Analysis of a chemotaxis model with indirect signal absorption

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Abstract

We consider the chemotaxis model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - vw, \\ w_t = -\delta w + u \end{cases}$$

in smooth, bounded domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, where $\delta > 0$ is a given parameter.

If either $n \leq 2$ or $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{3n}$ we show the existence of a unique global classical solution (u, v, w) and convergence of $(u(\cdot, t), v(\cdot, t), w(\cdot, t))$ towards a spatially constant equilibrium, as $t \to \infty$.

The proof of global existence for the case $n \leq 2$ relies on a bootstrap procedure. As a starting point we derive a functional inequality for a functional being sublinear in u, which appears to be novel in this context.
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1. Introduction

The model Organisms such as cells or bacteria may partially direct their movement towards an external chemical signal. This process is known as chemotaxis and corresponding mathematical

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models have been introduced by Keller and Segel [7] in the 1970s. The most prototypical system is

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v), \\
v_t = \Delta v - v + u,
\end{cases}
\tag{1.1}
$$

wherein u and v denote the cell/bacteria density and the concentration of the chemical signal, respectively. Its most striking feature is the possibility of chemotactic collapse; that is, the existence of solutions in space-dimension two [5,17] and higher [27] blowing up in finite time. In the past few decades mathematicians have analyzed several chemotaxis models; for a broader introduction we refer to the survey [2].

However, even simpler organisms may orient their movement towards a nutrient which is consumed rather than produced, leading to the model

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v), \\
v_t = \Delta v - uv.
\end{cases}
\tag{1.2}
$$

In space-dimensions one and two for any sufficiently smooth initial data classical solutions to (1.2) exist globally and converge to steady states [30], while in space-dimension three at least weak solutions have been constructed which become eventually smooth [20].

For higher space-dimensions n globality of classical solutions has been shown for sufficiently small values of $\|v_0\|_{L^\infty(\Omega)}$. Tao [19] proved that whenever the corresponding initial data are sufficiently smooth and satisfy $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{6(n+1)}$, then there exists a global classical solution of (1.2). In [1] this condition has then been improved; it is sufficient to require $\|v_0\|_{L^\infty(\Omega)} < \frac{\pi}{\sqrt{2(n+1)}}$.

In addition, chemotaxis-consumptions models have been embedded into more complex frameworks. For instance, coupled chemotaxis-fluid systems [14,26,29,28], systems with nonlinear diffusion and/or nonlinear chemotactic sensitivity [3,9,13,31] or systems with zeroth order terms accounting for logistic growth [10] or competition between species [23] have been analyzed.

However, models accounting for indirect consumption effects have apparently not been treated in mathematical literature yet. This stands in contrast to the case of signal production, where indirect effects have been studied for example in [4,16,22].

In the present work, we analyze a prototypical chemotaxis system with indirect consumption; that is, we study

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla v), & \text{in } \Omega \times (0, T), \\
v_t = \Delta v - vw, & \text{in } \Omega \times (0, T), \\
w_t = -\delta w + u, & \text{in } \Omega \times (0, T), \\
\partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega \times (0, T), \\
u(\cdot, 0) = u_0, v(\cdot, 0) = v_0, w(\cdot, 0) = w_0, & \text{in } \Omega
\end{cases}
\tag{P}
$$

for $T \in (0, \infty]$, a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, a parameter $\delta > 0$ and given initial data u_0, v_0, w_0.

Main ideas and results I: Global existence We start by stating a local existence result in Lemma 2.1 which already gives a criterion for global existence. In the following we improve the condition, it suffices to show an L^p bound for u for sufficiently high p (cf. Proposition 3.4). We will then proceed to gain such bounds.

At first glance, one might suspect that chemotaxis-consumption models such as (1.2) or (P) are easier to handle than chemotaxis-production models such as (1.1). After all, the comparison principle rapidly warrants that $0 \leq v \leq \|v_0\|_{L^\infty(\Omega)}$ (cf. Lemma 2.2 below). While indeed helpful, an L^∞-bound for v does not immediately solve all problems, since such a bound does not directly imply any bounds of ∇v, the term appearing in the first equation of (1.2) and (P). In addition, an important tool for analyzing (1.1) and variants thereof is to prove a certain functional inequality which simply does not seem to be available for chemotaxis-consumption models.

In many cases, for instance in [10,20,21], the authors utilize the functional

$$\int_\Omega u \log u + 2 \int_\Omega |\nabla \sqrt{v}|^2 \tag{1.3}$$

to handle problems similar to (1.2). The "worst" term appearing upon derivating $\int_\Omega u \log u$ is $\int_\Omega \nabla u \cdot \nabla v$, while upon derivating $\int_\Omega |\nabla \sqrt{v}|^2$ the term $-\frac{1}{2} \int_\Omega \nabla u \cdot \nabla v$ shows up. Hence by calculating the derivative of (1.3) these terms cancel out each other.

However, if we tried to follow this approach for the system (P) we would obtain

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega \nabla v \cdot \nabla w$$

instead. Even ignoring the fact that w might not be smooth enough to justify the calculation, it is not clear at all how to handle these terms.

Therefore it seems necessary to follow a different approach. In order to prove global existence, we will rely on functionals of the form

$$\int_\Omega u^p \varphi(v) \tag{1.4}$$

for certain functions φ and $p > 0$ (cf. Lemma 3.5).

For instance in [19,24] such functionals have been capitalized for $p > 1$. Indeed, for sufficiently small v_0 such an approach leads to success also for (P), see Proposition 3.6.

Functionals of the form of (1.4) have also already been studied with $p \in (0, 1)$ in [11,18], in both cases with $\varphi(s) = s^q$, $s \geq 0$, for some $q > 0$. However, in those works they have only helped to obtain weak solutions: The general idea is to obtain space-time bounds of expressions such as $|\nabla u^{\frac{p}{2}}|^2 \varphi(v)$; that is, one might then hope to construct (global in time) solutions $(u_\varepsilon, v_\varepsilon)$, $\varepsilon > 0$, to approximate problems and derive space-time-bounds of, for instance, $|\nabla u_\varepsilon^{\frac{p}{2}}|^2 \varphi(v_\varepsilon)$ independently of ε, allowing for the application of certain convergence theorems.

However, such bounds seemingly cannot be utilized to obtain global classical solutions. Here lies the crucial difference in the present problem; the special structure of (P) allows us to go further: In the quite simple but essential Lemma 3.7 we prove that space-time bounds for u imply uniform-in-time space bounds for w. This allows us (at least in space-dimension one and

two) to undertake a bootstrap procedure in Proposition 3.11: These bounds imply bounds for v in certain Sobolev spaces, which then imply improved space-time bounds for u, which again provide space estimates for w and so on.

Finally, we are able to prove

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded, smooth domain and $\beta \in (0, 1)$. Suppose that*

$$u_0 \in C^0(\overline{\Omega}), \quad v_0 \in W^{1,\infty}(\Omega) \quad and \quad w_0 \in C^\beta(\overline{\Omega}) \tag{1.5}$$

satisfy

$$u_0, v_0, w_0 \geq 0 \ in \ \overline{\Omega} \quad and \quad u_0 \not\equiv 0, \tag{1.6}$$

and if $n \geq 3$ also

$$\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{3n}.$$

Then there exists a global classical solution (u, v, w) of problem (P) which is uniquely determined by the inclusions

$$u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),$$

$$v \in \bigcap_{\theta > n} C^0([0, \infty); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$$

and

$$w \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{0,1}(\overline{\Omega} \times (0, \infty)).$$

Main ideas and results II: Large time behavior Having obtained global solutions we examine their large time behavior in Section 4.

The main challenge lies in the fact that the aforementioned bootstrap procedure for the case $n \leq 2$ only implies local-in-time boundedness of the solution components. Therefore we revise our arguments of Section 3 to show that ∇v is uniformly in time bounded in $L^\theta(\Omega)$ for some $\theta > n$, see Proposition 4.4.

Along with a very weak convergence result (Lemma 4.5) this allows us to deduce $v(\cdot, t) \to 0$ as $t \to \infty$ in $L^\infty(\Omega)$, see Lemma 4.6.

The results of Section 3 then allow us to find $t_0 > 0$ such that the solution to (P) with initial data $u(\cdot, t_0), v(\cdot, t_0), w(\cdot, t_0)$ is bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega) \times L^\infty(\Omega)$. Due to uniqueness this implies certain bounds for u, v and w as well. By using parabolic regularity theory we then improve this to bounds in certain Hölder spaces (Lemma 4.8).

Since we are also able to deduce a very weak convergence result for u in Lemma 4.9, we may use this regularity result in order to obtain convergence of u (Lemma 4.10) – which in turn together with the variations of constants formula implies convergence of w (Lemma 4.11).

In the end, we arrive at

Theorem 1.2. *Under the assumptions of Theorem 1.1 there exists $\alpha \in (0, 1)$ such that the solution (u, v, w) given by Theorem 1.1 fulfills*

$$u, v \in C_{\mathrm{loc}}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [1, \infty)) \quad and \quad w \in C_{\mathrm{loc}}^{\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [1, \infty)) \tag{1.7}$$

as well as

$$u(\cdot, t) \to \overline{u}_0 \text{ in } C^{2+\alpha}(\overline{\Omega}), \quad v(\cdot, t) \to 0 \text{ in } C^{2+\alpha}(\overline{\Omega}) \quad and \quad w(\cdot, t) \to \frac{\overline{u}_0}{\delta} \text{ in } C^{\alpha}(\overline{\Omega}),$$

$$as \ t \to \infty, \tag{1.8}$$

wherein

$$\overline{u}_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0.$$

2. Preliminaries

Henceforth we fix a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$.
We start by stating a local existence result.

