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Stokes and Navier-Stokes equations with Navier boundary conditions

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Abstract

We study the stationary Stokes and Navier-Stokes equations with nonhomogeneous Navier boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. We prove the existence and uniqueness of weak and strong solutions in $\boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{W}^{2,p}(\Omega)$ for all $1 < p < \infty$, considering minimal regularity on the friction coefficient α. Moreover, we deduce uniform estimates for the solution with respect to α which enables us to analyze the behavior of the solution when $\alpha \to \infty$.

Keywords: Stokes equations; Nonhomogeneous Navier boundary conditions; Weak solution; L^p-regularity; Navier-Stokes equations; Inf-sup condition

Contents

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1. Introduction

Let Ω be a bounded domain (open and connected) in $\mathbb{R}^3$ with boundary Γ, which might be disconnected, of class $\mathcal{C}^{1,1}$ (any other regularity of the boundary will be precised in the context). Let us consider the flow of a viscous fluid in Ω which is given by the stationary Navier-Stokes equations

$$-\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = \chi \ \text{ in } \Omega, \tag{1.1}$$

where $\boldsymbol{u}$ and π are the velocity field and the pressure of the fluid, respectively; $\boldsymbol{f}$ is the external force acting on the fluid and χ stands for the compressibility condition.

This equation, in a domain with boundary, has been studied extensively with the classical Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \ \text{ on } \Gamma,$$

which was formulated by G. Stokes in 1845. An alternative was suggested even before by C.L. Navier in 1823 [41]. Along with the usual impermeability condition

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \tag{1.2}$$

Navier proposed a slip boundary condition with friction which states that the tangent component of the fluid velocity, instead of being zero, should be proportional to the tangential component of the normal stress at the surface, *i.e.*,

$$2\,[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0} \ \text{ on } \Gamma, \tag{1.3}$$

where $\boldsymbol{n}$ is the unit outward normal vector on Γ, $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the strain tensor, the subscript $\boldsymbol{\tau}$ denotes the tangential component of a vector, *i.e.,* $\boldsymbol{v}_{\boldsymbol{\tau}} := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ and α is the scalar coefficient which measures the tendency of a fluid to slip over the boundary (in literature, it is called *friction coefficient* or the inverse of the *slip length*). Equations (1.2) and (1.3), jointly, are known as the *Navier (slip) boundary conditions* (with friction). Note that, formally, if $\alpha = \infty$, (1.2) and (1.3) are reduced to the Dirichlet boundary condition, which is also known as the *no-slip* condition, and when $\alpha = 0$, (1.2) and (1.3) are referred as the *Navier slip conditions without friction* (also called *full slip condition*).

Although the no-slip hypothesis seems to be in good agreement with experiments, it leads to certain rather surprising conclusions. One of them refers to the absence of collisions of rigid bodies immersed in a linearly viscous fluid [31]. In contrast with the no-slip condition, Navier boundary condition provides a solution which is, physically speaking, more acceptable and nearer to the reality, at least to some of the paradoxical phenomena, resulting from the no-slip condition (for instance, see [37]). For further discussion on Navier boundary condition, see [46], [24], [33] and the references therein.

In a 1959 paper, today classic, S. Agmon, A. Douglas & L. Nirenberg [1] devised a new methodology to address existence, uniqueness and regularity of solutions in L^p-spaces for a broad class of linear elliptic boundary-value equations. The method they introduced is based on the theory of potential, mainly on estimates of singular integrals by A.P. Calderón & A. Zygmund [17]. Their results were later generalized and adapted to linear elliptic boundary-value systems by various authors. The 1964 paper of these same authors [2] is possibly the most significant work in this direction. In the context of fluid mechanics, in particular, for the Stokes linear system with a homogeneous Dirichlet boundary condition, the pioneering work of L. Cattabriga [18] has resulted seminal in this area. In the current paper, we primarily develop a $W^{1,p}$-theory for the solutions of the non-homogeneous Navier-Stokes system (1.1) with the boundary conditions (1.2)-(1.3). As detailed below, to do this, we introduce a new methodology, which makes use of methods different from those developed in the previous classic works.

Let us briefly give here an overview of some related works. Concerning the nonstationary Navier-Stokes equations with Navier boundary condition, there are considerably many works, among other reasons, for studying the limiting viscosity case, *e.g.*, [19], [35], [13], [32], [16], [40]. On the contrary, for the stationary problem, comparatively less works are known. The first paper about basic existence and regularity results is by Solonnikov and Scadilov [48], where they treated the problem for $\alpha = 0$. They proved the existence of weak solutions in $\boldsymbol{H}^1(\Omega)$, which are regular (belongs to $\boldsymbol{H}^2_{loc}(\Omega)$) up to some part of the boundary (except in a neighborhood of the intersection of the two parts) for the stationary Stokes system in $\mathbb{R}^3$ with Dirichlet boundary condition on some part of the boundary and Navier boundary conditions (1.2)-(1.3), with $\alpha = 0$, on the other part. Also, it is worth mentioning the work of Beirão Da Veiga [12], where he proved existence of weak and strong solutions of generalized Stokes problem in $\mathbb{R}^3$ in the L^2-setting and with $\alpha \geq 0$ constant. He did not precise the dependence of the constant with respect to α in the estimate. Recently, Berselli [14] gave some results about very weak solutions, in general L^p-setting, in the special case of a flat domain in $\mathbb{R}^3$ and $\alpha = 0$, which is based on the regularity theory of Poisson equation. In the paper of Amrouche and Rejaiba [7], they proved the existence and regularity of weak, strong and very weak solutions in a bounded domain in $\mathbb{R}^3$ for all $p \in (1, \infty)$ for nonsmooth data, but for $\alpha = 0$ (full-slip condition). In [20], Conca studied a similar system in a smooth perforated bounded domain in $\mathbb{R}^2$, where he discussed the well-posedness of both, linear and nonlinear problems with (1.2)-(1.3), assuming $\alpha \geq 0$. He also proved some convergence results based on homogenization theory. In the work of Medková [39], we can find various other forms of Navier boundary conditions. Furthermore, the numerical study has been done in, *e.g.*, [49] (for $\alpha = 0$) and in [34] (for $\alpha \geq 0$ a function).

To the best of our knowledge, all the available works (in stationary and nonstationary problems) have let α be either a constant or a smooth function. In this article, we analyze the possible minimal regularity of α for the existence of weak and strong solutions in L^p-spaces for all $p \in (1, \infty)$, see (2.2), which provides a more general result. In fact, this is the first work available in the literature about general L^p-well-posedness theory for the Navier slip boundary condition with friction. It turns out that the problem becomes more interesting and difficult when the domain is axially symmetric, due to the presence of a nonzero kernel of the homogeneous problem consisting of the nonzero vectors $\boldsymbol{\beta}$ satisfying $\mathbb{D}\boldsymbol{\beta} = 0$ in Ω and $\boldsymbol{\beta} \cdot \boldsymbol{n} = 0$ on Γ. Deriving some nonstandard Korn-type inequalities (cf. Proposition 3.15), we discussed this nontrivial case in detail. Also, we have assumed the domain is merely $\mathcal{C}^{1,1}$ which might be optimal in most of the cases to obtain existence of solutions in $W^{1,p}$ and $W^{2,p}$ spaces for all $p \in (1, \infty)$. Note that the restriction that α is nonnegative is usual, in order to ensure the conservation of energy. However, mathematically speaking, we can take into account the negative values of α as well. Some authors have studied the evolution system with α negative (for example see [32], [38]), where there were no mathematical difficulties due to the availability of the Gronwall's inequality, which is not the case for the stationary problem. We also prove existence of weak solutions considering a more general right hand side of the form $\boldsymbol{L}^{r(p)} + \operatorname{div} \mathbb{L}^p$, where $r(p) < p$, than the one treated in [7] for the case $\alpha \equiv 0$.

The main novelty of the present work is that we try to find the precise dependence on α of the solution of the Stokes (S) and Navier-Stokes (NS) problems (defined in Section 2), in order to allow α to tend either to ∞ or 0 in (1.3), and then to figure out how the solution behaves. As far as we know, there is not previous work on these issues, even whether α is a smooth function or a constant. We prove that the solution is uniformly bounded with respect to α in Theorem 4.3 and Theorem 6.11, taking into account the geometry of the domain. The proof of Theorem 6.11 is interesting in the sense that it exploits the uniform $\boldsymbol{L}^2$-estimate which follows from the variational formulation and the observation given by Z. Shen [47] that for any $p > 2$, $\boldsymbol{W}^{1,p}$-estimate for (certain) elliptic equations is equivalent to the *weak reverse Hölder inequality* (6.10). Moreover, in Section 7, we prove that if α converges to 0, then the solution of the Stokes equations with Navier boundary conditions converges strongly to the solution of the Stokes equations corresponding to $\alpha = 0$, and if α tends to ∞, then the solution converges strongly to the solution of the Stokes equations with Dirichlet boundary condition. Though these results might seem predictable, their proofs are far from being trivial due to the fact that we need to derive the adequate bounds for the solution of the linear problem with respect to α.

We start with presenting the main results of our work in section 2. In section 3, we introduce the necessary functional framework. We deal with the linear problem in Hilbert space in section 4. First, we deduce the existence of a weak solution by using Lax-Milgram theorem and then, the weak formulation yields the α-independent estimates in $\boldsymbol{H}^1(\Omega)$. Later, the existence of a strong solution is deduced by using the classical method of difference quotients since it directly implies the uniform bounds in $\boldsymbol{H}^2(\Omega)$ with the help of the uniform H^1-estimates. Then, we study the L^p-theory in section 5, which provides a more general existence result (Theorem 5.4) for the solution of the Stokes problem. In section 6, we discuss the estimates of the solution which will be independent of the friction coefficient α in $\boldsymbol{W}^{1,p}(\Omega)$, with $p \neq 2$. In subsection 6.1, we deduce a first estimate, which later is improved in subsection 6.2. It is important to mention that the inf-sup condition, proved in Theorem 6.14, is an interesting result by itself, which arises from our work. Observe that we obtain uniform estimates for the solution for all $\alpha \in (0, \infty)$ when the domain is not axially symmetric. Otherwise, we need α being sufficiently large. This is natural because of the presence of the nontrivial kernel of the Stokes operator in an axially symmetric domain. In section 7, the limit problems are studied as mentioned above. Finally, the nonlinear problem is discussed in section 8. Apart from obtaining the existence of a weak solution by using the classical Galerkin method, the $\boldsymbol{W}^{1,p}$-existence result for $p > 2$ and the limiting cases are based on the theory developed for the linear problem. The existence of a weak solution in $\boldsymbol{W}^{1,p}$, with $p \in \left(\frac{3}{2}, 2\right)$, follows from the same construction given by Serre, see [45].

2. Main results

Before stating the main results, let us briefly introduce some notations, referring to the next sections for precise definitions and complete proofs. Since the case $\alpha \equiv 0$ in (1.3) has already been studied in [7], from here onwards, we consider that $\alpha \not\equiv 0$ on Γ, *i.e.,* if we do not state otherwise, we always assume

$$\alpha \geq 0 \quad \text{on } \Gamma, \quad \alpha > 0 \quad \text{on some } \Gamma_0 \subset \Gamma \text{ with } |\Gamma_0| > 0 \tag{2.1}$$

and the following regularity on α:

$$\alpha \in L^{t(p)}(\Gamma) \quad \text{with} \quad t(p) = \begin{cases} 2 & \text{if} \quad p = 2, \\ 2 + \varepsilon & \text{if} \quad \frac{3}{2} \leq p \leq 3, \, p \neq 2, \\ \frac{2}{3}\max\{p, p'\} + \varepsilon & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\varepsilon > 0$ is arbitrarily small and p' is the conjugate exponent of p. The idea is to choose $t(p)$ in such a way that the boundary integral $\int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}$ becomes well-defined for $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$. This is required for the notion of weak solution, see Lemma 3.9.

We also need the following exponent to define the space for the external force $\boldsymbol{f}$:

$$r(p) = \begin{cases} \max\left\{1, \frac{3p}{p+3}\right\} & \text{if} \quad p \neq \frac{3}{2}, \\ 1 + \varepsilon & \text{if} \quad p = \frac{3}{2}, \end{cases} \tag{2.3}$$

where $\varepsilon > 0$ is arbitrarily small. Here as well, the motivation to choose $r(p)$, as given before, is that the continuous embedding $W^{1,p'}(\Omega) \hookrightarrow L^{r(p)'}(\Omega)$ holds for all $p \in (1, \infty)$ which is essential to deduce the Green formula and define the notion of weak solution of our problem, see Lemma 3.6. Let $L^p_0(\Omega)$ denote the following space:

$$L^p_0(\Omega) := \left\{ v \in L^p(\Omega) : \int_\Omega v = 0 \right\}.$$

We use the term *axisymmetric* to define a nonempty set which is generated by rotation around an axis. We also introduce the vector

$$\boldsymbol{\beta}(x) = \boldsymbol{b} \times \boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^3 \tag{2.4}$$

when Ω is axisymmetric with respect to a constant vector $\boldsymbol{b} \in \mathbb{R}^3$.

We can always reduce the nonvanishing divergence problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases}$$

where $\mathbb{F}$ is a 3×3 matrix and $\boldsymbol{h}$ is a tangential vector on the boundary, (*i.e.,* $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ), to the case where div $\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, by solving the following Neumann problem:

$$\Delta \theta = \chi \quad \text{in } \Omega, \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on } \Gamma,$$

and hence, we perform the change of unknowns $\boldsymbol{w} = \boldsymbol{u} - \nabla \theta$ and $\Pi = \pi - \chi$ (we do not mention here the corresponding regularity results). Therefore, it is sufficient to study the following Stokes problem:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{S}$$

The first main result is the existence and uniqueness of weak and strong solutions of the Stokes problem (S).

Theorem 2.1 ***(Weak and strong solutions of Stokes problem)****. Let* $p \in (1, \infty)$*. If*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \ \mathbb{F} \in \mathbb{L}^p(\Omega), \ \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma),$$

where $t(p)$ *and* $r(p)$ *are defined in* (2.2) *and* (2.3)*, respectively, then the Stokes problem* (S) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$.

Moreover, if $\mathbb{F} = 0$ *and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ *if* $p \le \frac{3}{2}$ *and* $q = p$ *otherwise, then the solution* $(\boldsymbol{u}, \pi)$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Also, we obtain uniform bounds for the weak solution with respect to α of the problem (S) in $\boldsymbol{W}^{1,p}(\Omega)$ for all $p \in (1, \infty)$.

Theorem 2.2 ***(Stokes estimates)****. Let* $p \in (1, \infty)$ *and* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ *be the solution of the Stokes problem* (S) *given by Theorem* 2.1*. Then, it satisfies the following estimates:*

(i) *if* Ω *is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

(ii) *if* Ω *is axisymmetric and* $\alpha \ge \alpha_* > 0$*, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le \frac{C_p(\Omega)}{\min\{2, \alpha_*\}} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

The next theorem shows the existence of weak and strong solutions, with corresponding estimates, for the following Navier-Stokes problem:

$$\begin{cases} -\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{NS}$$

Theorem 2.3 ***(Weak and strong solutions of Navier-Stokes problem).*** *Let $p \in (\frac{3}{2}, \infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega),\ \mathbb{F} \in \mathbb{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

1. *Then, the problem (NS) has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$.*
2. *For any $p \in (1, \infty)$, if $\mathbb{F} = 0$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in W^{1-\frac{1}{q},q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \le \frac{3}{2}$ and $q = p$ otherwise, then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.
3. *For $p = 2$, the weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ satisfies the following estimates:*
a) *If Ω is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.5}$$

b) *If Ω is axisymmetric and*
(i) $\alpha \ge \alpha_* > 0$ on Γ, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le \frac{C(\Omega)}{\min\{2, \alpha_*\}} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.6}$$

(ii) $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

with $\boldsymbol{\beta}$ as in (2.4), then the solution $\boldsymbol{u}$ satisfies $\int_\Gamma \alpha \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ and

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \|\pi\|^2_{L^2(\Omega)} \le C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)^2. \tag{2.7}$$

In particular, if α is a constant, then $\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{2.8}$$

Remark 2.4. Note that in the case of $\boldsymbol{u}\cdot\boldsymbol{n}\neq 0$ on Γ, when Ω has multiply connected boundary, the existence of solutions of the Navier-Stokes equations with Dirichlet boundary condition is not yet clear in a complete generality, *e.g.*, see [36]. For this reason, although we are working with Navier boundary conditions, we do not consider Ω with multiply connected boundary either.

The last interesting result to mention is the strong convergence of (NS) to the Navier-Stokes equations with no-slip boundary condition when α tends to infinity.

Theorem 2.5 ***(Limit case for Navier-Stokes problem)****. Let $p\geq 2$, α be a constant and $(\boldsymbol{u}_\alpha,\pi_\alpha)$ be a solution of* (NS)*, where*

$$\boldsymbol{f}\in\boldsymbol{L}^{r(p)}(\Omega),\ \ \mathbb{F}\in\mathbb{L}^p(\Omega)\ \text{and}\ \boldsymbol{h}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Then, for any $q<p$ if $p\neq 2$ and for $q=2$ if $p=2$, we have

$$(\boldsymbol{u}_\alpha,\pi_\alpha)\to(\boldsymbol{u}_\infty,\pi_\infty)\quad \text{in}\quad \boldsymbol{W}^{1,q}(\Omega)\times L^q_0(\Omega)\quad \text{as}\quad \alpha\to\infty,$$

where $(\boldsymbol{u}_\infty,\pi_\infty)$ is a solution of the Navier-Stokes problem with Dirichlet boundary condition

$$\begin{cases}-\Delta\boldsymbol{u}_\infty+(\boldsymbol{u}_\infty\cdot\nabla)\boldsymbol{u}_\infty+\nabla\pi_\infty=\boldsymbol{f}+\operatorname{div}\mathbb{F} & \text{in }\ \Omega,\\ \operatorname{div}\boldsymbol{u}_\infty=0 & \text{in }\ \Omega,\\ \boldsymbol{u}_\infty=\boldsymbol{0} & \text{on }\ \Gamma.\end{cases}\tag{2.9}$$

3. Notations and preliminary results

In this section we review some of the basic notations and the functional framework that we shall require for the rest of the article. The vector fields and matrix fields (and the corresponding spaces) defined over Ω or over $\mathbb{R}^3$ are denoted by bold font and blackboard bold font, respectively. We follow the convention that C is an unspecified positive constant that may vary among inequalities, but not among equalities. Generally, C depends on Ω and the dependence of C on other parameters will be specified within parenthesis when it is necessary.

