u_2\|_H^2 - C_{\pi_\Gamma} \|v_1 - v_2\|_{H_\Gamma}^2 \geq -C_{\mathcal{P}} \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{H}}^2,
 \end{aligned}$$

from which the weak monotonicity. Moreover, using the Lipschitz continuity of  $\pi$  and  $\pi_\Gamma$ , a similar computation leads to

$$\begin{aligned}
 & \langle \mathcal{A}_\varepsilon(u, v) + \gamma_\lambda(u, v), (u, v) \rangle_{\mathcal{V}_\varepsilon} \\
 & = \int_D |\nabla u|^2 + \int_D \pi(u)u + \int_D \beta_\lambda(u)u + \varepsilon \int_\Gamma |\nabla_\Gamma v|^2 + \int_\Gamma \pi_\Gamma(v)v + \int_\Gamma \beta_{\Gamma,\lambda}(v)v \\
 & \geq \|\nabla u\|_H^2 + \varepsilon \|\nabla_\Gamma v\|_{H_\Gamma}^2 - C \left( 1 + \|(u, v)\|_{\mathcal{H}}^2 \right)
 \end{aligned}$$

for a positive constant  $C$ , from which we deduce the weak coercivity. Indeed, this is immediate if  $\varepsilon > 0$ ; if  $\varepsilon = 0$ , this follows from the fact that the norm  $\|\cdot\|_{\mathcal{V}_0}$  is equivalent to  $|\cdot|_{\mathcal{V}_0}$  (since  $\tau : V \rightarrow V_\Gamma^0$  is continuous). Finally, for any  $(\varphi, \psi) \in \mathcal{V}_\varepsilon$ , by the Lipschitz continuity of  $\pi$ ,  $\pi_\Gamma$ ,  $\beta_\lambda$  and  $\beta_{\Gamma,\lambda}$ , using the Hölder inequality and renominating the positive constant  $C$  at each passage we have

$$\begin{aligned}
 & \langle \mathcal{A}_{\varepsilon,\lambda}(u, v), (\varphi, \psi) \rangle_{\mathcal{V}_\varepsilon} = \int_D \nabla u \cdot \nabla \varphi + \int_D \pi(u)\varphi + \int_D \beta_\lambda(u)\varphi \\
 & + \varepsilon \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \psi + \int_\Gamma \pi_\Gamma(v)\psi + \int_\Gamma \beta_{\Gamma,\lambda}(v)\psi \\
 & \leq \|\nabla u\|_H \|\nabla \varphi\|_H + C(1 + \|u\|_H) \|\varphi\|_H + \frac{1}{\lambda} \|u\|_H \|\varphi\|_H \\
 & + \varepsilon \|\nabla_\Gamma v\|_{H_\Gamma} \|\nabla_\Gamma \psi\|_{H_\Gamma} + C(1 + \|v\|_{H_\Gamma}) \|\psi\|_{H_\Gamma} + \frac{1}{\lambda} \|v\|_{H_\Gamma} \|\psi\|_{H_\Gamma} \\
 & \leq C \left( \max\{1, \varepsilon\} + 1 + \frac{1}{\lambda} \right) \left( (1 + \|u\|_V) \|\varphi\|_V + (1 + \|v\|_{V_\Gamma^\varepsilon}) \|\psi\|_{V_\Gamma^\varepsilon} \right) \\
 & \leq C_{\varepsilon,\lambda} (1 + \|(u, v)\|_{\mathcal{V}_\varepsilon}) \|(\varphi, \psi)\|_{\mathcal{V}_\varepsilon},
 \end{aligned}$$

from which the boundedness follows.  $\square$

The previous lemma ensures that the approximated problem is well-posed according to the classical variational approach by Pardoux, Krylov and Rozovskiĭ (see [22,29,30]) in the Gelfand triple  $(\mathcal{V}_\varepsilon, \mathcal{H}, \mathcal{V}_\varepsilon^*)$ . Hence, for any  $\lambda \in (0, 1)$  there is a unique strong solution

$$(x_\lambda, y_\lambda) \in L^2(\Omega; C^0([0, T]; \mathcal{H})) \cap L^2(\Omega \times (0, T); \mathcal{V}_\varepsilon)$$

to the approximated problem (3.3)–(3.4). Moreover we can exhibit some a priori estimates, as it is shown in the following lemmata.

**Lemma 3.2.** *There exists a constant  $K$  such that the following inequality holds*

$$\begin{aligned} & \| (x_\lambda, y_\lambda) \|_{L^2(\Omega; C^0([0, T]; \mathcal{H}))}^2 + \| (x_\lambda, y_\lambda) \|_{L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon))}^2 + \| \beta_\lambda(x_\lambda)x_\lambda \|_{L^1(\Omega; L^1(0, T; L^1(D)))} \\ & + \| \beta_{\Gamma, \lambda}(y_\lambda)y_\lambda \|_{L^1(\Omega; L^1(0, T; L^1(\Gamma)))} \leq K \left( 1 + \| (x_0, y_0) \|_{L^2(\Omega; \mathcal{H})}^2 + \| \mathcal{B} \|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H}))}^2 \right). \end{aligned}$$

**Proof.** The proof relies on the application of the version of Itô formula introduced in [22]. Precisely, we have for every  $t \in [0, T]$  and  $\mathbb{P}$ -almost surely that

$$\begin{aligned} & \| (x_\lambda(t), y_\lambda(t)) \|_{\mathcal{H}}^2 + 2 \int_0^t \langle \mathcal{A}_{\varepsilon, \lambda}(x_\lambda, y_\lambda), (x_\lambda, y_\lambda) \rangle_{\mathcal{V}_\varepsilon} ds \\ & = \| (x_0, y_0) \|_{L^2(\Omega; \mathcal{H})}^2 + 2 \int_0^t (x_\lambda(s), y_\lambda(s)) \mathcal{B}(s) d\mathcal{W}_s + \int_0^t \| \mathcal{B}(s) \|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds, \end{aligned}$$

where the term  $(x_\lambda, y_\lambda)\mathcal{B}$  in the stochastic integral has to be intended as the operator

$$(x_\lambda, y_\lambda)\mathcal{B} : \Omega \times [0, T] \rightarrow \mathcal{L}_2(\mathcal{U}, \mathbb{R}), \quad u \mapsto ((x_\lambda(\omega, t), y_\lambda(\omega, t)), \mathcal{B}(\omega, t)u)_{\mathcal{H}},$$

for  $u \in \mathcal{U}$  and  $(\omega, t) \in \Omega \times [0, T]$ , so that the stochastic integral is defined in the usual way in the sense of Itô. Using the Lipschitzianity of  $\pi$  and  $\pi_\Gamma$  and the weak coercivity of  $\mathcal{A}_{\varepsilon, \lambda}$ , there exists a positive constant  $C$  independent of  $\lambda$  such that

$$\begin{aligned} & \| (x_\lambda(t), y_\lambda(t)) \|_{\mathcal{H}}^2 + 2C \int_0^t \| (x_\lambda(s), y_\lambda(s)) \|_{\mathcal{V}_\varepsilon}^2 ds - 2C \int_0^t \left( 1 + \| (x_\lambda(s), y_\lambda(s)) \|_{\mathcal{H}}^2 \right) ds \\ & + 2 \int_0^t \int_D \beta_\lambda(x_\lambda(s))x_\lambda(s) ds + 2 \int_0^t \int_\Gamma \beta_{\Gamma, \lambda}(y_\lambda(s))y_\lambda(s) ds \\ & \lesssim \| (x_0, y_0) \|_{L^2(\Omega; \mathcal{H})}^2 + 2 \int_0^t (x_\lambda(s), y_\lambda(s)) \mathcal{B}(s) d\mathcal{W}_s + \int_0^t \| \mathcal{B}(s) \|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds. \end{aligned}$$

Taking the supremum in time and expectation, thanks to the Gronwall lemma we get

$$\begin{aligned} & \mathbb{E} \|(x_\lambda(t), y_\lambda(t))\|_{C([0,T];\mathcal{H})}^2 + \mathbb{E} \|(x_\lambda, y_\lambda)\|_{L^2(0,T;\mathcal{V}_\varepsilon)}^2 \\ & + \mathbb{E} \int_0^T \int_D \beta_\lambda(x_\lambda(s)) x_\lambda(s) ds + \mathbb{E} \int_0^T \int_\Gamma \beta_{\Gamma,\lambda}(y_\lambda(s)) y_\lambda(s) ds \\ & \lesssim 1 + \mathbb{E} \|(x_0, y_0)\|_{L^2(\Omega;\mathcal{H})}^2 + \mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t (x_\lambda(s), y_\lambda(s)) \mathcal{B}(s) d\mathcal{W}_s \right| + \mathbb{E} \|\mathcal{B}(s)\|_{L^2(0,T;\mathcal{L}_2(\mathcal{U},\mathcal{H}))}^2. \end{aligned}$$

Thanks to the generalized Burkholder–Davis–Gundy inequality contained in [26, Lem. 4.3] and the definition of the operator  $(x_\lambda, y_\lambda)\mathcal{B}$  we get that for every  $\delta > 0$

$$\begin{aligned} \mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t (x_\lambda(s), y_\lambda(s)) \mathcal{B}(s) d\mathcal{W}_s \right| & \lesssim \mathbb{E} \left( \int_0^T \|(x_\lambda(s), y_\lambda(s))\|_{\mathcal{H}}^2 \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}^2 ds \right)^{1/2} \\ & \leq \delta \mathbb{E} \|(x_\lambda, y_\lambda)\|_{C([0,T];\mathcal{H})}^2 + K_\delta \mathbb{E} \int_0^T \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}^2 ds. \end{aligned}$$

From the arbitrariness of  $\delta$ , we conclude choosing  $\delta$  small enough.  $\square$

**Lemma 3.3.** *There exists  $\Omega' \in \mathcal{F}$  with  $\mathbb{P}(\Omega') = 1$  such that, for every  $\omega \in \Omega'$ , there exists a positive constant  $K = K(\omega)$  satisfying*

$$\begin{aligned} \|(x_\lambda, y_\lambda)(\omega)\|_{L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}_\varepsilon)}^2 & + \|\beta_\lambda(x_\lambda(\omega))x_\lambda(\omega)\|_{L^1((0,T) \times D)} \\ & + \|\beta_{\Gamma,\lambda}(y_\lambda(\omega))y_\lambda(\omega)\|_{L^1((0,T) \times \Gamma)} \leq K \quad \forall \lambda \in (0, 1). \end{aligned}$$

**Proof.** To shorten the notation, we will use the notation  $\mathcal{B} \cdot \mathcal{W}$  to mean the stochastic integral of  $\mathcal{B}$  with respect to  $\mathcal{W}$ . The approximated equation can be written as

$$\frac{d}{dt}((x_\lambda, y_\lambda) - \mathcal{B} \cdot \mathcal{W}) + \mathcal{A}_\varepsilon(x_\lambda, y_\lambda) + \gamma_\lambda(x_\lambda, y_\lambda) = 0 \quad \text{a.e. in } (0, T), \quad \mathbb{P}\text{-a.s.}$$

Moreover, thanks to (3.2) and the choice of  $\mathcal{Z}$  we have

$$\mathcal{B} \cdot \mathcal{W} \in L^2(0, T; \mathcal{V}_\varepsilon) \cap L^\infty(0, T; L^\infty(D) \times L^\infty(\Gamma)) \quad \mathbb{P}\text{-a.s.}$$

Let then  $\Omega' \in \mathcal{F}$  with  $\mathbb{P}(\Omega') = 1$  such that the two previous relations hold and fix  $\omega \in \Omega'$ . Testing (deterministically) the first equation by  $(x_\lambda, y_\lambda) - \mathcal{B} \cdot \mathcal{W}$ , we get

$$\begin{aligned}
& \frac{1}{2} \|(x_\lambda, y_\lambda)(t) - \mathcal{B} \cdot \mathcal{W}(t)\|_{\mathcal{H}}^2 + \int_0^t \langle A_\varepsilon(x_\lambda, y_\lambda)(s), (x_\lambda, y_\lambda)(s) - \mathcal{B} \cdot \mathcal{W}(s) \rangle_{\mathcal{V}_\varepsilon} ds \\
& + \int_0^t \int_D \beta_\lambda(x_\lambda(s))(x_\lambda(s) - B \cdot W(s)) ds + \int_0^t \int_\Gamma \beta_{\Gamma,\lambda}(y_\lambda(s))(y_\lambda(s) - B_\Gamma \cdot W_\Gamma(s)) ds \\
& = \frac{1}{2} \|(x_0, y_0)\|_{\mathcal{H}}^2 \quad \forall t \in [0, T].
\end{aligned}$$

Rearranging the terms, using the regularity of  $\mathcal{B} \cdot \mathcal{W}$ , the Young inequality and the Lipschitzianity of  $\pi$  and  $\pi_\Gamma$  we infer that

$$\begin{aligned}
& \frac{1}{2} \|(x_\lambda, y_\lambda)(t) - \mathcal{B} \cdot \mathcal{W}(t)\|_{\mathcal{H}}^2 + \int_0^t \|\nabla x_\lambda(s)\|_H^2 ds + \varepsilon \int_0^t \|\nabla_\Gamma y_\lambda(s)\|_{H_\Gamma}^2 ds \\
& + \int_0^t \int_D \beta_\lambda(x_\lambda(s))x_\lambda(s) ds + \int_0^t \int_\Gamma \beta_{\Gamma,\lambda}(y_\lambda(s))y_\lambda(s) ds \\
& = \frac{1}{2} \|(x_0, y_0)\|_{\mathcal{H}}^2 + \int_0^t \int_D \nabla x_\lambda(s) \cdot \nabla(B \cdot W)(s) ds + \varepsilon \int_0^t \int_\Gamma \nabla_\Gamma y_\lambda(s) \cdot \nabla_\Gamma(B_\Gamma \cdot W_\Gamma)(s) ds \\
& + \int_0^t \int_D \beta_\lambda(x_\lambda(s))B \cdot W(s) ds + \int_0^t \int_\Gamma \beta_{\Gamma,\lambda}(y_\lambda(s))B_\Gamma \cdot W_\Gamma(s) ds \\
& - \int_0^t \int_D \pi(x_\lambda(s))(x_\lambda(s) - B \cdot W(s)) ds - \int_0^t \int_\Gamma \pi_\Gamma(y_\lambda(s))(y_\lambda(s) - B_\Gamma \cdot W_\Gamma(s)) ds \\
& \leq \frac{1}{2} \|(x_0, y_0)\|_{\mathcal{H}}^2 + \frac{1}{2} \int_0^t \|\nabla x_\lambda(s)\|_H^2 ds + \frac{\varepsilon}{2} \int_0^t \|\nabla_\Gamma y_\lambda(s)\|_{H_\Gamma}^2 ds \\
& + \frac{1 \vee \varepsilon}{2} \|\mathcal{B} \cdot \mathcal{W}\|_{L^2(0,T;\mathcal{V}_\varepsilon)}^2 + \frac{1}{2} \int_0^t \int_D j^*(\beta_\lambda(x_\lambda(s))) ds + \frac{1}{2} \int_0^t \int_\Gamma j_\Gamma^*(\beta_{\Gamma,\lambda}(y_\lambda(s))) ds \\
& + \int_0^T \int_D j(2(B \cdot W)) + \int_0^T \int_\Gamma j_\Gamma(2(B_\Gamma \cdot W_\Gamma)) \\
& + \left( \mathcal{C}_P + \frac{1}{2} \right) \int_0^t \|(x_\lambda, y_\lambda)(s) - \mathcal{B} \cdot \mathcal{W}(s)\|_{\mathcal{H}}^2 ds + \frac{\mathcal{C}_P^2}{2} \|\mathcal{B} \cdot \mathcal{W}\|_{L^2(0,T;\mathcal{H})}^2.
\end{aligned}$$

Since  $B \cdot W \in L^\infty((0, T) \times D)$  and  $B_\Gamma \cdot W_\Gamma \in L^\infty((0, T) \times \Gamma)$ , by continuity of  $j$  and  $j_\Gamma$  we have that  $j(2(B \cdot W)) \in L^1((0, T) \times D)$  and  $j_\Gamma(2(B_\Gamma \cdot W_\Gamma)) \in L^1((0, T) \times \Gamma)$ . Recall now that



for any  $u \in \mathbb{R}$  it holds  $\beta_\lambda(u) \in \partial j((I + \lambda\beta)^{-1}u) = \beta((I + \lambda\beta)^{-1}u)$  and  $\beta_\lambda(u)(I + \lambda\beta)^{-1}u \geq 0$  so that the generalized Young inequality ensures

$$j((I + \lambda\beta)^{-1}u) + j^*(\beta_\lambda(u)) = \beta_\lambda(u)(I + \lambda\beta)^{-1}u \leq \beta_\lambda(u)u \quad \forall u \in \mathbb{R}$$

thanks to the contraction property of the resolvent operator. Hence, on the right-hand side we get

$$j^*(\beta_\lambda(x_\lambda)) \leq \beta_\lambda(x_\lambda)x_\lambda, \quad j_\Gamma^*(\beta_{\Gamma,\lambda}(y_\lambda)) \leq \beta_{\Gamma,\lambda}(y_\lambda)y_\lambda,$$

thanks to the positivity of  $j((I + \lambda\beta)^{-1}x_\lambda)$ . Rearranging the terms and using the Gronwall lemma we can conclude.  $\square$