Lemma 2.1. *Suppose $u_0, v_0, w_0 : \overline{\Omega} \to \mathbb{R}$ satisfy (1.5) for some $\beta \in (0, 1)$. Then there exist $T_{\max} \in (0, \infty)$ and functions*

$$u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \tag{2.1}$$

$$v \in \bigcap_{\theta > n} C^0([0, T_{\max}); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \tag{2.2}$$

and

$$w \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{0,1}(\overline{\Omega} \times (0, T_{\max})) \tag{2.3}$$

solving (P) classically and are such that if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^\theta(\Omega)} \right) = \infty \tag{2.4}$$

for all $\theta > n$. These functions are uniquely determined by the inclusions (2.1), (2.2) and (2.3) and can be represented by

$$u(\cdot, t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) \, \mathrm{d}s, \tag{2.5}$$

$$v(\cdot, t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} (v(\cdot, s) w(\cdot, s)) \, \mathrm{d}s, \tag{2.6}$$

and

$$w(\cdot, t) = e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)} u(\cdot, s) \, ds \tag{2.7}$$

for $t \in (0, T_{\max})$.

Proof. This can be shown by a fixed point argument as (inter alia) in [6, Theorem 3.1]. Let us briefly recall the main idea: Let $\theta > n$ be arbitrary. For sufficiently small $T > 0$ the map Φ given by

$$\Phi(u, v) = \begin{pmatrix} t \mapsto e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) \, ds \\ t \mapsto e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} v(\cdot, s) (\Psi(u))(\cdot, s) \, ds \end{pmatrix}$$

with

$$\Psi(u)) : t \mapsto e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-\sigma)} u(\cdot, \sigma) \, d\sigma$$

acts as a contraction on a certain closed subset of the Banach space

$$C^0([0, T]; C^0(\overline{\Omega})) \times C^0([0, T]; W^{1,\theta}(\Omega)).$$

By Banach's fixed point theorem one then obtains a unique tuple (u, v) such that (u, v, w) with $w := \Psi(u)$ satisfies (2.5), (2.6) and (2.7) for $t \in (0, T)$. Repeating this argument leads to the extensibility criterion (2.4).

In order to show that (2.1) and (2.2) hold, one uses parabolic regularity theory, similar as in for example [6]. Here it is important to note that Hölder regularity of u implies Hölder regularity of w (cf. Lemma 3.1 below). □

Lemma 2.2. *For any $u_0, v_0, w_0 : \overline{\Omega} \to \mathbb{R}$ satisfying (1.5) for some $\beta \in (0, 1)$ and (1.6) the solution (u, v, w) constructed in Lemma 2.1 fulfills*

$$u > 0, \quad v \geq 0, \quad v \leq \|v_0\|_{L^\infty(\Omega)} \quad and \quad w > 0$$

in $\overline{\Omega} \times (0, T_{\max})$, where $T_{\max}$ is given by Lemma 2.1. Furthermore, for all $t \in [0, T_{\max})$ we have

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 =: m. \tag{2.8}$$

Proof. By comparison we have $u > 0$ and $v \geq 0$ and then also $w \geq 0$, hence $-vw \leq 0$ and therefore also by comparison $v \leq \|v_0\|_{L^\infty(\Omega)}$ in $\overline{\Omega} \times (0, T_{\max})$.

Moreover, integrating the first equation in (P) over Ω yields (2.8). □

For the remainder of this article we fix $u_0, v_0, w_0 : \overline{\Omega} \to \mathbb{R}$ satisfying (1.5) and (1.6) for some $\beta \in (0, 1)$ and let always (u, v, w) and $T_{\max}$ be as in Lemma 2.1.

3. Global existence

3.1. Enhancing the extensibility criterion

We begin by providing some useful estimates.

Lemma 3.1. *There exists $C > 0$ such that for any function space*

$$X \in \bigcup_{p\in[1,\infty]} L^p(\Omega) \cup \bigcup_{\alpha\in[0,\beta]} C^\alpha(\overline{\Omega})$$

the inequality

$$\|w\|_{L^\infty((0,T_{\max});X)} \le C(1 + \|u\|_{L^\infty((0,T_{\max});X)})$$

holds.

Proof. By (2.7) we have

$$\sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_X \le \sup_{t\in(0,T_{\max})} \left(e^{-\delta t}\|w_0\|_X + \int_0^t e^{-\delta(t-s)}\|u(\cdot,s)\|_X\, \mathrm{d}s \right)$$

$$\le \sup_{t\in(0,T_{\max})} \left(\|w_0\|_X + \|u\|_{L^\infty((0,T_{\max});X)} \int_0^t e^{-\delta(t-s)}\, \mathrm{d}s \right)$$

$$\le \sup_{t\in(0,T_{\max})} \left(\|w_0\|_X + \frac{1}{\delta}\|u\|_{L^\infty((0,T_{\max});X)} \right),$$

such that the statement follows by setting $C := \max\{\frac{1}{\delta}, \|w_0\|_X\}$, which is finite as $w_0 \in X$ by (1.5). $\square$

Lemma 3.2. *Let $p \ge 1$. For all*

$$\theta \in \begin{cases} [1, \frac{np}{n-p}), & p \le n, \\ [1, \infty], & p > n \end{cases}$$

there exists a constant $C > 0$ such that

$$\|v\|_{L^\infty((0,T_{\max});W^{1,\theta}(\Omega))} \le C\left(1 + \|w\|_{L^\infty((0,T_{\max});L^p(\Omega))}\right)$$

holds.

Proof. Due to Hölder's inequality we may assume without loss of generality $\theta \geq p$.
Set

$$\gamma := -\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{\theta}\right). \tag{3.1}$$

Then we have for $p \leq n$

$$\gamma > -\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p} - \frac{n-p}{np}\right) = -\frac{1}{2} - \frac{n}{2} \cdot \frac{1}{n} = -1$$

and for $p > n$

$$\gamma \geq -\frac{1}{2} - \frac{n}{2} \cdot \left(\frac{1}{p} - \frac{1}{\infty}\right) = -\frac{1}{2} - \frac{n}{2p} > -1.$$

By known smoothing estimates for the Neumann Laplace semigroup (cf. [25, Lemma 1.3 (ii) and (iii)]) and Hölder's inequality there exist $c_1, c_2, \lambda > 0$ such that

$$\|\nabla e^{\sigma \Delta} \varphi\|_{L^\theta(\Omega)} \leq c_2(1 + \sigma^\gamma)e^{-\lambda \sigma}\|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in L^p(\Omega) \text{ and all } \sigma > 0$$

and

$$\|\nabla e^{\sigma \Delta} \varphi\|_{L^\theta(\Omega)} \leq c_1\|\nabla \varphi\|_{L^\infty(\Omega)} \quad \text{for all } \varphi \in L^\infty(\Omega) \text{ and all } \sigma > 0.$$

Therefore, for $t \in (0, T_{\max})$ we have by (2.6) and Lemma 2.2

$$\|\nabla v(\cdot, t)\|_{L^\theta(\Omega)}$$

$$\leq \|\nabla e^{t\Delta} v_0\|_{L^\theta(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta} v(\cdot, s)w(\cdot, s)\|_{L^\theta(\Omega)} \, ds$$

$$\leq 1\|\nabla v_0\|_{L^\infty(\Omega)} + c_2 \int_0^t (1 + (t-s)^\gamma e^{-\lambda(t-s)})\|v(\cdot, s)w(\cdot, s)\|_{L^p(\Omega)} \, ds$$

$$\leq c_1\|\nabla v_0\|_{L^\infty(\Omega)} + c_2\|v_0\|_{L^\infty(\Omega))}\|w\|_{L^\infty((0,T_{\max});L^p(\Omega))} \int_0^\infty (1 + s^\gamma)e^{-\lambda s} \, ds,$$

where $\gamma > -1$ warrants finiteness of the last integral therein. $\quad\square$

Lemma 3.3. *Let* $p, \theta, q \in [1, \infty]$ *with*

$$\frac{1}{p} + \frac{1}{\theta} = \frac{1}{q}.$$

For all $p' > 1$ with

$$p' \begin{cases} < \frac{nq}{n-q}, & q \leq n, \\ \leq \infty, & q > n \end{cases}$$

there exists $C > 0$ such that

$$\|u\|_{L^\infty((0,T_{\max});L^{p'}(\Omega))} \leq C \left(1 + \|u\|_{L^\infty((0,T_{\max});L^p(\Omega))} \|\nabla v\|_{L^\infty((0,T_{\max});L^\theta(\Omega))}\right)$$

holds.

Proof. Without loss of generality we assume $p' \geq q$. Define $\gamma > -1$ as in (3.1) with q instead of p and p' instead of θ.