The vector-valued Laplace operator of a vector field $\boldsymbol{v}=(v_1,v_2,v_3)$ is equivalently defined by

$$\Delta\boldsymbol{v}=2\operatorname{div}\mathbb{D}\boldsymbol{v}-\mathbf{grad}\operatorname{div}\boldsymbol{v}.$$

We denote by $\boldsymbol{\mathcal{D}}(\Omega)$ the space of smooth functions (infinitely differentiable) with compact support in Ω, and by $\boldsymbol{\mathcal{D}}'(\Omega)$ its dual space which is known as the space of distributions. Define

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega):=\{\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega):\ \operatorname{div}\boldsymbol{v}=0\text{ in }\Omega\}.$$

If $p\in(1,\infty)$, p' denotes the conjugate exponent of p, *i.e.*, $\frac{1}{p}+\frac{1}{p'}=1$. Also, for $p<3$, p^* denotes the Sobolev conjugate, *i.e.*, $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{3}$. For $p,r\in(1,\infty)$, we introduce the following space

$$\boldsymbol{H}^{r,p}(\mathrm{div},\Omega):=\left\{\boldsymbol{v}\in\boldsymbol{L}^r(\Omega):\ \operatorname{div}\boldsymbol{v}\in L^p(\Omega)\right\}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{r,p}(\mathrm{div},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|_{L^p(\Omega)}.$$

It can be proved that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^{r,p}(\mathrm{div},\Omega)$ (cf. [8, Lemma 13, (i)]). The closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}^{r,p}(\mathrm{div},\Omega)$ is denoted by $\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)$ and it can be characterized as

$$\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^{r,p}(\mathrm{div},\Omega) \colon\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \Gamma\right\}.$$

This characterization can be proved as it was done for the case $r = p = 2$, see [28, Theorem 2.6]. Also, for $p \in (1,\infty)$, the dual space of $\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)$, which is denoted by $[\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)]'$, can be characterized as follows (cf. [44, Proposition 1.0.4]):

Proposition 3.1. *Let Ω be a Lipschitz domain. A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)]'$ iff there exists $\boldsymbol{\psi} \in \boldsymbol{L}^{r'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi$. Moreover, we have the estimate:*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)]'} \leq \inf_{\boldsymbol{f}=\boldsymbol{\psi}+\nabla\chi} \max\{\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\chi\|_{L^{p'}(\Omega)}\}.$$

We also recall the following result (cf. [10, Theorem 3.5]):

Proposition 3.2. *Let $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ with $\mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)$, $\mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v}\cdot\boldsymbol{n} \in W^{1-\frac{1}{p},p}(\Gamma)$. Then $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\left(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)}\right).$$

Further, we need to introduce the following spaces:

$$\boldsymbol{V}_{\sigma,\tau}^p(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \colon\ \mathrm{div}\ \boldsymbol{v} = 0 \ \text{ in } \Omega \ \text{ and } \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma\right\}$$

equipped with the norm of $\boldsymbol{W}^{1,p}(\Omega)$,

$$\boldsymbol{H}_{\boldsymbol{\tau}}^1(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \colon \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma\right\},$$

and

$$\boldsymbol{E}^p(\Omega) := \left\{(\boldsymbol{v},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega) : -\Delta\boldsymbol{v} + \nabla\pi \in \boldsymbol{L}^{r(p)}(\Omega)\right\},$$

with $r(p)$ defined in (2.3). Note that $\boldsymbol{E}^p(\Omega)$ is a Banach space with the norm

$$\|(\boldsymbol{v},\pi)\|_{\boldsymbol{E}^p(\Omega)} := \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} + \|-\Delta\boldsymbol{v} + \nabla\pi\|_{\boldsymbol{L}^{r(p)}(\Omega)}.$$

Next, introducing the notation

$$\boldsymbol{\Lambda v} := \sum_{j=1}^{3} (v_{\boldsymbol{\tau}})_j \, \nabla_{\boldsymbol{\tau}} n_j,$$

where $\boldsymbol{v}_{\boldsymbol{\tau}} := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ and $\nabla_{\boldsymbol{\tau}}$ is the tangential gradient, we recall the following relations which show the equivalence of the following two boundary conditions: (1.3) and the *Navier-type boundary condition*

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}. \tag{3.1}$$

The name "Navier-type" for the above boundary condition comes from the equivalence relation (3.2) given below. The boundary condition (3.1) will be used later to prove some of our main results.

Lemma 3.3. [7, Appendix A] *For any $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$, we have the following identities:*

$$2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v} \cdot \boldsymbol{n}) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda v}\,,$$

$$\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = -\nabla_{\boldsymbol{\tau}}(\boldsymbol{v} \cdot \boldsymbol{n}) + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} + \boldsymbol{\Lambda v}\,.$$

Note that Ω of class $\mathcal{C}^{1,1}$ is sufficient and, in contrast with the relations given in [7], there is a change in the sign of the second relation. These are the correct identities.

Remark 3.4. The reason why is enough to consider Ω of class $\mathcal{C}^{1,1}$ to prove the above lemma is clear because in the proof given in [7, Appendix A], $\mathcal{C}^{2,1}$ regularity is not required anywhere. For example, in a domain of class $\mathcal{C}^{1,1}$, $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $W^{2,p}(\Omega)$ and $\boldsymbol{n} \in W^{1,\infty}(\Gamma)$. Then, all the calculations follow in the same way.

Remark 3.5. In the particular case $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ, for all $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$, we obtain

$$2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda v} \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\right)_{\boldsymbol{\tau}} + \boldsymbol{\Lambda v},$$

which implies that

$$2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda v}. \tag{3.2}$$

Note that in the case of a flat boundary, $\Lambda = 0$ and hence, the Navier slip and the Navier-type boundary conditions become equal, provided $\alpha = 0$.

Next, we give the following Green formula to define the tangential trace of the strain tensor of a vector field. The proof of the density result is similar to [29, Lemma 1.5.3.9], while the Green formula follows by proving it firstly for smooth functions by using integration by parts, and then extending the result by density (cf. [7, Lemma 2.4]).

Lemma 3.6. **(i)** *The space* $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ *is dense in* $\boldsymbol{E}^p(\Omega)$, *and*
(ii) *the linear mapping* $(\boldsymbol{v}, \pi) \mapsto [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$, *defined on* $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ *can be extended to a linear, continuous map from* $\boldsymbol{E}^p(\Omega)$ *to* $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. *Moreover, we have the following relation: for all* $(\boldsymbol{v}, \pi) \in \boldsymbol{E}^p(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$\int_\Omega (-\Delta \boldsymbol{v} + \nabla\pi) \cdot \boldsymbol{\varphi} = 2\int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} - 2\,\langle[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\varphi}\rangle_\Gamma\,, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ *will denote, from now onwards,* $\langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$.

Remark 3.7. 1. The following Green formula also can be obtained in the same way as it was done for (3.3), which will be used later: for $(\boldsymbol{v}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$ such that $-\mathrm{div}(2\mathbb{D}\boldsymbol{v} + \mathbb{F}) + \nabla\pi \in \boldsymbol{L}^{r(p)}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$\int_\Omega (-\mathrm{div}(2\mathbb{D}\boldsymbol{v} + \mathbb{F}) + \nabla\pi) \cdot \boldsymbol{\varphi} = 2\int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} + \int_\Omega \mathbb{F} : \nabla\boldsymbol{\varphi} - \langle[(2\mathbb{D}\boldsymbol{v} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\varphi}\rangle_\Gamma\,. \tag{3.4}$$

2. In fact, we can obtain Lemma 3.6 for any $\boldsymbol{v} \in \boldsymbol{F}^p(\Omega)$, where

$$\boldsymbol{F}^p(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) :\ \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{r(p)',p'}(\mathrm{div},\Omega)]'\right\}.$$

Thus, (3.2) can be extended to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ as follows: for Ω a bounded domain of class $\mathcal{C}^{1,1}$ and for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\Delta\boldsymbol{v} \in \boldsymbol{L}^{r(p)}(\Omega)$ and $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ,

$$2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} - 2\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma). \tag{3.5}$$

We will also need the following density result:

Lemma 3.8. *The space* $\left\{\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega) : \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'\right\}$ *is dense in* $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$.

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. There exists a sequence $\boldsymbol{u}_m \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that $\boldsymbol{u}_m \to \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$. Now consider the problem

$$\begin{cases} \Delta\chi_m &= \mathrm{div}\,\boldsymbol{u}_m & \text{in } \Omega \\ \frac{\partial\chi_m}{\partial\boldsymbol{n}} &= \boldsymbol{u}_m\cdot\boldsymbol{n} & \text{on } \Gamma. \end{cases}$$

Since Ω is of class $\mathcal{C}^{1,1}$, there exists a unique solution of the above problem $\chi_m \in H^2(\Omega) \cap L^2_0(\Omega)$. Also $\chi_m \to 0$ in $H^2(\Omega)$. Now, considering $\boldsymbol{v}_m = \boldsymbol{u}_m - \nabla\chi_m$, we have $\boldsymbol{v}_m \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ with $\Delta\boldsymbol{v}_m = \Delta\boldsymbol{u}_m - \nabla(\mathrm{div}\,\boldsymbol{u}_m) \in [\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'$ and $\boldsymbol{v}_m \to \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega)$. This completes the proof. $\square$

Lemma 3.9. *Let* $p \in (1,\infty)$. *For* $\alpha \in L^{t(p)}(\Gamma)$ *with* $t(p)$ *defined in (2.2),* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$, *the integral over the boundary* $\int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{\varphi}_{\boldsymbol{\tau}}$ *is well-defined.*

Proof. We use the following Sobolev embeddings $\boldsymbol{\varphi}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma) \hookrightarrow \boldsymbol{L}^m(\Gamma)$, where

$$\frac{1}{m} = \begin{cases} 1 - \frac{3}{2p} & \text{if } p > \frac{3}{2} \\ \text{any positive real number} < 1 & \text{if } p = \frac{3}{2} \\ 0 & \text{if } p < \frac{3}{2} \end{cases} \tag{3.6}$$

and $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{L}^s(\Gamma)$ with

$$\frac{1}{s} = \begin{cases} \frac{3}{2p} - \frac{1}{2} & \text{if } p < 3 \\ \text{any positive real number} < 1 & \text{if } p = 3 \\ 0 & \text{if } p > 3. \end{cases} \tag{3.7}$$

It is enough to show that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{m'}(\Gamma)$ by distinguishing four cases: $p = 2$, $\frac{3}{2} \le p \le 3$, $p > 3$ and $p < \frac{3}{2}$.

First, let us consider $p = 2$. Since $\alpha \in L^2(\Gamma)$, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ by Hölder inequality. But $\frac{1}{m'} = 1 - \frac{1}{m} = \frac{3}{4}$, *i.e.*, $q = m'$. So the integral is well-defined.

The other cases can be proved in the same way. $\square$

Definition 3.10. Given $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$, a function $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is called a weak solution of the Stokes system (S) if it satisfies that for all $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma . \tag{3.8}$$

Proposition 3.11. *Let* $p \in (1, \infty)$ *and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \textit{ and } \alpha \in L^{t(p)}(\Gamma)$$

with $r(p)$ *and* $t(p)$ *defined by* (2.3) *and* (2.2)*, respectively. Then, the following two statements are equivalent:*
(i) $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ *is a weak solution of* (S)*, in the sense of Definition* 3.10*, and*
(ii) *there exists* $\pi \in L^p_0(\Omega)$ *such that* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ *satisfies*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \ \operatorname{div} \boldsymbol{u} = 0 & \textit{in the sense of distributions,} \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \textit{in the sense of traces,} \\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \textit{in } \boldsymbol{W}^{-1/p,p}(\Gamma). \end{cases} \tag{3.9}$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ be a weak solution of (S). Choosing $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ as a test function in (3.8), we have

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi}.$$

Then, De Rham's theorem implies that there exists $\pi \in \mathcal{D}'(\Omega)$ such that

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F} \quad \text{in } \Omega, \tag{3.10}$$

and since $-\Delta \boldsymbol{u} - \boldsymbol{f} - \operatorname{div} \mathbb{F} \in \boldsymbol{W}^{-1,p}(\Omega)$, [6, Lemma 2.7] yields that $\pi \in L^p(\Omega)$, which is defined uniquely up to an additive constant. A different proof for the existence of a suitable pressure without using De Rham's theorem is established in [23, Theorem III.5.3] (see also [43]). Also, $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ implies $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Thus, it remains to prove that u satisfies the Navier boundary condition. As $-\operatorname{div}(2\mathbb{D}\boldsymbol{u} + \mathbb{F}) + \nabla \pi \in \boldsymbol{L}^{r(p)}(\Omega)$, taking dual product of equation (3.10) with $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$, and using the Green's formula (3.4), we have

$$\langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_\Gamma + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega). \tag{3.11}$$

Now, let $\boldsymbol{\mu} \in \boldsymbol{W}^{\frac{1}{p},p'}(\Omega)$. There exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. Then $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ and using (3.11), it follows

$$\begin{aligned}
\langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu} \rangle_\Gamma &= \langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\mu}_{\boldsymbol{\tau}} \rangle_\Gamma \\
&= \langle [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma = 0.
\end{aligned}$$

Hence,

$$[(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} \qquad \text{in } \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Conversely, if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies (3.9), then using the Green formula (3.4), we can easily deduce that $\boldsymbol{u}$ is a weak solution of (S), in the sense of Definition 3.10. □

The next lemma provides a general pressure estimate.

Lemma 3.12. *Suppose $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$. If $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is a weak solution of the Stokes system (S), then the pressure $\pi \in L^p_0(\Omega)$, whose existence follows from Proposition 3.11, satisfies:*

$$\|\pi\|_{L^p(\Omega)} \le C(\Omega, p) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\Delta \boldsymbol{u}\|_{\boldsymbol{W}^{-1,p}(\Omega)} \right). \tag{3.12}$$

Proof. Due to the properties of the gradient operator (cf. [6, ii) Corollary 2.5]), we can write,

$$\|\pi\|_{L^p(\Omega)} \le \|\nabla \pi\|_{\boldsymbol{W}^{-1,p}(\Omega)} \le C(\Omega) \|\Delta \boldsymbol{u} + \boldsymbol{f} + \operatorname{div} \mathbb{F}\|_{\boldsymbol{W}^{-1,p}(\Omega)}$$

where the last inequality comes from (3.10). This concludes the proof. □

The following two propositions offer some Korn-type inequalities which will be useful in the context.

Proposition 3.13. *Let Ω be a bounded Lipschitz domain. Then, for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, we have*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} \quad \textit{if } \Omega \textit{ is nonaxisymmetric}, \tag{3.13}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma)} \quad \textit{if } \Omega \textit{ is axisymmetric}. \tag{3.14}$$

More generally, if Ω is axisymmetric and $\alpha \in L^2(\Gamma)$ satisfies (2.1), then the following equivalence holds

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} + \|\sqrt{\alpha}\, \boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma_0)}. \tag{3.15}$$

Here, "$\simeq$" denotes the equivalence of two norms.

Proof. The inequality $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega)\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$ follows from [7, Lemma 3.3] and the reverse inequality is obvious, which gives (3.13).

To prove (3.14), it is enough to show the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C \left(\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^2(\Gamma)} \right)$$

which can be proved by classical contradiction argument (for example, see [22, Section 5.8.1, Theorem 1] or the following proof).

In order to show (3.15), we prove by contradiction, the following inequality:

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C(\Omega, \alpha) \left(\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \|\sqrt{\alpha}\, \boldsymbol{u}_{\boldsymbol{\tau}}\|^2_{\boldsymbol{L}^2(\Gamma_0)} \right).$$

Indeed, suppose that for all $m \in I\!N$, there exists $\boldsymbol{u}_m \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}_m \cdot \boldsymbol{n} = 0$ on Γ, $|||\boldsymbol{u}_m||| = 1$, with $|||\boldsymbol{u}||| := \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$, and

$$1 > m \left[\|\mathbb{D}\boldsymbol{u}_m\|^2_{\mathbb{L}^2(\Omega)} + \int_{\Gamma_0} \alpha |\boldsymbol{u}_m|^2 \right]. \tag{3.16}$$

Thus, $\{\boldsymbol{u}_m\}_m$ is a bounded sequence in $\boldsymbol{H}^1(\Omega)$ and there exist a subsequence, still denoted by $\{\boldsymbol{u}_m\}_m$, and $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$. This implies $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ and $\boldsymbol{u}_m \to \boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$; but, from (3.16), we deduce that $\mathbb{D}\boldsymbol{u} = \boldsymbol{0}$ in Ω which implies $\boldsymbol{u} = c\boldsymbol{\beta}$ for some $c \in \mathbb{R}$ and $\boldsymbol{\beta}$ as in (2.4). Also, $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma) \underset{\text{compact}}{\hookrightarrow} \boldsymbol{L}^2(\Gamma)$, and then $\boldsymbol{u}_m \to \boldsymbol{u}$ in $\boldsymbol{L}^2(\Gamma)$. For a.e. $\boldsymbol{x}$ on Γ, we have, up to a subsequence, $\boldsymbol{u}_m(\boldsymbol{x}) \to \boldsymbol{u}(\boldsymbol{x})$ and then, for a.e. $\boldsymbol{x}$ on Γ_0, $\sqrt{\alpha}\, \boldsymbol{u}_m(\boldsymbol{x}) \to \sqrt{\alpha}\, \boldsymbol{u}(\boldsymbol{x})$. However, from (3.16) we know that $\sqrt{\alpha}\, \boldsymbol{u}_m \to \boldsymbol{0}$ in $L^2(\Gamma_0)$. Consequently, since $\alpha > 0$ on Γ_0, we have $\boldsymbol{u} = \boldsymbol{0}$ almost everywhere on Γ_0, which implies that the constant c is equal to zero and then $\boldsymbol{u} = \boldsymbol{0}$ in Ω. Finally,

$$1 = |||\boldsymbol{u}_m||| = \|\boldsymbol{u}_m\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}_m\|_{\mathbb{L}^2(\Omega)} \to 0,$$

which is a contradiction. □

Remark 3.14. Let us consider, for given α, the kernel $\mathcal{T}_\alpha(\Omega)$ of the Stokes operator with Navier slip boundary conditions, *i.e.*, a function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ belongs to $\mathcal{T}_\alpha(\Omega)$ if there exists $\pi \in L^2_0(\Omega)$ such that $(\boldsymbol{u}, \pi)$ satisfies (S) in the weak sense of Definition 3.10, with $\boldsymbol{f} = \boldsymbol{0}$, $\boldsymbol{h} = \boldsymbol{0}$ and $\mathbb{F} = 0$. Then, we have the energy estimate

$$2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = 0,$$

with $\alpha \geq 0$ on Γ. Hence, $\mathbb{D}\boldsymbol{u} = 0$ in Ω implies that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x} + \boldsymbol{c}$ *a.e.* $\boldsymbol{x} \in \Omega$ (in fact, this identity holds for all $x \in \overline{\Omega}$ when $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{C}^0(\overline{\Omega})$), where $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ are arbitrary constant vectors. Further, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ yields $\boldsymbol{c} = \boldsymbol{0}$.

a) If $\alpha > 0$ on Γ_0, then $\boldsymbol{b} \times \boldsymbol{x} = \boldsymbol{0}$, for any $\boldsymbol{x} \in \Gamma_0$ and thus $\boldsymbol{b} = \boldsymbol{0}$, *i.e.,* $\mathcal{T}_\alpha(\Omega) = \{\boldsymbol{0}\}$.

b) If $\alpha \equiv 0$ on Γ, we can verify easily that

- **i)** $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x}$ if Ω is axisymmetric, *i.e.,* $\boldsymbol{b}$ is co-linear to the axis of Ω and $\dim \mathcal{T}_0(\Omega) = 1$.
- **ii)** $\boldsymbol{u} = \boldsymbol{0}$ if Ω is nonaxisymmetric, *i.e.,* $\mathcal{T}_0(\Omega) = \{\boldsymbol{0}\}$.