**Proposition 3.4.** *For any  $\omega \in \Omega'$  we can extract a subsequence  $\lambda' = \lambda'(\omega)$  of  $\lambda$  for which the following convergences hold as  $\lambda' \rightarrow 0$ :*

$$\begin{aligned} (x_{\lambda'}, y_{\lambda'})(\omega, \cdot) &\xrightarrow{*} (x, y)(\omega, \cdot) && \text{in } L^\infty(0, T; \mathcal{H}), \\ (x_{\lambda'}, y_{\lambda'})(\omega, \cdot) &\rightharpoonup (x, y)(\omega, \cdot) && \text{in } L^2(0, T; \mathcal{V}_\varepsilon), \\ \beta_{\lambda'}(x_{\lambda'}(\omega, \cdot)) &\rightharpoonup \xi(\omega, \cdot) && \text{in } L^1((0, T) \times D), \\ \beta_{\Gamma,\lambda'}(y_{\lambda'}(\omega, \cdot)) &\rightharpoonup \xi_\Gamma(\omega, \cdot) && \text{in } L^1((0, T) \times \Gamma), \\ (x_{\lambda'}, y_{\lambda'})(\omega, \cdot) &\rightarrow (x, y)(\omega, \cdot) && \text{in } L^2(0, T; \mathcal{H}). \end{aligned}$$

**Proof.** Let us fix  $\omega \in \Omega'$ . The first two convergences follow from Lemma 3.3. Regarding the third one, recall that for any  $u \in \mathbb{R}$  we have

$$j((I + \lambda\beta)^{-1}u) + j^*(\beta_\lambda(u)) = \beta_\lambda(u)(I + \lambda\beta)^{-1}u \leq \beta_\lambda(u)u \quad \forall u \in \mathbb{R}$$

thanks to the contraction property of the resolvent operator. From Lemma 3.3 and since  $j^*$  is superlinear at infinity, for any  $\omega \in \Omega'$  the sequence  $(\beta_\lambda(x_\lambda(\omega)))_\lambda$  turns out to be weakly relatively compact in  $L^1((0, T) \times D)$  thanks to the de la Vallée Poussin criterion along with Dunford–Pettis theorem. Hence, we can extract a subsequence  $\lambda'(\omega)$  which satisfies the required convergence. The same reasoning can be applied to  $\beta_{\Gamma,\lambda}$  to get the fourth convergence statement. Finally, it remains to show the strong convergence of  $(x_{\lambda'}, y_{\lambda'})$  in  $L^2(0, T; \mathcal{H})$ . To this end, going back to

$$\frac{d}{dt}((x_\lambda, y_\lambda) - \mathcal{B} \cdot \mathcal{W}) + \mathcal{A}_\varepsilon(x_\lambda, y_\lambda) + \gamma_\lambda(x_\lambda, y_\lambda) = 0 \quad \text{a.e. in } (0, T),$$

from Lemma 3.3, the boundedness of  $\mathcal{A}_\varepsilon$  and the fact that  $\mathcal{V}_\varepsilon^* \hookrightarrow \mathcal{Z}^*$ , we have

$$\begin{aligned} \|\mathcal{A}_\varepsilon(x_\lambda, y_\lambda)\|_{L^1(0,T;\mathcal{Z}^*)} &\lesssim 1 + \|(x_\lambda, y_\lambda)\|_{L^2(0,T;\mathcal{V}_\varepsilon)}, \\ \|\gamma_\lambda(x_\lambda, y_\lambda)\|_{L^1(0,T;\mathcal{Z}^*)} &\lesssim \|\gamma_\lambda(x_\lambda, y_\lambda)\|_{L^1(0,T;L^1(D) \times L^1(\Gamma))}. \end{aligned}$$

Hence, by Lemma 3.2,  $\|\frac{d}{dt}((x_\lambda, y_\lambda) - \mathcal{B} \cdot \mathcal{W})\|_{L^1(0,T;\mathcal{Z}^*)}$  is uniformly bounded in  $\lambda$  and we can apply Simon's theorem (see [33, Cor. 4, p. 85]) to get that  $(x_\lambda, y_\lambda)$  is relatively compact in  $L^2(0, T; \mathcal{H})$ . Then the weak convergence of  $(x_{\lambda'}, y_{\lambda'})$  towards  $(x, y)$  in  $L^2(0, T; \mathcal{V}_\varepsilon)$ , implies that

$$(x_{\lambda'}, y_{\lambda'}) \rightarrow (x, y) \quad \text{strongly in } L^2(0, T; \mathcal{H}),$$

which is the required convergence.  $\square$

### 3.2. The limit problem

Now we are ready for the proof of the well-posedness of the equation (3.1), where the noise enters the system in an additive fashion. We divide the proof in several steps:

**Identification of the limit.** Fix  $\omega \in \Omega'$ : in the sequel we do not emphasize the  $\omega$ -dependence as no confusion can arise. By Proposition 3.4,  $(x_{\lambda'}, y_{\lambda'}) \rightarrow (x, y)$  strongly in  $L^2(0, T; \mathcal{H})$ , and  $(x_{\lambda'}(t), y_{\lambda'}(t)) \rightarrow (x(t), y(t))$  for a.e.  $t \in [0, T]$  up to passing to a further subsequence. As for the  $\mathcal{A}_\varepsilon$ -part we write  $\mathcal{A}_\varepsilon = \mathcal{C}_\varepsilon + \mathcal{P}$ . Firstly, we employ weak convergence  $(x_{\lambda'}, y_{\lambda'}) \rightharpoonup (x, y)$  in  $L^2(0, T; \mathcal{V}_\varepsilon)$  to get

$$\int_0^t \mathcal{C}_\varepsilon(x_{\lambda'}(s), y_{\lambda'}(s)) ds \rightharpoonup \int_0^t \mathcal{C}_\varepsilon(x(s), y(s)) ds \quad \text{in } \mathcal{V}_\varepsilon^*$$

for all  $t \in [0, T]$ . This is straightforward setting  $\phi_0 \in \mathcal{V}_\varepsilon$  and choosing as a test function  $\phi := s \mapsto 1_{[0,t]}(s)\phi_0 \in L^2(0, T; \mathcal{V}_\varepsilon)$ . Secondly, we simply use the strong convergence  $(x_{\lambda'}, y_{\lambda'}) \rightarrow (x, y)$  in  $L^2(0, T; \mathcal{H})$  and the Lipschitz continuity of  $\mathcal{P}$  to get

$$\int_0^t \mathcal{P}(x_{\lambda'}(s), y_{\lambda'}(s)) ds \rightarrow \int_0^t \mathcal{P}(x(s), y(s)) ds \quad \text{in } \mathcal{H}$$

for every time  $t \in [0, T]$ . Regarding the monotone part we employ the weak convergence in  $L^1(0, T; L^1(D) \times L^1(\Gamma))$  of  $\gamma_{\lambda'}$  towards  $(\xi, \xi_\Gamma)$  to easily get that

$$\int_0^t \gamma_{\lambda'}(x_{\lambda'}(s), y_{\lambda'}(s)) ds \rightharpoonup \int_0^t (\xi(s), \xi_\Gamma(s)) ds \quad \text{in } L^1(D) \times L^1(\Gamma)$$

for all  $t \in [0, T]$ . From the very definition of  $\mathcal{Z}$  it follows that both  $\mathcal{V}_\varepsilon^* \hookrightarrow \mathcal{Z}^*$  and  $L^1(D) \times L^1(\Gamma) \hookrightarrow \mathcal{Z}^*$ . Hence, summing up the previous convergences we get the limit equation

$$\begin{aligned} (x(t), y(t)) + \int_0^t \mathcal{A}_\varepsilon(x(s), y(s)) ds + \int_0^t (\xi(s), \xi_\Gamma(s)) ds \\ = (x_0, y_0) + \int_0^t \mathcal{B}(s) d\mathcal{W}_s, \quad \text{in } \mathcal{Z}^* \end{aligned} \tag{3.5}$$

for almost every  $t \in [0, T]$ . Actually, equation (3.5) can be written as an equality also in  $\mathcal{V}_\varepsilon^* \cap (L^1(D) \times L^1(\Gamma))$ . Indeed, all the terms except for the third one on the left hand side belongs to  $\mathcal{V}_\varepsilon^*$  and all the terms apart from  $\mathcal{A}_\varepsilon(x(s), y(s))$  take values in  $L^1(D) \times L^1(\Gamma)$ .

Since  $\mathcal{A}_\varepsilon(x, y)$ ,  $(\xi, \xi_\Gamma)$  and  $\mathcal{B} \cdot \mathcal{W}$  belong to  $L^1(0, T; \mathcal{Z}^*)$ , we also deduce that  $(x, y) \in C([0, T]; \mathcal{Z}^*)$  and then (3.5) holds for every  $t \in [0, T]$ . Moreover, from a result due to Strauss [36, Thm. 2.1], it follows that the solution is also weakly continuous in  $\mathcal{H}$ , i.e.  $(x, y) \in C_w([0, T]; \mathcal{H})$ .

It remains to show that  $(\xi, \xi_\Gamma)$  belongs to  $(\beta(x), \beta_\Gamma(y))$  a.e. in  $(0, T) \times D$ . Let us use the strong convergence of  $(x_{\lambda'}, y_{\lambda'}) \rightarrow (x, y)$  in  $L^2(0, T; \mathcal{H})$  to extract a subsequence (still denoted with  $\lambda'$ ) so that both  $(I + \lambda'\beta)^{-1}x_{\lambda'}$  and  $(I + \lambda'\beta_\Gamma)^{-1}y_{\lambda'}$  would converge a.e. in  $(0, T) \times D$  to  $x$  and  $y$ , respectively. At this point remember that  $\beta_{\lambda'}(x_{\lambda'}) \in \beta((I + \lambda'\beta)^{-1}x_{\lambda'})$  almost everywhere in  $(0, T) \times D$  and that

$$\int_0^T \int_D \beta_{\lambda'}(x_{\lambda'})(I + \lambda'\beta)^{-1}x_{\lambda'} \leq \int_0^T \int_D \beta_{\lambda'}(x_{\lambda'})x_{\lambda'} < K_\omega$$

because of Lemma 3.2 and the fact that  $\omega \in \Omega'$ . Then we are in position to apply Brezis' lemma [8, Thm. 18, p. 126] and we get the identification  $\xi \in \beta(x)$ . The same reasoning holds for the monotone operator  $\beta_\Gamma$  on the boundary and the claim is proved.

Finally, using the lower semicontinuity of the convex integrands for the weak convergence and Lemma 3.3, we have that the limit solution satisfies

$$\begin{aligned} \int_0^T \int_D (j(x) + j^*(\xi)) &\leq \liminf_{\lambda' \rightarrow 0} \int_0^T \int_D (j((I + \lambda'\beta)^{-1}x_{\lambda'}) + j^*(\beta_{\lambda'}(x_{\lambda'}))) \\ &= \liminf_{\lambda' \rightarrow 0} \int_0^T \int_D \beta_{\lambda'}(x_{\lambda'})(I + \lambda'\beta)^{-1}x_{\lambda'} \leq C_1, \\ \int_0^T \int_\Gamma (j_\Gamma(y) + j_\Gamma^*(\xi_\Gamma)) &\leq \liminf_{\lambda' \rightarrow 0} \int_0^T \int_\Gamma (j_\Gamma((I + \lambda'\beta_\Gamma)^{-1}y_{\lambda'}) + j_\Gamma^*(\beta_{\Gamma, \lambda'}(y_{\lambda'}))) \\ &= \liminf_{\lambda' \rightarrow 0} \int_0^T \int_\Gamma \beta_{\Gamma, \lambda'}(y_{\lambda'})(I + \lambda'\beta_\Gamma)^{-1}y_{\lambda'} \leq C_2, \end{aligned}$$

where  $C_1, C_2$  are constants which depend only on  $\omega$ .

**Uniqueness.** Here we show a conditional uniqueness result for equation (3.5) with  $\omega$  fixed. By contradiction, suppose that  $(x_i, y_i, \xi_i, \xi_{\Gamma, i})$  satisfy equation (3.5) and are such that  $(x_i, y_i) \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}_\varepsilon)$ ,  $(\xi_i, \xi_{\Gamma, i}) \in L^1(0, T; L^1(D) \times L^1(\Gamma))$ ,  $j(x_i) + j^*(\xi_i) \in L^1((0, T) \times D)$ ,  $j_\Gamma(y_i) + j_\Gamma^*(\xi_{\Gamma, i}) \in L^1((0, T) \times \Gamma)$ . Setting  $(\bar{x}, \bar{y}) = (x_1, y_1) - (x_2, y_2)$ ,  $(\bar{\xi}, \bar{\xi}_\Gamma) = (\xi_1, \xi_{\Gamma, 1}) - (\xi_2, \xi_{\Gamma, 2})$  and  $\bar{\mathcal{P}} := \mathcal{P}(x_1, y_1) - \mathcal{P}(x_2, y_2)$  we have that

$$(\bar{x}(t), \bar{y}(t)) + \int_0^t \mathcal{C}_\varepsilon(\bar{x}(s), \bar{y}(s)) ds + \int_0^t \bar{\mathcal{P}}(s) ds + \int_0^t (\bar{\xi}(s), \bar{\xi}_\Gamma(s)) ds = 0 \quad \forall t \in [0, T] \quad (3.6)$$

and it is enough to show that  $(\bar{x}, \bar{y}) = 0$  and  $(\bar{\xi}, \bar{\xi}_\Gamma) = 0$  for every  $t \in [0, T]$ . The idea is to smooth out the equation, recover some structure condition and then use strong convergence to pass to the limit in the approximation. To this end, we multiply the above equation by a power (high enough) of the resolvent of  $\mathcal{C}_\varepsilon$ . Remember that the strong formulation of  $\mathcal{C}_\varepsilon$  is

$$\mathcal{C}_\varepsilon(u, v) := (-\Delta u, \partial_{\mathbf{n}} u - \varepsilon \Delta_\Gamma v)$$

and that we are able to show regularizing properties of the resolvent operator associated to  $\mathcal{C}_\varepsilon$ , which are presented in detail in the Appendix. In particular, we can prove (see Corollary A.6) that there is  $m \in \mathbb{N}$  so that  $(I + \delta \mathcal{C}_\varepsilon)^{-m}$  is ultracontractive, i.e. that it is well-defined, linear and continuous from  $L^1(D) \times L^1(\Gamma)$  to  $L^\infty(D) \times L^\infty(\Gamma)$ . If we use the notation  $(u, v)^\delta = (u^\delta, v^\delta) := (I + \delta \mathcal{C}_\varepsilon)^{-m}(u, v)$  for any  $(u, v)$  for which it makes sense and for any  $\delta > 0$ , then we have

$$(\bar{x}, \bar{y})^\delta(t) + \int_0^t \mathcal{C}_\varepsilon(\bar{x}, \bar{y})^\delta(s) ds + \int_0^t \bar{\mathcal{P}}^\delta(s) ds + \int_0^t (\bar{\xi}, \bar{\xi}_\Gamma)^\delta(s) ds = 0$$

for every  $t \in [0, T]$ . Applying energy estimates (with  $\omega$  still fixed) of the form  $e^{-rt} \|\cdot\|_{\mathcal{H}}^2$ , with  $r > 0$ , we get

$$\begin{aligned} e^{-rt} \|(\bar{x}, \bar{y})^\delta(t)\|_{\mathcal{H}}^2 &+ 2 \int_0^t e^{-rs} \|\nabla \bar{x}^\delta(s)\|_H^2 ds + 2\varepsilon \int_0^t e^{-rs} \|\nabla_\Gamma \bar{y}^\delta(s)\|_{H_\Gamma}^2 ds \\ &+ r \int_0^t e^{-rs} \|(\bar{x}, \bar{y})^\delta(s)\|_{\mathcal{H}}^2 ds + 2 \int_0^t e^{-rs} (\bar{\mathcal{P}}^\delta(s), (\bar{x}, \bar{y})^\delta(s))_{\mathcal{H}} ds \\ &+ 2 \int_0^t e^{-rs} ((\bar{\xi}, \bar{\xi}_\Gamma)^\delta(s), (\bar{x}, \bar{y})^\delta(s))_{\mathcal{H}} ds = 0. \end{aligned}$$

By the monotonicity of  $\mathcal{C}_\varepsilon$  and the Lipschitz continuity of  $\mathcal{P}$ , if we choose  $r \geq 2C_{\mathcal{P}}$  we easily get

$$e^{-rt} \|(\bar{x}, \bar{y})^\delta(t)\|_{\mathcal{H}}^2 + 2 \int_0^t e^{-rs} ((\bar{\xi}, \bar{\xi}_\Gamma)^\delta(s), (\bar{x}, \bar{y})^\delta(s))_{\mathcal{H}} ds \leq 0. \quad (3.7)$$