Again relying on known smoothing estimates for the Neumann Laplace semigroup (cf. [25, Lemma 1.3 (i) and (iv)]) we can find $c_1, c_2, \lambda > 0$ such that

$$\|e^{\sigma\Delta}\varphi\|_{L^{p'}(\Omega)} \leq c_1 \|\varphi\|_{L^{p'}(\Omega)} \quad \text{for all } \varphi \in L^{p'}(\Omega) \text{ and all } \sigma > 0$$

and

$$\|e^{\sigma\Delta}\nabla \cdot \varphi\|_{L^{p'}(\Omega)} \leq c_2(1+\sigma^\gamma)e^{-\lambda\sigma}\|\varphi\|_{L^q(\Omega)} \quad \text{for all } \varphi \in L^q(\Omega) \text{ and all } \sigma > 0,$$

hence by (2.5) we have for $t \in (0, T_{\max})$

$$\|u(\cdot,t)\|_{L^{p'}(\Omega)} \leq \|e^{t\Delta}u_0\|_{L^{p'}(\Omega)} + \int_0^t \|e^{(t-s)\Delta}\nabla \cdot (u(\cdot,t)\nabla v(\cdot,t))\|_{L^{p'}(\Omega)} \, ds$$

$$\leq c_1 \|u_0\|_{L^{p'}(\Omega)} + c_2 \int_0^t (1+(t-s)^\gamma)e^{\lambda(t-s)}\|u(\cdot,t)\nabla v(\cdot,t)\|_{L^q(\Omega)} \, ds$$

$$\leq c_1 \|u_0\|_{L^{p'}(\Omega)}$$

$$+ c_2 \|u\|_{L^\infty((0,T_{\max});L^p(\Omega))} \|\nabla v\|_{L^\infty((0,T_{\max});L^\theta(\Omega))} \int_0^\infty (1+s^\gamma)e^{-\lambda s} \, ds.$$

Finiteness of the last integral therein is again guaranteed by $\gamma > -1$. $\quad\square$

Equipped with these estimates we are able to improve our extensibility criterion of Lemma 2.1.

Proposition 3.4. *Let $p \geq \max\{2, n\}$. If there exists $c_0 > 0$ such that*

$$\int_\Omega u^p < c_0 \quad \text{in } [0, T_{\max})$$

then $T_{\max} = \infty$. In that case we furthermore have

$$\{(u(\cdot, t), v(\cdot, t), w(\cdot, t)) : t \in (0, T_{\max})\} \text{ is bounded in } L^\infty(\Omega) \times W^{1,\infty}(\Omega) \times L^\infty(\Omega). \quad (3.2)$$

Proof. In view of Lemma 2.1 it suffices to show (3.2).

Set $\theta := 2p$. As $\frac{np}{(n-p)_+} = \infty$, Lemma 3.1 and Lemma 3.2 assert boundedness of $\{v(\cdot, t) : t \geq 0\}$ in $W^{1,\theta}(\Omega)$.

Since

$$q := \frac{p\theta}{p+\theta} = \frac{2p^2}{3p} = \frac{2}{3}p > \max\left\{1, \frac{n}{2}\right\}$$

fulfills $\frac{1}{q} = \frac{1}{p} + \frac{1}{\theta}$ and

$$\frac{nq}{(n-q)_+} > \frac{n \cdot \frac{n}{2}}{n - \frac{n}{2}} = n$$

we may invoke Lemma 3.3 to obtain boundedness of $S := \{u(\cdot, t) : t \geq 0\}$ in $L^{p'}(\Omega)$ for some $p' > n$.

By again using Lemma 3.1 and Lemma 3.2 we obtain boundedness of $\{v(\cdot, t) : t \geq 0\}$ in $W^{1,\infty}(\Omega)$. Then $q' := p'$ fulfills $\frac{1}{p'} + \frac{1}{\infty} = \frac{1}{q'}$ and $q' > n \geq 1$, hence Lemma 3.3 implies boundedness of S in $L^\infty(\Omega)$. Therefore the statement follows by a final application of Lemma 3.1. $\quad \square$

3.2. Global existence for small $\|v_0\|_{L^\infty(\Omega)}$

The estimates in this as well as in the following subsection will rely heavily on the following functional inequality.

Lemma 3.5. *Let $p \in (0, 1) \cup (1, \infty)$, $\sigma_p := \mathrm{sign}(p - 1)$ and $\varphi \in C^2([0, \|v_0\|_{L^\infty(\Omega)}])$ with $\varphi > 0$. Then for all $\eta_1, \eta_2 > 0$*

$$\frac{\sigma_p}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p \varphi(v) + |p - 1| \int_\Omega u^{p-2} |\nabla u|^2 \varphi(v)[1 - \eta_1 - \eta_2]$$

$$\leq \int_\Omega u^p |\nabla v|^2 g_\varphi(v) - \frac{\sigma_p}{p} \int_\Omega u^p v \varphi'(v) w$$

holds, where

$$g_\varphi(v) := \frac{|p - 1|}{4\eta_2} \varphi(v) + |\varphi'(v)| + \frac{1}{\eta_1 |p - 1|} \frac{\varphi'^2(v)}{\varphi(v)} - \frac{\sigma_p}{p} \varphi''(v).$$

Proof. By integrating by parts we obtain

$$
\frac{\sigma_p}{p} \frac{d}{dt} \int_\Omega u^p \varphi(v) = -\sigma_p \int_\Omega \nabla(u^{p-1}\varphi(v)) \cdot \nabla u + \sigma_p \int_\Omega \nabla(u^{p-1}\varphi(v)) \cdot (u\nabla v)
$$

$$
- \frac{\sigma_p}{p} \int_\Omega \nabla(u^p \varphi'(v)) \cdot \nabla v - \frac{\sigma_p}{p} \int_\Omega u^p v\varphi'(v)w
$$

$$
= -|p-1| \int_\Omega u^{p-2}|\nabla u|^2 \varphi(v) - \sigma_p \int_\Omega u^{p-1}\varphi'(v)\nabla u \cdot \nabla v
$$

$$
+ |p-1| \int_\Omega u^{p-1}\varphi(v)\nabla u \cdot \nabla v + \sigma_p \int_\Omega u^p |\nabla v|^2 \varphi'(v)
$$

$$
- \sigma_p \int_\Omega u^{p-1}\varphi'(v)\nabla u \cdot \nabla v - \frac{\sigma_p}{p} \int_\Omega u^p |\nabla v|^2 \varphi''(v)
$$

$$
- \frac{\sigma_p}{p} \int_\Omega u^p v\varphi'(v)w
$$

in $(0, T_{\max})$.

Herein we use Young's inequality to conclude

$$
\left| -2\sigma_p \int_\Omega u^{p-1}\varphi'(v)\nabla u \cdot \nabla v \right| \leq \eta_1|p-1| \int_\Omega u^{p-2}|\nabla u|^2 \varphi(v) + \frac{1}{\eta_1|p-1|} \int_\Omega u^p |\nabla v|^2 \cdot \frac{\varphi'^2(v)}{\varphi(v)}
$$

and

$$
\left| |p-1| \int_\Omega u^{p-1}\varphi(v)\nabla u \cdot \nabla v \right| \leq \eta_2|p-1| \int_\Omega u^{p-2}|\nabla u|^2 \varphi(v) + \frac{|p-1|}{4\eta_2} \int_\Omega u^p |\nabla v|^2 \varphi(v)
$$

in $(0, T_{\max})$.

Combining these estimates with $\sigma_p\varphi' \leq |\varphi'|$ already completes the proof. $\quad\square$

A first application of this Lemma is

Proposition 3.6. *If* $v_0 \leq \frac{1}{3\max\{2,n\}}$, *then* $T_{\max} = \infty$ *and* (3.2) *holds.*

Proof. We follow an idea of [19, Lemma 3.1].

Without loss of generality suppose $v_0 \not\equiv 0$. Let $p := \max\{2, n\}$, $I := [0, \|v_0\|_{L^\infty(\Omega)}]$ and

$$
\varphi : I \to \mathbb{R}, \quad s \mapsto e^{\gamma s^2},
$$

where

$$\gamma := \frac{p-1}{12p\|v_0\|_{L^\infty(\Omega)}^2} > 0.$$

Then we have

$$\varphi'(s) = 2\gamma s e^{\gamma s^2} \geq 0 \quad \text{and} \quad \varphi''(s) = [2\gamma + (2\gamma s)^2]e^{\gamma s^2} \geq 2\gamma e^{\gamma s^2} > 0$$

for $s \in I$, such that Lemma 3.5 yields for $\eta_1 = \eta_2 = \frac{1}{2}$

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \varphi(v) \leq \frac{1}{p}\int_\Omega u^p |\nabla v|^2 \left[\frac{p(p-1)}{2}\varphi(v) + p\varphi'(v) + \frac{2p}{p-1}\frac{\varphi'^2(v)}{\varphi(v)} - \varphi''(v)\right].$$

As $0 \leq v \leq \|v_0\|_{L^\infty(\Omega)}$ by Lemma 2.2, we have

$$\frac{p(p-1)\varphi(v)}{2\varphi''(v)} \leq \frac{p(p-1)}{4\gamma} = 3p^2\|v_0\|_{L^\infty(\Omega)}^2 \leq \frac{1}{3} \quad \text{in } (0, T_{\max}),$$

$$\frac{p\varphi'(v)}{\varphi''(v)} \leq \frac{2p\gamma v e^{\gamma v^2}}{2\gamma e^{\gamma v^2}} \leq p\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{3} \quad \text{in } (0, T_{\max})$$

and

$$\frac{2p\varphi'^2(v)}{(p-1)\varphi(v)\varphi''(v)} \leq \frac{2p(2\gamma v)^2}{2(p-1)\gamma} \leq \frac{4p\gamma}{p-1}\|v_0\|_{L^\infty(\Omega)}^2 = \frac{1}{3} \quad \text{in } (0, T_{\max}),$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \varphi(v) \leq 0 \quad \text{in } (0, T_{\max}).$$