Proposition 3.15. *Let Ω be a Lipschitz bounded domain. For Ω axisymmetric, we have the following inequalities: for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ,*

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C \left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} \right)^2 \right] \tag{3.17}$$

and

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C \left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} \right)^2 \right], \tag{3.18}$$

with $\boldsymbol{\beta}$ as in (2.4).

Proof. **(i)** First, recall from (2.4) that $\boldsymbol{\beta} \in \mathcal{C}^\infty(\mathbb{R}^3)$ and $\mathbb{D}\boldsymbol{\beta} = 0$ in $\mathbb{R}^3$. Then, (3.17) follows from the following result [7, Lemma 3.3]:

$$\inf_{\boldsymbol{w} \in \mathcal{T}_0(\Omega)} \|\boldsymbol{u} + \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} \leq C(\Omega) \left(\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma |\boldsymbol{u} \cdot \boldsymbol{n}|^2 \right), \tag{3.19}$$

where $\mathcal{T}_0(\Omega)$ is the kernel of the Stokes operator with Navier boundary conditions corresponding to $\alpha \equiv 0$ (cf. Remark 3.14). Since Ω is axisymmetric, $\boldsymbol{w} = c\boldsymbol{\beta}$ for some $c \in \mathbb{R}$, therefore $\inf_{\boldsymbol{w} \in \mathcal{T}_0(\Omega)} \|\boldsymbol{u} + \boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} = \inf_{c \in \mathbb{R}} \|\boldsymbol{u} + c\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}$ and this infimum is attained at

$$c = \frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}} \left(\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} \right).$$

Then (3.17) follows from

$$\left\| \boldsymbol{u} - \frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}} \left(\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} \right) \boldsymbol{\beta} \right\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}} \left(\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} \right)^2 .$$

(ii) Now, we prove the inequality (3.18) by the same contradiction argument as in (3.15). Let us assume that for all $m \in I\!N$, there exists $\boldsymbol{u}_m \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{u}_m \cdot \boldsymbol{n} = 0$ on Γ, $|||\boldsymbol{u}_m||| = 1$, where $|||\boldsymbol{u}||| := \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$, and

$$1 > m \left[\|\mathbb{D}\boldsymbol{u}_m\|^2_{\mathbb{L}^2(\Omega)} + \left(\int_\Gamma \boldsymbol{u}_m \cdot \boldsymbol{\beta} \right)^2 \right]. \tag{3.20}$$

Thus, $\{\boldsymbol{u}_m\}_m$ is a bounded sequence in $\boldsymbol{H}^1(\Omega)$, and then, there exist a subsequence, which we still call it $\{\boldsymbol{u}_m\}_m$ and $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ so that $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$. This implies that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ and $\boldsymbol{u}_m \to \boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$; but, from (3.20), we have

$$\mathbb{D}\boldsymbol{u}_m \to 0 \ \text{ in } \ \boldsymbol{L}^2(\Omega) \quad \text{ and } \quad \int_\Gamma \boldsymbol{u}_m \cdot \boldsymbol{\beta} \to 0.$$

Then, $\mathbb{D}\boldsymbol{u} = 0$ in Ω which implies that $\boldsymbol{u} = c\boldsymbol{\beta}$ for some $c \in \mathbb{R}$. But, $\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ is compactly embedded in $\boldsymbol{L}^2(\Gamma)$, and then, $\boldsymbol{u}_m \to \boldsymbol{u}$ in $\boldsymbol{L}^2(\Gamma)$. Therefore, we have $\boldsymbol{u}_m \cdot \boldsymbol{\beta} \to \boldsymbol{u} \cdot \boldsymbol{\beta}$ in $\boldsymbol{L}^2(\Gamma)$, which yields $c\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Gamma)} = \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$. This implies $c = 0$, and hence, $\boldsymbol{u} = \boldsymbol{0}$ in Ω. Finally,

$$1 = |||\boldsymbol{u}_m||| = \|\boldsymbol{u}_m\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}_m\|_{\mathbb{L}^2(\Omega)} \to 0,$$

which is a contradiction. □

4. Stokes equations: L^2-theory

In this section, we study the well-posedness, in the Hilbertian case, of solutions of the Stokes problem (S). First, we prove the existence and uniqueness of the weak solution.

Theorem 4.1 *(**Existence in $\boldsymbol{H}^1(\Omega)$**). Let Ω be a Lipschitz bounded domain, and*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \mathbb{F} \in \mathbb{L}^2(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ and } \alpha \in L^2(\Gamma),$$

where $\alpha > 0$ on $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| > 0$. Then, the Stokes problem (S) *has a unique weak solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ in the sense of Definition 3.10 which satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\alpha)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{4.1}$$

Proof. The existence of a unique weak solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ of (S) follows from the Lax-Milgram theorem. The bilinear form

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} \tag{4.2}$$

is clearly continuous on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ since

$$\begin{aligned}|a(\boldsymbol{u}, \boldsymbol{\varphi})| &\leq \max\{2, \|\alpha\|_{L^2(\Gamma)}\}\left(\|\mathbb{D}\boldsymbol{u}\|_{L^2(\Omega)}\|\mathbb{D}\boldsymbol{\varphi}\|_{L^2(\Omega)} + \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^4(\Gamma)}\|\boldsymbol{\varphi}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^4(\Gamma)}\right)\\ &\leq C(\alpha, \Omega)\,\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Also it is coercive on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ due to Proposition 3.13. Moreover, the linear form $\boldsymbol{\ell} : \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \to \mathbb{R}$, defined as

$$\boldsymbol{\ell}(\boldsymbol{\varphi}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}$$

is continuous on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Hence, the Lax-Milgram theorem gives the existence of a unique $\boldsymbol{u} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ satisfying (3.8). This completes the proof. The estimate (4.1) follows easily from the variational formulation (3.8). $\square$

Remark 4.2. Note that if $\alpha > 0$ on some $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| > 0$, then we get the uniqueness of the solution of the Stokes problem (S). However, for the case $\alpha \equiv 0$ on Γ, there is a nontrivial kernel when Ω is axisymmetric (cf. [7, Theorem 3.4]). See Remark 3.14 for more details.

In the next theorem, we improve the estimate (4.1) with respect to α in some particular cases.

Theorem 4.3 ***(Estimates in $\boldsymbol{H}^1(\Omega)$).*** *With the same assumptions on $\boldsymbol{f}, \mathbb{F}, \boldsymbol{h}$ and α as in Theorem 4.1, the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ of the Stokes problem (S) satisfies the following estimates:*
a) *if Ω is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{4.3}$$

b) *if Ω is axisymmetric and*
(i) $\alpha \geq \alpha_* > 0$ on Γ, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{4.4}$$

(ii) $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition

$$\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_{\Omega} \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_{\Gamma} = 0, \tag{4.5}$$

then the solution $\boldsymbol{u}$ *satisfies* $\int_{\Gamma} \alpha \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ *and*

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_{\Gamma} \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 + \|\pi\|^2_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)^2. \tag{4.6}$$

In particular, if α *is a nonzero constant, then* $\int_{\Gamma} \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ *and*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{4.7}$$

Remark 4.4. Note that in the case of Ω axisymmetric, if α is a nonzero constant, we can use the estimate (4.4) with $\alpha = \alpha_*$. In particular, if $\alpha = \frac{1}{n}, n \in \mathbb{N}^*$, the corresponding solution $(\boldsymbol{u}_n, \pi_n)$ satisfies

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + \|\pi_n\|_{L^2(\Omega)} \leq nC(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right).$$

However, this estimate is not optimal when we suppose (4.5). In fact, because of $\int_{\Gamma} \boldsymbol{u}_n \cdot \boldsymbol{\beta} = 0$, we have a better estimate by using (4.7):

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + \|\pi_n\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right),$$

where $C(\Omega)$ does not depend on n. This means that if $\alpha \to 0$, then (4.7) is a better estimate than (4.4).

Proof. The solution $\boldsymbol{u}$ satisfies

$$2\int_{\Omega} |\mathbb{D}\boldsymbol{u}|^2 + \int_{\Gamma} \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}. \tag{4.8}$$

a) If Ω is nonaxisymmetric, then the estimate (3.13) shows that the norm $\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$ is equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and from (4.8), it follows

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right). \tag{4.9}$$

Then the estimate (4.3) follows from (4.9), together with the pressure estimate (3.12).
b) If Ω is axisymmetric and

(i) $\alpha \geq \alpha_* > 0$, then the estimate (3.14) implies

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}} \left(2 \int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \right).$$

Hence, the estimate (4.4) follows from (4.8).

(ii) $\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition (4.5), then from $a(\boldsymbol{u}, \boldsymbol{\varphi}) = \boldsymbol{\ell}(\boldsymbol{\varphi})$, we get

$$\begin{aligned} 2 \int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 &= \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} + k\boldsymbol{\beta}) - \int_\Omega \mathbb{F} : \nabla(\boldsymbol{u} + k\boldsymbol{\beta}) + \langle \boldsymbol{h}, \boldsymbol{u} + k\boldsymbol{\beta}\rangle_\Gamma \quad \forall k \in \mathbb{R} \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \inf_{k\in\mathbb{R}} \|\boldsymbol{u} + k\boldsymbol{\beta}\|_{\boldsymbol{H}^1(\Omega)}. \end{aligned}$$

Further, from Korn inequality and the inequality (3.19), we know that

$$\inf_{k\in\mathbb{R}} \|\boldsymbol{u} + k\boldsymbol{\beta}\|^2_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega) \left(\inf_{k\in\mathbb{R}} \|\boldsymbol{u} + k\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \right) \leq C(\Omega)\, \|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)},$$

which yields

$$2 \int_\Omega |\mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}.$$

This implies

$$\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)$$

and then

$$\int_\Gamma \alpha |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right)^2$$

which proves the inequality (4.6).

Moreover, if α is a nonzero constant, the variational formulation (3.8) gives $\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$. Therefore, (3.18) shows that the norm $\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}$ is equivalent to the full norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and (4.7) is a consequence of (4.8). □

Next, we discuss the strong solutions of the system (S) and the corresponding bounds which do not depend on α.

Theorem 4.5 ***(Existence and estimate in $\boldsymbol{H}^2(\Omega)$).*** *Assume that α is a constant. If*

$$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) \textit{ and } \boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma),$$

then the solution $(\boldsymbol{u}, \pi)$ *of the Stokes problem* (S) *with* $\mathbb{F} = 0$ *belongs to* $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$. *Also, it satisfies the following estimates:*
(i) *if* Ω *is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right). \tag{4.10}$$

(ii) *if* Ω *is axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha\}}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right). \tag{4.11}$$

If moreover, $\boldsymbol{f}$ *and* $\boldsymbol{h}$ *satisfy the condition:*

$$\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_{\Gamma} = 0,$$

then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right). \tag{4.12}$$

Remark 4.6. 1. We will show the existence of $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ for more general α, not necessarily constant, in Theorem 5.10.

2. It is not sensible to consider a nonzero $\mathbb{F} \in \mathbb{H}^1(\Omega)$ for the strong solution since we are considering any function $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ in the RHS.

Proof. Method I: If α is a constant and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and therefore $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. So, using the regularity result for strong solutions of the Stokes system with full-slip boundary condition [7, Theorem 4.1], we get that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.

Note that, with this method, we do not obtain the estimate for $\boldsymbol{u}$, independent of α, by using the results in [7]. Thus, we will use the fundamental long method which we explain below.

Method II: Here, we follow the method of difference quotients as in the book of L.C. Evans [22]. Without loss of generality, we consider $\boldsymbol{h} = \boldsymbol{0}$, for the facility of calculations. Also, let us denote the k-difference quotient of size h by

$$D_k^h \boldsymbol{u}(x) := \frac{\boldsymbol{u}(x + h\boldsymbol{e}_k) - \boldsymbol{u}(x)}{h},$$

where e_k is the canonical basis element of $\mathbb{R}^3$, $k = 1, 2, 3$ and $h \in \mathbb{R}$.

Interior regularity: The fact that the unique solution $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ of (S) belongs to $\boldsymbol{H}^2_{loc}(\Omega) \times H^1_{loc}(\Omega)$ with the corresponding local estimates (4.10)-(4.12) is proved by using difference quotients and Theorem 4.3, in the same way as it was done for the Dirichlet boundary condition case, since the proof does not depend on the boundary conditions. Thus, we do not repeat it. For another approach to the interior regularity, we also refer to [23, Theorem IV.4.1].

Boundary regularity: Note that the solution $(\boldsymbol{u}, \pi)$ satisfies the following variational formulation: for all $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega)$,

$$2\int_{\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_{\Gamma} \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} - \int_{\Omega} \pi \operatorname{div}\boldsymbol{\varphi} = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\varphi}. \tag{4.13}$$

Case 1. $\Omega = B(0,1) \cap \mathbb{R}^3_+$**:** First, we consider the case when Ω is a half ball. Set $V := B(0, \frac{1}{2}) \cap \mathbb{R}^3_+$ and choose a cut-off function $\zeta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\begin{cases} \zeta \equiv 1 \text{ on } B(0, \frac{1}{2}), \ \zeta \equiv 0 \text{ on } \mathbb{R}^3 \setminus B(0,1), \\ 0 \le \zeta \le 1. \end{cases}$$

So $\zeta \equiv 1$ on V and vanishes on the curved part of Γ.

(i) Tangential regularity of the velocity: Let $h > 0$ be small and $\boldsymbol{\varphi} = -D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})$, with $k = 1, 2$. Clearly, $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega)$. So, substituting $\boldsymbol{\varphi}$ into the identity (4.13), we obtain

$$\begin{aligned} &2\int_{\Omega} \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 + 2\int_{\Omega} D_k^h \mathbb{D}\boldsymbol{u} : 2\zeta \nabla\zeta\, D_k^h \boldsymbol{u} + \int_{\Gamma} \alpha \zeta^2 |D_k^h \boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\ &- \int_{\Omega} \pi \operatorname{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})) = \int_{\Omega} \boldsymbol{f} \cdot (-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})). \end{aligned} \tag{4.14}$$

Now, we estimate the different terms. In this proof, from here on, the constant C might depend on ζ which we do not mention. Working on the second term in the left hand side (LHS) of (4.14), we get

$$\left| \int_{\Omega} D_k^h \mathbb{D}\boldsymbol{u} : 2\zeta \nabla\zeta\, D_k^h \boldsymbol{u} \right| \le C \left[\epsilon \int_{\Omega} \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 + \frac{1}{\epsilon} \int_{\Omega} |D_k^h \boldsymbol{u}|^2 \right], \tag{4.15}$$

by using Cauchy's inequality with ε. Similarly, for the fourth term in the LHS of (4.14), we have

$$\left| \int_{\Omega} \pi \operatorname{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})) \right| \le \epsilon \int_{\Omega} |\operatorname{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))|^2 + \frac{C}{\epsilon} \int_{\Omega} |\pi|^2.$$

On the other hand,

$$\begin{aligned} \operatorname{div}(D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})) &= D_k^{-h} \operatorname{div}(\zeta^2 D_k^h \boldsymbol{u}) = D_k^{-h}(2\zeta \nabla\zeta \cdot D_k^h \boldsymbol{u}) + D_k^{-h}(\zeta^2 \underbrace{\operatorname{div}(D_k^h \boldsymbol{u})}_{=0}) \\ &= D_k^{-h}(2\zeta \nabla\zeta) \cdot D_k^h \boldsymbol{u}(x - he_k) + 2\zeta \nabla\zeta \cdot D_k^{-h} D_k^h \boldsymbol{u} \end{aligned}$$

which implies that

$$\int_{\Omega} |\operatorname{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))|^2 \le C \left(\int_{\Omega} |D_k^h \boldsymbol{u}|^2 + \int_{\Omega} \zeta^2 |D_k^{-h} D_k^h \boldsymbol{u}|^2 \right)$$

$$\leq C\left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2\right).$$

Therefore,

$$\left|\int_\Omega \pi \ \mathrm{div}(D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))\right| \leq \epsilon\left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2\right) + \frac{C}{\epsilon}\int_\Omega |\pi|^2. \tag{4.16}$$

For the right hand side, proceeding in the same way, we derive

$$\left|\int_\Omega \boldsymbol{f}\cdot(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))\right| \leq \epsilon \int_\Omega |D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})|^2 + \frac{C}{\epsilon}\int_\Omega |\boldsymbol{f}|^2,$$

and since

$$\int_\Omega |D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})|^2 \leq C\int_\Omega |\nabla(\zeta^2 D_k^h \boldsymbol{u})|^2 \leq C\left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2\right),$$

we get

$$\left|\int_\Omega \boldsymbol{f}\cdot(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}))\right| \leq \epsilon\left(\int_\Omega |D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2\right) + \frac{C}{\epsilon}\int_\Omega |\boldsymbol{f}|^2. \tag{4.17}$$

Hence, incorporating (4.15), (4.16) and (4.17) in (4.14), along with the fact that $\alpha \geq 0$, yields

$$\begin{aligned}
&2\int_\Omega \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 \\
&\leq \epsilon\left(\int_\Omega \zeta^2 |\mathbb{D} D_k^h \boldsymbol{u}|^2 + \int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2\right) + \frac{C_1}{\epsilon}\left(\int_\Omega |\boldsymbol{f}|^2 + \int_\Omega |\pi|^2\right) + C_2\int_\Omega |D_k^h \boldsymbol{u}|^2 \\
&\leq \epsilon\int_\Omega \zeta^2 |\nabla D_k^h \boldsymbol{u}|^2 + \frac{C_1}{\epsilon}\left(\int_\Omega |\boldsymbol{f}|^2 + \int_\Omega |\pi|^2\right) + C_2\int_\Omega |D_k^h \boldsymbol{u}|^2.
\end{aligned} \tag{4.18}$$

Furthermore, we see that

$$\begin{aligned}
\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\leq C\left(\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbb{D}(\zeta D_k^h \boldsymbol{u})\|^2_{\mathbb{L}^2(\Omega)}\right) \\
&\leq C\left(\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\zeta\, D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\zeta\, \mathbb{D} D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)}\right) \\
&\leq C\left(\|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\zeta\, \mathbb{D} D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)}\right)
\end{aligned}$$

and concerning the first term in the right hand side of (4.18), we rewrite as,

$$\|\zeta \nabla D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} = \|\nabla(\zeta D_k^h \boldsymbol{u}) - \nabla \zeta D_k^h \boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \le \|\nabla(\zeta D_k^h \boldsymbol{u})\|^2_{\mathbb{L}^2(\Omega)} + C\, \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\le C\left(\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right).$$

Combining these inequalities with (4.18), we have

$$\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \le \epsilon \|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{C_1}{\epsilon}\left(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\pi\|^2_{L^2(\Omega)}\right) + C_2 \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Choosing ε sufficiently small, we obtain

$$\|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(V)} \le \|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \le C\left(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\pi\|^2_{L^2(\Omega)} + \|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right)$$

for $k = 1, 2$. Then, for sufficiently small $|h| \neq 0$, we conclude that $\partial^2 \boldsymbol{u}/\partial x_i \partial x_j$ belongs to $\boldsymbol{L}^2(V)$ for all $i, j = 1, 2, 3$ except $i = j = 3$, with their corresponding estimates by using Theorem 4.3.