Thanks to Lemma A.7 we know that  $(\bar{x}, \bar{y})^\delta(t) \rightarrow (\bar{x}, \bar{y})(t)$  in  $\mathcal{H}$  for all  $t \in [0, T]$ . This yields the convergence of  $\|(\bar{x}, \bar{y})^\delta(t)\|_{\mathcal{H}}^2$  towards  $\|(\bar{x}, \bar{y})(t)\|_{\mathcal{H}}^2$ . Concerning the second term, let us carefully rewrite it using the projection of the resolvent map defined in the Appendix. To shorten the notation here we denote by  $(\bar{x}, \bar{y})_i^\delta$  the projection on the  $i$ -th coordinate:  $p_i \circ (I + \delta \mathcal{C}_\varepsilon)^{-m}(\bar{x}, \bar{y})$ ,  $i = 1, 2$  (the same for  $(\bar{\xi}, \bar{\xi}_\Gamma)_i^\delta$ ). Hence we have

$$\begin{aligned}
 ((\bar{x}, \bar{y})^\delta, (\bar{\xi}, \bar{\xi}_\Gamma)^\delta)_{\mathcal{H}} &= ((I + \delta C_\varepsilon)^{-m}(\bar{x}, \bar{y}), (I + \delta C_\varepsilon)^{-m}(\bar{\xi}, \bar{\xi}_\Gamma)^\delta)_{\mathcal{H}} \\
 &= (((\bar{x}, \bar{y})_1^\delta, (\bar{x}, \bar{y})_2^\delta), ((\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta, (\bar{\xi}, \bar{\xi}_\Gamma)_2^\delta))_{\mathcal{H}} \\
 &= ((\bar{x}, \bar{y})_1^\delta, (\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta)_H + ((\bar{x}, \bar{y})_2^\delta, (\bar{\xi}, \bar{\xi}_\Gamma)_2^\delta)_{H_\Gamma} \\
 &= \int_D (\bar{x}, \bar{y})_1^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta + \int_\Gamma (\bar{x}, \bar{y})_2^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_2^\delta.
 \end{aligned}$$

Since  $(\bar{x}, \bar{y})^\delta \rightarrow (\bar{x}, \bar{y})$  in  $L^2(0, T; \mathcal{H})$  and  $(\bar{\xi}, \bar{\xi}_\Gamma)^\delta \rightarrow (\bar{\xi}, \bar{\xi}_\Gamma)$  in  $L^1(0, T; L^1(D) \times L^1(\Gamma))$ , we can extract a subsequence of  $\delta$ , still denoted by the same symbol, such that  $(\bar{x}, \bar{y})^\delta \rightarrow (\bar{x}, \bar{y})$  and  $(\bar{\xi}, \bar{\xi}_\Gamma)^\delta \rightarrow (\bar{\xi}, \bar{\xi}_\Gamma)$  a.e. in  $(0, T) \times D \times \Gamma$ . From the continuity of the projection map  $p_1$  we also know that  $(\bar{x}, \bar{y})_1^\delta \rightarrow \bar{x}$ ,  $(\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta \rightarrow \bar{\xi}$  and the same holds for  $p_2$ . The strong character of the convergences also guarantee the convergence of the products:

$$\begin{aligned}
 (\bar{x}, \bar{y})_1^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta &\longrightarrow \bar{x} \bar{\xi} \quad \text{a.e. in } (0, t) \times D, \\
 (\bar{x}, \bar{y})_2^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_2^\delta &\longrightarrow \bar{y} \bar{\xi}_\Gamma \quad \text{a.e. in } (0, t) \times \Gamma.
 \end{aligned}$$

Moreover, thanks to the assumptions (H1), (H2) or (H3 $_{\varepsilon>0}$ )–(H3 $_{\varepsilon=0}$ ) on the compatibility of the potentials  $j, j_\Gamma$  and Corollary A.10 in the Appendix, the families  $(\bar{x}, \bar{y})_1^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta$  and  $(\bar{x}, \bar{y})_2^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_2^\delta$ , with  $i = 1, 2$ , are uniformly integrable: hence, using Vitali's theorem we infer that  $(\bar{x}, \bar{y})_1^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_1^\delta \rightarrow \bar{x} \bar{\xi}$  and  $(\bar{x}, \bar{y})_2^\delta (\bar{\xi}, \bar{\xi}_\Gamma)_2^\delta \rightarrow \bar{y} \bar{\xi}_\Gamma$  in  $L^1((0, t) \times D)$  and  $L^1((0, t) \times \Gamma)$ , respectively. Letting then  $\delta \searrow 0$  in (3.7), we get that

$$e^{-rt} \|(\bar{x}, \bar{y})(t)\|_{\mathcal{H}}^2 + 2 \int_0^t \int_D e^{-rs} \bar{x} \bar{\xi} + 2 \int_0^t \int_\Gamma e^{-rs} \bar{y} \bar{\xi}_\Gamma \leq 0.$$

From the monotonicity of  $(\beta, \beta_\Gamma)$  we deduce that  $(\bar{x}, \bar{y})(t) = 0$  for all  $t \in [0, T]$ . Then also  $\int_0^t (\bar{\xi}, \bar{\xi}_\Gamma)(s) ds = 0$  for all  $t \in [0, T]$  from equation (3.6) and we get the uniqueness due to the arbitrariness of  $t \in [0, T]$ .

**Regularity in  $\omega$ .** So far we have worked with  $\omega \in \Omega'$  fixed. Now we are interested in showing the regularity of the solution with respect to  $\omega$ , starting from the issue of measurability. We follow the same ideas developed in [26], which we briefly resume here for the convenience of the reader, with obvious adaptations to our setting. The key point is the uniqueness result obtained for the solution, which guarantees that the subsequence  $\lambda'$  actually does not depend on  $\omega \in \Omega'$ . The proof relies on a standard argument: from any subsequence of  $\lambda$  we can extract a further subsequence  $\lambda'$ , depending on  $\omega$ , such that all the convergences obtained in Proposition 3.4 take place. Since the limit is unique, then the same results holds for the entire sequence  $\lambda$  which does not depend on  $\omega$  anymore.

From Proposition 3.4 we know that  $(x_\lambda, y_\lambda)(\omega, \cdot) \rightarrow (x, y)(\omega, \cdot)$  strongly in  $L^2(0, T; \mathcal{H})$ , which implies the convergence of  $(x_\lambda, y_\lambda)(\omega, t)$  in  $\mathbb{P} \otimes dt$ -measure to the same limit. Then, along a subsequence we have convergence  $\mathbb{P} \otimes dt$ -a.e. and this is enough to ensure the predictability of the limit  $(x, y)$  (since  $(x_\lambda, y_\lambda)$  is adapted with continuous trajectories). Concerning the singular

part  $(\xi, \xi_\Gamma)$  we have to be more careful. First, we set  $\xi_\lambda := \beta_\lambda(x_\lambda)$ ,  $\xi_{\Gamma,\lambda} := \beta_{\Gamma,\lambda}(y_\lambda)$  and define for any  $g \in L^\infty((0, T) \times D)$ ,  $h \in L^\infty((0, T) \times \Gamma)$

$$F_\lambda(\omega) := \int_0^T \int_D \xi_\lambda(\omega, s, z) g(s, z) dz ds, \quad F(\omega) := \int_0^T \int_D \xi(\omega, s, z) g(s, z) dz ds,$$

$$F_{\Gamma,\lambda}(\omega) := \int_0^T \int_\Gamma \xi_{\Gamma,\lambda}(\omega, s, \zeta) h(s, \zeta) d\zeta ds, \quad F_\Gamma(\omega) := \int_0^T \int_\Gamma \xi_\Gamma(\omega, s, \zeta) h(s, \zeta) d\zeta ds.$$

Then, since  $\xi_\lambda \rightarrow \xi$  and  $\xi_{\Gamma,\lambda} \rightarrow \xi_\Gamma$  in  $L^1((0, T) \times D)$  and  $L^1((0, T) \times \Gamma)$ , respectively, we have that  $F_\lambda \rightarrow F$  and  $F_{\Gamma,\lambda} \rightarrow F_\Gamma$   $\mathbb{P}$ -almost surely in  $\Omega$ . What we are going to show now is the following. We first prove that  $F_\lambda$  and  $F_{\Gamma,\lambda}$  converge weakly in  $L^1(\Omega)$  to  $F$  and  $F_\Gamma$ , respectively, so that  $(\xi_\lambda, \xi_{\Gamma,\lambda}) \rightharpoonup (\xi, \xi_\Gamma)$  in  $L^1(\Omega; L^1(0, T; L^1(D) \times L^1(\Gamma)))$ . Then, using the Mazur lemma, we will infer measurability of the limit  $(\xi, \xi_\Gamma)$ .

Let us show some details for  $F_\lambda$ ; the same technique can be adopted for  $F_{\Gamma,\lambda}$ . Firstly, for any  $l \in L^\infty(\Omega)$ , setting  $j_0(\cdot) := j^*(\cdot/M)$  and  $M = 1/[(\|g\|_{L^\infty((0,T) \times D)} \vee 1)(\|l\|_{L^\infty(\Omega)} \vee 1)]$ , we use Jensen's inequality to get

$$\mathbb{E} j_0(F_\lambda l) = \mathbb{E} j_0 \left( \int_0^T \int_D \xi_\lambda(\omega, s, z) g(s, z) l(\omega) dz ds \right) \leq \mathbb{E} \int_0^T \int_D j^*(\xi_\lambda(\omega, s, z)) dz ds.$$

The last term is bounded due to Lemma 3.2: then by the de la Vallée Poussin criterion the sequence  $F_\lambda l$  is uniformly integrable. This yields the strong convergence  $F_\lambda l \rightarrow Fl$  in  $L^1(\Omega)$  thanks to Vitali's theorem. From the arbitrariness of the function  $l$  we get weak convergence in  $L^1(\Omega)$  of  $(F_\lambda)_\lambda$ , thus the weak convergence of  $\xi_\lambda$  to  $\xi$  in  $L^1(\Omega \times (0, T) \times D)$ . Applying Mazur's lemma we extract a convex combination  $(\tilde{\xi}_\lambda)_\lambda$  of  $(\xi_\lambda)_\lambda$  which converge strongly to  $\xi$  in  $L^1(\Omega \times (0, T) \times D)$ . To conclude note that  $\tilde{\xi}_\lambda$  are predictable (as finite convex combination of predictable processes), hence the limit  $\xi$  is a predictable  $L^1(D)$ -valued process.

Once we have the measurability of the limit solution  $(x, y, \xi, \xi_\Gamma)$ , we are interested in showing some estimates in expectation. To this end, we just use lower semicontinuity of the norms with respect to the weak convergence and Fatou's lemma:

$$\mathbb{E} \|(x, y)\|_{L^2(0,T;\mathcal{V}_e)}^2 \leq \mathbb{E} \liminf_{\lambda \rightarrow 0} \|(x_\lambda, y_\lambda)\|_{L^2(0,T;\mathcal{V}_e)}^2 \leq \liminf_{\lambda \rightarrow 0} \mathbb{E} \|(x_\lambda, y_\lambda)\|_{L^2(0,T;\mathcal{V}_e)}^2,$$

$$\mathbb{E} \|(x, y)\|_{L^\infty(0,T;\mathcal{H})}^2 \leq \mathbb{E} \liminf_{\lambda \rightarrow 0} \|(x_\lambda, y_\lambda)\|_{L^\infty(0,T;\mathcal{H})}^2 \leq \liminf_{\lambda \rightarrow 0} \mathbb{E} \|(x_\lambda, y_\lambda)\|_{L^\infty(0,T;\mathcal{H})}^2,$$

$$\mathbb{E} \|\xi\|_{L^1(0,T;L^1(D))}^2 \leq \mathbb{E} \liminf_{\lambda \rightarrow 0} \|\xi_\lambda\|_{L^1(0,T;L^1(D))}^2 \leq \liminf_{\lambda \rightarrow 0} \mathbb{E} \|\xi_\lambda\|_{L^1(0,T;L^1(D))}^2,$$

$$\mathbb{E} \|\xi_\Gamma\|_{L^1(0,T;L^1(\Gamma))}^2 \leq \mathbb{E} \liminf_{\lambda \rightarrow 0} \|\xi_{\Gamma,\lambda}\|_{L^1(0,T;L^1(\Gamma))}^2 \leq \liminf_{\lambda \rightarrow 0} \mathbb{E} \|\xi_{\Gamma,\lambda}\|_{L^1(0,T;L^1(\Gamma))}^2.$$

Thanks to Lemma 3.2 and Proposition 3.4 the right hand side of all the above inequalities is bounded uniformly in  $\lambda$ , hence

$$(x, y) \in L^2(\Omega; L^\infty(0, T; \mathcal{H})) \cap L^2(\Omega \times (0, T); \mathcal{V}_\varepsilon),$$

$$\xi \in L^1(\Omega \times (0, T) \times D), \quad \xi_\Gamma \in L^1(\Omega \times (0, T) \times \Gamma),$$

and we have the required regularity. Finally, as we did at the beginning of Subsection 3.2, the weak lower semicontinuity of the convex integrands, Lemma 3.2 and the fact that the subsequence  $\lambda$  does not depend on  $\omega$  ensure also that

$$j(x) + j^*(\xi) \in L^1(\Omega \times (0, T) \times D), \quad j(y) + j_\Gamma^*(\xi_\Gamma) \in L^1(\Omega \times (0, T) \times \Gamma).$$

### 3.3. Continuous dependence on the data

Here we prove a continuous dependence result for equation (3.1) for every  $\varepsilon \geq 0$ .

We consider  $(x_0^i, y_0^i) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H})$  and  $\mathcal{B}_i \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))$  for  $i = 1, 2$ : let  $(x_i, y_i, \xi_i, \xi_{\Gamma,i})$  be any corresponding strong solutions to (3.1),  $i = 1, 2$ . Setting  $(x, y) := (x_1, y_1) - (x_2, y_2)$ ,  $(\xi, \xi_\Gamma) := (\xi_1, \xi_{\Gamma,1}) - (\xi_2, \xi_{\Gamma,2})$ ,  $\eta := \mathcal{P}(x_1, y_1) - \mathcal{P}(x_2, y_2)$ ,  $x_0 := x_0^1 - x_0^2$ ,  $y_0 := y_0^1 - y_0^2$  and  $\mathcal{B} := \mathcal{B}_1 - \mathcal{B}_2$ , we have

$$(x(t), y(t)) + \int_0^t \mathcal{C}_\varepsilon(x(s), y(s)) ds + \int_0^t \eta(s) ds + \int_0^t (\xi(s), \xi_\Gamma(s)) ds = (x_0, y_0) + \int_0^t \mathcal{B}(s) d\mathcal{W}_s$$

for every  $t \in [0, T]$ ,  $\mathbb{P}$ -almost surely. Now, the idea is to proceed as in Subsection 3.2 when we proved uniqueness for the limit problem: from now on, we refer to the Appendix for any useful properties on the resolvent of  $\mathcal{C}_\varepsilon$ . If  $m \in \mathbb{N}$  is given by Corollary A.6, using again the notation  $(h_1, h_2)^\delta = (h_1^\delta, h_2^\delta) := (I + \delta \mathcal{C}_\varepsilon)^{-m} h$  for any  $h$  for which it makes sense, we have

$$(x, y)^\delta(t) + \int_0^t \mathcal{C}_\varepsilon(x, y)^\delta(s) ds + \int_0^t \eta^\delta(s) ds + \int_0^t (\xi, \xi_\Gamma)^\delta(s) ds = (x_0, y_0)^\delta + \int_0^t \mathcal{B}^\delta(s) d\mathcal{W}_s.$$

Thanks to the smoothing properties of  $(I + \delta \mathcal{C}_\varepsilon)^{-m}$ , we can apply the classical Itô's formula for  $e^{-rt} \|\cdot\|_{\mathcal{H}}^2$  to get

$$\begin{aligned} & e^{-rt} \|(x, y)^\delta(t)\|_{\mathcal{H}}^2 + 2 \int_0^t e^{-rs} \|\nabla x^\delta(s)\|_H^2 ds + 2\varepsilon \int_0^t e^{-rs} \|\nabla_\Gamma y^\delta(s)\|_{H_\Gamma}^2 ds \\ & + r \int_0^t e^{-rs} \|(x, y)^\delta(s)\|_{\mathcal{H}}^2 ds + 2 \int_0^t e^{-rs} (\eta^\delta(s), (x, y)^\delta(s))_{\mathcal{H}} ds \\ & + 2 \int_0^t e^{-rs} ((\xi, \xi_\Gamma)^\delta(s), (x, y)^\delta(s))_{\mathcal{H}} ds \\ & = \|(x_0, y_0)^\delta\|_{\mathcal{H}}^2 + \int_0^t e^{-rs} \|\mathcal{B}^\delta(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds + 2 \int_0^t e^{-rs} (x, y)^\delta(s) \mathcal{B}^\delta(s) d\mathcal{W}_s. \end{aligned}$$