Since $\varphi \geq 1$ we obtain upon integrating

$$\int_\Omega u^p \leq \int_\Omega u^p \varphi(v) \leq \int_\Omega u_0^p \varphi(v_0) \quad \text{in } (0, T_{\max}),$$

therefore we may apply Proposition 3.4 to obtain the statement. □

3.3. Global existence for $n \leq 2$

The following lemma exploits the special structure of (P) and is a key ingredient for our further analysis. Space-time bounds of u can be turned into space bounds of w:

Lemma 3.7. *For all $p \in [1, \infty)$ there exists $C > 0$ such that for all $T \in (0, T_{\max}]$ we have*

$$\sup_{t\in(0,T)} \|w(\cdot, t)\|_{L^p(\Omega)} \leq C\left(1 + \|u\|_{L^p(\Omega\times(0,T))}\right).$$

Proof. Fix $p \in [1, \infty)$ and let $p' := \frac{p}{p-1}$. By using (2.7) and Hölder's inequality we obtain

$$\|w(\cdot, t)\|_{L^p(\Omega)} \leq e^{-\delta t} \|w_0\|_{L^p(\Omega)} + \int_0^t e^{-\delta(t-s)} \|u(\cdot, s)\|_{L^p(\Omega)} \, ds$$

$$\leq \|w_0\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega \times (0,t))} \left(\int_0^t e^{-\delta p' s} \, ds \right)^{\frac{1}{p'}}$$

$$\leq \|w_0\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega \times (0,T))} (\delta p')^{-\frac{1}{p'}}$$

for $t \in (0, T)$, hence the statement follows for $C := \max\{\|w_0\|_{L^p(\Omega)}, (\delta p')^{-\frac{1}{p'}}\}$. $\quad \square$

We proceed to gain space-time bounds for u:

Lemma 3.8. *There exists $p_0 \in (0, 1)$ such that for all $p \in (0, p_0)$ there is $C > 0$ with*

$$\int_0^{T_{\max}} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 < C.$$

Proof. The function $\varphi : I := [0, \|v_0\|_{L^\infty(\Omega)}] \to \mathbb{R}$ defined by

$$\varphi(s) := 1 + \|v_0\|_{L^\infty(\Omega)}^2 - s^2, \quad s \in I,$$

satisfies

$$\varphi \geq 1, \quad \varphi' \leq 0 \quad \text{and} \quad \varphi'' \leq -1.$$

Therefore, for all $s \in I$ we have

$$g_p(s) := \frac{3}{4}(1 - p)\varphi(s) + |\varphi'(s)| + \frac{3}{1-p} \frac{(\varphi')^2(s)}{\varphi(s)} + \frac{1}{p}\varphi''(s)$$

$$\leq \frac{3}{4}(1 - p)\|\varphi\|_{L^\infty(I)} + \|\varphi'\|_{L^\infty(I)} + \frac{3}{1-p}\|\varphi'\|_{L^\infty(I)}^2 - \frac{1}{p} \to -\infty$$

as $p \searrow 0$, hence there exists $p_0 \in (0, 1)$ such that $g_p(s) \leq 0$ for all $s \in I$ and all $p < p_0$. Lemma 3.5 with $\eta_1 = \eta_2 = \frac{1}{3}$ then yields for $p \in (0, p_0)$

$$-\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \varphi(v) + \frac{1-p}{3} \int_\Omega u^{p-2} |\nabla u|^2 \varphi(v) \leq \int_\Omega u^p |\nabla v|^2 g_p(v) \leq 0 \quad \text{in } (0, T_{\max}),$$

as $v(\overline{\Omega} \times [0, T_{\max})) \subset I$ by Lemma 2.2.

Thus, upon integrating over $(0, T)$, $T \in (0, T_{\max})$, we obtain by using Hölder's inequality

$$\frac{1-p}{3} \int_0^T \int_\Omega u^{p-2} |\nabla u|^2 \leq \frac{1-p}{3} \int_0^T \int_\Omega u^{p-2} |\nabla u|^2 \varphi(v)$$

$$\leq \frac{1}{p} \int_\Omega u^p(\cdot, T) \varphi(v) - \frac{1}{p} \int_\Omega u_0^p \varphi(v)$$

$$\leq \frac{1}{p} \|\varphi\|_{L^\infty(I)} \int_\Omega u^p(\cdot, T)$$

$$\leq \frac{1}{p} \|\varphi\|_{L^\infty(I)} m^p |\Omega|^{1-p},$$

hence the statement follows by setting $C := \frac{3}{(1-p)p} \|\varphi\|_{L^\infty(I)} m^p |\Omega|^{1-p}$ and using the monotone convergence theorem. $\square$

Lemma 3.9. *For $p \in (0, 2]$ there exists $C > 0$ with*

$$\int_\Omega u^{\frac{2}{n}+p} \leq C \left(1 + \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \right) \quad in \ (0, T_{\max}).$$

Proof. Set $p' := \frac{2}{p} \cdot (\frac{2}{n} + p) > 2$. As

$$\frac{2}{p'} \in (0, 1) \quad \text{and} \quad \left(\frac{1}{2} - \frac{1}{n} \right) \cdot \frac{2}{p'} + \frac{1 - \frac{2}{p'}}{\frac{2}{p}} = \frac{1}{p'} \left(\frac{n-2}{n} + \frac{p}{2} \cdot p' - p \right) = \frac{1}{p'}$$

we may invoke the Gagliardo–Nirenberg inequality to obtain $C_1, C_2 > 0$ such that

$$\int_\Omega \psi^{\frac{2}{n}+p} = \|\psi^{\frac{p}{2}}\|_{L^{p'}}^{p'}$$

$$\leq C_1 \|\nabla \psi^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \|\psi^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{p'-2} + C_2 \|\psi^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{p'}$$

$$= C_1 \int_\Omega |\nabla \psi^{\frac{p}{2}}|^2 \left(\int_\Omega \psi \right)^{\frac{p(p'-2)}{2}} + C_2 \left(\int_\Omega \psi \right)^{\frac{pp'}{2}}$$

holds for all nonnegative $\psi : \Omega \to \mathbb{R}$ with $\psi^{\frac{p}{2}} \in W^{1,2}(\Omega)$.

The statement follows then by taking $\psi = u(\cdot, t)$, $t \in (0, T_{\max})$, and Lemma 2.2. $\square$

Lemma 3.10. *For $p \in (0, 2]$ there exists $C_p > 0$ such that*

$$\frac{\sigma_p}{p} \frac{d}{dt} \int_\Omega u^p + \frac{|p-1|}{p^2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \leq C_p \int_\Omega |\nabla v|^{2+np} + C_p \quad \text{in } (0, T_{\max}),$$

where $\sigma_p := \text{sign}(p-1)$.

Proof. Without loss of generality let $p \in (0, 2] \setminus \{1\}$. By applying Lemma 3.5 with $\varphi \equiv 1$ and $\eta_1 = \eta_2 = \frac{1}{4}$ we obtain

$$\frac{\sigma_p}{p} \frac{d}{dt} \int_\Omega u^p + \frac{|p-1|}{2} \int_\Omega u^{p-2} |\nabla u|^2 \leq |p-1| \int_\Omega u^p |\nabla v|^2 \quad \text{in } (0, T_{\max}). \tag{3.3}$$

According to Lemma 3.9 we may find $C_p' > 0$ such that

$$\int_\Omega u^{\frac{2}{n}+p} \leq C_p' \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + C_p' \quad \text{in } (0, T_{\max}),$$

hence Young's inequality (with exponents $\frac{\frac{2}{n}+p}{p}, \frac{\frac{2}{n}+p}{\frac{2}{n}} = \frac{2+np}{2}$) implies the existence of $C_p'' > 0$ satisfying

$$\int_\Omega u^p |\nabla v|^2 \leq \frac{1}{p^2 C_p'} \int_\Omega u^{\frac{2}{n}+p} + C_p'' \int_\Omega |\nabla v|^{2+np}$$

$$\leq \frac{1}{p^2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{1}{p^2} + C_p'' \int_\Omega |\nabla v|^{2+np} \tag{3.4}$$

in $(0, T_{\max})$. The statement follows by combining (3.3) with (3.4), due to the pointwise equality $|\nabla u^{\frac{p}{2}}|^2 = \frac{p^2}{4} u^{p-2} |\nabla u|^2$ and by setting $C_p := |p-1| \max\{\frac{1}{p^2}, C_p''\}$. $\quad \square$

For $n \leq 2$ we may now apply a bootstrap procedure to achieve globality of (u, v, w). A combination of Lemma 3.8, Lemma 3.9 and Lemma 3.7 serves as a starting point, while Lemma 3.10 and Lemma 3.7 are the main ingredients for improving bounds for u step by step.

Proposition 3.11. *If $n \leq 2$, then $T_{\max} = \infty$; that is, the solution (u, v, w) is global in time.*

Proof. Suppose $T_{\max} < \infty$.