(ii) Tangential regularity of the pressure: Now, we deduce the tangential regularity of the pressure in terms of the above derivatives of $\boldsymbol{u}$. Indeed, from the Stokes equations, we get

$$\frac{\partial}{\partial x_i}(\nabla \pi) = \frac{\partial}{\partial x_i}(\boldsymbol{f} + \Delta \boldsymbol{u}) = \frac{\partial \boldsymbol{f}}{\partial x_i} + \mathrm{div}(\nabla \frac{\partial \boldsymbol{u}}{\partial x_i}),$$

for $i = 1, 2$. Since there is no term of the form $\partial^2 \boldsymbol{u}/\partial x_3^2$, by preceding arguments, we obtain $\nabla \frac{\partial \pi}{\partial x_i} = \frac{\partial}{\partial x_i}(\nabla \pi) \in \boldsymbol{H}^{-1}(V)$. Furthermore, as we already know $\frac{\partial \pi}{\partial x_i} \in H^{-1}(V)$, and then, Nečas inequality implies that $\frac{\partial \pi}{\partial x_i} \in L^2(V)$, which also satisfies the usual estimate.

(iii) Normal regularity: For the complete regularity of the solution, it remains to study the derivatives of $\boldsymbol{u}$ and π in the direction of $\boldsymbol{e}_3$. Differentiating the divergence equation with respect to x_3 and from the third component of the Stokes equations, we get respectively

$$\frac{\partial^2 u_3}{\partial x_3^2} = -\sum_{i=1}^{2} \frac{\partial^2 u_i}{\partial x_i \partial x_3} \in L^2(V) \quad \text{and} \quad \frac{\partial \pi}{\partial x_3} = f_3 + \Delta u_3 \in L^2(V)$$

which proves that $\pi \in H^1(V)$. Finally, for $i = 1, 2$, we can write the ith equation of the system in the form

$$\frac{\partial^2 u_i}{\partial x_3^2} = -\sum_{j=1}^{2} \frac{\partial^2 u_i}{\partial x_j^2} - f_i + \frac{\partial \pi}{\partial x_i} \in L^2(V)$$

and this implies that $u_i \in H^2(V)$. Hence, apart from the regularity of $\boldsymbol{u}$ and π, we obtain the existence of a constant $C = C(\Omega) > 0$ independent of α such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(V)} + \|\pi\|_{H^1(V)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

Case 2. General domain: Now, we drop the assumption that Ω is a half ball and consider the general case. In this part, we follow the strategy in [48] (in the same way as it was done in [12]). Since Γ is $\mathcal{C}^{1,1}$, for any $x_0 \in \Gamma$, we can assume, upon relabelling the coordinate axes, that

$$\Omega \cap B(x_0, r) = \left\{x \in B(x_0, r) : x_3 > H(x')\right\}$$

for some $r > 0$ and $H : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^{1,1}$. We denote here $x' = (x_1, x_2)$. Let us now introduce the change of variable

$$y = (x_1, x_2, x_3 - H(x')) := \phi(x), \quad i.e., \quad x = (y_1, y_2, y_3 + H(y')) := \phi^{-1}(y)$$

which flattens the boundary locally. We choose $s > 0$ small such that the half ball $\Omega' := B(0, s) \cap \mathbb{R}^3_+$ lies in $\phi(\Omega \cap B(x_0, r))$. Let us define $V' := B(0, s/2) \cap \mathbb{R}^3_+$. We also introduce the new unknown variable

$$\boldsymbol{u}'(y) = \left(u_1(x), u_2(x), u_3(x) - \frac{\partial H}{\partial x_1} u_1(x) - \frac{\partial H}{\partial x_2} u_2(x)\right).$$

It is easy to see $\boldsymbol{u}' \in \boldsymbol{H}^1(\Omega')$ and $\boldsymbol{u}' \cdot \boldsymbol{n} = 0$ on $\partial\Omega' \cap \partial\mathbb{R}^3_+$. The last relation is true because of $\frac{\partial H}{\partial y_i}(0, 0) = 0$, for $i = 1, 2$. With this transformation, it follows, for $i, j = 1, 2$,

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u'_i}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial u'_i}{\partial y_3}, \qquad \frac{\partial u_i}{\partial x_3} = \frac{\partial u'_i}{\partial y_3}$$

$$\frac{\partial u_3}{\partial x_j} = \frac{\partial u'_3}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial u'_3}{\partial y_3} + \sum_{k=1}^{2}\left[\frac{\partial u'_k}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial u'_k}{\partial y_3}\right], \qquad \frac{\partial u_3}{\partial x_3} = \frac{\partial u'_3}{\partial y_3} + \sum_{k=1}^{2}\frac{\partial H}{\partial y_k}\frac{\partial u'_k}{\partial y_3}.$$

Next, we consider the variational formulation (4.13) under this change of variable. From here on, the calculation is exactly the same as it was done in [48] (or in [12]), hence, we do not repeat it. Note that the boundary term remains unchanged, *i.e.,*

$$\int_{\Gamma} \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \int_{\Gamma'} \alpha \boldsymbol{u}'_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}'_{\boldsymbol{\tau}}.$$

Therefore, following exactly the same method as in [12, page 1099], we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(V)} \le C(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

where $V = \phi^{-1}(V')$.

Now, since Γ is compact, we can cover Γ with finitely many sets $\{V_i\}$ as above. Thus, summing the resulting estimates, along with the interior estimate, we get $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and its corresponding estimate. □

5. Stokes equations: L^p-theory

5.1. General solution in $W^{1,p}(\Omega)$

In this subsection, we study the regularity of weak solutions of the Stokes problem (S). We begin with recalling some useful results. For more details about the next theorem, which was introduced, independently, by Babŭska [11] and Brezzi [15], see [28, Lemma 4.1].

Theorem 5.1. *Let X and M be two reflexive Banach spaces and X' and M' be their dual spaces. Let $a(v, w)$ be a continuous bilinear form defined on $X \times M$ and $A \in \mathcal{L}(X; M')$ and $A' \in \mathcal{L}(M; X')$ be the continuous linear operators, associated to $a(v, w)$, defined by*

$$\forall v \in X, \quad \forall w \in M, \quad a(v, w) = \langle Av, w\rangle = \left\langle v, A'w\right\rangle$$

and $V = Ker\ A$. Then, the following statements are equivalent:
(i) There exists $C = C(\Omega) > 0$ such that

$$\inf_{\substack{w\in M\\ w\neq 0}} \sup_{\substack{v\in X\\ v\neq 0}} \frac{a(v, w)}{\|v\|_X\ \|w\|_M} \geq C. \tag{5.1}$$

(ii) The operator $A : X/V \to M'$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of A^{-1}.
(iii) The operator $A' : M \to V^0$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of $(A')^{-1}$, where V^0 is the polar set, defined by

$$V^0 := \{g \in X' : \ \langle g, v\rangle = 0 \quad \forall v \in V\}.$$

Remark 5.2. As a consequence, if the Inf-Sup condition (5.1) is satisfied, then we have the following properties:

i) If $V = \{0\}$, then for any $f \in X'$, there exists a unique $w \in M$ such that,

$$\forall v \in X, \quad a(v, w) = \langle f, v\rangle \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta}\|f\|_{X'}. \tag{5.2}$$

ii) If $V \neq \{0\}$, then for any $f \in X'$, satisfying the compatibility condition,

$\forall v \in V,\ \langle f, v\rangle = 0$, there exists a unique $w \in M$ such that (5.2) holds.
iii) For any $g \in M'$, $\exists v \in X$, unique up to an additive element of V such that,

$$\forall w \in M, \quad a(v, w) = \langle g, w\rangle \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta}\|g\|_{M'}.$$

Next, we introduce the kernel:

$$\boldsymbol{K}^p_T(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) :\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\ \boldsymbol{v} = \mathbf{0} \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

Thanks to [10, Corollary 4.1], we know that this kernel is trivial iff Ω is simply connected. Otherwise, it is of finite dimension and spanned by the functions $\widetilde{\nabla} q_j^T$, $1 \leq j \leq J$, where each $q_j^T \in W^{2,p}(\Omega^o)$ is the unique solution up to an additive constant of the problem:

$$\begin{cases} -\Delta q_j^T &= 0 \quad \text{in } \Omega^o, \\ \partial_n q_j^T &= 0 \quad \text{on } \Gamma, \\ [q_j^T]_k &= c \quad \text{and} \quad [\partial_n q_j^T]_k = 0, \quad 1 \leq k \leq J, \\ \left\langle \partial_n q_j^T, 1 \right\rangle_{\Sigma_k} &= \delta_{jk}, \quad 1 \leq k \leq J, \end{cases}$$

where c is any constant. Recall that Σ_j are the cuts in Ω such that the open set $\Omega^o = \Omega \backslash \bigcup_{j=1}^{J} \Sigma_j$ is simply connected, $[\cdot]_j$ denotes the jump of a function over Σ_j, $\langle \cdot, \cdot \rangle_{\Sigma_j}$ is the duality bracket over Σ_j and $\widetilde{\nabla} q$ is an extension of ∇q from $L^p(\Omega^o)$ to $L^p(\Omega)$, for any function $q \in W^{1,p}(\Omega^o)$ (observe that this extension is different from the gradient of q in $\mathcal{D}'(\Omega)$). For more details, see Notation 3.9 and Lemma 3.10 in [4].

Also, recall the following inf-sup condition (see [10, Lemma 4.4]):

Lemma 5.3. *There exists a constant $C > 0$, depending only on Ω and p, such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq \mathbf{0}}} \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}}{\|\boldsymbol{\xi}\|_{\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}(\Omega)}} \geq C, \tag{5.3}$$

where

$$\boldsymbol{V}^{p'}(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega) : \ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0 \quad \forall\, 1 \leq j \leq J \right\}.$$

Theorem 5.4. *Let $p \in (1, \infty)$, $\boldsymbol{\ell} \in [\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the problem:*

$$\textit{find } \boldsymbol{u} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega) \textit{ such that for any } \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega), \quad a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle \tag{5.4}$$

has a unique solution, where a is defined in (4.2).

Proof. First, let us consider $p \geq 2$. Since $[\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]' \hookrightarrow [\boldsymbol{V}^{2}_{\sigma,\tau}(\Omega)]'$, by Lax-Milgram theorem, there exists a unique $\boldsymbol{u} \in \boldsymbol{V}^{2}_{\sigma,\tau}(\Omega)$ satisfying

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{2}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{2}_{\sigma,\tau}(\Omega)} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{2}_{\sigma,\tau}(\Omega). \tag{5.5}$$

We want to show that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Since the inf-sup condition (5.1) is known for the bilinear form

$$b(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}$$

with the suitable spaces X and M given in (5.3), we use another formulation of problem (5.5).

To do so, let us consider any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$ and from Remark 3.7, point 2, we have the Green formula, for $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$,

$$- \langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_\Omega = 2 \int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} - 2 \langle [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_\Gamma . \tag{5.6}$$

Also, recall the following Green formula from [7, Lemma 2.3],

$$- \langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_\Omega = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} - \langle \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma . \tag{5.7}$$

Note that, extension of [7, Lemma 2.3] for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$ is straightforward. Therefore, we obtain from (5.6) and (5.7),

$$2 \int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} + 2 \langle [(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\varphi} \rangle_\Gamma - \langle \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma . \tag{5.8}$$

In particular, (5.8) also holds for $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ with $\Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$. Next, plugging in the relation (3.5) in (5.8) gives, for $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ with $\Delta \boldsymbol{v} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$,

$$2 \int_\Omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} - 2 \langle \Lambda \boldsymbol{v}, \boldsymbol{\varphi} \rangle_\Gamma . \tag{5.9}$$

Now, due to the density Lemma 3.8, relation (5.9) is true for any $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Therefore, (5.5) becomes,

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^2_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi}. \tag{5.10}$$

Now, we are in position to prove that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and for that, we consider different cases.
(i) $2 < p \leq 3$:

1st Step: Since $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_1}(\Gamma)$ with $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But, $\boldsymbol{L}^{q_1}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_1}, p_1}(\Gamma)$ with $p_1 = \frac{3}{2} q_1 \; > 2$, *i.e.*,

$$\frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right).$$

Therefore, as $\boldsymbol{W}^{\frac{1}{p_1}, p_1'}(\Gamma) \hookrightarrow \boldsymbol{L}^{q_1'}(\Gamma)$ with $\frac{4}{3} < q_1' < 4$ and $\boldsymbol{\Lambda} \boldsymbol{u} \in \boldsymbol{L}^4(\Gamma)$, the mapping

$$\langle \boldsymbol{L}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi} \quad \text{for } \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}(\Omega) \tag{5.11}$$

defines an element in the dual space of $\boldsymbol{V}^{s_1'}(\Omega)$, with $s_1 = \min\{p_1, p\}$. From the inf-sup condition (5.3) and using Remark 5.2, there exists a unique $\boldsymbol{v} \in \boldsymbol{V}^{s_1}_{\sigma,\tau}(\Omega)$ such that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} = \langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{V}^{s_1'}(\Omega)]' \times \boldsymbol{V}^{s_1'}(\Omega)} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}(\Omega). \tag{5.12}$$

In order to show that $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{u}$, we extend (5.12) to any test function $\boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$. Since $\widetilde{\nabla} q_j^T \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \hookrightarrow \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$ and using (5.10), we get

$$\begin{aligned}\left\langle \boldsymbol{L}, \widetilde{\nabla} q_j^T \right\rangle_{[\boldsymbol{V}^{s_1'}(\Omega)]' \times \boldsymbol{V}^{s_1'}(\Omega)} &= \left\langle \boldsymbol{\ell}, \widetilde{\nabla} q_j^T \right\rangle_{[\boldsymbol{V}^{s_1'}(\Omega)]' \times \boldsymbol{V}^{s_1'}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot (\widetilde{\nabla} q_j^T)_{\boldsymbol{\tau}} + 2\int_\Gamma \boldsymbol{\Lambda u} \cdot \widetilde{\nabla} q_j^T \\ &= \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\widetilde{\nabla} q_j^T \; = 0.\end{aligned}$$

Hence, for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$, we set $\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \sum_j \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} \widetilde{\nabla} q_j^T$, which implies that

$$\langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{L}, \tilde{\boldsymbol{\varphi}}\rangle$$

and also $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{V}^{s_1'}(\Omega)$. Therefore, (5.12) yields

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} = \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\tilde{\boldsymbol{\varphi}} = \langle \boldsymbol{L}, \tilde{\boldsymbol{\varphi}}\rangle = \langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle .$$

Finally, we get that $\boldsymbol{v} \in \boldsymbol{V}^{s_1}_{\sigma,\tau}(\Omega)$ satisfies

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} = \langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega). \tag{5.13}$$

Since $\boldsymbol{V}^2_{\sigma,\tau}(\Omega) \hookrightarrow \boldsymbol{V}^{s_1'}_{\sigma,\tau}(\Omega)$, we deduce from (5.10) that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} = \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{\varphi} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$$

which implies that

$$\mathbf{curl}\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{v} \quad \text{ in } \Omega. \tag{5.14}$$

Then, since $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega) \hookrightarrow \boldsymbol{L}^{s_1}(\Omega)$, $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^{s_1}(\Omega)$, $\operatorname{div}\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, we deduce from Proposition 3.2 that $\boldsymbol{u} \in \boldsymbol{W}^{1,s_1}(\Omega)$. If $p_1 \geq p$, the proof is complete. Otherwise, $s_1 = p_1$ and we proceed to the next step.

2nd Step: As $s_1 = p_1 < 3$, we have $\boldsymbol{u} \in \boldsymbol{V}^{p_1}_{\sigma,\tau}(\Omega)$ and therefore, $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^m(\Gamma)$, with $\frac{1}{m} = \frac{3}{2p_1} - \frac{1}{2}$. So, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{q_2}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p_2},p_2}(\Gamma)$, where $\frac{1}{q_2} = \frac{1}{m} + \frac{1}{2+\varepsilon}$ and $p_2 = \frac{3}{2}q_2 > p_1$. Setting $a = \frac{2}{3}\left(\frac{1}{2+\varepsilon} - \frac{1}{2}\right) < 0$, we get,

$$\frac{1}{p_2} = \frac{1}{p_1} + a.$$

Thus, as $\boldsymbol{W}^{\frac{1}{p_2},p_2'}(\Gamma) \hookrightarrow \boldsymbol{L}^{m'}(\Gamma)$ and $\boldsymbol{\Lambda u} \in \boldsymbol{L}^m(\Gamma)$, the mapping $\boldsymbol{L}$ in (5.11),

$$\langle \boldsymbol{L}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{V}^{s_2'}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{s_2'}_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} + 2\int_\Gamma \boldsymbol{\Lambda u} \cdot \boldsymbol{\varphi} \quad \text{for } \boldsymbol{\varphi} \in \boldsymbol{V}^{s_2'}(\Omega)$$

defines an element in the dual of $\boldsymbol{V}^{s_2'}(\Omega)$, with $s_2 = \min\{p_2, p\}$. Hence, analogous to the previous step, there exists a unique $\boldsymbol{v} \in \boldsymbol{V}^{s_2}_{\sigma,\tau}(\Omega)$ such that (5.13) holds for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{s_2'}_{\sigma,\tau}(\Omega)$ and then we conclude (5.14). Thus, we get $\boldsymbol{u} \in \boldsymbol{L}^{p_1^*}(\Omega) \hookrightarrow \boldsymbol{L}^{s_2}(\Omega)$, **curl** $\boldsymbol{u} \in \boldsymbol{L}^{s_2}(\Omega)$, div $\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, which implies that $\boldsymbol{u} \in \boldsymbol{W}^{1,s_2}(\Omega)$. If $p_2 \geq p$, we are done. Otherwise, $s_2 = p_2$.