We want to pass to the limit as  $\delta \searrow 0$  in the previous expression and obtain in this way an Itô's formula for the limit processes. To this aim, the terms on the left hand side have already been handled in Subsection 3.2 when dealing with uniqueness of the limit problem. Moreover, by virtue of Lemma A.7, the dominated convergence theorem and the ideal property of  $\mathcal{L}_2(\mathcal{U}, \mathcal{H})$  in  $\mathcal{L}(\mathcal{U}, \mathcal{H})$  we have that  $\|(x_0, y_0)^\delta\|_{\mathcal{H}}^2 \rightarrow \|(x_0, y_0)\|_{\mathcal{H}}^2$  and  $\int_0^t e^{-rs} \|\mathcal{B}^\delta(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \rightarrow \int_0^t e^{-rs} \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds$  as  $\delta \searrow 0$ . Hence, we only need to check the convergence of the stochastic integral: note that, using the notation  $f^r$  to indicate the process  $f^r(t) := e^{-\frac{r}{2}t} f(t)$  for any process  $f$ , thanks to the Burkholder–Davis–Gundy inequality we have

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t e^{-rs} [(x, y)^\delta(s) \mathcal{B}^\delta(s) - (x, y)(s) \mathcal{B}(s)] d\mathcal{W}_s \right| \\ & \lesssim \mathbb{E} \left[ \left( \int_0^T e^{-2rs} \|(x, y)^\delta(s)\|_{\mathcal{H}}^2 \|\mathcal{B}^\delta(s) - \mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \right)^{1/2} \right] \\ & \quad + \mathbb{E} \left[ \left( \int_0^T e^{-2rs} \|(x, y)^\delta(s) - (x, y)(s)\|_{\mathcal{H}}^2 \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \right)^{1/2} \right] \\ & \lesssim \|(x, y)^{\delta, r}\|_{L^2(\Omega; L^\infty(0, T; \mathcal{H}))} \|\mathcal{B}^{\delta, r} - \mathcal{B}^r\|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \\ & \quad + \mathbb{E} \left[ \left( \int_0^T e^{-2rs} \|(x, y)^\delta(s) - (x, y)(s)\|_{\mathcal{H}}^2 \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \right)^{1/2} \right] \end{aligned}$$

and the last term converges to 0 as  $\delta \searrow 0$ . Indeed, the contraction properties of the resolvent and the fact that  $\mathcal{B}^{\delta, r} \rightarrow \mathcal{B}^r$  in  $L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))$  ensures that

$$\begin{aligned} & \|(x, y)^{\delta, r}\|_{L^2(\Omega; L^\infty(0, T; \mathcal{H}))} \|\mathcal{B}^{\delta, r} - \mathcal{B}^r\|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \\ & \leq \|(x, y)^r\|_{L^2(\Omega; L^\infty(0, T; \mathcal{H}))} \|\mathcal{B}^{\delta, r} - \mathcal{B}^r\|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \rightarrow 0. \end{aligned}$$

For the second term, note that by Lemma A.7 we have that  $(x, y)^\delta \rightarrow (x, y)$  in  $\mathcal{H}$  almost everywhere in  $\Omega \times [0, T]$ ; moreover, the contraction of the resolvent yields

$$\|(x, y)^\delta\|_{\mathcal{H}}^2 \|\mathcal{B}\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 \leq \|(x, y)\|_{\mathcal{H}}^2 \|\mathcal{B}\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 \leq \|(x, y)\|_{L^\infty(0, T; \mathcal{H})}^2 \|\mathcal{B}\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2,$$

and since the right-hand side belongs to  $L^1(\Omega \times (0, T))$ , by the Lebesgue dominated convergence theorem we easily get also that

$$\mathbb{E} \left[ \left( \int_0^T e^{-2rs} \|(x, y)^\delta(s) - (x, y)(s)\|_{\mathcal{H}}^2 \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \right)^{1/2} \right] \rightarrow 0,$$



from which the required convergence follows. Consequently, taking everything into account, we infer that the following Itô's formula holds:

$$\begin{aligned} & \| (x, y)^r(t) \|_{\mathcal{H}}^2 + 2 \int_0^t \| \nabla x^r(s) \|_H^2 ds + 2\varepsilon \int_0^t \| \nabla_{\Gamma} y^r(s) \|_{H_{\Gamma}}^2 ds + r \int_0^t \| (x, y)^r(s) \|_{\mathcal{H}}^2 ds \\ & + 2 \int_0^t (\eta^r(s), (x, y)^r(s))_{\mathcal{H}} ds + 2 \int_0^t \int_D \xi^r(s) x^r(s) ds + 2 \int_0^t \int_{\Gamma} \xi_{\Gamma}^r(s) y^r(s) ds \\ & = \| (x_0, y_0) \|_{\mathcal{H}}^2 + \int_0^t \| \mathcal{B}^r(s) \|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds + 2 \int_0^t (x, y)^r(s) \mathcal{B}^r(s) d\mathcal{W}_s. \end{aligned} \quad (3.8)$$

Using the Lipschitz continuity of  $\mathcal{P}$ , the Gronwall lemma, the monotonicity of  $(\beta, \beta_{\Gamma})$  and the weak coercivity of  $\mathcal{C}_{\varepsilon}$ , taking supremum in time and expectations we get

$$\begin{aligned} & \| (x, y)^r(t) \|_{L^2(\Omega; L^{\infty}(0, T; \mathcal{H}))}^2 + r \| (x, y)^r \|_{L^2(\Omega; L^2(0, T; \mathcal{H}))}^2 + \| (x, y)^r \|_{L^2(\Omega; L^2(0, T; \mathcal{V}_{\varepsilon}))}^2 \\ & \lesssim \| (x_0, y_0) \|_{L^2(\Omega; \mathcal{H})}^2 + \| \mathcal{B}^r \|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H}))}^2 + \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t (x, y)^r(s) \mathcal{B}^r(s) d\mathcal{W}_s \right|. \end{aligned}$$

Estimating the last term as in the proof of Lemma 3.2, we deduce that

$$\begin{aligned} & \| (x, y)^r(t) \|_{L^2(\Omega; L^{\infty}(0, T; \mathcal{H}))} + \sqrt{r} \| (x, y)^r \|_{L^2(\Omega; L^2(0, T; \mathcal{H}))} + \| (x, y)^r \|_{L^2(\Omega; L^2(0, T; \mathcal{V}_{\varepsilon}))} \\ & \lesssim \| (x_0, y_0) \|_{L^2(\Omega; \mathcal{H})} + \| \mathcal{B}^r \|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H}))}, \end{aligned} \quad (3.9)$$

where the desired continuous dependence relation for the problem with additive noise follows choosing  $r = 0$ . This implies that the solution  $(x, y, \xi, \xi_{\Gamma})$  built as limit of Yosida approximations of the problem is indeed the unique strong solution to the problem.

### 3.4. Extension to general additive noise

Here we conclude the proof of existence of a solution to (3.1), with general additive noise: namely, we remove the hypothesis (3.2) and we assume just that

$$\mathcal{B} \in L^2 \left( \Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})) \right).$$

To do it, we follow the same ideas as in [26, Prop. 5.1]. Using the notation of the Appendix, for any  $\delta \in (0, 1)$  and  $m$  given by Corollary A.6 we have

$$\mathcal{B}^{\delta} := (I + \delta \mathcal{C}_{\varepsilon})^{-m} \mathcal{B} \in L^2 \left( \Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{Z})) \right)$$

for a certain Hilbert space  $\mathcal{Z} \hookrightarrow \mathcal{V}_{\varepsilon} \cap L^{\infty}(D) \times L^{\infty}(\Gamma)$ . Hence, the problem with additive noise  $\mathcal{B}^{\delta}$  admits a unique solution  $(x^{\delta}, y^{\delta}, \xi^{\delta}, \xi_{\Gamma}^{\delta})$  for what we have already proved.

Since by contraction of the resolvent  $\|\mathcal{B}^\delta\|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \leq \|\mathcal{B}\|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))}$  for every  $\delta \in (0, 1)$ , by Lemma 3.2 and the weak lower semicontinuity of the norms we have that

$$\|(x^\delta, y^\delta)\|_{L^2(\Omega; L^\infty(0, T; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon))}^2 \leq K_1$$

for a positive constant  $K_1$ , for every  $\delta \in (0, 1)$ . Moreover, by the continuous dependence result proved in the previous section, for every  $\delta, \eta \in (0, 1)$  we have

$$\|(x^\delta, y^\delta) - (x^\eta, y^\eta)\|_{L^2(\Omega; L^\infty(0, T; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon))} \lesssim \|\mathcal{B}^\delta - \mathcal{B}^\eta\|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \rightarrow 0$$

as  $\delta, \eta \searrow 0$ , so that  $\{(x^\delta, y^\delta)\}_\delta$  is Cauchy in  $L^2(\Omega; L^\infty(0, T; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon))$ . Concerning the term  $\xi^\delta$  recall that there exists a constant  $C > 0$  such that for every  $\delta \in (0, 1)$

$$\begin{aligned} \|\xi^\delta x^\delta\|_{L^1(\Omega \times (0, T) \times D)} &= \mathbb{E} \int_0^T \int_D (j(x^\delta) + j^*(\xi^\delta)) \\ &\leq \liminf_{\lambda \rightarrow 0} \mathbb{E} \int_0^T \int_D \left( j((I + \lambda\beta)^{-1} x_\lambda^\delta) + j^*(\beta_\lambda(x_\lambda^\delta)) \right) \\ &= \liminf_{\lambda \rightarrow 0} \mathbb{E} \int_0^T \int_D \beta_\lambda(x_\lambda^\delta) (I + \lambda\beta)^{-1} x_\lambda^\delta \leq C. \end{aligned}$$

From the same computation for  $\xi_\Gamma^\delta$  we get the existence of a positive constant  $K_2$  such that for every  $\delta \in (0, 1)$  it holds

$$\|\xi^\delta x^\delta\|_{L^1(\Omega \times (0, T) \times D)} + \|\xi_\Gamma^\delta y^\delta\|_{L^1(\Omega \times (0, T) \times \Gamma)} \leq K_2$$

Now, using again that

$$\xi^\delta x^\delta = j(x^\delta) + j^*(\xi^\delta), \quad \xi_\Gamma^\delta y^\delta = j_\Gamma(y^\delta) + j_\Gamma^*(\xi_\Gamma^\delta),$$

since  $j^*$  and  $j_\Gamma^*$  are superlinear, the de la Vallée-Poussin and Dunford–Pettis theorems ensure that  $\{\xi^\delta\}_\delta$  and  $\{\xi_\Gamma^\delta\}_\delta$  are relatively weakly compact in  $L^1(\Omega \times (0, T) \times D)$  and  $L^1(\Omega \times (0, T) \times \Gamma)$ , respectively. Taking this information into account, we deduce that (along a subsequence)

$$\begin{aligned} (x^\delta, y^\delta) &\rightharpoonup (x, y) && \text{in } L^2(\Omega; L^\infty(0, T; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon)), \\ \xi^\delta &\rightharpoonup \xi && \text{in } L^1(\Omega \times (0, T) \times D), \\ \xi_\Gamma^\delta &\rightharpoonup \xi_\Gamma && \text{in } L^1(\Omega \times (0, T) \times \Gamma). \end{aligned}$$

Using these convergences, the weak lower semicontinuity of the convex integrands and the strong-weak closure of the maximal monotone graphs as in the passage to the limit in  $\lambda$ , it is not difficult to check that  $(x, y, \xi, \xi_\Gamma)$  is a strong solution for the problem with additive noise.

Finally, let us point out how to prove the strong continuity in  $\mathcal{H}$  of the trajectories of  $(x, y)$ . Arguing as in Itô's formula (3.8) and using the fact that the stochastic integral has continuous trajectories, it follows that  $t \mapsto \|(x, y)(t)\|_{\mathcal{H}}^2$  is continuous on a set of probability 1. Since  $\mathcal{H}$  is reflexive and  $(x, y) \in C_w^0([0, T]; \mathcal{H})$ , the strong continuity follows by a classical criterion. For a more detailed argument, we refer here to [24].

### 3.5. Well-posedness with multiplicative noise

We collect here the conclusion of the proof of Theorem 2.3, showing that the original problem (2.1) is well-posed also with multiplicative noise.

Given a progressively measurable process  $(z, w) \in L^2(\Omega; L^2(0, T; \mathcal{H}))$ , thanks to the hypotheses on  $\mathcal{B}$ , we have that  $\mathcal{B}(\cdot, \cdot, z, w) \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))$  and is progressively measurable. By the well-posedness result with additive noise proved in the previous sections, the problem

$$d(x_t, y_t) + \mathcal{A}_\varepsilon(x_t, y_t) dt + \gamma(x_t, y_t) dt \ni \mathcal{B}(t, z_t, w_t) d\mathcal{W}_t, \quad (x(0), y(0)) = (x_0, y_0),$$

is well-posed. Hence, it is well-defined the map

$$\begin{aligned} \mathcal{S}_\varepsilon : L^2(\Omega; L^2(0, T; \mathcal{H})) &\rightarrow L^2(\Omega; C^0([0, T]; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon)), \\ \mathcal{S}_\varepsilon : (z, w) &\mapsto (x, y). \end{aligned}$$

It is clear that any fixed point  $(x, y)$  for  $\mathcal{S}_\varepsilon$  (together with its corresponding  $(\xi, \xi_\Gamma)$ ) is a strong solution to (2.1). If we introduce for any  $p \in [1, +\infty]$  the norms on  $L^p(0, T)$  given by  $\|f\|_{L_r^p(0, T)} := \|f^r\|_{L^p(0, T)}$ , for  $f \in L^p(0, T)$ , using (3.9) and the Lipschitz continuity of  $\mathcal{B}$ , it is immediate to check that

$$\begin{aligned} \|\mathcal{S}_\varepsilon(z_1, w_1) - \mathcal{S}_\varepsilon(z_2, w_2)\|_{L^2(\Omega; L_r^2(0, T; \mathcal{H}))} &\lesssim \frac{1}{\sqrt{r}} \|\mathcal{B}(z_1, w_1) - \mathcal{B}(z_2, w_2)\|_{L^2(\Omega; L_r^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \\ &\lesssim \frac{1}{\sqrt{r}} \|(z_1, w_1) - (z_2, w_2)\|_{L^2(\Omega; L_r^2(0, T; \mathcal{H}))}, \end{aligned}$$

where the implicit constant is independent of  $r$ . Consequently, there exists  $r$  large enough such that  $\mathcal{S}_\varepsilon$  is a strict contraction on  $L^2(\Omega; L_r^2(0, T; \mathcal{H}))$ . By Banach fixed point theorem, there exists a unique fixed point  $(x, y)$  for  $\mathcal{S}_\varepsilon$ : this implies that (2.1) has a unique strong solution.

Finally, the last thing we have to prove is that the solution map

$$\begin{aligned} \Lambda_\varepsilon : L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H}) &\rightarrow L^2(\Omega; C^0([0, T]; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_\varepsilon)), \\ \Lambda_\varepsilon : (x_0, y_0) &\mapsto (x, y), \end{aligned}$$

where  $(x, y)$  is the unique solution to (2.1), is Lipschitz continuous. To this aim, given  $(x_0^i, y_0^i) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H})$  and setting  $(x_i, y_i) := \Lambda_\varepsilon(x_0^i, y_0^i)$  for  $i = 1, 2$ , rewriting (3.9) with the choice  $\mathcal{B}_i := \mathcal{B}(x_i, y_i)$ ,  $i = 1, 2$ , and using the Lipschitz continuity of  $\mathcal{B}$ , we have

$$\begin{aligned}
& \| (x_1, y_1) - (x_2, y_2) \|_{L^2(\Omega; C^0([0, T]; \mathcal{H}) \cap L^2(\Omega; L_r^2(0, T; \mathcal{V}_\varepsilon))} + \sqrt{r} \| (x_1, y_1) - (x_2, y_2) \|_{L^2(\Omega; L_r^2(0, T; \mathcal{H}))} \\
& \lesssim \| (x_0^1, y_0^1) - (x_0^2, y_0^2) \|_{L^2(\Omega; \mathcal{H})} + \| \mathcal{B}(x_1, y_1) - \mathcal{B}(x_2, y_2) \|_{L^2(\Omega; L_r^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} \\
& \lesssim \| (x_0^1, y_0^1) - (x_0^2, y_0^2) \|_{L^2(\Omega; \mathcal{H})} + \| (x_1, y_1) - (x_2, y_2) \|_{L^2(\Omega; L_r^2(0, T; \mathcal{H}))}
\end{aligned}$$

where again the implicit constant does not depend on  $r$ . Consequently, the assertion follows choosing  $r$  large enough and from the fact that  $\|\cdot\|_{L_r^p(0, T)}$  is equivalent to the usual norm of  $L^p(0, T)$  for every  $r \geq 0$ .

#### 4. The asymptotic behaviour as $\varepsilon \searrow 0$

This last section is devoted to the proof of the asymptotic result as  $\varepsilon \searrow 0$  contained in Theorem 2.4.