By Lemma 3.8 we may find $p_0 \in (0, 1)$ such that

$$\int_0^{T_{\max}} \int_\Omega |\nabla u^{\frac{p_0}{2}}|^2 < C_1$$

for some $C_1 > 0$.

As $\frac{2}{n} \geq 1$ a combination of Lemma 3.9 and the Hölder inequality implies

$$\int_\Omega u^{1+p_0} \leq C_2 \left(1 + \int_\Omega |\nabla u^{\frac{p_0}{2}}|^2 \right) \quad \text{in } (0, T_{\max})$$

for some $C_2 > 0$, we conclude

$$\int_0^{T_{\max}} \int_\Omega u^{1+p_0} \leq C_2 \left(T_{\max} + C_1 \right).$$

Hence, by Lemma 3.7 there is $c_0 > 0$ with

$$\int_\Omega w^{1+p_0} \leq c_0 \quad \text{in } (0, T_{\max}). \tag{3.5}$$

Set $a_0 := 2 - (1 + p_0) \in (0, 1)$ and

$$p_{k+1} := \begin{cases} \frac{p_k}{a_0}, & \frac{p_k}{a_0} < 1, \\ \frac{p_k + 1}{2}, & \frac{p_k}{a_0} \geq 1 \end{cases}$$

for $k \in \mathbb{N}_0$. Note that $p_k \in (0, 1)$ and $p_k \to 1$ for $k \to \infty$, hence there exists $k_0 \in \mathbb{N}$ with $p_{k_0} > \frac{1}{2}$. We next show by induction that for each $k \in \mathbb{N}_0$ there exists $c_k > 0$ such that

$$\int_\Omega w^{1+p_k} \leq c_k \quad \text{in } (0, T_{\max}). \tag{3.6}$$

Let $k = 0$, then (3.6) is exactly (3.5), hence suppose (3.6) holds for some $k \in \mathbb{N}_0$. As

$$2 + np_{k+1} \leq 2(1 + p_{k+1})$$
$$\leq 2 \left(1 + \frac{p_k}{a_0} \right)$$
$$< 2 \left(1 + \frac{1 - a_0 + p_k}{a_0} \right)$$
$$= 2 + \frac{2(1 + p_k) - 2a_0}{a_0}$$
$$= \frac{2(1 + p_k)}{2 - (1 + p_0)}$$
$$< \frac{2(1 + p_k)}{2 - (1 + p_k)} \leq \frac{n(1 + p_k)}{(n - (1 + p_k))_+}$$

we may apply Lemma 3.2 to obtain $c'_{k+1} > 0$ with

$$\int_\Omega |\nabla v|^{2+np_{k+1}} \leq c'_{k+1} \quad \text{in } (0, T_{\max}).$$

By applying Lemma 3.10 and integrating over $(0, T)$ for $T \in (0, T_{\max})$ we then obtain

$$-\frac{1}{p_{k+1}} \int_\Omega u^{p_{k+1}}(\cdot, T) + \frac{1}{p_{k+1}} \int_\Omega u_0^{p_{k+1}} + \frac{1 - p_{k+1}}{p_{k+1}^2} \int_0^T \int_\Omega |\nabla u^{\frac{p_{k+1}}{2}}|^2 \leq C_{p_{k+1}} (1 + c'_k) T \quad (3.7)$$

with $C_{p_{k+1}}$ as in Lemma 3.10.

Since $p_{k+1} \in (0, 1)$ we have $\int_\Omega u^{p_{k+1}} < c''_{k+1}$ in $(0, T_{\max})$ for some $c''_{k+1} > 0$ by Lemma 2.2. As $u_0 \geq 0$ and $T_{\max} < \infty$ by assumption, (3.7) implies

$$\int_0^T \int_\Omega |\nabla u^{\frac{p_{k+1}}{2}}|^2 \leq \frac{p_{k+1}^2}{1 - p_{k+1}} \left((1 + c'_{k+1}) C_{p_{k+1}} T_{\max} + \frac{c''_{k+1}}{p_{k+1}} \right) < \infty \quad \text{for all } T \in (0, T_{\max}).$$

By using Lemma 3.9 and Lemma 3.7 we then obtain (3.6) for $k + 1$ instead of k and some $c_{k+1} > 0$.

Finally, (3.6) for $k = k_0$ and Lemma 3.2 assert boundedness of $\{\nabla v(\cdot, t) : t \in (0, T_{\max}\}$ in $L^6(\Omega)$, since

$$6 = \frac{2(1 + \frac{1}{2})}{2 - (1 + \frac{1}{2})} < \frac{2(1 + p_{k_0})}{2 - (1 + p_{k_0})} \leq \frac{n(1 + p_{k_0})}{(n - (1 + p_{k_0}))_+},$$

such that by another application of Lemma 3.10 and Hölder's inequality $(2 + 2n \leq 6)$ we obtain $C > 0$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^2 < C \quad \text{in } (0, T_{\max}).$$

However, this contradicts Proposition 3.4. $\quad\square$

Remark 3.12. Proposition 3.11 does not yet claim any uniform-in-time boundedness of any solution component. That the solution is indeed bounded if $n \leq 2$ will be seen in Proposition 4.4 and Lemma 4.8 below.

Remark 3.13. Apart from Proposition 3.11 all statements in this subsection hold for all $n \in \mathbb{N}$. However, for $n \geq 3$ the lemmata above (at least in the form stated) are not sufficient to prove $T_{\max} = \infty$ also for higher dimensions: Lemma 3.8 and Lemma 3.9 imply

$$\int_\Omega u^{\frac{2}{n}+p} < C(T) \quad \text{in } (0, T)$$

for $T \in (0, T_{\max})$ and some $p \in (0, 1)$, $C(T) > 0$, but for $p < \frac{n-2}{n}$ this does not improve on boundedness in $L^1(\Omega)$, which is already known (Lemma 2.2).

Theorem 1.1 is now an immediate consequence of the propositions above:

Proof of Theorem 1.1. Local existence and uniqueness have been shown in Lemma 2.1, while $T_{\max} = \infty$ has been proved in Proposition 3.6 and Proposition 3.11 for the cases $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{3\max\{2,n\}}$ and $n \le 2$, respectively. $\quad\square$

4. Large time behavior

4.1. A sufficient condition

We show convergence of the solution towards a spatial constant equilibrium, if additionally the following condition is satisfied. That is, if one is able to show this for a set of parameters not discussed here, the statements in the following subsections still apply.

Condition 4.1. The solution (u, v, w) is global in time and there exist $\theta > n$ and $C > 0$ with

$$\int_\Omega |\nabla v|^\theta < C \quad \text{in } (0, \infty).$$

In the remainder of this subsection we will show that if $n \le 2$ or $n \ge 2$ and $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{3n}$ this is always the case.

Lemma 4.2. Let $n \le 2$, $p \in (0, 1)$ and set $u^* := \frac{1}{|\Omega|} \int_\Omega u^{\frac{p}{2}}$. Then there exists $C > 0$ such that

$$\int_\Omega \left| u^{\frac{p}{2}} - u^* \right|^{\frac{2}{p} \cdot (1+p)} \le C \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \quad \text{in } (0, \infty).$$

Proof. Set $p' := \frac{2}{p} \cdot (\frac{2}{n} + p) > 2$. As

$$\frac{2}{p'} \in (0, 1) \quad \text{and} \quad \left(\frac{1}{2} - \frac{1}{n}\right) \cdot \frac{2}{p'} + \frac{1 - \frac{2}{p'}}{\frac{2}{p}} = \frac{1}{p'}\left(\frac{n-2}{n} + \frac{p}{2} \cdot p' - p\right) = \frac{1}{p'}$$

By the Gagliardo-Nirenberg inequality there exist $C_1, C_2 > 0$ such that

$$\|\psi\|_{L^{p'}}^{p'} \le C_1 \|\nabla\psi\|_{L^2(\Omega)}^2 \|\psi\|_{L^{\frac{2}{p}}(\Omega)}^{p'-2} + C_2 \|\psi\|_{L^2(\Omega)}^{p'} \quad \text{for all } \psi \in W^{1,2}(\Omega).$$

Additionally, Poincaré's and Hölder's ($\frac{2}{p} > 2$, as $p \in (0, 1)$) inequalities yield the existence of $C_3 > 0$ such that

$$\|\psi\|_{L^2(\Omega)}^{p'} = \|\psi\|_{L^2(\Omega)}^2 \|\psi\|_{L^2(\Omega)}^{p'-2}$$

$$\le C_3 \|\nabla\psi\|_{L^2(\Omega)}^2 \|\psi\|_{L^{\frac{2}{p}}(\Omega)}^{p'-2} \quad \text{for all } \psi \in W^{1,2}(\Omega) \text{ with } \int_\Omega \psi = 0.$$

By combining these estimates we may find $C' > 0$ such that

$$\|\psi\|_{L^{p'}}^{p'} \leq C' \|\nabla\psi\|_{L^2(\Omega)}^2 \|\psi\|_{L^{\frac{2}{p}}(\Omega)}^{p'-2} \quad \text{for all } \psi \in W^{1,2}(\Omega) \text{ with } \int_\Omega \psi = 0. \tag{4.1}$$

Since u^* is constant in space we have $\nabla u^{\frac{p}{2}} = \nabla(u^{\frac{p}{2}} - u^*)$. As boundedness of $u^{\frac{p}{2}} - u^*$ in $L^{\frac{2}{p}}(\Omega)$ is implied by Lemma 2.2 and the assumption $n \leq 2$ warrants $p' \leq \frac{2}{p} \cdot (1 + p)$, by taking $\psi = u^{\frac{p}{2}}(\cdot, t) - u^*$, $t \in (0, T_{\max})$, in (4.1) and employing Hölder's inequality we obtain the statement. $\quad\square$

Lemma 4.3. *Let $n \leq 2$. Then there exist $\theta > n$ and $C > 0$ such that Condition 4.1 is fulfilled.*

Proof. By Proposition 3.11 the solution is global in time.