(k+1)th Step: For $k \geq 1$, we construct p_{k+1} inductively, which satisfies

$$\frac{1}{p_{k+1}} = \frac{1}{p_k} + a.$$

This shows that $\left\{\frac{1}{p_k}\right\}_k$ being an arithmetic sequence with difference $a < 0$, there exists $k^* \geq 1$ such that

$$\frac{1}{p_{k^*+1}} \leq \frac{1}{p} < \frac{1}{p_{k^*}}.$$

In particular, it suffices to take $k^* = \left[-\frac{1}{a}\left(\frac{1}{2} - \frac{1}{p}\right)\right] + 1$ where $[s]$ denotes the integer part of s. Therefore, we obtain $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ for $2 < p \leq 3$, and conclude that there exists a unique $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$, (5.10) holds, where the duality bracket $\langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{V}^2_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega)}$ is now replaced by $\langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]' \times \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}$.

(ii) $p > 3$: From the previous case, we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$, which implies $\boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^s(\Gamma)$ for all $s \in (1, \infty)$. Now, $\alpha \in L^{\frac{2}{3}p+\varepsilon}(\Gamma)$ yields $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$, where $\frac{1}{q} = \frac{1}{s} + \frac{1}{\frac{2}{3}p+\epsilon}$. Choosing $s > 1$ suitably, we can get $q = \frac{2}{3}p$ and hence, $\boldsymbol{L}^q(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Since $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma) \hookrightarrow \boldsymbol{L}^{q'}(\Gamma) \hookrightarrow \boldsymbol{L}^{s'}(\Gamma)$ and $\boldsymbol{\Lambda u} \in \boldsymbol{L}^s(\Gamma)$, the mapping $\boldsymbol{L}$ in (5.11) defines an element in the dual of $\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$. Then, there exists a unique $\boldsymbol{v} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ such that (5.13) holds for any $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ and we deduce (5.14). Therefore, we obtain similarly $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Hence, $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ solves the problem (5.4) for all $2 \leq p < \infty$.

Finally for $1 < p < 2$, let us consider the operator $A \in \mathcal{L}(\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega), (\boldsymbol{V}^p_{\sigma,\tau}(\Omega))')$ associated to the bilinear form a, defined by $\langle A\xi, \varphi\rangle = a(\xi, \varphi)$. As we proved above, for $p' \geq 2$, the operator

A is an isomorphism from $\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$ to $(\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega))'$. Then the adjoint operator, which is equal to A, is an isomorphism from $\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ to $(\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega))'$ for $p < 2$. This means that the operator A is also an isomorphism for $p < 2$ and we finish the proof. □

As a consequence of the above theorem, we obtain the following important inf-sup condition.

Proposition 5.5. *For all $p \in (1, \infty)$ and $\alpha \in L^{t(p)}(\Gamma)$, there exists a constant $\gamma = \gamma(\Omega, p, \alpha) > 0$ such that*

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{u}\in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)\\ \boldsymbol{u}\neq 0}} \frac{2\int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int\limits_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}}}{\|\boldsymbol{u}\|_{\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq \gamma. \tag{5.15}$$

Further, for any $\boldsymbol{\ell} \in [\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'$, the unique solution $\boldsymbol{u} \in \boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ of the variational problem:

$$2\int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int\limits_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega),$$

given by Theorem 5.4, satisfies the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \frac{1}{\gamma} \|\boldsymbol{\ell}\|_{[\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'}. \tag{5.16}$$

Remark 5.6. The inf-sup condition (5.15) will be improved in Theorem 6.14, where we obtain that the above continuity constant γ does not depend on α.

Proof. Using the equivalence (i) and (ii) in Theorem 5.1, we obtain the inf-sup condition (5.15) from Theorem 5.4. The estimate (5.16) follows immediately from (5.15). □

Finally, Theorem 5.4 enables us to deduce the following existence-uniqueness result for the weak solution of the Stokes problem for all $1 < p < \infty$.

Corollary 5.7 ***(Existence and uniqueness in $\boldsymbol{W}^{1,p}(\Omega)$).*** Let $p \in (1, \infty)$ and

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^{p}(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

Then, the Stokes problem (S) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ which satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C(\Omega, \alpha, p) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right). \tag{5.17}$$

Remark 5.8. In the above corollary, the existence of $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and the corresponding estimate (5.17), for $p > 2$, can be deduced directly by using the regularity result in [7, Theorem 3.7], by taking $\alpha \boldsymbol{u}_{\boldsymbol{\tau}}$ as the source term in the right hand side. However, Theorem 5.4 is required to obtain the existence of solution of (S) for $p < 2$ where we need that the general operator A (defined as $\langle A\xi, \varphi\rangle = a(\xi, \varphi)$) is an isomorphism from $\boldsymbol{V}^{p}_{\sigma,\tau}(\Omega)$ to $(\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega))'$ for $p > 2$.

Furthermore, note that the problem (5.4) in Theorem 5.4 is more general than the Stokes problem (3.8) since the problem (5.4) consists of a general right hand side $\boldsymbol{\ell} \in [\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)]'$, hence the existence of solution for the problem (5.4) with $p > 2$ does not follow from [7].

Proof. The existence of a unique solution is immediate from Theorem 5.4 with

$$\langle \boldsymbol{\ell}, \boldsymbol{\varphi}\rangle := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega).$$

The estimate (5.17) follows from (5.16) and the pressure estimate (3.12). □

Remark 5.9. i) All the previous results, where we have assumed $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega)$, hold also true for $\boldsymbol{f} \in [\boldsymbol{H}_0^{(r(p))', p'}(\mathrm{div}, \Omega)]'$, which is clear from the characterization of the space in Proposition 3.1, as the gradient term can be absorbed in the pressure term.

ii) We also want to emphasize that in this work, our assumption on α is quite steep. This regularity is required in order to ensure that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{-\frac{1}{q}, q}(\Gamma)$ for some q so that eventually we can use our tools. But we will see later (Subsection 7.4) that we may suppose α less regular in some cases.

iii) Note that even in the case $\alpha \equiv 0$, we are considering here more general Stokes problem than in [7].

5.2. Strong solution in $W^{2,p}(\Omega)$

Concerning the existence of a strong solution, we prove the following result:

Theorem 5.10 ***(Existence in*** $\boldsymbol{W}^{2,p}(\Omega)$*). Let $p \in (1, \infty)$. Then, for*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p}, p}(\Gamma) \textit{ and } \alpha \in W^{1-\frac{1}{q}, q}(\Gamma)$$

with $q > \frac{3}{2}$ if $p \le \frac{3}{2}$ and $q = p$ otherwise, the solution $(\boldsymbol{u}, \pi)$ of the Stokes problem (S) with $\mathbb{F} = 0$, given by Corollary 5.7, belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ which also satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \le C(\Omega, \alpha, p) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p}, p}(\Gamma)} \right).$$

Remark 5.11. As for the Stokes problem with Dirichlet boundary condition, we observe that the regularity $\boldsymbol{W}^{2,p}(\Omega)$ for the Stokes problem with Navier boundary condition holds if the domain Ω is only $\mathcal{C}^{1,1}$, unlike the case of Navier-type boundary condition for which the domain regularity $\mathcal{C}^{2,1}$ seems to be necessary, even though this last hypothesis does not appear explicitly in the paper [9] (cf. Theorem 4.8). Our above regularity result improves [7, Theorem 4.1] which supposes that the domain is $\mathcal{C}^{2,1}$.

Proof. The proof is done essentially by using the existence of weak solutions and a bootstrap argument. Clearly, the data $\boldsymbol{f}$, $\boldsymbol{h}$ and α satisfy the hypothesis of Corollary 5.7. Hence, there exists a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S).

(i) $1 < p \le \frac{3}{2}$**:** We also have the following embeddings: $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{r(q)}(\Omega)$, $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{q},q}(\Gamma)$ and $W^{1-\frac{1}{\frac{3}{2}+\epsilon},\frac{3}{2}+\epsilon}(\Gamma) \hookrightarrow L^{2+\tilde{\varepsilon}}(\Gamma)$, where $q = p^*$, with $q \in (\frac{3}{2}, 3]$ and $\tilde{\varepsilon} > 0$ is an arbitrarily small number. These inclusions show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,q}(\Omega) \times L^q(\Omega)$ by using Corollary 5.7. Therefore, $\boldsymbol{u} \in \boldsymbol{W}^{1,q}(\Omega) \hookrightarrow \boldsymbol{L}^{q^*}(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$. Also, since $\alpha \in W^{1-\frac{1}{\frac{3}{2}+\epsilon},\frac{3}{2}+\epsilon}(\Gamma)$, we can consider $\alpha \in W^{1,\frac{3}{2}+\varepsilon}(\Omega)$ by using the lift operator. Hence, from Sobolev inequality $\alpha \in L^{(\frac{3}{2}+\varepsilon)^*}(\Omega)$ and $\nabla\alpha \in \boldsymbol{L}^{\frac{3}{2}+\varepsilon}(\Omega)$. Then, for all $i, j = 1, 2, 3$,

$$\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_1}(\Omega), \text{ where } \frac{1}{q_1} = \frac{1}{\frac{3}{2}+\varepsilon} - \frac{1}{3} + \frac{1}{q}$$

and

$$\frac{\partial \alpha}{\partial x_j} u_i \in L^{q_2}(\Omega), \text{ where } \frac{1}{q_2} = \frac{1}{\frac{3}{2}+\varepsilon} + \frac{1}{q^*}.$$

But $q_1 = q_2 > p$ and thus $\frac{\partial}{\partial x_j}(\alpha u_i) = \frac{\partial \alpha}{\partial x_j} u_i + \alpha \frac{\partial u_i}{\partial x_j} \in L^p(\Omega)$. This implies that $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ or, in other words, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Therefore, the regularity result of the Stokes system with full-slip boundary condition [7, Theorem 4.1] implies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Note that it is possible to prove the aforementioned existence result for strong solutions only for $\mathcal{C}^{1,1}$ domain since the problem (S) takes the form of an uniformly elliptic operator with complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [2].

(ii) $p > \frac{3}{2}$**:** First, we assume $p < 3$. Then, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^s(\Omega)$ for all $s \in (1, \infty)$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Also, since $\alpha \in W^{1-\frac{1}{p},p}(\Gamma)$, we can consider $\alpha \in W^{1,p}(\Omega)$. Consequently, $\alpha \in L^{p^*}(\Omega)$ and $\nabla\alpha \in \boldsymbol{L}^p(\Omega)$. Therefore, for all $i, j = 1, 2, 3$,

$$\frac{\partial \alpha}{\partial x_j} u_i \in L^{q_2}(\Omega), \quad \text{where } \frac{1}{q_2} = \frac{1}{p} + \frac{1}{s}$$

and

$$\alpha \frac{\partial u_i}{\partial x_j} \in L^{q_3}(\Omega), \quad \text{where } \frac{1}{q_3} = \frac{1}{p^*} + \frac{1}{3} = \frac{1}{p}.$$

Clearly, $q_2 < q_3$ and then $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^{q_2}(\Omega)$, where $q_2 \in (\frac{3}{2}, p)$. This implies that $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,q_2}(\Omega)$ and hence, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{q_2},q_2}(\Gamma)$. Again, we have $\boldsymbol{u} \in \boldsymbol{W}^{2,q_2}(\Omega)$, where $q_2 \in (\frac{3}{2}, p)$, by the regularity result.

Now, $\boldsymbol{u} \in \boldsymbol{W}^{2,q_2}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\nabla \boldsymbol{u} \in \boldsymbol{W}^{1,q_2}(\Omega) \hookrightarrow \boldsymbol{L}^{q_2^*}(\Omega)$. So, for all i, j,

$$\frac{\partial \alpha}{\partial x_j} u_i \in L^p(\Omega) \text{ and } \alpha \frac{\partial u_i}{\partial x_j} \in L^{q_4}(\Omega), \quad \text{where } \frac{1}{q_4} = \frac{1}{p^*} + \frac{1}{q_2^*} = \frac{1}{p} + \frac{1}{q_2} - \frac{2}{3}.$$

Since $q_4 > p$, $\frac{\partial}{\partial x_j}(\alpha u_i) \in L^p(\Omega)$, and then, $\alpha \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Thus, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ and the regularity result for the Stokes system with full-slip boundary condition [7, Theorem 4.1] implies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

The case for $p \geq 3$ follows by applying a similar argument. Indeed, as $\alpha \in L^\infty(\Gamma)$, one gets $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ and [7, Theorem 4.1] can be applied directly. □

6. Uniform estimates

6.1. First estimate

We can deduce some estimates giving a precise dependence of the weak solution of (S) on the friction coefficient α in some particular cases. Then, we attain a better estimate than (5.17). Note that the following result is not optimal with respect to α and will be improved in Theorem 6.11.

Proposition 6.1. *Let $p > 2$. With the same assumptions on $\boldsymbol{f}, \mathbb{F}, \boldsymbol{h}$ and α as in Corollary 5.7, the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of problem (S) satisfies the following bounds:*
a) *if Ω is nonaxisymmetric, then*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \\ &\leq C(\Omega, p)\left(1 + \|\alpha\|^2_{L^{t(p)}(\Gamma)}\right)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right). \end{aligned} \tag{6.1}$$

b) *if Ω is axisymmetric and*

(*i*) *$\alpha \geq \alpha_* > 0$ on Γ, then*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \\ &\leq \frac{C(\Omega,p)}{\min\{2,\alpha_*\}}\left(1 + \|\alpha\|^2_{L^{t(p)}(\Gamma)}\right)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right). \end{aligned}$$

(*ii*) *$\boldsymbol{f}, \mathbb{F}$ and $\boldsymbol{h}$ satisfy the condition:*

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0,$$

then

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \\ &\leq C(\Omega, p)\left(1 + \|\alpha\|^2_{L^{t(p)}(\Gamma)}\right)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right). \end{aligned}$$

Proof. We only prove (6.1) since the other inequalities follow in the same way. Assume that Ω is nonaxisymmetric.

(i) $2 < p < 3$**:** From the proof of Lemma 3.9, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^q(\Gamma)$ with $\frac{1}{q} = \frac{3}{2p} - \frac{1}{2} + \frac{1}{2+\epsilon} < \frac{3}{2p}$ and $\boldsymbol{L}^q(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Therefore, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, but from the relation

$$\boldsymbol{L}^q(\Gamma) \underset{\text{compact}}{\hookrightarrow} \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \underset{\text{continuous}}{\hookrightarrow} \boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

we have that for any $\delta > 0$, there exists a constant $C(\delta) > 0$, with $C(\delta) \to \infty$ as $\delta \to 0$, such that

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq \delta\, \|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Gamma)} + C(\delta)\|\boldsymbol{v}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{L}^q(\Gamma). \tag{6.2}$$

Choosing $\boldsymbol{v} = \alpha \boldsymbol{u}_{\boldsymbol{\tau}}$ in (6.2) and using Hölder inequality and trace theorem, we get

$$\begin{aligned}\|\alpha \boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} &\leq \delta\, \|\alpha \boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^q(\Gamma)} + C(\delta)\, \|\alpha \boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{L}^{4/3}(\Gamma)} \\ &\leq \delta\, \|\alpha\|_{L^{2+\epsilon}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + C(\delta)\, \|\alpha\|_{L^2(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

The $W^{1,p}$-regularity result of the Stokes system with full-slip boundary condition [7, Corollary 3.8] yields

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p_0(\Omega)} &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|\alpha \boldsymbol{u}_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) \\ &\qquad + \delta\|\alpha\|_{L^{2+\epsilon}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + C(\delta)\|\alpha\|_{L^2(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned}$$

Then, choosing $\delta > 0$ such that $1 - \delta C\|\alpha\|_{L^{2+\epsilon}(\Gamma)} = \frac{1}{2}$, we obtain

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p_0(\Omega)} \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right) + C\, \|\alpha\|_{L^2(\Gamma)}\|\alpha\|_{L^{2+\epsilon}(\Gamma)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq C(1 + \|\alpha\|^2_{L^{(2+\epsilon)}(\Gamma)})\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\right).\end{aligned}$$

(ii) $p \geq 3$**:** The analysis is exactly similar to the previous case. □

Remark 6.2. We can also extend the above estimates of Proposition 6.1 for $p < 2$ by applying a duality argument in the same way as it was done in Proposition 6.10 and Proposition 6.12.

6.2. Second estimate

In this subsection, we prove one of the main results of this work. The estimates in Proposition 6.1 are improved with respect to α and for all $p \in (1, \infty)$.

First, we discuss the estimate for $p > 2$ with $\boldsymbol{f} = \boldsymbol{h} = \boldsymbol{0}$, similar to (4.3) or (4.4).

Theorem 6.3 *(Estimates in* $\boldsymbol{W^{1,p}(\Omega)}$*, with* $\boldsymbol{p > 2}$ *and RHS* $\mathbb{F}$*). Let* $p > 2$*,* $\mathbb{F} \in \mathbb{L}^p(\Omega)$ *and* $\alpha \in L^{t(p)}(\Gamma)$*. Then, the solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ *of (S) with* $\boldsymbol{f} = \boldsymbol{0}$ *and* $\boldsymbol{h} = \boldsymbol{0}$ *satisfies the following estimates:*

(i) *if* Ω *is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\Omega)\ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \tag{6.3}$$

(ii) *if* Ω *is axisymmetric and* $\alpha \geq \alpha_* > 0$*, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*)\ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}.$$

The proof of the above theorem uses the weak Reverse Hölder inequality and the steps are similar to the ones of the Laplace-Robin problem, discussed in [5], although they are not the same because of the pressure term in our current problem.

Before the main proof of Theorem 6.3, we require some additional results and tools.

Since Ω is of class $\mathcal{C}^{1,1}$, there exists some $r_0 > 0$ such that for any $x_0 \in \Gamma$, there exist a coordinate system (x', x_3), which is isometric to the usual coordinate system (which involves rotation and/or translation) and a $\mathcal{C}^{1,1}$ function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$B(x_0, r_0) \cap \Omega = \left\{(x', x_3) \in B(x_0, r_0) : x_3 > \psi(x')\right\} \tag{6.4}$$

and

$$B(x_0, r_0) \cap \Gamma = \left\{(x', x_3) \in B(x_0, r_0) : x_3 = \psi(x')\right\}.$$

Now consider any ball $B(x_0, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x_0, 2r) \subset \Omega$ or $x_0 \in \Gamma$. In some places, we may write B instead of $B(x_0, r)$ provided there is no ambiguity, and $aB := B(x_0, ar)$ for $a > 0$. Also, for any integrable function f on a domain ω, we use the usual notation to denote the average value of f on ω by

$$⨍_{\omega} f = \frac{1}{|\omega|} \int_{\omega} f.$$

The following lemma is proved in [27, Lemma 0.5] (see also [26, Proposition 1.1, Chapter V]). We may replace cubes by balls as well in the following result, see for example [21, Proposition 3.7].