##### 4.1. The additive noise case

Let  $(x_\varepsilon, y_\varepsilon, \xi_\varepsilon, \xi_{\Gamma, \varepsilon})$  and  $(x, y, \xi, \xi_\Gamma)$  be the unique solutions to (2.1) with additive noise  $\mathcal{B}$  in the cases  $\varepsilon > 0$  and  $\varepsilon = 0$ , respectively. Moreover, for any  $\delta \in (0, 1)$ , let  $(x_\varepsilon^\delta, y_\varepsilon^\delta, \xi_\varepsilon^\delta, \xi_{\Gamma, \varepsilon}^\delta)$  and  $(x^\delta, y^\delta, \xi^\delta, \xi_\Gamma^\delta)$  be the unique solutions to (2.1) with additive noise  $(I + \delta \mathcal{C}_\varepsilon)^{-m} \mathcal{B}$  in the cases  $\varepsilon > 0$  and  $\varepsilon = 0$ , respectively, where  $m$  is given by Corollary A.6.

Assume for a moment the validity of Theorem 2.4 with additive noise given by

$$(I + \delta \mathcal{C}_\varepsilon)^{-m} \mathcal{B} \in L^2\left(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{Z}))\right), \quad \delta \in (0, 1).$$

If  $\delta \in (0, 1)$  is fixed, the quadruplets  $(x_\varepsilon^\delta, y_\varepsilon^\delta, \xi_\varepsilon^\delta, \xi_{\Gamma, \varepsilon}^\delta)$  converge as  $\varepsilon \searrow 0$  to the limit  $(x^\delta, y^\delta, \xi^\delta, \xi_\Gamma^\delta)$  in the sense of (2.8)–(2.12) in Theorem 2.4. Moreover, from Section 3.4 we also know that, as  $\delta \searrow 0$ ,

$$(x_\varepsilon^\delta, y_\varepsilon^\delta) \rightarrow (x_\varepsilon, y_\varepsilon), \quad (x^\delta, y^\delta) \rightarrow (x, y) \quad \text{in } L^2(\Omega; C^0([0, T]; \mathcal{H})) \cap L^2(\Omega; L^2(0, T; \mathcal{V}_0)).$$

More specifically, for every  $p \in [1, 2)$ , by the continuous dependence property in the cases  $\varepsilon > 0$  and  $\varepsilon = 0$  we have

$$\begin{aligned}
& \| (x_\varepsilon, y_\varepsilon) - (x, y) \|_{L^p(\Omega; L^2(0, T; \mathcal{H}))} \\
& \leq \| (x_\varepsilon, y_\varepsilon) - (x_\varepsilon^\delta, y_\varepsilon^\delta) \|_{L^p(\Omega; L^2(0, T; \mathcal{H}))} + \| (x_\varepsilon^\delta, y_\varepsilon^\delta) - (x^\delta, y^\delta) \|_{L^p(\Omega; L^2(0, T; \mathcal{H}))} \\
& \quad + \| (x^\delta, y^\delta) - (x, y) \|_{L^p(\Omega; L^2(0, T; \mathcal{H}))} \\
& \lesssim \| \mathcal{B} - \mathcal{B}^\delta \|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))} + \| (x_\varepsilon^\delta, y_\varepsilon^\delta) - (x^\delta, y^\delta) \|_{L^p(\Omega; L^2(0, T; \mathcal{H}))} \\
& \quad + \| \mathcal{B} - \mathcal{B}^\delta \|_{L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))},
\end{aligned}$$

where the implicit constant is independent of both  $\varepsilon$  and  $\delta$ . Since we know that  $\mathcal{B}^\delta \rightarrow \mathcal{B}$  in  $L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))$  as  $\delta \searrow 0$ , in order to prove (2.8) it is not restrictive to work with  $(I + \delta \mathcal{C}_\varepsilon)^{-m} \mathcal{B}$  with  $\delta > 0$ . In other words, this shows that it is sufficient to consider  $\mathcal{B} \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{Z})))$ . A similar argument applies to (2.9)–(2.10). As far as

(2.11)–(2.12) are concerned, it is enough to show, respectively, that  $(\xi_\varepsilon, \xi_{\Gamma,\varepsilon})_\varepsilon$  is relatively compact in  $L^1(D) \times L^1(\Gamma)$  and that  $(\varepsilon^{1/2} \nabla_\Gamma y_\varepsilon)_\varepsilon$  is bounded in  $L^2(\Omega \times (0, T); H_\Gamma)$ : these in turn follow immediately from Itô's formula for the solutions  $(x_\varepsilon, y_\varepsilon, \xi_\varepsilon, \xi_{\Gamma,\varepsilon})$ , as usual.

Taking these remarks into account, we assume that  $\mathcal{B} \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{Z})))$ .

By definition of strong solution, we know that

$$\begin{aligned} (x_\varepsilon(t), y_\varepsilon(t)) &+ \int_0^t \mathcal{C}_\varepsilon(x_\varepsilon(s), y_\varepsilon(s)) ds + \int_0^t \mathcal{P}(x_\varepsilon(s), y_\varepsilon(s)) ds + \int_0^t (\xi_\varepsilon(s), \xi_{\Gamma,\varepsilon}(s)) ds \\ &= (x_0, y_0) + \int_0^t \mathcal{B}(s) d\mathcal{W}_s \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (4.1)$$

Proceeding as in Section 3.3 with the choice  $r = 0$ , the following Itô's formula holds for every  $t \in [0, T]$ ,  $\mathbb{P}$ -almost surely:

$$\begin{aligned} &\|(x_\varepsilon, y_\varepsilon)(t)\|_{\mathcal{H}}^2 + 2 \int_0^t \|\nabla x_\varepsilon(s)\|_H^2 ds + 2\varepsilon \int_0^t \|\nabla_\Gamma y_\varepsilon(s)\|_{H_\Gamma}^2 ds \\ &+ 2 \int_0^t (\mathcal{P}(x_\varepsilon(s), y_\varepsilon(s)), (x_\varepsilon, y_\varepsilon)(s))_{\mathcal{H}} ds + 2 \int_0^t \int_D \xi_\varepsilon(s) x_\varepsilon(s) ds + \int_0^t \int_\Gamma \xi_{\Gamma,\varepsilon}(s) y_\varepsilon(s) ds \\ &= \|(x_0, y_0)\|_{\mathcal{H}}^2 + \int_0^t \|\mathcal{B}(s)\|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds + 2 \int_0^t (x_\varepsilon, y_\varepsilon)(s) \mathcal{B}(s) d\mathcal{W}_s. \end{aligned}$$

Taking supremum in time and expectations, estimating the stochastic integral as in the proof of Lemma 3.2 and using the Gronwall lemma together with the Lipschitzianity of  $\mathcal{P}$ , we easily deduce that there exists  $C > 0$ , independent of  $\varepsilon$ , such that

$$\begin{aligned} &\|(x_\varepsilon, y_\varepsilon)\|_{L^2(\Omega; L^\infty(0, T; \mathcal{H}))} \leq C, \\ &\|\nabla x_\varepsilon\|_{L^2(\Omega; L^2(0, T; H))} \leq C, \quad \varepsilon^{1/2} \|\nabla_\Gamma y_\varepsilon\|_{L^2(\Omega; L^2(0, T; H_\Gamma))} \leq C, \\ &\|\xi_\varepsilon x_\varepsilon\|_{L^1(\Omega \times (0, T) \times D)} \leq C, \quad \|\xi_{\Gamma,\varepsilon} y_\varepsilon\|_{L^1(\Omega \times (0, T) \times \Gamma)} \leq C. \end{aligned}$$

Similarly, proceeding as in Lemma 3.3 and owing to (3.2), we infer that there exists  $\Omega' \in \mathcal{F}$  with  $\mathbb{P}(\Omega') = 1$  such that, for every  $\omega \in \Omega'$ , there is  $C_\omega > 0$  (independent of  $\varepsilon \in (0, 1)$ ) such that

$$\begin{aligned} &\|(x_\varepsilon, y_\varepsilon)(\omega)\|_{L^\infty(0, T; \mathcal{H})} \leq C_\omega, \\ &\|\nabla x_\varepsilon(\omega)\|_{L^2(0, T; H)} \leq C_\omega, \quad \varepsilon^{1/2} \|\nabla_\Gamma y_\varepsilon(\omega)\|_{L^2(0, T; H_\Gamma)} \leq C_\omega, \\ &\|\xi_\varepsilon(\omega) x_\varepsilon(\omega)\|_{L^1((0, T) \times D)} \leq C_\omega, \quad \|\xi_{\Gamma,\varepsilon}(\omega) y_\varepsilon(\omega)\|_{L^1((0, T) \times \Gamma)} \leq C_\omega. \end{aligned}$$

Since  $(\xi_\varepsilon, \xi_{\Gamma,\varepsilon}) \in \gamma(x_\varepsilon, y_\varepsilon)$  almost everywhere, we have

$$j^*(\xi_\varepsilon) \leq j(x_\varepsilon) + j^*(\xi_\varepsilon) = \xi_\varepsilon x_\varepsilon, \quad j_\Gamma^*(\xi_{\Gamma,\varepsilon}) \leq j(y_\varepsilon) + j_\Gamma^*(\xi_{\Gamma,\varepsilon}) = \xi_{\Gamma,\varepsilon} y_\varepsilon,$$

so that the families  $j^*(\xi_\varepsilon)$  and  $j_\Gamma^*(\xi_{\Gamma,\varepsilon})$  are uniformly bounded in  $L^1((0, T) \times D)$  and  $L^1((0, T) \times \Gamma)$ , respectively. Furthermore, since

$$\frac{d}{dt}((x_\varepsilon, y_\varepsilon) - \mathcal{B} \cdot \mathcal{W}) + \mathcal{A}_\varepsilon(x_\varepsilon, y_\varepsilon) + (\xi_\varepsilon, \xi_{\Gamma,\varepsilon}) = 0,$$

if  $\mathcal{Z}$  is as in the proof of Proposition 3.4, by difference we also infer that

$$\left\| \frac{d}{dt}((x_\varepsilon, y_\varepsilon)(\omega) - \mathcal{B} \cdot \mathcal{W}(\omega)) \right\|_{L^1(0, T; \mathcal{Z}^*)} \leq C_\omega.$$

We deduce that for every  $\omega \in \Omega'$  there is a subsequence  $\varepsilon' = \varepsilon'(\omega)$  along which we have

$$(x_{\varepsilon'}, y_{\varepsilon'}) \rightarrow (\tilde{x}, \tilde{y}) \quad \text{in } L^2(0, T; \mathcal{H}), \quad (4.2)$$

$$(x_{\varepsilon'}, y_{\varepsilon'}) \xrightarrow{*} (\tilde{x}, \tilde{y}) \quad \text{in } L^\infty(0, T; \mathcal{H}), \quad (4.3)$$

$$(x_{\varepsilon'}, y_{\varepsilon'}) \rightharpoonup (\tilde{x}, \tilde{y}) \quad \text{in } L^2(0, T; \mathcal{V}_0), \quad (4.4)$$

$$(\xi_{\varepsilon'}, \xi_{\Gamma,\varepsilon'}) \rightharpoonup (\tilde{\xi}, \tilde{\xi}_\Gamma) \quad \text{in } L^1(0, T; L^1(D) \times L^1(\Gamma)). \quad (4.5)$$

Moreover, it is also clear that

$$\varepsilon \nabla_\Gamma y_\varepsilon \rightarrow 0 \quad \text{in } L^2\left(\Omega; L^2(0, T; H_\Gamma)\right). \quad (4.6)$$

At this point, repeating exactly the same argument contained in Section 3.2, we infer that  $\varepsilon'$  is independent of  $\omega$  and that the limit processes satisfy

$$(\tilde{x}, \tilde{y}) \in L^2\left(\Omega; C^0([0, T]; \mathcal{H})\right) \cap L^2\left(\Omega; L^2(0, T; \mathcal{V}_0)\right),$$

$$(\tilde{\xi}, \tilde{\xi}_\Gamma) \in L^1\left(\Omega; L^1(0, T; L^1(D) \times L^1(\Gamma))\right),$$

$$j(\tilde{x}) + j^*(\tilde{\xi}) \in L^1(\Omega \times (0, T) \times D), \quad j_\Gamma(\tilde{y}) + j_\Gamma^*(\tilde{\xi}_\Gamma) \in L^1(\Omega \times (0, T) \times \Gamma),$$

$$(\tilde{x}(t), \tilde{y}(t)) + \int_0^t \mathcal{C}_0(\tilde{x}(s), \tilde{y}(s)) ds + \int_0^t \mathcal{P}(\tilde{x}(s), \tilde{y}(s)) ds + \int_0^t (\tilde{\xi}(s), \tilde{\xi}_\Gamma(s)) ds$$

$$= (x_0, y_0) + \int_0^t \mathcal{B}(s) d\mathcal{W}_s \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

Since the problem (3.1) is well-posed for  $\varepsilon = 0$  (in particular, it admits a unique solution), we deduce that  $(\tilde{x}, \tilde{y}) = (x, y)$  and  $(\tilde{\xi}, \tilde{\xi}_\Gamma) = (\xi, \xi_\Gamma)$ .

The next thing that we have to show is that the convergences (2.8)–(2.12) hold. To this aim, note that (2.8) follows from (4.2) and the fact that  $(x_\varepsilon, y_\varepsilon)$  is uniformly bounded in

$L^2(\Omega \times (0, T); \mathcal{H})$ , while (2.12) coincides with (4.6). Moreover, (2.9)–(2.11) follow from (4.3)–(4.5) using the fact that  $\varepsilon'$  is independent of  $\omega$  and Vitali's convergence theorem, via a similar argument to the one performed in Section 3.2 when dealing with the regularity in  $\omega$ .

#### 4.2. The multiplicative noise case

Let us consider now the case of multiplicative noise. As before, let  $(x_\varepsilon, y_\varepsilon, \xi_\varepsilon, \xi_{\Gamma, \varepsilon})$  and  $(x, y, \xi, \xi_\Gamma)$  be the unique solutions to (2.1) with multiplicative noise  $\mathcal{B}$  in the cases  $\varepsilon > 0$  and  $\varepsilon = 0$ , respectively, i.e.

$$d(x_\varepsilon, y_\varepsilon) + \mathcal{A}_\varepsilon(x_\varepsilon, y_\varepsilon) dt + (\xi_\varepsilon, \xi_{\Gamma, \varepsilon}) dt = \mathcal{B}(t, x_\varepsilon, y_\varepsilon) d\mathcal{W}_t, \quad (x_\varepsilon(0), y_\varepsilon(0)) = (x_0, y_0)$$

and

$$d(x, y) + \mathcal{A}_0(x, y) dt + (\xi, \xi_\Gamma) dt = \mathcal{B}(t, x, y) d\mathcal{W}_t, \quad (x(0), y(0)) = (x_0, y_0).$$

Furthermore, note that the linear growth assumption on  $\mathcal{B}$  together with the regularity of  $(x, y)$  implies that  $\mathcal{B}(\cdot, \cdot, x, y) \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(\mathcal{U}, \mathcal{H})))$ . Consequently, for every  $\varepsilon > 0$ , there exists a unique solution  $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon, \tilde{\xi}_\varepsilon, \tilde{\xi}_{\Gamma, \varepsilon})$  to the problem with additive noise  $\mathcal{B}(\cdot, \cdot, x, y)$ , i.e.

$$d(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) + \mathcal{A}_\varepsilon(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) dt + (\tilde{\xi}_\varepsilon, \tilde{\xi}_{\Gamma, \varepsilon}) dt = \mathcal{B}(t, x, y) d\mathcal{W}_t, \quad (\tilde{x}_\varepsilon(0), \tilde{y}_\varepsilon(0)) = (x_0, y_0).$$

Now, the validity of Theorem 2.4 in the case of additive noise (already proved) guarantees that  $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon, \tilde{\xi}_\varepsilon, \tilde{\xi}_{\Gamma, \varepsilon}) \rightarrow (x, y, \xi, \xi_\Gamma)$  in the sense of the convergences (2.8)–(2.12).