Lemma 3.8 allows us to fix $p \in (0, 1)$ and $C_0 > 0$ such that

$$\frac{2}{p} \in \mathbb{N} \quad \text{and} \quad \int_0^\infty \int_\Omega |\nabla u^{\frac{p}{2}}|^2 < C_0.$$

Let u^* be as in Lemma 4.2 and set

$$\tilde{w} : \overline{\Omega} \times [0, \infty) \to \mathbb{R}, \quad (x, t) \mapsto w(x, t) - \int_0^t e^{-\delta(t-s)} \left(u(x, s) - \left| u^{\frac{p}{2}}(x, s) - u^*(s) \right|^{\frac{2}{p}} \right) ds.$$

The representation formula (2.7), Hölder's inequality and Lemma 4.2 yield for $t \in (0, \infty)$

$$\int_\Omega |\tilde{w}(\cdot, t)|^{p+1} = \int_\Omega \left| e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)} \left| u^{\frac{p}{2}}(\cdot, s) - u^*(s) \right|^{\frac{2}{p}} ds \right|^{p+1}$$

$$\leq 2^p e^{-(p+1)\delta t} \int_\Omega w_0^{p+1} + 2^p \int_\Omega \left(\int_0^t e^{-\delta(t-s)} \left| u^{\frac{p}{2}}(\cdot, s) - u^*(s) \right|^{\frac{2}{p}} ds \right)^{p+1}$$

$$\leq 2^p \int_\Omega w_0^{p+1} + 2^p \int_\Omega \left[\left(\int_0^t e^{-\frac{(p+1)\delta}{p}(t-s)} \right)^p \int_0^t \left| u^{\frac{p}{2}} - u^* \right|^{\frac{2}{p} \cdot (1+p)} \right]$$

$$\leq 2^p \int_\Omega w_0^{p+1} + 2^p \left(\int_{-\infty}^t e^{-\frac{(p+1)\delta}{p}(t-s)} \right)^p \int_0^\infty \int_\Omega \left| u^{\frac{p}{2}} - u^* \right|^{\frac{2}{p} \cdot (1+p)}$$

$$\leq 2^p \int_\Omega w_0^{p+1} + 2^p \left(\frac{p}{(p+1)\delta} \right)^p \int_0^\infty C' \int_\Omega |\nabla u^{\frac{p}{2}}|^2$$

$$\leq 2^p \int_\Omega w_0^{p+1} + 2^p \left(\frac{p}{(p+1)\delta} \right)^p C' C_0 =: C_1. \tag{4.2}$$

As by Hölder's inequality and Lemma 2.2 there is $C_2 > 0$ with $|u^*| \leq C_2$ in $(0, T_{\max})$, we may further estimate (using the binomial theorem, note that $\frac{2}{p} \in \mathbb{N}$, and Jensen's inequality) for $k \in \{1, \ldots, \frac{2}{p}\}$

$$\left| u - \left| u^{\frac{p}{2}} - u^* \right|^{\frac{2}{p}} \right|^{1+\frac{p}{2}} \leq \left(\sum_{k=1}^{\frac{2}{p}} \binom{\frac{2}{p}}{k} u^{\frac{p}{2} \cdot (\frac{2}{p} - k)} (u^*)^k \right)^{1+\frac{p}{2}}$$

$$\leq \left(\sum_{k=1}^{\frac{2}{p}} C_2^k \binom{\frac{2}{p}}{k} u^{1 - \frac{kp}{2}} \right)^{1+\frac{p}{2}} \leq \sum_{k=1}^{\frac{2}{p}} D_k u^{q_k}$$

in $(0, T_{\max})$ for

$$D_k := \left(\frac{2}{p} \right)^{\frac{p}{2}} \left(C_2^k \binom{\frac{2}{p}}{k} \right)^{1+\frac{p}{2}} > 0 \quad \text{and}$$

$$q_k := \left(1 - \frac{kp}{2} \right) \left(1 + \frac{p}{2} \right) = 1 + (1-k)\frac{p}{2} - \frac{kp^2}{4} < 1.$$

Because of $\int_{-\infty}^t \delta e^{-\delta(t-s)} \, ds = 1$ we may apply Jensen's inequality to conclude that

$$\int_\Omega |w - \tilde{w}|^{1+\frac{p}{2}} = \int_\Omega \left| \int_0^t e^{-\delta(t-s)} \left(u(\cdot, s) - \left| u^{\frac{p}{2}}(\cdot, s) - u^*(s) \right|^{\frac{2}{p}} \right) ds \right|^{1+\frac{p}{2}}$$

$$\leq \delta^{-(1+\frac{p}{2})} \int_\Omega \left(\int_{-\infty}^t \delta e^{-\delta(t-s)} \mathbb{1}_{(0,t)}(s) \left| u(\cdot, s) - \left| u^{\frac{p}{2}}(\cdot, s) - u^*(s) \right|^{\frac{2}{p}} \right| ds \right)^{1+\frac{p}{2}}$$

$$\leq \delta^{-(1+\frac{p}{2})} \int_{-\infty}^t \left(\delta e^{-\delta(t-s)} \mathbb{1}_{(0,t)}(s) \sum_{k=1}^{\frac{2}{p}} D_k \int_\Omega u^{q_k}(\cdot, s) \right) ds \leq C_3 \tag{4.3}$$

holds in $(0, T_{\max})$ for some $C_3 > 0$ due to Lemma 2.2, as $q_k < 1$ for $k \in \{1, \ldots, \frac{2}{p}\}$.

As another application of Hölder's inequality gives

$$\int_\Omega |\tilde{w}|^{1+\frac{p}{2}} \leq C_4 \int_\Omega |\tilde{w}|^{1+p}$$

for some $C_4 > 0$, we obtain by combining (4.2) and (4.3)

$$\|w(\cdot,t)\|_{L^{1+\frac{p}{2}}(\Omega)} \le \|\tilde{w}(\cdot,t)\|_{L^{1+\frac{p}{2}}(\Omega)} + \|w(\cdot,t) - \tilde{w}(\cdot,t)\|_{L^{1+\frac{p}{2}}(\Omega)} \le (C_4 C_1)^{\frac{1}{1+\frac{p}{2}}} + C_3^{\frac{1}{1+\frac{p}{2}}}$$

for $t \in (0, T_{\max})$.

The statement follows by applying Lemma 3.2, as $\frac{2(1+\frac{p}{2})}{2-(1+\frac{p}{2})} > 2$ (since $1 + \frac{p}{2} > 1 - \frac{p}{2}$) and $\frac{1(1+\frac{p}{2})}{(1-(1+\frac{p}{2}))_+} > 1$. $\quad\square$

Proposition 4.4. *If $n \le 2$ or $n \ge 2$ and $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{3n}$, then Condition 4.1 is fulfilled.*

Proof. For $n \le 2$ this is a consequence of Lemma 4.3 while for $n > 2$ and $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{3n}$ this already has been shown in Proposition 3.6. $\quad\square$

4.2. Convergence of v

We begin by stating that v converges at least in some very weak sense.

Lemma 4.5. *If $T_{\max} = \infty$, then*

$$\int_0^\infty \int_\Omega vw < \infty. \tag{4.4}$$

Proof. Integrating the second equation in (P) over $(0, T) \times \Omega$ (for any $T > 0$) yields

$$\int_\Omega v(\cdot, T) - \int_\Omega v_0 = \int_0^T \int_\Omega \Delta v - \int_0^T \int_\Omega vw$$

and as $\int_\Omega \Delta v = 0$ due to $\partial_\nu v = 0$ on $\partial\Omega$ and $v \ge 0$ by Lemma 2.2 we have

$$\int_0^T \int_\Omega vw \le \int_\Omega v_0 \quad \text{for all } T > 0,$$

which already implies (4.4). $\quad\square$

Lemma 4.6. *If Condition 4.1 is fulfilled, then $v(\cdot, t) \to 0$ in $C^0(\overline{\Omega})$ for $t \to \infty$.*

Proof. By (4.4) and since $vw \ge 0$ there exists an increasing sequence $(t_k)_{k\in\mathbb{N}} \subset (0, \infty)$ such that $t_k \to \infty$ for $k \to \infty$ with

$$\int_\Omega v(\cdot, t_k) w(\cdot, t_k) \to 0 \quad \text{for } k \to \infty. \tag{4.5}$$

Condition 4.1 and the embedding $W^{1,\theta}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$ for all $\theta > n$ warrant that we may choose a subsequence of $(t_k)_{k \in \mathbb{N}}$ – which we also denote by $(t_k)_{k \in \mathbb{N}}$ for convenience – along which

$$v(\cdot, t_k) \to v_\infty \quad \text{in } C^0(\overline{\Omega}) \text{ for } k \to \infty$$

for some $v_\infty \in C^0(\overline{\Omega})$. As $v \geq 0$ by Lemma 2.2 we have $v_\infty \geq 0$.