Lemma 6.4. *Let* f, g, h *be nonnegative functions in* $L^1(Q_0)$*, where* Q_0 *is a cube in* $\mathbb{R}^n$*,* $Q_R(x_0)$ *is a cube centered at* x_0 *with sides* $2R$ *and let* $\beta \in \mathbb{R}^+$*. There exists* δ_0 *such that if for some* $\delta \leq \delta_0$*, the following inequality*

$$⨍_{Q_R(x_0)} f \leq C(\delta) \left[R^{-\beta} ⨍_{Q_{2R}(x_0)} g + ⨍_{Q_{2R}(x_0)} h \right] + \delta ⨍_{Q_{2R}(x_0)} f$$

holds for all $x_0 \in Q_0$ *and* $R < \frac{1}{2} dist(x_0, \partial Q_0)$*, then there exists a constant* $C > 0$ *such that*

$$\int_{Q_R(x_0)} f \leq C \left[R^{-\beta} \int_{Q_{2R}(x_0)} g + \int_{Q_{2R}(x_0)} h \right]$$

for all $x_0 \in Q_0$ *and* $R < \frac{1}{2} dist(x_0, \partial Q_0)$.

Next, we deduce the Caccioppoli inequality for the Stokes problem, up to the boundary.

Lemma 6.5 *(**Caccioppoli inequality**). Let* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ *satisfy*

$$2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} = - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{H}^1_{\boldsymbol{\tau}}(\Omega). \tag{6.5}$$

Then, there exists a constant $C > 0$*, independent of* α*, such that for all* $x_0 \in \overline{\Omega}$ *and* $0 < r < \frac{r_0}{2}$*, we have*

$$\int_{B\cap\Omega} |\nabla \boldsymbol{u}|^2 \leq C \left(\frac{1}{r^2} \int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \int_{2B\cap\Omega} |\mathbb{F}|^2 \right), \tag{6.6}$$

where r_0 *is given in* (6.4).

Proof. We will use the following identity several times: for any "smooth enough" $\boldsymbol{v}$ and any symmetric matrix $\mathbb{M}$, $\int_\omega \mathbb{D}\boldsymbol{v} : \mathbb{M} = \int_\omega \nabla \boldsymbol{v} : \mathbb{M}$. In particular, for any "smooth enough" $\boldsymbol{v}$ and $\boldsymbol{\varphi}$,

$$\int_\omega \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi} = \int_\omega \nabla \boldsymbol{v} : \mathbb{D}\boldsymbol{\varphi}.$$

(i) Pressure estimate: Let $\pi_0 = \fint_{2B\cap\Omega} \pi$. Since $\pi \in L^2_0(\Omega)$, consider $\boldsymbol{v} \in \boldsymbol{H}^1_0(2B \cap \Omega)$ which satisfies

$$\int_{2B\cap\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = \int_{2B\cap\Omega} (\pi - \pi_0) \operatorname{div} \boldsymbol{\varphi} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{H}^1_0(2B \cap \Omega).$$

As Ω is connected, we get

$$\|\pi - \pi_0\|_{L^2(2B\cap\Omega)} \leq C \|\nabla(\pi - \pi_0)\|_{\boldsymbol{H}^{-1}(2B\cap\Omega)} = C \|\boldsymbol{v}\|_{\boldsymbol{H}^1_0(2B\cap\Omega)},$$

where the constant C depends only on Ω, not on r (cf. [27, comment before Remark 1.7, Part II]). But from (6.5), we obtain (extending $\boldsymbol{\varphi}$ by 0 outside $2B \cap \Omega$, we may consider $\boldsymbol{\varphi} \in \boldsymbol{H}^1_0(\Omega)$ and replacing π by $\pi - \pi_0$ since $\pi - \pi_0$ also satisfies (S)) that

$$\int_{2B\cap\Omega} \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} + \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{H}^1_0(2B \cap \Omega).$$

Now, putting $\boldsymbol{\varphi} = \boldsymbol{v}$ yields

$$\|\pi - \pi_0\|_{L^2(2B\cap\Omega)} \leq C(\Omega) \left(\|\nabla \boldsymbol{u}\|_{\mathbb{L}^2(2B\cap\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(2B\cap\Omega)}\right). \tag{6.7}$$

(ii) Caccioppoli inequality: Let us consider a cut-off function $\eta \in C_c^\infty(2B)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B \quad \text{and} \quad |\nabla \eta| \leq \frac{C}{r} \text{ in } 2B. \tag{6.8}$$

Choosing $\boldsymbol{\varphi} = \eta^2 \boldsymbol{u}$ in (6.5), we have

$$2 \int_{2B\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}(\eta^2 \boldsymbol{u}) + \int_{2B\cap\Gamma} \alpha \eta^2 |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 - \int_{2B\cap\Omega} (\pi - \pi_0) \operatorname{div}(\eta^2 \boldsymbol{u}) = - \int_{2B\cap\Omega} \mathbb{F} : \nabla(\eta^2 \boldsymbol{u})$$

and using the fact that div $\boldsymbol{u} = 0$ in Ω, it follows

$$\begin{aligned} & 2 \int_{2B\cap\Omega} \eta^2 |\mathbb{D}\boldsymbol{u}|^2 + \int_{2B\cap\Gamma} \alpha \eta^2 |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\ = & -4 \int_{2B\cap\Omega} \mathbb{D}\boldsymbol{u} : \eta \nabla \eta \boldsymbol{u} + 2 \int_{2B\cap\Omega} (\pi - \pi_0) \eta \nabla \eta \boldsymbol{u} - \int_{2B\cap\Omega} \mathbb{F} : \eta^2 \nabla \boldsymbol{u} - 2 \int_{2B\cap\Omega} \mathbb{F} : \eta \nabla \eta \boldsymbol{u}, \end{aligned}$$

where $\nabla \eta \boldsymbol{u}$ is the matrix $\nabla \eta \otimes \boldsymbol{u}$. Next, by using Young's inequality on the RHS, we obtain

$$\begin{aligned} & 2 \int_{2B\cap\Omega} \eta^2 |\mathbb{D}\boldsymbol{u}|^2 + \int_{2B\cap\Gamma} \alpha \eta^2 |\boldsymbol{u}_{\boldsymbol{\tau}}|^2 \\ \leq \; & \varepsilon \int_{2B\cap\Omega} \eta^2 |\mathbb{D}\boldsymbol{u}|^2 + C_\varepsilon \int_{2B\cap\Omega} |\boldsymbol{u}|^2 |\nabla \eta|^2 + \varepsilon \int_{2B\cap\Omega} \eta^2 |\pi - \pi_0|^2 + C_\varepsilon \int_{2B\cap\Omega} |\nabla \eta|^2 |\boldsymbol{u}|^2 \\ & + \varepsilon \int_{2B\cap\Omega} \eta^2 |\nabla \boldsymbol{u}|^2 + C_\varepsilon \int_{2B\cap\Omega} \eta^2 |\mathbb{F}|^2 + \varepsilon \int_{2B\cap\Omega} \eta^2 |\mathbb{F}|^2 + C_\varepsilon \int_{2B\cap\Omega} |\nabla \eta|^2 |\boldsymbol{u}|^2. \end{aligned}$$

Since $\alpha \geq 0$ and by choosing $\varepsilon > 0$ suitably and using the properties (6.8), we get

$$\int_{B\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 \leq \frac{C}{r^2} \int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \varepsilon \int_{2B\cap\Omega} |\pi - \pi_0|^2 + \varepsilon \int_{2B\cap\Omega} |\nabla \boldsymbol{u}|^2 + C \int_{2B\cap\Omega} |\mathbb{F}|^2,$$

where the constant $C > 0$ is independent of α. Now, using the pressure estimate (6.7), we have

$$\int_{B\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 \leq \frac{C}{r^2} \int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \varepsilon \int_{2B\cap\Omega} |\nabla \boldsymbol{u}|^2 + C \int_{2B\cap\Omega} |\mathbb{F}|^2.$$

Next, adding the term $\int_{B\cap\Omega} |\boldsymbol{u}|^2$ in both sides, choosing $r \leq 1$ (as Ω is bounded, we can do this) and using Korn inequality, we obtain

$$\int_{B\cap\Omega} |\nabla \boldsymbol{u}|^2 \leq \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(B\cap\Omega)} \leq C(\Omega)\left(\frac{1}{r^2}\int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \int_{2B\cap\Omega} |\mathbb{F}|^2\right) + \varepsilon \int_{2B\cap\Omega} |\nabla \boldsymbol{u}|^2.$$

Therefore, using Lemma 6.4 with $\beta = 2$, we achieve the desired estimate (6.6). □

We further state the following boundary Hölder estimate which can be proved in the same way as it was done in [27, Theorem 2.8 (a), Part II], since we have the corresponding Caccioppoli inequality (6.6) as in [27, Theorem 2.2, Part II].:

Proposition 6.6. *Let $\gamma \in (0, 1)$ and suppose that $(\boldsymbol{v}, z) \in \boldsymbol{H}^1(B(x_0, r) \cap \Omega) \times L^2(B(x_0, r) \cap \Omega)$ satisfies*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla z = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{v} = 0 & \text{in } B(x_0, r) \cap \Omega \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } B(x_0, r) \cap \Gamma \end{cases}$$

for some $x_0 \in \Gamma$ and $0 < r < r_0$. Then for any $x, y \in B(x_0, r/2) \cap \Omega$, we have

$$|\boldsymbol{v}(x) - \boldsymbol{v}(y)| \leq C \left(\frac{|x-y|}{r}\right)^{\gamma} \left(\fint_{B(x_0,r)\cap\Omega} |\boldsymbol{v}|^2 \right)^{1/2}, \tag{6.9}$$

where $C > 0$ depends on Ω, but is independent of α.

Lemma 6.7 ***(Weak reverse Hölder inequality)**. Let $p \geq 2$. Then for any $B(x_0, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x_0, 2r) \subset \Omega$ or $x_0 \in \Gamma$, the following weak reverse Hölder inequality holds:*
(i) *if $B(x_0, 2r) \subset \Omega$, then*

$$\left(\fint_{B(x_0,r)} |\nabla \boldsymbol{u}|^p \right)^{1/p} \leq C \left(\fint_{B(x_0,2r)} |\nabla \boldsymbol{u}|^2 \right)^{1/2}, \tag{6.10}$$

whenever $\boldsymbol{u} \in H^1(B(x, 2r))$ satisfies $-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}$, $\operatorname{div} \boldsymbol{u} = 0$ in $B(x, 2r)$.
(ii) *if $x_0 \in \Gamma$, then*

$$\left(\fint_{B(x_0,r)\cap\Omega} \left(|\nabla \boldsymbol{u}|^p + |\boldsymbol{u}|^p\right) \right)^{1/p} \leq C \left(\fint_{B(x_0,2r)\cap\Omega} \left(|\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2\right) \right)^{1/2}, \tag{6.11}$$

whenever $\boldsymbol{u} \in \boldsymbol{H}^1(B(x_0, 2r) \cap \Omega)$ satisfies

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } B(x_0, 2r) \cap \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \alpha \boldsymbol{u}_{\boldsymbol{\tau}} + 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } B(x_0, 2r) \cap \Gamma. \end{cases}$$

The constant $C > 0$ at most depends on Ω and p.

Proof. Case (i): $B(x_0, 2r) \subset \Omega$.
The weak reverse Hölder inequality (6.10) holds for any $p \geq 2$, by the following interior estimates for Stokes operator [30, Theorem 2.7 (1)]:

$$\sup_{B(x_0,r)} |\nabla \boldsymbol{u}| \leq C \left(\fint_{B(x_0,2r)} |\nabla \boldsymbol{u}|^2 \right)^{1/2}.$$

Case (ii): $x_0 \in \Gamma$.
From the interior gradient estimate for the Stokes problem, we can write (e.g. see [30, Theorem 2.7, (3)])

$$|\nabla \boldsymbol{u}(x)| \leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |\boldsymbol{u}|^2 \right)^{1/2},$$

for any $x \in (B(x_0, r) \cap \Omega)$ where $\delta(x) = \mathrm{dist}(x, \Gamma)$ and $c > 0$ is chosen such that $B(x, 2c\delta(x)) \subsetneq (B(x_0, 2r) \cap \Omega)$. Now, for fixed $y \in B(x_0, 2c\delta(x))$, let $\boldsymbol{v}(x) = \boldsymbol{u}(x) - \boldsymbol{u}(y)$. Then $-\Delta \boldsymbol{v} + \nabla z = \boldsymbol{0}$, $\mathrm{div}\, \boldsymbol{v} = 0$ in $B(x, 2c\delta(x))$ and thus, we may write from the above argument

$$|\nabla \boldsymbol{v}(x)| \leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |\boldsymbol{v}|^2 \right)^{1/2},$$

which implies, along with the boundary Hölder estimate (6.9), that

$$\begin{aligned}
|\nabla \boldsymbol{u}(x)| &\leq \frac{C}{\delta(x)} \left(\fint_{B(x,c\delta(x))} |\boldsymbol{u}(z) - \boldsymbol{u}(y)|^2 \mathrm{d}z \right)^{1/2} \\
&= \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\int_{B(x,c\delta(x))} |\boldsymbol{u}(z) - \boldsymbol{u}(y)|^2 \mathrm{d}z \right)^{1/2} \\
&\leq \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left[\int_{B(x,2c\delta(x))} \left(\frac{|z-y|}{r} \right)^{2\gamma} \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right) \mathrm{d}z \right]^{1/2} \\
&\leq \frac{C}{\delta(x)^{1+\frac{3}{2}}} \left(\fint_{B(x_0,2r)\cap\Omega} |\boldsymbol{u}|^2 \right)^{1/2} \frac{1}{r^\gamma} \left(\int_{B(x,2c\delta(x))} |z-y|^{2\gamma} \mathrm{d}z \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C_\gamma}{(\delta(x))^{1+\frac{3}{2}}}\left(\fint_{B(x_0,2r)\cap\Omega}|\boldsymbol{u}|^2\right)^{1/2}\frac{1}{r^\gamma}(\delta(x))^{\gamma+\frac{3}{2}} \\
&= C_\gamma\,\frac{(\delta(x))^{\gamma-1}}{r^\gamma}r^{-3/2}\left(\int_{B(x_0,2r)\cap\Omega}|\boldsymbol{u}|^2\right)^{1/2} \\
&\leq C_\gamma\,\frac{(\delta(x))^{\gamma-1}}{r^\gamma}r^{1-3/2}\left(\int_{B(x_0,2r)\cap\Omega}|\boldsymbol{u}|^6\right)^{1/6} \\
&\leq C_\gamma\left(\frac{r}{\delta(x)}\right)^{1-\gamma}\left(\fint_{B(x_0,2r)\cap\Omega}|\nabla\boldsymbol{u}|^2+|\boldsymbol{u}|^2\right)^{1/2}.
\end{aligned}
$$

Since $\gamma \in (0, 1)$ is arbitrary, we have

$$
|\nabla\boldsymbol{u}(x)| \leq C_\gamma\left(\frac{r}{\delta(x)}\right)^{\gamma}\left(\fint_{B(x_0,2r)\cap\Omega}|\nabla\boldsymbol{u}|^2+|\boldsymbol{u}|^2\right)^{1/2}.
$$

Finally, this yields, by choosing γ so that $p\gamma < 1$, that

$$
\left(\fint_{B(x_0,r)\cap\Omega}|\nabla\boldsymbol{u}|^p\right)^{1/p} \leq C_p\left(\fint_{B(x_0,2r)\cap\Omega}|\nabla\boldsymbol{u}|^2+|\boldsymbol{u}|^2\right)^{1/2}.
$$

This completes the proof. □

With the following abstract lemma which is proved in [25, Theorem 2.2], we are now in a position to prove Theorem 6.3.

Lemma 6.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $p > 2$. Let $G \in L^2(\Omega)$ and $f \in L^q(\Omega)$ for some $2 < q < p$. Suppose that for each ball B with the property that $|B| \leq \beta|\Omega|$ and either $2B \subset \Omega$ or B centers on Γ, there exist two integrable functions G_B and R_B on $2B \cap \Omega$ such that $|G| \leq |G_B| + |R_B|$ on $2B \cap \Omega$ and*

$$
\left(\fint_{2B\cap\Omega}|R_B|^p\right)^{1/p} \leq C_1\left[\left(\fint_{\gamma B\cap\Omega}|G|^2\right)^{1/2} + \sup_{B'\supset B}\left(\fint_{B'\cap\Omega}|f|^2\right)^{1/2}\right] \tag{6.12}
$$

and

$$\left(\fint_{2B\cap\Omega} |G_B|^2 \right)^{1/2} \leq C_2 \sup_{B'\supset B} \left(\fint_{B'\cap\Omega} |f|^2 \right)^{1/2}, \tag{6.13}$$

where $C_1, C_2 > 0$ and $0 < \beta < 1 < \gamma$. Then, we have

$$\left(\fint_{\Omega} |G|^q \right)^{1/q} \leq C \left[\left(\fint_{\Omega} |G|^2 \right)^{1/2} + \left(\fint_{\Omega} |f|^q \right)^{1/q} \right], \tag{6.14}$$

where $C > 0$ depends only on $C_1, C_2, n, p, q, \beta, \gamma$ and Ω.

Proof of Theorem 6.3. Given any ball B with either $2B \subset \Omega$ or B centers on Γ, let $\varphi \in C_c^\infty(8B)$ be a cut-off function such that $0 \leq \varphi \leq 1$ and

$$\varphi = \begin{cases} 1 & \text{in } 4B \\ 0 & \text{outside } 8B \end{cases}$$

and we decompose $(\boldsymbol{u}, \pi) = (\boldsymbol{v}, \pi_1) + (\boldsymbol{w}, \pi_2)$, where $(\boldsymbol{v}, \pi_1), (\boldsymbol{w}, \pi_2) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ satisfy

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \pi_1 = \operatorname{div}(\varphi \mathbb{F}), \quad \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad 2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = -[(\varphi \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} \text{ on } \Gamma \end{cases} \tag{6.15}$$

and

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \pi_2 = \operatorname{div}((1-\varphi) \mathbb{F}), \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad 2\,[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{w}_{\boldsymbol{\tau}} = -[((1-\varphi) \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} \text{ on } \Gamma. \end{cases} \tag{6.16}$$

From the weak formulation of (6.15), we get

$$\int_\Omega |\nabla \boldsymbol{v}|^2 + \int_\Gamma \alpha |\boldsymbol{v}_{\boldsymbol{\tau}}|^2 = -\int_\Omega \varphi \mathbb{F} : \nabla \boldsymbol{v},$$

which implies

$$\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} \leq \|\varphi \mathbb{F}\|_{\mathbb{L}^2(\Omega)} \tag{6.17}$$

and

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega, \alpha_*) \|\varphi \mathbb{F}\|_{\mathbb{L}^2(\Omega)}. \tag{6.18}$$

Note that the above constant C is independent of α_* when Ω is nonaxisymmetric (cf. Theorem 4.3).