Let us now estimate the quantities  $x_\varepsilon - \tilde{x}_\varepsilon$  and  $y_\varepsilon - \tilde{y}_\varepsilon$  by taking the difference of the respective equations. Employing once more Itô's formula yields, for every  $t \in [0, T]$ ,  $\mathbb{P}$ -almost surely,

$$\begin{aligned} & \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(t) \|_{\mathcal{H}}^2 + 2 \int_0^t \| \nabla (x_\varepsilon - \tilde{x}_\varepsilon)(s) \|_H^2 ds + 2\varepsilon \int_0^t \| \nabla_\Gamma (y_\varepsilon - \tilde{y}_\varepsilon)(s) \|_{H_\Gamma}^2 ds \\ & + 2 \int_0^t (\mathcal{P}(x_\varepsilon(s), y_\varepsilon(s)) - \mathcal{P}(\tilde{x}_\varepsilon(s), \tilde{y}_\varepsilon(s)), (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s))_{\mathcal{H}} ds \\ & + 2 \int_0^t \int_D (\xi_\varepsilon - \tilde{\xi}_\varepsilon)(s) (x_\varepsilon - \tilde{x}_\varepsilon)(s) ds + \int_0^t \int_\Gamma (\xi_{\Gamma, \varepsilon} - \tilde{\xi}_{\Gamma, \varepsilon})(s) (y_\varepsilon - \tilde{y}_\varepsilon)(s) ds \\ & = \int_0^t \| \mathcal{B}(s, x_\varepsilon(s), y_\varepsilon(s)) - \mathcal{B}(s, x(s), y(s)) \|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \\ & + 2 \int_0^t (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s) (\mathcal{B}(s, x_\varepsilon(s), y_\varepsilon(s)) - \mathcal{B}(s, x(s), y(s))) d\mathcal{W}_s. \end{aligned}$$

Using the monotonicity of  $\beta$  and  $\beta_\Gamma$ , the Lipschitz-continuity of  $\mathcal{P}$  and  $\mathcal{B}$  we have

$$\begin{aligned} & \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(t) \|_{\mathcal{H}}^2 + 2 \int_0^t \| \nabla (x_\varepsilon - \tilde{x}_\varepsilon)(s) \|_H^2 ds + 2\varepsilon \int_0^t \| \nabla_\Gamma (y_\varepsilon - \tilde{y}_\varepsilon)(s) \|_{H_\Gamma}^2 ds \\ & \leq 2C_{\mathcal{P}} \int_0^t \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s) \|_{\mathcal{H}}^2 ds + C_{\mathcal{B}}^2 \int_0^t \| (x_\varepsilon - x, y_\varepsilon - y)(s) \|_{\mathcal{H}}^2 ds \\ & + 2 \left| \int_0^t (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s) (\mathcal{B}(s, x_\varepsilon(s), y_\varepsilon(s)) - \mathcal{B}(s, x(s), y(s))) d\mathcal{W}_s \right|. \end{aligned}$$

Arguing as at the end of the proof of Lemma 3.2 using the Burkholder–Davis–Gundy and Young inequalities, (see also [24, Lemma 4.1]), we have, for every  $p \in [1, 2)$

$$\begin{aligned} & \mathbb{E} \sup_{r \leq t} \left| \int_0^r (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s) (\mathcal{B}(s, x_\varepsilon(s), y_\varepsilon(s)) - \mathcal{B}(s, x(s), y(s))) d\mathcal{W}_s \right|^{p/2} \\ & \leq \frac{1}{2} \mathbb{E} \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon) \|_{L^\infty(0, t; \mathcal{H})}^p \\ & + \frac{1}{2} \mathbb{E} \left( \int_0^t \| \mathcal{B}(s, x_\varepsilon(s), y_\varepsilon(s)) - \mathcal{B}(s, x(s), y(s)) \|_{\mathcal{L}_2(\mathcal{U}, \mathcal{H})}^2 ds \right)^{p/2}. \end{aligned}$$

Hence, taking power  $p/2$ , supremum in time and expectations in the previous inequality, by the Lipschitz-continuity of  $\mathcal{B}$  we infer that

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon) \|_{L^\infty(0, t; \mathcal{H})}^p + 2 \mathbb{E} \left( \int_0^t \| \nabla (x_\varepsilon - \tilde{x}_\varepsilon)(s) \|_H^2 ds \right)^{p/2} \\ & + 2\varepsilon^{p/2} \mathbb{E} \left( \int_0^t \| \nabla_\Gamma (y_\varepsilon - \tilde{y}_\varepsilon)(s) \|_{H_\Gamma}^2 ds \right)^{p/2} \\ & \leq 2C_{\mathcal{P}} \mathbb{E} \left( \int_0^t \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s) \|_{\mathcal{H}}^2 ds \right)^{p/2} \\ & + \frac{3C_{\mathcal{B}}^p}{2} \mathbb{E} \left( \int_0^t \| (x_\varepsilon - x, y_\varepsilon - y)(s) \|_{\mathcal{H}}^2 ds \right)^{p/2} \end{aligned}$$



$$\begin{aligned} &\leq \left( 2C_{\mathcal{P}} + \frac{3C_{\mathcal{B}}^p}{2^{1-p/2}} \right) \mathbb{E} \left( \int_0^t \| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon)(s) \|_{\mathcal{H}}^2 ds \right)^{p/2} \\ &\quad + \frac{3C_{\mathcal{B}}^p}{2^{1-p/2}} \mathbb{E} \left( \int_0^T \| (\tilde{x}_\varepsilon - x, \tilde{y}_\varepsilon - y)(s) \|_{\mathcal{H}}^2 ds \right)^{p/2}. \end{aligned}$$

The Gronwall lemma implies then

$$\begin{aligned} &\| (x_\varepsilon - \tilde{x}_\varepsilon, y_\varepsilon - \tilde{y}_\varepsilon) \|_{L^p(\Omega; L^\infty(0, T; \mathcal{H}))} + \| \nabla (x_\varepsilon - \tilde{x}_\varepsilon) \|_{L^p(\Omega; L^2(0, T; H))} \\ &\quad + \varepsilon^{1/2} \| \nabla_\Gamma (y_\varepsilon - \tilde{y}_\varepsilon) \|_{L^p(\Omega; L^2(0, T; H_\Gamma))} \\ &\lesssim \| (\tilde{x}_\varepsilon - x, \tilde{y}_\varepsilon - y) \|_{L^p(\Omega; L^2(0, T; \mathcal{H}))}, \end{aligned}$$

where the right-hand side converges to 0 thanks to the already proved convergences of  $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)$ . At this point, by a usual argument comparison argument, we can infer that  $(x_\varepsilon, y_\varepsilon) \rightarrow (x, y)$  in the sense of convergences (2.8)–(2.12). The weak convergence of  $(\xi_\varepsilon, \xi_{\Gamma, \varepsilon})$  to  $(\xi, \xi_\Gamma)$  follows again as in the case of additive noise, employing the de la Vallée-Poussin and Dunford–Pettis theorems.

## Appendix A. An auxiliary result

Let  $\varepsilon \geq 0$  be fixed. We introduce the operator

$$C_\varepsilon : \mathcal{H} \rightarrow \mathcal{H}, \quad C_\varepsilon(u, v) := (-\Delta u, \partial_{\mathbf{n}} u - \varepsilon \Delta_\Gamma v), \quad (u, v) \in D(C_\varepsilon), \quad (\text{A.7})$$

where

$$D(C_\varepsilon) := \begin{cases} \{ (u, v) \in H^2(D) \times H^2(\Gamma) : v = \tau u \} & \text{if } \varepsilon > 0, \\ \{ (u, v) \in H^{3/2}(D) \times H^1(\Gamma) : \Delta u \in L^2(D), v = \tau u \} & \text{if } \varepsilon = 0. \end{cases} \quad (\text{A.8})$$

It is clear  $C_\varepsilon$  is a well-defined linear operator on  $\mathcal{H}$  if  $\varepsilon > 0$ . If  $\varepsilon = 0$ , this is still true since the conditions  $-\Delta u \in H$  and  $u \in H^{3/2}(D)$  imply that  $\partial_{\mathbf{n}} u \in H_\Gamma$  by [21, Thm. 2.27]. Note that  $C_\varepsilon$  is the strong formulation of the linear component of the operator  $\mathcal{A}_\varepsilon$  on  $\mathcal{H}$ .

**Lemma A.1** (Maximal monotonicity). *The operator  $C_\varepsilon$  is maximal monotone on  $\mathcal{H}$  and, consequently, its resolvent  $(I + \delta C_\varepsilon)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$  is a linear contraction<sup>1</sup> for every  $\delta > 0$ .*

**Proof.** It is immediate to see that  $C_\varepsilon$  is monotone. Hence, we only have to check that  $R(I + C_\varepsilon) = \mathcal{H}$ . Let then  $(f, g) \in \mathcal{H}$ : since  $C_\varepsilon$  is coercive on  $\mathcal{V}_\varepsilon$ , by the Lax–Milgram lemma there exists a couple  $(u, v) \in \mathcal{V}_\varepsilon$  such that

<sup>1</sup> Throughout the Appendix, by the term “contraction” we mean a 1-Lipschitz continuous function, i.e. a non-expansive operator.

$$\int_D u\varphi + \int_\Gamma v\psi + \int_D \nabla u \cdot \nabla \varphi + \varepsilon \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \psi = \int_D f\varphi + \int_\Gamma g\psi \quad \forall (\varphi, \psi) \in \mathcal{V}_\varepsilon. \quad (\text{A.9})$$

Taking  $\varphi \in C_c^\infty(D)$  and  $\psi = 0$  in the previous variational formulation, one can see that  $u$  satisfies in  $D$  the following PDE (in the sense of distributions on  $D$ )

$$u - \Delta u = f. \quad (\text{A.10})$$

We recover immediately that also  $-\Delta u \in H$  by difference, and the previous equation holds in  $H$ . Moreover, for any  $\psi \in C^\infty(\Gamma)$ , since the trace operator  $\tau$  has a right-inverse which is linear and continuous from  $H^{k-1/2}(\Gamma)$  to  $H^k(D)$  for every  $k \geq 1$  (see [21, Thm. 2.24]), thanks to the usual Sobolev embeddings results there is  $\varphi \in C^\infty(\overline{D})$  such that  $\psi = \tau\varphi$ : choosing  $(\varphi, \psi)$  in (A.9), integrating by parts on  $D$  and taking (A.10) into account, we see that  $v$  satisfies (in the sense of distributions on the boundary  $\Gamma$ )

$$v + \partial_{\mathbf{n}} u - \varepsilon \Delta_\Gamma v = g. \quad (\text{A.11})$$

Assume now  $\varepsilon > 0$ . By definition of  $\mathcal{V}_\varepsilon$ , we know that  $v = \tau u \in H^1(\Gamma)$ : hence, thanks to [21, Thm. 3.2] we also deduce that  $u \in H^{3/2}(D)$ . Using the result contained in [21, Thm. 2.27] and the facts that  $u \in H^{3/2}(D)$  and  $-\Delta u \in H$ , we deduce that  $\partial_{\mathbf{n}} u \in H_\Gamma$  as well. By comparison in (A.11) we have that  $v - \varepsilon \Delta_\Gamma v \in H_\Gamma$ , so that  $v \in H^2(\Gamma)$  by elliptic regularity on the boundary and (A.11) holds in  $H_\Gamma$ . Finally, since  $\Delta u \in H$  and  $v \in H^2(\Gamma) \hookrightarrow H^{3/2}(\Gamma)$ , thanks again to [21, Thm. 3.2] we deduce that  $u \in H^2(D)$ . Hence,  $(u, v) \in D(\mathcal{C}_\varepsilon)$  and this completes the proof in the case  $\varepsilon > 0$ ; moreover, note that the two results [21, Thm. 2.27 and 3.2] that we have used ensure also that

$$\|(u, v)\|_{H^2(D) \times H^2(\Gamma)} \lesssim \|(f, g)\|_{\mathcal{H}}.$$

Assume  $\varepsilon = 0$ . By definition of  $\mathcal{V}_\varepsilon$ , now we have  $u \in H^1(D)$  and  $v \in H^{1/2}(\Gamma)$ . Moreover, by comparison in (A.11) we have  $\partial_{\mathbf{n}} u \in H_\Gamma$ : thanks to [21, Thm. 3.2] and the facts that  $\Delta u \in H$  and  $\partial_{\mathbf{n}} u \in H_\Gamma$ , we deduce that  $u \in H^{3/2}(D)$ , and consequently also  $v \in H^1(\Gamma)$ . Hence,  $(u, v) \in D(\mathcal{C}_\varepsilon)$  and the proof is complete. As before, owing to the results [21, Thm. 2.27 and 3.2] we also have that

$$\|(u, v)\|_{H^{3/2}(D) \times H^1(\Gamma)} \lesssim \|(f, g)\|_{\mathcal{H}}. \quad \square$$

**Remark A.2.** Note that the first part of the proof of the previous lemma ensures that  $\mathcal{C}_\varepsilon$  can be extended to a linear operator  $\mathcal{C}_\varepsilon : \mathcal{V}_\varepsilon \rightarrow \mathcal{V}_\varepsilon^*$ , given by the left-hand side of (A.9), which is still maximal monotone in the sense of Minty–Browder theory (see [4, Ch. 2]).

**Lemma A.3 (Regularity).** *Let  $\delta > 0$  and  $k \in \mathbb{N}$ . Then,*

$$(I + \delta \mathcal{C}_\varepsilon)^{-1} \in \begin{cases} \mathcal{L}(H^k(D) \times H^k(\Gamma), H^{k+2}(D) \times H^{k+2}(\Gamma)) & \text{if } \varepsilon > 0, \\ \mathcal{L}(H^k(D) \times H^k(\Gamma), H^{k+3/2}(D) \times H^{k+1}(\Gamma)) & \text{if } \varepsilon = 0. \end{cases} \quad (\text{A.12})$$

**Proof.** It is not restrictive to assume that  $\delta = 1$ . In Lemma A.1 we have already proved the case  $k = 0$ : let us only show the case  $k = 1$ , since for a general  $k$  it follows by induction. Let  $(f, g) \in H^1(D) \times H^1(\Gamma)$  and  $(u, v) = (I + \mathcal{C}_\varepsilon)^{-1}(f, g) \in D(\mathcal{C}_\varepsilon)$ .

If  $\varepsilon > 0$ , by (A.10) we have  $u - \Delta u \in H^1(D)$ : since by definition of  $D(\mathcal{C}_\varepsilon)$  we also know that  $v = \tau u \in H^2(\Gamma)$ , it follows from [21, Thm. 3.2] that  $u \in H^{5/2}(D)$ . Consequently,  $\partial_n u \in H^1(\Gamma)$ : hence, by comparison in (A.11) we have  $v - \varepsilon \Delta_\Gamma v \in H^1(\Gamma)$ . By elliptic regularity on the boundary we deduce that  $v \in H^3(\Gamma)$ . Combining this information with the fact that  $u - \Delta u \in H^1(D)$ , again by [21, Thm. 3.2] we also have  $u \in H^3(D)$ . Taking into account [21, Thm. 2.27 and 3.2], it is clear also that

$$\|(u, v)\|_{H^3(D) \times H^3(\Gamma)} \lesssim \|(f, g)\|_{H^1(D) \times H^1(\Gamma)}.$$

If  $\varepsilon = 0$ , since  $v \in H^1(\Gamma)$ , by difference in (A.11) we have  $\partial_n u \in H^1(\Gamma)$ ; this information, together with the fact that  $-\Delta u \in H^1(D)$  by difference in (A.10), implies that  $u \in H^{5/2}(D)$ . Consequently,  $v = \tau u \in H^2(\Gamma)$ . Finally, again by [21, Thm. 2.27 and 3.2], we have that

$$\|(u, v)\|_{H^{5/2}(D) \times H^2(\Gamma)} \lesssim \|(f, g)\|_{H^1(D) \times H^1(\Gamma)}. \quad \square$$

**Lemma A.4** (Extension to  $L^1$ ). *Let  $\delta > 0$ . The resolvent  $(I + \delta \mathcal{C}_\varepsilon)^{-1}$  can be uniquely extended to a linear contraction from  $L^1(D) \times L^1(\Gamma)$  to itself. Moreover, one has that  $(I + \delta \mathcal{C}_\varepsilon)^{-1} \in \mathcal{L}(L^1(D) \times L^1(\Gamma), W^{1,q}(D) \times W^{1-1/q,q}(\Gamma))$  for every  $q \in [1, \frac{N}{N-1})$ .*

**Proof.** Since  $\delta > 0$  is fixed throughout the proof, we do not use notation for the dependence on  $\delta$  of the quantities that we introduce. Given  $(f, g) \in L^1(D) \times L^1(\Gamma)$ , let us consider  $\{(f_n, g_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  such that  $(f_n, g_n) \rightarrow (f, g)$  in  $L^1(D) \times L^1(\Gamma)$  as  $n \rightarrow \infty$ , and  $(u_n, v_n) := (I + \delta \mathcal{C}_\varepsilon)^{-1}(f_n, g_n)$ . Moreover, let  $\{\rho_k\}_{k \in \mathbb{N}}$  be a sequence of smooth Lipschitz-continuous increasing functions on  $\mathbb{R}$  approximating pointwise the maximal monotone graph

$$\text{sign} : \mathbb{R} \rightarrow 2^{\mathbb{R}}, \quad \text{sign}(x) := \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ [-1, 1] & \text{if } x = 0. \end{cases}$$

For example, one can take  $\rho_k(x) = \tanh(kx)$ ,  $x \in \mathbb{R}$ . Now, we consider equation (A.9) with respect to  $n, m \in \mathbb{N}$ , take the difference and test by  $(\rho_k(u_n - u_m), \rho_k(v_n - v_m)) \in \mathcal{V}_\varepsilon$ , obtaining

$$\begin{aligned} & \int_D (u_n - u_m) \rho_k(u_n - u_m) + \int_\Gamma (v_n - v_m) \rho_k(v_n - v_m) \\ & \quad + \delta \int_D \rho'_k(u_n - u_m) |\nabla(u_n - u_m)|^2 + \varepsilon \delta \int_\Gamma \rho'_k(v_n - v_m) |\nabla_\Gamma(v_n - v_m)|^2 \\ & = \int_D (f_n - f_m) \rho_k(u_n - u_m) + \int_\Gamma (g_n - g_m) \rho_k(v_n - v_m). \end{aligned}$$

Using monotonicity and the fact that  $|\rho_k| \leq 1$ , letting  $k \rightarrow \infty$ , by the dominated convergence theorem it is immediate to see that

$$\int_D |u_n - u_m| + \int_\Gamma |v_n - v_m| \leq \int_D |f_n - f_m| + \int_\Gamma |g_n - g_m| :$$

since  $(f_n, g_n) \rightarrow (f, g)$  in  $L^1(D) \times L^1(\Gamma)$ , this implies that  $\{(u_n, v_n)\}_{n \in \mathbb{N}}$  is Cauchy in  $L^1(D) \times L^1(\Gamma)$ . Hence, there is  $(u, v) \in L^1(D) \times L^1(\Gamma)$  (which is independent of the approximating sequence  $\{(f_n, g_n)\}_{n \in \mathbb{N}}$ ) such that

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } L^1(D) \times L^1(\Gamma).$$

This proves that  $(I + \delta \mathcal{C}_\varepsilon)^{-1}$  extends uniquely to a linear contraction on  $L^1(D) \times L^1(\Gamma)$  and the first part of the lemma is proved.