Claim 1: The limit v_∞ is constant.

Proof. Suppose v_∞ is not constant, then

$$\lim_{k \to \infty} \frac{1}{|\Omega|} \int_\Omega v(\cdot, t_k) = \frac{1}{|\Omega|} \int_\Omega v_\infty < \|v_\infty\|_{L^\infty(\Omega)},$$

hence there exists $k_0 \in \mathbb{N}$ such that

$$\frac{1}{|\Omega|} \int_\Omega v(\cdot, t_{k_0}) < \|v_\infty\|_{L^\infty(\Omega)}.$$

Set $\overline{v}(\cdot, t) := e^{t\Delta} v(\cdot, t_{k_0})$. It is well known (see for instance [25, Lemma 1.3 (i)]) that

$$\left\| \overline{v}(\cdot, t) - \frac{1}{|\Omega|} \int_\Omega v(\cdot, t_{k_0}) \right\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty,$$

hence there exist $k_1 > k_0$ and $\varepsilon > 0$ such that

$$\overline{v}(\cdot, t) \leq \|v_\infty\|_{L^\infty(\Omega)} - \varepsilon$$

in $(t_{k_1} - t_{k_0}, \infty)$. Note that $t_{k_1} \geq t_{k_0}$ as $(t_k)_{k \in \mathbb{N}}$ is increasing.

Moreover, $\underline{v}(\cdot, t) := v(\cdot, t + t_{k_0}), t \geq 0$, defines a subsolution $\underline{v}$ of

$$\begin{cases} v_t = \Delta v, & \text{in } \Omega \times (0, \infty), \\ \partial_\nu v = 0, & \text{on } \partial\Omega \times (0, \infty), \\ v(\cdot, 0) = v(\cdot, t_{k_0}), & \text{in } \Omega, \end{cases}$$

since $vw \geq 0$ by Lemma 2.2.

Therefore by comparison we have

$$v(\cdot, t + t_{k_0}) = \underline{v}(\cdot, t) \leq \overline{v}(\cdot, t)$$

for $t \geq 0$. However, this implies

$$\|v_\infty\|_{L^\infty(\Omega)} = \lim_{k_1 \leq k \to \infty} \|v(\cdot, t_k)\|_{L^\infty(\Omega)} \leq \|v_\infty\|_{L^\infty(\Omega)} - \varepsilon,$$

which is a contradiction, hence v_∞ is constant.

Claim 2: The limit fulfills $v_\infty \equiv 0$.

Proof. Suppose $v_\infty \not\equiv 0$, then by the first claim $v_\infty \equiv C$ for some $C > 0$, thus we may choose $k_2 \in \mathbb{N}$ such that $v(\cdot, t_k) \geq \frac{C}{2}$ for all $k \geq k_2$. Then (4.5) implies (as $w \geq 0$ by Lemma 2.2)

$$0 \leq \lim_{k_2 \leq k \to \infty} \int_\Omega w(\cdot, t_k) \leq \lim_{k_2 \leq k \to \infty} \frac{2}{C} \int_\Omega v(\cdot, t_k) w(\cdot, t_k) = 0,$$

hence

$$\lim_{k \to \infty} \int_\Omega w(\cdot, t_k) = 0. \tag{4.6}$$

However, by (2.7) and Lemma 2.2 we have for $k \in \mathbb{N}$

$$\int_\Omega w(\cdot, t_k) = e^{-\delta t_k} \int_\Omega w_0 + \int_\Omega \int_0^{t_k} e^{-\delta(t_k - s)} u(\cdot, s) \, ds \geq \frac{m}{\delta}[1 - e^{-\delta t_k}] \geq \frac{m}{\delta}[1 - e^{-\delta t_1}] > 0,$$

which contradicts (4.6); hence $v_\infty \equiv 0$.

Claim 3: The statement holds.

Proof. Let $\varepsilon > 0$. By the second claim we may choose $k' \in \mathbb{N}$ such that $\|v(\cdot, t_{k'})\|_{C^0(\overline{\Omega})} < \varepsilon$. Therefore, Lemma 2.2 (for initial data $u(\cdot, t_{k'}), v(\cdot, t_{k'}), w(\cdot, t_{k'})$) implies

$$\|v(\cdot, t)\|_{C^0(\overline{\Omega})} < \varepsilon \quad \text{for } t \geq t_{k'},$$

thus the claim and hence the statement follow. □

4.3. Boundedness of u

Lemma 4.6 allows us to show boundedness of u, which is an important step towards proving convergence.

Lemma 4.7. *If Condition 4.1 is fulfilled, then* $\{u(\cdot, t) : t \geq 0\}$ *is bounded in* $L^\infty(\Omega)$.

Proof. By Lemma 4.6 there exists $t_0 > 0$ such that $v(\cdot, t_0) \leq \frac{1}{3 \max\{2, n\}}$. Proposition 3.6 then states that the solution $(\tilde{u}, \tilde{v}, \tilde{w})$ of (P) with initial data

$$\tilde{u}_0 := u(\cdot, t_0), \quad \tilde{v}_0 := v(\cdot, t_0) \quad \text{and} \quad \tilde{w}_0 := w(\cdot, t_0)$$

is bounded: There is $C > 0$ with $\tilde{u}(\cdot, t) \leq C$ for all $t > 0$. As by uniqueness $\tilde{u}(\cdot, t) = u(\cdot, t + t_0)$ and since $u \in C^0(\overline{\Omega} \times [0, t_0])$ the statement follows. □

Lemma 4.8. *If Condition [4.1] is fulfilled, then there exist $\alpha_0 \in (0,1)$ and $C_0 > 0$ such that for all $t_0 \geq 1$ we have*

$$u, v \in C^{2+\alpha_0, 1+\frac{\alpha_0}{2}}(\overline{\Omega} \times [t_0, t_0 + 1]) \quad and \quad w \in C^{\alpha_0, 1+\frac{\alpha_0}{2}}(\overline{\Omega} \times [t_0, t_0 + 1]))$$

with

$$\max \left\{ \|u\|_{C^{2+\alpha_0, 1+\frac{\alpha_0}{2}}(\overline{\Omega}\times[t_0,t_0+1])}, \; \|v\|_{C^{2+\alpha_0, 1+\frac{\alpha_0}{2}}(\overline{\Omega}\times[t_0,t_0+1])}, \; \|w\|_{C^{\alpha_0, 1+\frac{\alpha_0}{2}}(\overline{\Omega}\times[t_0,t_0+1])} \right\} \leq C_0.$$

Proof. Lemma [4.7] and Lemma [3.4] assert the existence of $M > 0$ such that

$$\max \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)}, \; \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right\} \leq M$$

for all $t \geq 0$. Therefore, the statement is mainly a consequence of known parabolic regularity theory (and Lemma [3.1]). Nonetheless, we choose to include a short proof here. For this purpose we at first fix $0 < t_1 < t_2 < t_3 < t_4 < t_5 < 1$.

As [15, Theorem 1.3] warrants that there exist $\alpha_1 \in (0,1)$ and $C_1 > 0$ such that

$$u \in C^{\alpha_1, \frac{\alpha_1}{2}}(\overline{\Omega} \times [t', t'+1]) \quad \text{with} \quad \|u\|_{C^{\alpha_1, \frac{\alpha_1}{2}}(\overline{\Omega}\times[t',t'+1])} \leq C_1 \qquad \text{for all } t' \geq t_1,$$

by Lemma [3.1] there exists $C_2 > 0$ with

$$w(\cdot, t') \in C^{\alpha_2}(\overline{\Omega}) \quad \text{and} \quad \|w(\cdot, t')\|_{C^{\alpha_2}(\overline{\Omega})} \leq C_2 \qquad \text{for all } t' \geq t_2,$$

where $\alpha_2 := \alpha_1$.

Then [8, Theorem IV.5.3] implies the existence of $\alpha_3 \in (0,1)$ and $C_3 > 0$ such that

$$v \in C^{2+\alpha_3, \frac{2+\alpha_3}{2}}(\overline{\Omega} \times [t', t'+1]) \quad \text{with} \quad \|v\|_{C^{2+\alpha_3, \frac{2+\alpha_3}{2}}(\overline{\Omega}\times[t',t'+1])} \leq C_3 \qquad \text{for all } t' \geq t_3.$$

This in turn allows us to employ first [12, Theorem 1.1] and then again [8, Theorem IV.5.3] to obtain $\alpha_4, \alpha_5 \in (0,1)$ and $C_4, C_5 > 0$ such that

$$u \in C^{1+\alpha_4, \frac{1+\alpha_4}{2}}(\overline{\Omega} \times [t', t'+1]) \quad \text{with} \quad \|u\|_{C^{1+\alpha_4, \frac{1+\alpha_4}{2}}(\overline{\Omega}\times[t',t'+1])} \leq C_4 \qquad \text{for all } t' \geq t_4$$

and

$$u \in C^{2+\alpha_5, \frac{2+\alpha_5}{2}}(\overline{\Omega} \times [t', t'+1]) \quad \text{with} \quad \|u\|_{C^{2+\alpha_5, \frac{2+\alpha_5}{2}}(\overline{\Omega}\times[t',t'+1])} \leq C_5 \qquad \text{for all } t' \geq t_5.$$