(i) First, we consider the case $4B \subset \Omega$. We want to apply Lemma 6.8 with $G = |\nabla \boldsymbol{u}|$, $G_B = |\nabla \boldsymbol{v}|$ and $R_B = |\nabla \boldsymbol{w}|$. It is easy to see that

$$|G| \leq |G_B| + |R_B|.$$

In order to verify (6.12) and (6.13), note that the estimate (6.17) yields

$$\frac{1}{|2B|}\int_{2B} |G_B|^2 = \frac{1}{|2B|}\int_{2B} |\nabla \boldsymbol{v}|^2 \leq \frac{1}{|2B\cap\Omega|}\int_{\Omega} |\nabla \boldsymbol{v}|^2 \leq \frac{1}{|2B\cap\Omega|}\int_{\Omega} |\varphi\mathbb{F}|^2$$
$$\leq \frac{C(\Omega)}{|8B\cap\Omega|}\int_{8B\cap\Omega} |\mathbb{F}|^2,$$

where in the last inequality, we used that $|8B\cap\Omega| \leq C(\Omega)|2B\cap\Omega|$. This estimate holds since Ω is a Lipschitz domain and thus, it satisfies the interior cone condition. This allows us to deduce the estimate (6.13).

Next, from (6.16), we observe that $-\Delta \boldsymbol{w} + \nabla\pi_2 = \mathbf{0}$, $\operatorname{div} \boldsymbol{w} = 0$ in $4B$. Hence, by the weak reverse Hölder inequality in Lemma 6.7 (using $2B$ instead of B), we have

$$\left(\fint_{2B} |\nabla \boldsymbol{w}|^p\right)^{1/p} \leq C_p(\Omega)\left(\fint_{4B} |\nabla \boldsymbol{w}|^2\right)^{1/2},$$

which implies, together with (6.17), that

$$\left(\fint_{2B} |R_B|^p\right)^{1/p} \leq C_p(\Omega)\left(\fint_{4B} |\nabla \boldsymbol{w}|^2\right)^{1/2} \leq C_p(\Omega)\left[\left(\fint_{4B} |\nabla \boldsymbol{u}|^2\right)^{1/2} + \left(\fint_{4B} |\nabla \boldsymbol{v}|^2\right)^{1/2}\right]$$
$$\leq C_p(\Omega)\left(\fint_{4B} |G|^2\right)^{1/2} + \left(\fint_{8B\cap\Omega} |\mathbb{F}|^2\right)^{1/2}.$$

This yields (6.12). So, from (6.14), it follows that

$$\left(\fint_{\Omega} |\nabla \boldsymbol{u}|^q\right)^{1/q} \leq C_p(\Omega)\left[\left(\fint_{\Omega} |\nabla \boldsymbol{u}|^2\right)^{1/2} + \left(\fint_{\Omega} |\mathbb{F}|^q\right)^{1/q}\right]$$

for any $2 < q < p$, where $C_p(\Omega) > 0$ does not depend on α.

Observe that the weak reverse Hölder condition (6.10) has the self-improving property, that is, if $\boldsymbol{u}$ satisfies (6.10) for some $p > 2$, then it satisfies (6.10) for some $\overline{p} > p$. This implies the above estimate also holds for any $q \in (2, \tilde{p})$ for some $\tilde{p} > p$, and in particular, for $q = p$. Then, we deduce (6.3) from the $\boldsymbol{L}^2$-estimate (4.3).

(ii) Next, let us consider B centered on Γ. Now, we apply again Lemma 6.8 with $G = |\boldsymbol{u}| + |\nabla \boldsymbol{u}|$, $G_B = |\boldsymbol{v}| + |\nabla \boldsymbol{v}|$ and $R_B = |\boldsymbol{w}| + |\nabla \boldsymbol{w}|$. Obviously, $|G| \leq |G_B| + |R_B|$ and by (6.18)

$$\fint_{2B\cap\Omega} |G_B|^2 \leq \fint_{2B\cap\Omega} (|\boldsymbol{v}|^2 + |\nabla \boldsymbol{v}|^2) \leq \frac{1}{|2B\cap\Omega|} \|\boldsymbol{v}\|^2_{H^1(\Omega)} \leq \frac{C(\Omega,\alpha_*)}{|2B\cap\Omega|} \int_\Omega |\varphi\mathbb{F}|^2$$
$$\leq \frac{C(\Omega,\alpha_*)}{|8B\cap\Omega|} \int_{8B\cap\Omega} |\mathbb{F}|^2$$

which yields (6.13). Also, $(\boldsymbol{w}, \pi_2)$ satisfies the problem

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla\pi_2 = \boldsymbol{0}, \quad \operatorname{div} \boldsymbol{w} = 0 & \text{in } 4B\cap\Omega \\ \boldsymbol{w}\cdot\boldsymbol{n} = 0, \quad \alpha\boldsymbol{w}_{\boldsymbol{\tau}} + 2[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } 4B\cap\Gamma. \end{cases}$$

By the weak reverse Hölder inequality (6.11) and the estimate (6.18), we can then write,

$$\begin{aligned}
&\left(\fint_{2B\cap\Omega} |R_B|^p\right)^{1/p} \\
&\leq \left(\frac{1}{|2B\cap\Omega|}\int_{2B\cap\Omega} ((|\boldsymbol{w}| + |\nabla\boldsymbol{w}|)^2)^{p/2}\right)^{1/p} \\
&\leq C_p(\Omega)\left(\frac{1}{|4B\cap\Omega|}\int_{4B\cap\Omega} (|\boldsymbol{w}|^2 + |\nabla\boldsymbol{w}|^2)\right)^{1/2} \\
&\leq C_p(\Omega)\left[\left(\frac{1}{|4B\cap\Omega|}\int_{4B\cap\Omega} (|\boldsymbol{u}|^2 + |\nabla\boldsymbol{u}|^2)\right)^{1/2} + \left(\frac{1}{|4B\cap\Omega|}\int_{4B\cap\Omega} (|\boldsymbol{v}|^2 + |\nabla\boldsymbol{v}|^2)\right)^{1/2}\right] \\
&\leq C_p(\Omega)\left(\frac{1}{|4B\cap\Omega|}\int_{4B\cap\Omega} |G|^2\right)^{1/2} + C_p(\Omega,\alpha_*)\left(\frac{1}{|8B\cap\Omega|}\int_{8B\cap\Omega} |\mathbb{F}|^2\right)^{1/2}
\end{aligned}$$

which yields (6.12). Thus we get from (6.14),

$$\left(\fint_\Omega (|\boldsymbol{u}| + |\nabla\boldsymbol{u}|)^q\right)^{1/q} \leq C_p(\Omega,\alpha_*)\left[\left(\fint_\Omega (|u| + |\nabla u|)^2\right)^{1/2} + \left(\fint_\Omega |\mathbb{F}|^q\right)^{1/q}\right]$$

for any $2 < q < p$ where $C_p(\Omega,\alpha_*) > 0$ does not depend on α. This completes the proof together with the previous case. □

The next proposition will be used to study the complete Stokes problem (S). We will improve the following result in Proposition 6.12, where we consider data which are less regular.

Proposition 6.9 *(Estimates in $\boldsymbol{W}^{1,p}(\Omega)$, with $p > 2$ and RHS $\boldsymbol{f}$). Let $p > 2$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S), with $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$, satisfies the following estimates:*
(i) *if Ω is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}$$

(ii) *if Ω is axisymmetric and $\alpha \ge \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega, \alpha_*)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. The result follows by using the same argument as in Theorem 6.3 and the pressure estimate (3.12). Hence, we do not repeat it. □

Proposition 6.10 *(Estimates in $\boldsymbol{W}^{1,p}(\Omega)$ for RHS $\mathbb{F}$). Let $p \in (1, \infty)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$ satisfies the following estimates:*
(i) *if Ω is nonaxisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega)\, \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \tag{6.19}$$

(ii) *if Ω is axisymmetric and $\alpha \ge \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega, \alpha_*)\, \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{6.20}$$

Proof. For $p > 2$, the estimates (6.19) and (6.20) are proved in Theorem 6.3. Now, suppose that $1 < p < 2$. We prove it in two steps. Also, without loss of generality, we consider that Ω is nonaxisymmetric.

(i) First, we prove that

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le C_p(\Omega) \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{6.21}$$

We write

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \mathbb{G} \in \mathbb{L}^{p'}(\Omega)} \frac{\left|\int_\Omega \nabla \boldsymbol{u} : \mathbb{G}\right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}}, \tag{6.22}$$

and for any matrix $\mathbb{G} \in (\mathcal{D}(\Omega))^{3\times 3}$, let $(\boldsymbol{v}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ be the solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \tilde{\pi} = \operatorname{div} \mathbb{G}, \quad \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{v} + \mathbb{G})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

Since $p' > 2$, from Theorem 6.3, we have

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \le C_p(\Omega) \|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}.$$

Also, if $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the solution of (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$, by using the weak formulation of the problems from which $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy, we obtain

$$-\int_\Omega \mathbb{F} : \nabla \boldsymbol{v} = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = -\int_\Omega \mathbb{G} : \nabla \boldsymbol{u},$$

which implies

$$\left| \int_\Omega \boldsymbol{G} : \nabla \boldsymbol{u} \right| \leq \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C_p(\Omega) \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} \|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}$$

and hence, (6.21) follows from (6.22).

(ii) Next, we prove that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}. \tag{6.23}$$

Similarly to the previous step, we write

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi}|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)}}. \tag{6.24}$$

From Proposition 6.9, we get for any $\boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)$, the unique solution $(\boldsymbol{w}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L_0^{p'}(\Omega)$ of the problem

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla \tilde{\pi} = \boldsymbol{\varphi}, \quad \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad 2[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{w}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{6.25}$$

which satisfies

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\, \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{6.26}$$

Therefore, using the weak formulation of the problems from which $\boldsymbol{u}$ and $\boldsymbol{w}$ satisfy, we get

$$\begin{aligned} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \tilde{\pi}) &= 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{w} - 2\int_\Gamma \boldsymbol{u} \cdot (\mathbb{D}\boldsymbol{w})\boldsymbol{n} \\ &= 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{w} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{w}_{\boldsymbol{\tau}} = -\int_\Omega \mathbb{F} : \nabla \boldsymbol{w}, \end{aligned}$$

which implies (6.23) from the relations (6.24) and (6.26).

For the pressure estimate, we have (3.12). This completes the proof. □

Now, we study the complete problem (S).

Theorem 6.11 *(Complete estimates in $W^{1,p}(\Omega)$). Let $p \in (1,\infty)$ and*

$$f \in L^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), h \in W^{-\frac{1}{p},p}(\Gamma), \alpha \in L^{t(p)}(\Gamma).$$

Then the solution $(u,\pi) \in W^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S) satisfies the following estimates:
(i) *if Ω is nonaxisymmetric, then*

$$\|u\|_{W^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega)\left(\|f\|_{L^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|h\|_{W^{-\frac{1}{p},p}(\Gamma)}\right) \tag{6.27}$$

(ii) *if Ω is axisymmetric and $\alpha \ge \alpha_* > 0$, then*

$$\|u\|_{W^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega,\alpha_*)\left(\|f\|_{L^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|h\|_{W^{-\frac{1}{p},p}(\Gamma)}\right). \tag{6.28}$$

To prove the above theorem, we also need the following proposition:

Proposition 6.12 *(Estimates in $W^{1,p}(\Omega)$ with RHS f and h). Let $p \in (1,\infty)$,*

$$f \in L^{r(p)}(\Omega), h \in W^{-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \alpha \in L^{t(p)}(\Gamma).$$

Then the solution $(u,\pi) \in W^{1,p}(\Omega) \times L^p_0(\Omega)$ of (S), with $\mathbb{F} = 0$, satisfies the following estimates:
(i) *if Ω is nonaxisymmetric, then*

$$\|u\|_{W^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega)\left(\|f\|_{L^{r(p)}(\Omega)} + \|h\|_{W^{-\frac{1}{p},p}(\Gamma)}\right)$$

(ii) *if Ω is axisymmetric and $\alpha \ge \alpha_* > 0$, then*

$$\|u\|_{W^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C_p(\Omega,\alpha_*)\left(\|f\|_{L^{r(p)}(\Omega)} + \|h\|_{W^{-\frac{1}{p},p}(\Gamma)}\right).$$

Proof. Without loss of generality, we only consider the case Ω is nonaxisymmetric. The proof is similar to that of Proposition 6.10 with obvious modifications.
(i) For proving

$$\|\nabla u\|_{L^p(\Omega)} \le C_p(\Omega)\left(\|f\|_{L^{r(p)}(\Omega)} + \|h\|_{W^{-\frac{1}{p},p}(\Gamma)}\right), \tag{6.29}$$

we write

$$\|\nabla u\|_{L^p(\Omega)} = \sup_{0 \neq \mathbb{G} \in \mathbb{L}^{p'}(\Omega)} \frac{\left|\int_\Omega \nabla u : \mathbb{G}\right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}}. \tag{6.30}$$

For any matrix $\mathbb{G} \in (\mathcal{D}(\Omega))^{3\times 3}$, let $(v,\tilde{\pi}) \in W^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ be the solution of

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \tilde{\pi} = \operatorname{div} \mathbb{G}, \quad \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{v} + \mathbb{G})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{v}_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

which satisfies

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega) \|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)},$$

by using Proposition 6.10. Also, if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ is a solution of (S), with $\mathbb{F} = 0$, by using the weak formulation of the problems from which $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy, we get

$$-\int_\Omega \mathbb{G} : \nabla \boldsymbol{u} = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \langle \boldsymbol{h}, \boldsymbol{v} \rangle_\Gamma .$$

This implies, together with the embedding $\boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{(r(p))'}(\Omega)$ for all $p \in (1, \infty)$ (which follows from the definition of $r(p)$), that

$$\begin{aligned} \left| \int_\Omega \mathbb{G} : \nabla \boldsymbol{u} \right| &\leq \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^{(r(p))'}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \|\boldsymbol{v}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \\ &\leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)}. \end{aligned}$$

Therefore, (6.29) follows from (6.30).

(ii) Next, we prove the following bound:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \tag{6.31}$$

as it was done for (6.23). Knowing that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi}|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)}},$$

there exists a unique $(\boldsymbol{w}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}_0(\Omega)$ of the problem (6.25) for any $\boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)$ satisfying the estimate (6.26). Thus, we can write

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \tilde{\pi}) = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{w} + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{w}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \langle \boldsymbol{h}, \boldsymbol{w} \rangle_\Gamma$$

which yields (6.31). The pressure estimate can be obtained from (3.12). □

Proof of Theorem 6.11. Let $\boldsymbol{u}_1 \in \boldsymbol{W}^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} -\Delta \boldsymbol{u}_1 + \nabla \pi_1 = \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u}_1 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_1 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_1 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{1\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

given by Proposition 6.10, and $\boldsymbol{u}_2 \in \boldsymbol{W}^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} -\Delta \boldsymbol{u}_2 + \nabla \pi_2 = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u}_2 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_2 \cdot \boldsymbol{n} = 0, \quad 2\,[(\mathbb{D}\boldsymbol{u}_2)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{2\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases}$$

given by Proposition 6.12. Then, $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1, \pi_1) + (\boldsymbol{u}_2, \pi_2)$ is the solution of the problem (S) which also satisfies the estimates (6.27) and (6.28). $\square$

Remark 6.13. Note that it is also possible to deduce a uniform estimate (6.27) in the case when Ω is axisymmetric, α is a constant with no strict positive lower bound α_* and the condition (4.5) is satisfied. Indeed, we may use the L^2-estimate (4.7) in (6.18) and carry forward all consequent results.

In the next result, we improve the dependence of the continuity constant γ of the inf-sup condition (5.15) on the parameters, and show that it is actually independent of α.

Theorem 6.14. *Let $p \in (1, \infty)$ and $\alpha \in L^{t(p)}(\Gamma)$. We have the following inf-sup condition:*

$$\inf_{\substack{\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega) \\ \boldsymbol{u} \neq 0}} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \frac{\left| 2 \int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int\limits_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} \right|}{\|\boldsymbol{u}\|_{\boldsymbol{V}^p_{\sigma,\tau}(\Omega)} \, \|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)}} \geq C(\Omega, p),$$

when either (i) Ω is non-axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$.

Proof. It follows the same proof as in Proposition 6.12. Indeed, let $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ and $\boldsymbol{u} \neq \boldsymbol{0}$. Then, by Korn inequality, $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$.

(i) First, we write

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0 \neq \mathbb{G} \in \mathbb{L}^{p'}(\Omega)} \frac{\left| \int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{G} \right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}} = \sup_{0 \neq \mathbb{G} \in \mathbb{L}^{p'}_s(\Omega)} \frac{\left| \int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{G} \right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}}, \tag{6.32}$$

where $\mathbb{L}^{p'}_s(\Omega)$ is the space of all symmetric matrices in $\mathbb{L}^{p'}(\Omega)$. For the last equality, note that any matrix $\mathbb{G}$ can be decomposed as $\mathbb{G} = \frac{1}{2}(\mathbb{G} + \mathbb{G}^T) + \frac{1}{2}(\mathbb{G} - \mathbb{G}^T)$. Then, we have $\int\limits_\Omega \mathbb{D}\boldsymbol{u} : (\mathbb{G} - \mathbb{G}^T) = 0$, and denoting $\mathbb{K} = \frac{1}{2}(\mathbb{G} + \mathbb{G}^T)$, we have $\mathbb{K} \in \mathbb{L}^{p'}_s(\Omega)$ and $\|\mathbb{K}\|_{\mathbb{L}^{p'}(\Omega)} \leq 2\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}$, which proves that

$$\sup_{0\neq \mathbb{G}\in \mathbb{L}^{p'}(\Omega)} \frac{\left|\int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{G}\right|}{\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}} \leq \sup_{0\neq \mathbb{K}\in \mathbb{L}_s^{p'}(\Omega)} \frac{\left|\int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{K}\right|}{\|\mathbb{K}\|_{\mathbb{L}^{p'}(\Omega)}}.$$

The reverse inequality in the above relation is clear.