Let us focus on the second part. First of all, we need to prove an auxiliary result, which is a generalization of the classical elliptic regularity theorems by Stampacchia (see [35]). Namely, for every  $p > N$  and  $h_0, \dots, h_N \in L^p(D)$ , by the Lax–Milgram lemma there is a weak solution  $(z, w) \in \mathcal{V}_\varepsilon$  such that

$$\int_D z\varphi + \int_\Gamma w\psi + \delta \int_D \nabla z \cdot \nabla \varphi + \varepsilon \delta \int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma \psi = \int_D h_0 \varphi + \sum_{i=1}^N \int_D h_i \frac{\partial \varphi}{\partial x_i} \quad (\text{A.13})$$

for every  $(\varphi, \psi) \in \mathcal{V}_\varepsilon$ . Let us prove that  $(z, w) \in L^\infty(D) \times L^\infty(\Gamma)$  and that there exists  $C > 0$  such that

$$\|w\|_{L^\infty(\Gamma)} \leq \|z\|_{L^\infty(D)} \leq C \sum_{i=0}^N \|h_i\|_{L^p(D)}.$$

For every  $k \in \mathbb{N}$ , we introduce the Lipschitz function

$$G_k : \mathbb{R} \rightarrow \mathbb{R}, \quad G_k(t) := \begin{cases} t + k & \text{if } t < -k, \\ 0 & \text{if } -k \leq t \leq k, \\ t - k & \text{if } t > k. \end{cases}$$

Testing (A.13) by  $(G_k(z), G_k(w)) \in \mathcal{V}_\varepsilon$ , setting  $A_k := \{|z| \geq k\} \subseteq D$  we get

$$\int_D G_k(z)z + \int_\Gamma G_k(w)w + \delta \int_{A_k} |\nabla G_k(z)|^2 + \varepsilon \delta \int_{|w| \geq k} |\nabla_\Gamma w|^2 = \int_{A_k} h_0 G_k(z) + \sum_{i=1}^N \int_{A_k} h_i \frac{\partial z}{\partial x_i}.$$

Using the Young inequality, the fact that  $|G_k(z)| \leq |z|$  and the monotonicity of  $G_k$ , we deduce that

$$\|G_k(z)\|_H^2 + \frac{\delta}{2} \int_{A_k} |\nabla G_k(z)|^2 \leq \int_{A_k} h_0 G_k(z) + \frac{1}{2} \sum_{i=1}^N \int_{A_k} |h_i|^2,$$

which can be rewritten as

$$\|G_k(z)\|_{H^1(D)}^2 \leq \max \left\{ 1, \frac{2}{\delta} \right\} \int_{A_k} h_0 G_k(z) + \max \left\{ \frac{1}{2}, \frac{1}{\delta} \right\} \sum_{i=1}^N \int_{A_k} |h_i|^2.$$

Let us consider first the case  $N \geq 3$ . If we set  $2^* := \frac{2N}{N-2}$  and  $2_* := \frac{2^*}{2^*-1} = \frac{2N}{N+2}$ , the Sobolev embedding  $H^1(D) \hookrightarrow L^{2^*}(D)$  on the left-hand side and the Hölder inequality on the right-hand side yield

$$\left( \int_{A_k} |G_k(z)|^{2^*} \right)^{\frac{2}{2^*}} \leq C \left( \int_{A_k} |h_0|^{2_*} \right)^{\frac{1}{2_*}} \left( \int_{A_k} |G_k(z)|^{2^*} \right)^{\frac{1}{2^*}} + C \sum_{i=1}^N \int_{A_k} |h_i|^2$$

for a positive constant  $C$ , from which, thanks to the Young inequality we have

$$\left( \int_{A_k} |G_k(z)|^{2^*} \right)^{\frac{2}{2^*}} \leq C \left( \int_{A_k} |h_0|^{2_*} \right)^{\frac{2}{2_*}} + 2C \sum_{i=1}^N \int_{A_k} |h_i|^2.$$

Using the fact that  $h_i \in L^p(D)$  for  $i = 0, \dots, N$ , that  $p > 2_*$  (since  $p > N$ ) we deduce

$$\left( \int_{A_k} |G_k(z)|^{2^*} \right)^{\frac{2}{2^*}} \leq 2C \left[ \|h_0\|_{L^p(D)}^2 |A_k|^{\frac{2}{2_*} - \frac{2}{p}} + |A_k|^{1 - \frac{2}{p}} \sum_{i=1}^N \|h_i\|_{L^p(D)}^2 \right].$$

Now, for every  $h > k$  we have  $A_h \subseteq A_k$  and  $G_k(u) \geq h - k$  on  $A_h$  so that

$$(h - k)^2 |A_h|^{\frac{2}{2^*}} \leq 2C \sum_{i=0}^N \|h_i\|_{L^p(D)}^2 \left( |A_k|^{\frac{2}{2_*} - \frac{2}{p}} + |A_k|^{1 - \frac{2}{p}} \right).$$

Renominating the constant  $C$ , since it is not restrictive to assume that  $|A_k| < 1$ , it follows

$$|A_k| \leq C \left( \sum_{i=0}^N \|h_i\|_{L^p}^2 \right)^{\frac{2^*}{2}} \frac{|A_k|^\alpha}{(h - k)^{2^*}}, \quad \alpha := \frac{2^*}{2} \min \left\{ \frac{2}{2_*} - \frac{2}{p}, 1 - \frac{2}{p} \right\}.$$

Now, using the fact that  $p > N$ , it is a standard matter to see that

$$\alpha = \frac{2^*}{2} \min \left\{ \frac{N+2}{N} - \frac{2}{p}, 1 - \frac{2}{p} \right\} = \frac{N}{N-2} \left( 1 - \frac{2}{p} \right) = \frac{1-2/p}{1-2/N} > 1.$$

If  $N = 2$ , then we know that  $H^1(D) \hookrightarrow L^r(D)$  for all  $r \in [1, +\infty)$ : using this fact, we repeat the same argument replacing  $2_*$  and  $2^*$  by an arbitrary  $q \in (1, 2)$  and its conjugate exponent  $q' = \frac{q}{q-1}$ , respectively. With such a choice, the same computations yield

$$\alpha = \frac{q'}{2} \min \left\{ \frac{2}{q} - \frac{2}{p}, 1 - \frac{2}{p} \right\} = \frac{q}{2(q-1)} \left( 1 - \frac{2}{p} \right) = \frac{q(p-2)}{2p(q-1)}.$$

It is easily seen that  $\alpha > 1$  if and only if  $q < \frac{2p}{p+2}$ : since the fact that  $p > 2$  implies that  $\frac{2p}{p+2} \in (1, 2)$ , we can choose  $q \in (1, \frac{2p}{p+2})$ , getting  $\alpha > 1$  also in the case  $N = 2$ , as desired. By [35, Lem. 4.1], we can conclude that  $z \in L^\infty(D)$  and  $\|z\|_{L^\infty(D)} \leq C \sum_{i=0}^N \|h_i\|_{L^p(D)}$ , suitably renominating the positive constant  $C$ . Moreover, since  $w = \tau z$ , we also have that  $w \in L^\infty(\Gamma)$  and  $\|w\|_{L^\infty(\Gamma)} \leq \|z\|_{L^\infty(D)}$ .

We are now ready to complete the proof of the lemma. Testing (A.13) by  $(u_n, v_n)$ , recalling the definition of  $(u_n, v_n)$  we have

$$\int_D h_0 u_n + \sum_{i=1}^N \int_D h_i \frac{\partial u_n}{\partial x_i} = \int_D f_n z + \int_\Gamma g_n w \leq C \sum_{i=0}^N \|h_i\|_{L^p(D)} (\|f_n\|_{L^1(D)} + \|g_n\|_{L^1(\Gamma)});$$

taking into account that  $h_0, \dots, h_N \in L^p(D)$  are arbitrary, we deduce that

$$\left\| \left( u_n, \frac{\partial u_n}{\partial x_1}, \dots, \frac{\partial u_n}{\partial x_N} \right) \right\|_{L^q(D)^{N+1}} \leq C \|(f_n, g_n)\|_{L^1(D) \times L^1(\Gamma)},$$

where  $q := \frac{p}{p-1}$  is the conjugate exponent of  $p$ . Since  $(f_n, g_n) \rightarrow (f, g)$  in  $L^1(D) \times L^1(\Gamma)$ , recalling that the operator  $C_\varepsilon$  is linear and  $p > N$ , we have that  $u_n \rightarrow u$  in  $W^{1,q}(D)$  for every  $q \in [1, \frac{N}{N-1})$ , and consequently  $v_n \rightarrow v$  in  $W^{1-1/q,q}(\Gamma)$ . This ensures that  $v = \tau u$ ; moreover, letting  $n \rightarrow \infty$  we have  $\|(u, v)\|_{W^{1,q}(D) \times W^{1-1/q,q}(\Gamma)} \leq C \|(f, g)\|_{L^1(D) \times L^1(\Gamma)}$ , from which the thesis follows.  $\square$

**Lemma A.5** (Extension to  $L^q$ ,  $q > 1$ ). Let  $\delta > 0$  and  $q \in [1, \frac{N}{N-1})$ . Then the resolvent  $(I + \delta C_\varepsilon)^{-1}$  can be uniquely extended to a linear contraction from  $L^q(D) \times L^q(\Gamma)$  to itself. Moreover, for every  $k \in \mathbb{N}$ , one has that

$$(I + \delta C_\varepsilon)^{-1} \in \begin{cases} \mathcal{L}(W^{k,q}(D) \times W^{k,q}(\Gamma), W^{k+2,q}(D) \times W^{k+2-1/q,q}(\Gamma)) & \text{if } \varepsilon > 0, \\ \mathcal{L}(W^{k,q}(D) \times W^{k,q}(\Gamma), W^{k+1,q}(D) \times W^{k+1-1/q,q}(\Gamma)) & \text{if } \varepsilon = 0. \end{cases}$$

**Proof.** The fact that  $(I + \delta C_\varepsilon)^{-1}$  can be extended to a contraction on  $L^q(D) \times L^q(\Gamma)$  can be showed in exactly the same way as in the proof of Lemma A.4: the only difference is the choice of  $\{\rho_k\}_{k \in \mathbb{N}}$ . Here, one should take  $\rho_k(t) : \mathbb{R} \rightarrow \mathbb{R}$  smooth, increasing, Lipschitz continuous such that  $\rho_k(0) = 0$  and  $\rho_k(t) = |t|^{q-2}t$  if  $|t| \geq \frac{1}{k}$ .

Let us focus on the regularity result. We only show the case  $k = 0$ , since one can easily generalize by induction to any  $k \in \mathbb{N}$ . Let then  $(f, g) \in L^q(D) \times L^q(\Gamma)$  and let us consider  $(u, v) := (I + \delta C_\varepsilon)^{-1}(f, g)$ . By Lemma A.4 we have that  $u \in W^{1,q}(D)$ ,  $v \in W^{1-1/q,q}(\Gamma)$ ,  $v = \tau u$  and  $u - \delta \Delta u = f$  in the sense of distributions on  $D$ . Hence, owing to [21, Thm. 2.27] we deduce that  $\partial_n u \in W^{-1/q,q}(\Gamma)$ , so that we can write  $v + \partial_n u - \delta \Delta_\Gamma v = g$  in the sense of distributions on  $\Gamma$ . If  $\varepsilon > 0$ , by elliptic regularity on the boundary we deduce that  $v \in W^{2-1/q,q}(\Gamma)$ : consequently, thanks to [21, Thm. 3.2], we infer that  $u \in W^{2,q}(D)$ . If  $\varepsilon = 0$ , we have by difference that  $\partial_n u \in L^q(\Gamma) \hookrightarrow W^{-1/2q,q}(\Gamma)$ : hence, the result [21, Thm. 3.2] ensures that  $u \in W^{1+1/2q,q}(D) \hookrightarrow W^{1,q}(D)$ , and consequently  $v \in W^{1-1/q,q}(\Gamma)$ .  $\square$

**Corollary A.6** (Ultracontractivity). *There exists  $m \in \mathbb{N}$  such that, for every  $\delta > 0$ ,*

$$(I + \delta \mathcal{C}_\varepsilon)^{-m} \in \mathcal{L} \left( L^1(D) \times L^1(\Gamma), L^\infty(D) \times L^\infty(\Gamma) \right).$$

**Proof.** It easily follows from Lemmas A.4–A.5 and the Sobolev embeddings theorems.  $\square$

**Lemma A.7** (Asymptotics as  $\delta \searrow 0$ ). *Let  $\varepsilon \geq 0$  and  $(u_\delta, v_\delta) := (I + \delta \mathcal{C}_\varepsilon)^{-1}(f, g)$  for any  $(f, g)$  for which it makes sense. Then, as  $\delta \searrow 0$ , we have*

$$\begin{aligned} (u_\delta, v_\delta) &\rightarrow (f, g) \quad \text{in } L^1(D) \times L^1(\Gamma) && \text{if } (f, g) \in L^1(D) \times L^1(\Gamma), \\ (u_\delta, v_\delta) &\rightarrow (f, g) \quad \text{in } \mathcal{H} && \text{if } (f, g) \in \mathcal{H}, \\ (u_\delta, v_\delta) &\rightarrow (f, g) \quad \text{in } \mathcal{V}_\varepsilon && \text{if } (f, g) \in \mathcal{V}_\varepsilon. \end{aligned}$$

**Proof.** We start with the case  $(f, g) \in \mathcal{H}$ : testing (A.9) by  $(u_\delta, v_\delta)$  and using the Young inequality we easily deduce that

$$\frac{1}{2} \|u_\delta\|_H^2 + \frac{1}{2} \|v_\delta\|_{H_\Gamma}^2 + \delta \|\nabla u_\delta\|_H^2 + \varepsilon \delta \|\nabla_\Gamma v_\delta\|_{H_\Gamma}^2 \leq \frac{1}{2} \|f\|_H^2 + \frac{1}{2} \|g\|_{H_\Gamma}^2. \quad (\text{A.14})$$

It follows (for a subsequence, which we still denote by  $\delta$ ) that

$$(u_\delta, v_\delta) \rightharpoonup (u, v) \quad \text{in } \mathcal{H}, \quad \delta(u_\delta, v_\delta) \rightarrow 0 \quad \text{in } \mathcal{V}_\varepsilon,$$

where by a standard density argument  $(u, v) = (f, g)$ . Moreover, we also have that

$$\limsup_{\delta \searrow 0} \|(u_\delta, v_\delta)\|_{\mathcal{H}} \leq \|(f, g)\|_{\mathcal{H}},$$

which implies that  $(u_\delta, v_\delta) \rightarrow (f, g)$  in  $\mathcal{H}$  for the original sequence.

If  $(f, g) \in L^1(D) \times L^1(\Gamma)$ , we introduce  $\{(f_n, g_n)\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  such that  $(f_n, g_n) \rightarrow (f, g)$  in  $L^1(D) \times L^1(\Gamma)$  as  $n \rightarrow \infty$ : let  $(u_{n,\delta}, v_{n,\delta}) := (I + \delta \mathcal{C}_\varepsilon)^{-1}(f_n, g_n)$ . Using the fact that  $(I + \delta \mathcal{C}_\varepsilon)^{-1}$  is a contraction on  $L^1(D) \times L^1(\Gamma)$  (see Lemma A.4), we have

$$\begin{aligned} \|(u_\delta, v_\delta) - (f, g)\|_{L^1(D) \times L^1(\Gamma)} &\leq \|(u_\delta, v_\delta) - (u_{n,\delta}, v_{n,\delta})\|_{L^1(D) \times L^1(\Gamma)} \\ &\quad + \|(u_{n,\delta}, v_{n,\delta}) - (f_n, g_n)\|_{L^1(D) \times L^1(\Gamma)} + \|(f_n, g_n) - (f, g)\|_{L^1(D) \times L^1(\Gamma)} \\ &\leq 2 \|(f, g) - (f_n, g_n)\|_{L^1(D) \times L^1(\Gamma)} + C \|(u_{n,\delta}, v_{n,\delta}) - (f_n, g_n)\|_{\mathcal{H}} \end{aligned}$$

for a positive constant  $C$  independent of  $n$  and  $\delta$ . Now, for any  $\eta > 0$ , there is  $n \in \mathbb{N}$  such that the first term on the right-hand side of the previous expression is controlled by  $\eta$ : for such an  $n$ , thanks to what we have already proved, there is  $\delta$  such that the second term is less or equal than  $\eta$ . Hence, the right-hand side can be made smaller than  $2\eta$  and the claim is proved.