Finally, the asserted regularity of w follows from (2.7) and the third line in (P). $\square$

4.4. Convergence of u and w

Again, we start by stating a rather weak convergence result:

Lemma 4.9. *If $T_{\max} = \infty$, then*

$$\int_1^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} < \infty. \tag{4.7}$$

Proof. By multiplying the second equation in (P) by v and integrating over $(0, T) \times \Omega$ (for $T > 0$) we obtain

$$\frac{1}{2} \int_\Omega v^2(\cdot, T) - \frac{1}{2} \int_\Omega v_0^2 = \frac{1}{2} \frac{d}{dt} \int_0^T \int_\Omega v^2 = - \int_0^T \int_\Omega |\nabla v|^2 - \int_0^T \int_\Omega v^2 w,$$

hence (as $v, w \geq 0$ by Lemma 2.2)

$$\int_0^T \int_\Omega |\nabla v|^2 \leq \frac{1}{2} \int_\Omega v_0^2 \quad \text{for all } T > 0. \tag{4.8}$$

Furthermore, we have by the first equation in (P), by integrating by parts and using Young's inequality

$$-\frac{d}{dt} \int_\Omega \log u = - \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \frac{\nabla u \cdot \nabla v}{u} \leq -\frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{1}{2} \int_\Omega |\nabla v|^2$$

in $(0, \infty)$. After integration over $(1, T)$ for $T > 1$ this yields

$$\int_1^T \int_\Omega \frac{|\nabla u|^2}{u^2} \leq 2 \int_\Omega \log u(\cdot, T) - 2 \int_\Omega \log u(\cdot, 1) + \int_1^T \int_\Omega |\nabla v|^2$$

$$\leq 2m - 2 \log \inf_{x \in \Omega} u(x, 1) \cdot |\Omega| + \frac{1}{2} \int_\Omega v_0^2$$

by Lemma 2.2 (note that $\log s \leq s$ for $s > 0$), as $-\log$ is decreasing, $u(\cdot, 1) > 0$ in $\overline{\Omega}$ by Lemma 2.2 and due to (4.8). An immediate consequence thereof is (4.7). $\quad\square$

Lemma 4.10. *If Condition 4.1 is fulfilled, then for all $\alpha \in (0, \alpha_0)$ with α_0 as in Lemma 4.8*

$$u(\cdot, t) \to \overline{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0 \quad \text{in } C^{2+\alpha}(\overline{\Omega}) \text{ for } 1 < t \to \infty$$

holds.

Proof. Suppose there are $\varepsilon > 0$ and a sequence $(t_k)_{k \in \mathbb{N}} \subset (0, \infty)$ with $t_k \to \infty$ and

$$\|u(\cdot, t_k) - \overline{u}_0\|_{C^{2+\alpha}(\overline{\Omega})} \geq \varepsilon \quad \text{for } k \in \mathbb{N}. \tag{4.9}$$

As $C^{2+\alpha_0}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{2+\alpha}(\overline{\Omega})$, Lemma 4.8 allows us to find a subsequence $(t_{k_j})_{j \in \mathbb{N}}$ of $(t_k)_{k \in \mathbb{N}}$ along which

$$u(\cdot, t_{k_j}) \to u_\infty \quad \text{in } C^{2+\alpha}(\overline{\Omega}) \text{ as } j \to \infty$$

for some $u_\infty \in C^{2+\alpha}(\overline{\Omega})$.

Hölder's and Poincaré's inequalities as well as (4.7) imply

$$\int_1^\infty \int_\Omega |u - \overline{u}_0| \leq |\Omega|^{\frac{1}{2}} \int_1^\infty \int_\Omega (u - \overline{u}_0)^2$$

$$\leq C \int_1^\infty \int_\Omega |\nabla u|^2$$

$$\leq C \|u\|^2_{L^\infty(\Omega \times (0,\infty))} \int_1^\infty \int_\Omega \frac{|\nabla u|^2}{u^2} < \infty$$

for some $C > 0$. As

$$\sup_{T \geq 1} \|u\|_{C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [T, T+1])} < \infty$$

by Lemma 4.8 for all $T \geq 1$, the map $[1, \infty) \ni t \mapsto \int_\Omega |u(\cdot, t) - \overline{u}_0|$ is uniformly continuous. However, this implies $u_\infty = \overline{u}_0$, which contradicts (4.9). $\quad\square$

Lemma 4.11. *If Condition 4.1 is fulfilled, then $w(\cdot, t) \to \frac{\overline{u}_0}{\delta}$ in $C^0(\overline{\Omega})$ for $t \to \infty$.*

Proof. Let $\varepsilon > 0$. According to Lemma 4.10 we may choose $t_1 > 0$ such that

$$\|u(\cdot, t) - \overline{u}_0\|_{C^0(\overline{\Omega})} < \frac{\varepsilon \delta}{3} \quad \text{for all } t > t_1.$$

Furthermore, there are $t_2, t_3 > 0$ such that

$$e^{-\delta t} \|w_0\|_{C^0(\overline{\Omega})} < \frac{\varepsilon}{3} \quad \text{for all } t > t_2$$

and

$$\frac{\|u - \overline{u}_0\|_{C^0(\overline{\Omega} \times (0, t_1))}}{\delta} e^{-\delta(t - t_1)} < \frac{\varepsilon}{3} \quad \text{for all } t > t_3.$$

Let

$$\tilde{w} : \overline{\Omega} \times [0, \infty) \to \mathbb{R}, \quad (x, t) \mapsto w(x, t) - \frac{\overline{u}_0}{\delta}[1 - e^{-\delta t}],$$

then we have for $t > t_0 := \max\{t_1, t_2, t_3\}$ by the representation formula (2.7)

$$\|\tilde{w}(\cdot, t)\|_{C^0(\overline{\Omega})} = \left\| w(\cdot, t) - \int_0^t e^{-\delta(t-s)} \overline{u}_0 \, ds \right\|_{C^0(\overline{\Omega})}$$

$$\leq e^{-\delta t} \|w_0\|_{C^0(\overline{\Omega})} + \int_0^t e^{-\delta(t-s)} \|u(\cdot, s) - \overline{u}_0\|_{C^0(\overline{\Omega})} \, ds$$

$$< \frac{\varepsilon}{3} + \|u - \overline{u}_0\|_{C^0(\overline{\Omega} \times (0, t_1))} \int_0^{t_1} e^{-\delta(t-s)} \, ds + \frac{\varepsilon \delta}{3} \int_{t_1}^t e^{-\delta(t-s)} \, ds$$

$$= \frac{\varepsilon}{3} + \|u - \overline{u}_0\|_{C^0(\overline{\Omega} \times (0, t_1))} \frac{1}{\delta}[e^{-\delta(t-t_1)} - e^{-\delta t}] + \frac{\varepsilon \delta}{3\delta}[1 - e^{-\delta(t-t_1)}]$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude

$$\lim_{t \to \infty} \|\tilde{w}(\cdot, t)\|_{C^0(\overline{\Omega})} = 0$$

and therefore

$$\lim_{t \to \infty} \left\| w(\cdot, t) - \frac{\overline{u}_0}{\delta} \right\|_{C^0(\overline{\Omega})} \leq \lim_{t \to \infty} \|\tilde{w}(\cdot, t)\|_{C^0(\overline{\Omega})} + \lim_{t \to \infty} \left\| \frac{\overline{u}_0}{\delta} e^{-\delta t} \right\|_{C^0(\overline{\Omega})}$$

$$= 0 + 0 = 0. \quad \square$$

4.5. Improving the type of convergence. Proof of Theorem 1.2

Proposition 4.12. *Suppose Condition 4.1 is fulfilled. Let α_0 be as in Lemma 4.8 and $\alpha \in (0, \alpha_0)$, then (1.8) holds.*

Proof. The statements for u have been shown in Lemma 4.10.

Suppose there exist $\varepsilon > 0$ and a sequence $(t_k)_{k \in \mathbb{N}}$ with $t_k \to \infty$ and

$$\|v(\cdot, t_k) - 0\|_{C^{2+\alpha}(\overline{\Omega})} \geq \varepsilon \quad \text{for } k \in \mathbb{N}. \tag{4.10}$$

By Lemma 4.8 we could then choose a subsequence $(t_{k_j})_{j \in \mathbb{N}}$ of $(t_k)_{k \in \mathbb{N}}$ along which

$$v(\cdot, t_{k_j}) \to v_\infty \quad \text{in } C^{2+\alpha}(\overline{\Omega}) \text{ as } j \to \infty$$

for some $v_\infty \in C^{2+\alpha}(\overline{\Omega})$. However, Lemma 4.6 implies $v_\infty \equiv 0$, which contradicts (4.10).

The statement for w can be shown analogously. □

Finally we are able to prove Theorem 1.2:

Proof of Theorem 1.2. Condition 4.1 is fulfilled by Lemma 4.4.

Let α_0 be as in Lemma 4.8 and $\alpha \in (0, \alpha_0)$. As

$$\bigcup_{T \geq 1} \overline{\Omega} \times [T, T+1] = \overline{\Omega} \times [1, \infty)$$

and $\overline{\Omega} \times [T, T+1]$ is compact for all $T \geq 1$, (1.7) is satisfied, while (1.8) has been shown in Proposition 4.12. □

Acknowledgment

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