Finally, let  $(f, g) \in \mathcal{V}_\varepsilon$ : for what we have already proved, we know that  $(u_\delta, v_\delta) \rightarrow (f, g)$  in  $\mathcal{H}$  as  $\delta \searrow 0$ . Moreover, we have

$$(u_\delta, v_\delta) + \delta \mathcal{C}_\varepsilon(u_\delta, v_\delta) = (f, g) \quad \text{in } \mathcal{H} :$$

taking the scalar product in  $\mathcal{H}$  with  $\mathcal{C}_\varepsilon(u_\delta, v_\delta)$  in the previous expression, using the fact that  $(f, g) \in \mathcal{V}_\varepsilon$  and integrating by parts we get

$$\int_D |\nabla u_\delta|^2 + \varepsilon \int_\Gamma |\nabla_\Gamma v_\delta|^2 + \delta \|\mathcal{C}_\varepsilon(u_\delta, v_\delta)\|_{\mathcal{H}}^2 = \int_D \nabla f \cdot \nabla u_\delta + \varepsilon \int_\Gamma \nabla_\Gamma g \cdot \nabla_\Gamma v_\delta.$$

The Young inequality yields then

$$\|\nabla u_\delta\|_H^2 + \varepsilon \|\nabla_\Gamma v_\delta\|_{H_\Gamma}^2 + 2\delta \|\mathcal{C}_\varepsilon(u_\delta, v_\delta)\|_{\mathcal{H}}^2 \leq \|\nabla f\|_H^2 + \varepsilon \|\nabla_\Gamma g\|_{H_\Gamma}^2,$$

which together with (A.14) implies that

$$|(u_\delta, v_\delta)|_{\mathcal{V}_\varepsilon}^2 \leq |(f, g)|_{\mathcal{V}_\varepsilon}^2.$$

We deduce that

$$(u_\delta, v_\delta) \rightharpoonup (f, g) \quad \text{in } \mathcal{V}_\varepsilon, \quad \limsup_{\delta \searrow 0} |(u_\delta, v_\delta)|_{\mathcal{V}_\varepsilon} \leq |(f, g)|_{\mathcal{V}_\varepsilon},$$

from which  $(u_\delta, v_\delta) \rightarrow (f, g)$  in  $\mathcal{V}_\varepsilon$  as well.  $\square$

**Lemma A.8** (*Maximum principle*). *Let  $c_1, c_2 > 0$  and  $(f, g) \in L^1(D) \times L^1(\Gamma)$  with  $f \leq c_1$  and  $g \leq c_2$  almost everywhere on  $D$  and  $\Gamma$ , respectively; if  $(u, v) := (I + \delta \mathcal{C}_\varepsilon)^{-1}(f, g)$  then*

$$u \leq \max\{c_1, c_2\} \quad \text{a.e. on } D, \quad v \leq \max\{c_1, c_2\} \quad \text{a.e. on } \Gamma.$$

**Proof.** Setting  $c := \max\{c_1, c_2\}$ , we introduce the Lipschitz function  $\rho(t) := (t - c)_+$ ,  $t \in \mathbb{R}$ : testing the corresponding variational formulation (A.9) by  $(\rho(u), \rho(v)) \in \mathcal{V}_\varepsilon$  we have

$$\int_D \rho(u)u + \int_\Gamma \rho(v)v + \delta \int_D \rho'(u)|\nabla u|^2 + \varepsilon \delta \int_\Gamma \rho'(v)|\nabla_\Gamma v|^2 = \int_D f\rho(u) + \int_\Gamma g\rho(v).$$

Using the definition of  $\rho$ , monotonicity and the hypotheses on  $f$  and  $g$  we infer that

$$\int_D |\rho(u)|^2 + \int_\Gamma |\rho(v)|^2 \leq \int_D (f - c)\rho(u) + \int_\Gamma (g - c)\rho(v) \leq 0,$$

from which  $\rho(u) = 0$  and  $\rho(v) = 0$ . Hence,  $u \leq c$  and  $v \leq c$  almost everywhere.  $\square$

We introduce the projections on the first and second component, respectively, as

$$p_1 : L^1(D) \times L^1(\Gamma) \rightarrow L^1(D), \quad p_2 : L^1(D) \times L^1(\Gamma) \rightarrow L^1(\Gamma).$$



Let now  $\delta > 0$ : we set

$$\begin{aligned} J_{\varepsilon,\delta}^1 &:= p_1 \circ (I + \delta \mathcal{C}_\varepsilon)^{-1} : L^1(D) \times L^1(\Gamma) \rightarrow L^1(D), \\ J_{\varepsilon,\delta}^2 &:= p_2 \circ (I + \delta \mathcal{C}_\varepsilon)^{-1} : L^1(D) \times L^1(\Gamma) \rightarrow L^1(\Gamma). \end{aligned}$$

Owing to Lemma A.4, it is well-clear that  $J_{\varepsilon,\delta}^i$  is a linear continuous operator for  $i = 1, 2$  and that for every  $(f, g) \in L^1(D) \times L^1(\Gamma)$  by linearity we have

$$(I + \delta \mathcal{C}_\varepsilon)^{-1}(f, g) = \left( J_{\varepsilon,\delta}^1(f, g), J_{\varepsilon,\delta}^2(f, g) \right) = \left( J_{\varepsilon,\delta}^1(f, 0) + J_{\varepsilon,\delta}^1(0, g), J_{\varepsilon,\delta}^2(f, 0) + J_{\varepsilon,\delta}^2(0, g) \right).$$

**Lemma A.9** (Convexity inequality). *Let  $(f, g) \in L^1(D) \times L^1(\Gamma)$  and  $\Phi, \Psi : \mathbb{R} \rightarrow [0, +\infty)$  two proper convex and lower semicontinuous functions with  $(\Phi(f), \Psi(g)) \in L^1(D) \times L^1(\Gamma)$ . Then, for every  $\delta > 0$  we have that*

$$\begin{aligned} \Phi\left(J_{\varepsilon,\delta}^1(f, 0)\right) + \Psi\left(J_{\varepsilon,\delta}^1(0, g)\right) &\leq J_{\varepsilon,\delta}^1(\Phi(f), \Psi(g)) \quad \text{a.e. in } D, \\ \Phi\left(J_{\varepsilon,\delta}^2(f, 0)\right) + \Psi\left(J_{\varepsilon,\delta}^2(0, g)\right) &\leq J_{\varepsilon,\delta}^2(\Phi(f), \Psi(g)) \quad \text{a.e. in } \Gamma. \end{aligned}$$

**Proof.** We introduce the operators

$$\begin{aligned} L_{\varepsilon,\delta}^1 : L^1(D) &\rightarrow L^1(D), & L_{\varepsilon,\delta}^1(f) &:= J_{\varepsilon,\delta}^1(f, 0), & f &\in L^1(D), \\ G_{\varepsilon,\delta}^1 : L^1(\Gamma) &\rightarrow L^1(D), & G_{\varepsilon,\delta}^1(g) &:= J_{\varepsilon,\delta}^1(0, g), & g &\in L^1(\Gamma). \end{aligned}$$

Then, by Lemma A.4 it is a standard matter to see that  $L_{\varepsilon,\delta}^1$  and  $G_{\varepsilon,\delta}^1$  are linear contractions. Moreover, Lemma A.8 ensures that they are sub-Markovian operators in the sense of [20, Def. 3.1]: hence, the generalized Jensen inequality contained in [20, Thm. 3.4] implies that a.e. on  $D$

$$\Phi\left(J_{\varepsilon,\delta}^1(f, 0)\right) \leq J_{\varepsilon,\delta}^1(\Phi(f), 0), \quad \Psi\left(J_{\varepsilon,\delta}^1(0, g)\right) \leq J_{\varepsilon,\delta}^1(0, \Psi(g)).$$

The first thesis follows summing the two inequalities, while the second can be easily proved with the other (obvious) choice of  $L_{\varepsilon,\delta}^2$  and  $G_{\varepsilon,\delta}^2$ .  $\square$

**Corollary A.10.** *For every  $(f, g) \in \mathcal{V}_\varepsilon$  and for every  $(h, \ell) \in L^1(D) \times L^1(\Gamma)$  such that  $j(f) + j^*(h) \in L^1(D)$  and  $j_\Gamma(g) + j_\Gamma^*(\ell) \in L^1(\Gamma)$ , the families*

$$\left\{ J_{\varepsilon,\delta}^1(f, g) J_{\varepsilon,\delta}^1(h, \ell) \right\}_{\delta>0} \quad \text{and} \quad \left\{ J_{\varepsilon,\delta}^2(f, g) J_{\varepsilon,\delta}^2(h, \ell) \right\}_{\delta>0}$$

*are uniformly integrable on  $D$  and  $\Gamma$ , respectively.*

**Proof.** Using linearity, Young's inequality, the symmetry of  $j$ ,  $j_\Gamma$  and Lemma A.9 we have

$$\begin{aligned}
 & \pm J_{\varepsilon,\delta}^1(f, g) J_{\varepsilon,\delta}^1(h, \ell) \leq \left( \pm J_{\varepsilon,\delta}^1(f, 0) \pm J_{\varepsilon,\delta}^1(0, g) \right) \left( J_{\varepsilon,\delta}^1(h, 0) + J_{\varepsilon,\delta}^1(0, \ell) \right) \\
 & = \pm J_{\varepsilon,\delta}^1(f, 0) J_{\varepsilon,\delta}^1(h, 0) \pm J_{\varepsilon,\delta}^1(0, g) J_{\varepsilon,\delta}^1(0, \ell) \pm J_{\varepsilon,\delta}^1(0, g) J_{\varepsilon,\delta}^1(h, 0) \pm J_{\varepsilon,\delta}^1(f, 0) J_{\varepsilon,\delta}^1(0, \ell) \\
 & \leq j \left( \pm J_{\varepsilon,\delta}^1(f, 0) \right) + j^* \left( J_{\varepsilon,\delta}^1(h, 0) \right) + j_\Gamma \left( \pm J_{\varepsilon,\delta}^1(0, g) \right) + j_\Gamma^* \left( J_{\varepsilon,\delta}^1(0, \ell) \right) \\
 & \quad + j^* \left( J_{\varepsilon,\delta}^1(h, 0) \right) + j_\Gamma^* \left( J_{\varepsilon,\delta}^1(0, \ell) \right) + j \left( \pm J_{\varepsilon,\delta}^1(0, g) \right) + j_\Gamma \left( \pm J_{\varepsilon,\delta}^1(f, 0) \right) \\
 & \lesssim 1 + j \left( J_{\varepsilon,\delta}^1(f, 0) \right) + j^* \left( J_{\varepsilon,\delta}^1(h, 0) \right) + j_\Gamma \left( J_{\varepsilon,\delta}^1(0, g) \right) + j_\Gamma^* \left( J_{\varepsilon,\delta}^1(0, \ell) \right) \\
 & \quad + j^* \left( J_{\varepsilon,\delta}^1(h, 0) \right) + j_\Gamma^* \left( J_{\varepsilon,\delta}^1(0, \ell) \right) + j \left( J_{\varepsilon,\delta}^1(0, g) \right) + j_\Gamma \left( J_{\varepsilon,\delta}^1(f, 0) \right) \\
 & \leq 1 + J_{\varepsilon,\delta}^1(j(f), j_\Gamma(g)) + 2J_{\varepsilon,\delta}^1(j^*(h), j_\Gamma^*(\ell)) + j \left( J_{\varepsilon,\delta}^1(0, g) \right) + j_\Gamma \left( J_{\varepsilon,\delta}^1(f, 0) \right).
 \end{aligned}$$

Now, since  $(j(f), j_\Gamma(g)), (j^*(h), j_\Gamma^*(\ell)) \in L^1(D) \times L^1(\Gamma)$ , by Lemma A.7 the sum of the first three terms on the right-hand side converge in  $L^1(D)$  to  $1 + j(f) + 2j^*(h)$  as  $\delta \searrow 0$ . Hence, the first thesis follows if we are able to prove that

$$j \left( J_{\varepsilon,\delta}^1(0, g) \right) + j_\Gamma \left( J_{\varepsilon,\delta}^1(f, 0) \right)$$

is uniformly integrable on  $D$ . To this aim, we need to distinguish whether (H1), (H2) or (H3 $_{\varepsilon>0}$ )–(H3 $_{\varepsilon=0}$ ) is in order. Firstly, if we assume hypothesis (H1), the fact that  $(j(f), j_\Gamma(g)) \in L^1(D) \times L^1(\Gamma)$  implies that also  $(j_\Gamma(f), j(g)) \in L^1(D) \times L^1(\Gamma)$ : consequently, by Lemma A.9, the two terms are bounded by  $J_{\varepsilon,\delta}^1(j_\Gamma(f), j(g))$ , which converges in  $L^1(D)$  thanks to Lemma A.7. Secondly, let us assume (H2). The facts that  $j_\Gamma$  controls  $j$  and  $j_\Gamma(g) \in L^1(\Gamma)$  imply that  $j(g) \in L^1(\Gamma)$ , so that by Lemma A.9 the first term is handled by  $J_{\varepsilon,\delta}^1(0, j(g))$ , which converges in  $L^1(\Gamma)$  by Lemma A.7. Moreover,  $f \in H^1(D)$  and the Sobolev embeddings ensure that

$$\begin{cases} H^1(D) \hookrightarrow L^p(D) & \forall p \in [1, +\infty) & \text{if } N = 2, \\ H^1(D) \hookrightarrow L^{\frac{2N}{N-2}}(D) & & \text{if } N > 2 : \end{cases}$$

hence, hypothesis (H2) implies that  $j_\Gamma(f) \in L^1(D)$ , so that the second term is bounded by  $J_{\varepsilon,\delta}^1(j_\Gamma(f), 0)$ , which converges in  $L^1(D)$  by Lemma A.7. Finally, let us assume (H3). Since  $j$  controls  $j_\Gamma$  and  $j(f) \in L^1(D)$ , we have also  $j_\Gamma(f) \in L^1(D)$ : hence, by Lemma A.9 the first term is handled by  $J_{\varepsilon,\delta}^1(j_\Gamma(f), 0)$ , which converges in  $L^1(D)$  by Lemma A.7. Let us focus on the first term  $j \left( J_{\varepsilon,\delta}^1(0, g) \right)$ . If  $\varepsilon > 0$ , we have  $g \in H^1(\Gamma)$  and by the Sobolev embeddings (since  $\Gamma$  has dimension  $N - 1$ )

$$\begin{cases} H^1(\Gamma) \hookrightarrow L^\infty(\Gamma) & & \text{if } N = 2, \\ H^1(\Gamma) \hookrightarrow L^p(\Gamma) & \forall p \in [1, +\infty) & \text{if } N = 3, \\ H^1(\Gamma) \hookrightarrow L^{\frac{2(N-1)}{N-3}}(\Gamma) & & \text{if } N > 3. \end{cases}$$

Hence, hypothesis  $(H3_{\varepsilon>0})$  ensures that  $j(g) \in L^1(\Gamma)$ , so that by Lemma A.9 we have  $j\left(J_{\varepsilon,\delta}^1(0, g)\right) \leq J_{\varepsilon,\delta}^1(0, j(g))$ , which converges in  $L^1(D)$  by Lemma A.7. Similarly, if  $\varepsilon = 0$  then  $g \in H^{1/2}(\Gamma)$  and by the Sobolev embeddings we have

$$\begin{cases} H^{1/2}(\Gamma) \hookrightarrow L^p(\Gamma) & \forall p \in [1, +\infty) & \text{if } N = 2, \\ H^{1/2}(\Gamma) \hookrightarrow L^{\frac{2(N-1)}{N-2}}(\Gamma) & & \text{if } N > 2. \end{cases}$$

Consequently,  $(H3_{\varepsilon=0})$  ensures again that  $j(g) \in L^1(\Gamma)$ , and we can conclude as in the case  $\varepsilon > 0$ .

We have proved that  $\pm J_{\varepsilon,\delta}^1(f, g)J_{\varepsilon,\delta}^1(h, \ell)$ , hence also  $|J_{\varepsilon,\delta}^1(f, g)J_{\varepsilon,\delta}^1(h, \ell)|$ , is bounded by a family which converges in  $L^1(D)$  as  $\delta \searrow 0$ , from which the uniform integrability follows. The argument for the family  $\left\{J_{\varepsilon,\delta}^2(f, g)J_{\varepsilon,\delta}^2(h, \ell)\right\}_{\delta>0}$  is exactly the same, and this completes the proof.  $\square$

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