Now, for any $\mathbb{G} \in \mathbb{L}_s^{p'}(\Omega)$, let $(\boldsymbol{\varphi}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L_0^{p'}(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta\boldsymbol{\varphi} + \nabla\tilde{\pi} = \operatorname{div}\mathbb{G}, \quad \operatorname{div}\boldsymbol{\varphi} = 0 & \text{in } \Omega, \\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0, \quad \left[(2\mathbb{D}\boldsymbol{\varphi} + \mathbb{G})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\boldsymbol{\varphi}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{6.33}$$

Since we have either (i) Ω is nonaxisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, the solution also satisfies the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\|\mathbb{G}\|_{\mathbb{L}^{p'}(\Omega)}, \tag{6.34}$$

by using Proposition 6.10. Also, taking $\boldsymbol{u}$ as a test function in the weak formulation of (6.33), we obtain

$$2\int\limits_\Omega \mathbb{D}\boldsymbol{\varphi} : \mathbb{D}\boldsymbol{u} + \int\limits_\Gamma \alpha\boldsymbol{\varphi}_{\boldsymbol{\tau}}\cdot\boldsymbol{u}_{\boldsymbol{\tau}} = -\int\limits_\Omega \mathbb{G} : \nabla\boldsymbol{u} = -\int\limits_\Omega \mathbb{G} : \mathbb{D}\boldsymbol{u}, \tag{6.35}$$

where in the last equality, we used that $\mathbb{G}$ is a symmetric matrix. Thus, from (6.32), (6.34) and (6.35), we get

$$\|\mathbb{D}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_{\sigma,\boldsymbol{\tau}}^{p'}(\boldsymbol{\Omega}) \\ \boldsymbol{\varphi}\neq\boldsymbol{0}}} \frac{\left|2\int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int\limits_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{\varphi}_{\boldsymbol{\tau}}\right|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}}.$$

(ii) Similarly to the estimate (6.23), to prove

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_p(\Omega) \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_{\sigma,\boldsymbol{\tau}}^{p'}(\boldsymbol{\Omega}) \\ \boldsymbol{\varphi}\neq\boldsymbol{0}}} \frac{\left|2\int\limits_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int\limits_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{\varphi}_{\boldsymbol{\tau}}\right|}{\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)}}, \tag{6.36}$$

we write

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \sup_{0\neq\boldsymbol{w}\in \boldsymbol{L}^{p'}(\Omega)} \frac{|\int\limits_\Omega \boldsymbol{u}\cdot\boldsymbol{w}|}{\|\boldsymbol{w}\|_{\boldsymbol{L}^{p'}(\Omega)}}. \tag{6.37}$$

So, for any $\boldsymbol{w} \in \boldsymbol{L}^{p'}(\Omega)$, the unique solution $(\boldsymbol{\varphi}, \tilde{\pi}) \in \boldsymbol{W}^{1,p'}(\Omega) \times L_0^{p'}(\Omega)$ of the Stokes problem

$$\begin{cases} -\Delta\boldsymbol{\varphi} + \nabla\tilde{\pi} = \boldsymbol{w}, & \operatorname{div}\boldsymbol{\varphi} = 0 & \text{in } \Omega, \\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0, & 2\left[(\mathbb{D}\boldsymbol{\varphi})\boldsymbol{n}\right]_{\boldsymbol{\tau}} + \alpha\boldsymbol{\varphi}_{\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{6.38}$$

satisfies

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C_p(\Omega)\ \|\boldsymbol{w}\|_{\boldsymbol{L}^{p'}(\Omega)}, \tag{6.39}$$

by using Proposition 6.9. Therefore, taking $\boldsymbol{u}$ as a test function in the weak formulation of (6.38), we get

$$2\int_\Omega \mathbb{D}\boldsymbol{\varphi} : \mathbb{D}\boldsymbol{u} + \int_\Gamma \alpha\boldsymbol{\varphi}_{\boldsymbol{\tau}}\cdot\boldsymbol{u}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{w},$$

which yields (6.36) from (6.37) and (6.39). □

Remark 6.15. If we consider the operator of the form $\operatorname{div}(A(x)\nabla\boldsymbol{u})$ instead of $\Delta\boldsymbol{u}$ in the first equation of the Stokes system (S), we may obtain the following improved $\boldsymbol{W}^{2,p}$-estimate as it is done in [5, Theorem 3.1]:

Let $p \in (1,\infty)$, α be a constant and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. Then the solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ of (S) with $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ satisfies the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C_p(\Omega,\alpha_*)\ \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

7. Limiting cases

Our goal in this section is to study the limiting behavior of the solution of (S), when the friction coefficient α goes to 0 or ∞.

7.1. α tends to 0

Theorem 7.1. *Let $p \in (1,\infty)$, Ω be a nonaxisymmetric bounded domain and $(\boldsymbol{u}_\alpha,\pi_\alpha) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ be the solution of (S), with*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \quad \text{and} \quad \alpha \in L^{t(p)}(\Gamma).$$

If $\alpha \to 0$ in $L^{t(p)}(\Gamma)$, then we have the convergence

$$(\boldsymbol{u}_\alpha,\pi_\alpha) \to (\boldsymbol{u}_0,\pi_0) \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p_0(\Omega),$$

where $(\boldsymbol{u}_0,\pi_0)$ satisfies, in the sense of distributions, the following Stokes problem with Navier boundary conditions:

$$\begin{cases} -\Delta\boldsymbol{u}_0 + \nabla\pi_0 = \boldsymbol{f} + \operatorname{div}\mathbb{F}, & \operatorname{div}\boldsymbol{u}_0 = 0 & \text{in } \Omega, \\ \boldsymbol{u}_0\cdot\boldsymbol{n} = 0, & [(2\mathbb{D}\boldsymbol{u}_0 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{on } \Gamma, \end{cases} \tag{7.1}$$

which corresponds to the case $\alpha = 0$.

Proof. Let $\alpha \to 0$ in $L^{t(p)}(\Gamma)$. This means that there does not exist any $\alpha_* > 0$ such that $\alpha \geq \alpha_*$ on Γ. Now, from the estimate (6.27), it is clear that $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for all $p \in (1, \infty)$. Then, there exists $(\boldsymbol{u}_0, \pi_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0) \quad \text{weakly in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

It can be easily proved that $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique weak solution of the Stokes problem (7.1). Indeed, being a weak solution of (S), $\boldsymbol{u}_\alpha$ satisfies the weak formulation (3.8). Now, as shown in Lemma 3.9, $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ implies $(\boldsymbol{u}_\alpha)_{\boldsymbol{\tau}} \rightharpoonup (\boldsymbol{u}_0)_{\boldsymbol{\tau}}$ in $\boldsymbol{L}^s(\Gamma)$, where s satisfies (3.7). Also $\alpha \to 0$ in $L^{t(p)}(\Gamma)$ gives $\alpha(\boldsymbol{u}_\alpha)_{\boldsymbol{\tau}} \rightharpoonup \boldsymbol{0}$ in $\boldsymbol{L}^{m'}(\Gamma)$, with m defined in (3.6). Hence, in the weak formulation (3.8), the boundary term in the left hand side goes to 0. Finally, passing to the limit, we deduce

$$2\int_\Omega \mathbb{D}\boldsymbol{u}_0 : \mathbb{D}\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega). \tag{7.2}$$

Satisfying this variational formulation (7.2) is equivalent to saying that $(\boldsymbol{u}_0, \pi_0)$ satisfies (7.1) in the sense of distributions, as shown in Proposition 3.11. Note that, the system (7.1) has a unique weak solution if the domain is nonaxisymmetric (cf. Remark 3.14).

Now, by using the variational formulations for the systems (7.1) and (S), we obtain that $(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)$ is a weak solution of the following system, in the sense of Definition 3.10,

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) + \nabla(\pi_\alpha - \pi_0) = \boldsymbol{0}, \quad \operatorname{div}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) = 0 & \text{in } \Omega, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_0) \cdot \boldsymbol{n} = 0, \quad 2\,[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)_{\boldsymbol{\tau}} = -\alpha(\boldsymbol{u}_0)_{\boldsymbol{\tau}} & \text{on } \Gamma, \end{cases}$$

for which employing the estimate of Theorem 6.11, the Hölder inequality and the trace theorem, yields

$$\begin{aligned} \|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha - \pi_0\|_{L^p(\Omega)} &\leq C(\Omega)\, \|\alpha(\boldsymbol{u}_0)_{\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \\ &\leq C(\Omega)\, \|\alpha\|_{L^{t(p)}(\Gamma)} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}. \end{aligned}$$

Therefore, $\boldsymbol{u}_\alpha - \boldsymbol{u}_0$ and $\pi_\alpha - \pi_0$ both tend to zero in the same rate as α. □

Remark 7.2. We can prove also the above theorem for Ω axisymmetric and α constant, provided the compatibility condition (4.5), with the help of the estimate (4.7) and the Remark 6.13. Indeed, to expect the limiting system to be (7.1), we must assume the compatibility condition, since this is the necessary condition for the existence of a solution of the system (7.1).

7.2. α tends to ∞

Next, we study the behavior of $\boldsymbol{u}_\alpha$, where α is a constant and grows to ∞.

Theorem 7.3. *Let $p \in (1, \infty)$ and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the solution of (S), with*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \text{ a constant.}$$

(i) *If $\alpha \to \infty$, then we have the convergence*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \textit{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega),$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ is the unique solution of the Stokes problem with Dirichlet boundary condition:

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u}_\infty = 0 \ \ \textit{in}\ \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{u}_\infty = \boldsymbol{0} \ \ \textit{on}\ \Gamma. \end{cases} \tag{7.3}$$

(ii) *Moreover, for any $q < p$ if $p \neq 2$ and $q = 2$ if $p = 2$, we obtain the strong convergence*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \textit{in} \quad \boldsymbol{W}^{1,q}(\Omega) \times L^q(\Omega).$$

Proof. Since $\alpha \to \infty$, we can consider $\alpha \geq 1$.

(i) From the estimates (6.27) or (6.28), we see that $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for all $p \in (1, \infty)$, hence there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{weakly in } \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

On the other hand, we can also write the system (S) as follows:

$$\begin{cases} -\Delta \boldsymbol{u}_\alpha + \nabla \pi_\alpha = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u}_\alpha = 0 \ \text{ in } \Omega, \\ \qquad\qquad \boldsymbol{u}_\alpha = \frac{1}{\alpha}\left(\boldsymbol{h} - [(2\mathbb{D}\boldsymbol{u}_\alpha + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}}\right) \ \text{ on } \Gamma. \end{cases} \tag{7.4}$$

Observe that, the condition $\boldsymbol{u}_\alpha \cdot \boldsymbol{n} = 0$ on Γ is included in the above system, because of the assumption $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ (see paragraph 3 of Section 2). Passing to the limit in (7.4) as $\alpha \to \infty$, we obtain that $(\boldsymbol{u}_\infty, \pi_\infty)$ is the solution of the Stokes problem with Dirichlet boundary condition (7.3).

Indeed, passing to the limit in the first two equations of (7.4), we obtain what we desire. For the boundary condition, we have that $2[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}}$ is bounded in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, since $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{E}^p(\Omega)$ and by using the Green formula (3.3). Hence, taking the limit as $\alpha \to \infty$ in the boundary condition of (7.4), we obtain the boundary condition of (7.3).

(ii) Now, to show the strong convergence, we know from the variational formulations of the systems (7.3) and (S), that $(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ is a weak solution of the problem

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) + \nabla(\pi_\alpha - \pi_\infty) = \boldsymbol{0}, \quad \operatorname{div}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) = 0 & \text{in } \Omega, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \cdot \boldsymbol{n} = 0, \quad 2\,[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}} = \boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma, \end{cases}$$

and then, the Green formula (3.3) yields, choosing $(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ as a test function,

$$2\int_\Omega |\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)|^2 + \alpha \int_\Gamma |(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}}|^2 = \langle \boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

As $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_\infty$ in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ weakly and $\boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, this shows the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$. The strong convergence for the pressure term follows from the estimate (3.12).

Next, since $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$, we have $\nabla \boldsymbol{u}_\alpha \to \nabla \boldsymbol{u}_\infty$ almost everywhere in Ω. Further, we know that $\nabla \boldsymbol{u}_\alpha$ is a bounded sequence in $\boldsymbol{L}^p(\Omega)$. Therefore, the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_\infty$ in $\boldsymbol{W}^{1,q}(\Omega)$ for any $q < p$ follows, cf. [3, Lemma 1.2.3, Chapter 1]. This completes the proof. $\square$

7.3. α less regular

Theorem 7.4. *Let*

$$\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega), \mathbb{F} \in \mathbb{L}^2(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \text{ and } \alpha \in L^{\frac{4}{3}}(\Gamma).$$

Then the Stokes problem (S) has a solution $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$.

Proof. (i) First, let us consider that Ω is nonaxisymmetric. Using Theorem II.4.2 in [42], we know that there exists a sequence $\chi_k \in W^{1,\frac{4}{3}}(\Omega)$ such that $\chi_k|_\Gamma \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$. Thanks to the density of $\mathcal{D}(\overline{\Omega})$ in $W^{1,\frac{4}{3}}(\Omega)$, we deduce the existence of a sequence $\tilde{\alpha}_k \in \mathcal{D}(\overline{\Omega})$ such that for $\alpha_k := \tilde{\alpha}_k|_\Gamma$, $\alpha_k \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$. If $(\boldsymbol{u}_k, \pi_k) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ is the solution of the problem (S) corresponding to α_k, due to the estimate (4.3) satisfied by $(\boldsymbol{u}_k, \pi_k)$, there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ such that

$$(\boldsymbol{u}_k, \pi_k) \rightharpoonup (\boldsymbol{u}, \pi) \quad \text{in} \quad \boldsymbol{H}^1(\Omega) \times L^2(\Omega).$$

This implies $(-\Delta \boldsymbol{u}_k + \nabla \pi_k) \rightharpoonup (-\Delta \boldsymbol{u} + \nabla \pi)$ in $\boldsymbol{H}^{-1}(\Omega)$. Similarly, $\operatorname{div} \boldsymbol{u}_k \rightharpoonup \operatorname{div} \boldsymbol{u}$ in $\boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{u}_k \cdot \boldsymbol{n} \rightharpoonup \boldsymbol{u} \cdot \boldsymbol{n}$ in $H^{\frac{1}{2}}(\Gamma)$. Thus, we obtain, in the sense of distributions,

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F} \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma.$$

Next, from the Green formula (3.3), we have $[(\mathbb{D}\boldsymbol{u}_k)\boldsymbol{n}]_{\boldsymbol{\tau}} \rightharpoonup [(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}$ in $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. Moreover, as $\alpha_k \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$ and $(\boldsymbol{u}_k)_{\boldsymbol{\tau}} \rightharpoonup \boldsymbol{u}_{\boldsymbol{\tau}}$ in $L^4(\Gamma)$, it follows $\alpha_k(\boldsymbol{u}_k)_{\boldsymbol{\tau}} \rightharpoonup \alpha \boldsymbol{u}_{\boldsymbol{\tau}}$ in $\boldsymbol{L}^1(\Gamma)$. Therefore, passing to the limit in the Navier boundary condition satisfied by $\boldsymbol{u}_k$,

$$[(2\mathbb{D}\boldsymbol{u}_k + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha_k \boldsymbol{u}_{k\boldsymbol{\tau}} = \boldsymbol{h} \text{ on } \Gamma,$$

it yields, $[(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h}$ on Γ. Hence, $(\boldsymbol{u}, \pi)$ becomes the solution of the Stokes problem (S).

(ii) Note that, when Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, we can find a sequence $\alpha_k \in \mathcal{D}(\overline{\Omega})$ such that $\alpha_k \geq \alpha_*$ in Ω and $\alpha_k|_\Gamma \to \alpha$ in $L^{\frac{4}{3}}(\Gamma)$. So, we can use the estimate (4.4) and obtain the same result. $\square$

Remark 7.5. For $\alpha \in L^{\frac{4}{3}}(\Gamma)$ and $\boldsymbol{h} = \boldsymbol{0}$, the solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfies the additional property: $[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{L}^1(\Gamma)$.

Remark 7.6. Let $\boldsymbol{f} = \boldsymbol{0}$, $\mathbb{F} = 0$ and $\boldsymbol{h} \in \boldsymbol{L}^{\frac{4}{3}}(\Gamma)$ with $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ.
(i) If $\alpha \in L^2(\Gamma)$, then we have $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ by Theorem 4.1. This implies that $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^{\frac{4}{3}}(\Gamma)$. Hence, $[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{L}^{\frac{4}{3}}(\Gamma)$. We may now use complex interpolation between weak and strong

solutions of the Stokes problem by treating $\alpha \boldsymbol{u}_{\boldsymbol{\tau}}$ as a source term on the right hand side (cf. [7]), and we can consider the map

$$T : \boldsymbol{h} \to \boldsymbol{u}$$

$$\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \to \boldsymbol{W}^{1,p}(\Omega)$$

$$\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \to \boldsymbol{W}^{2,p}(\Omega).$$

After interpolating, this yields that for $\boldsymbol{h} \in \boldsymbol{L}^{\frac{4}{3}}(\Gamma)$, the solution $\boldsymbol{u} \in \boldsymbol{W}^{1+\frac{1}{4},\frac{4}{3}}(\Omega)$.
(ii) If $\alpha \in L^{\frac{4}{3}}(\Gamma)$, we have $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, from Theorem 7.4. Then, $\alpha \boldsymbol{u}_{\boldsymbol{\tau}} \in \boldsymbol{L}^1(\Omega)$ and from here, we cannot improve the regularity any more.

8. Navier-Stokes equations

Finally, we consider the nonlinear problem and study the existence of weak and strong solutions for the Navier-Stokes system (NS).

Definition 8.1. Given $\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$, a function $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is called a weak solution of the Navier-Stokes system (NS) if it satisfies: for all $\boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\varphi}) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_\Gamma , \tag{8.1}$$

where $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{w}$.

Theorem 8.2. *Let $p \in (1,\infty)$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma),$$

where $r(p)$ and $t(p)$ are defined by (2.3) and (2.2), respectively. Then the following two statements are equivalent:
(i) *$\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ is a weak solution of (NS) and,*
(ii) *there exists $\pi \in L^p(\Omega)$ such that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies:*

$$\begin{cases} -\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} + \operatorname{div}\mathbb{F}, \ \operatorname{div}\boldsymbol{u} = 0 & \text{in the sense of distributions,} \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{in the sense of traces,} \\ 2[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{h} & \text{in } \boldsymbol{W}^{-1/p,p}(\Gamma). \end{cases}$$

The proof is standard and very similar to that of Proposition 3.11, hence we omit it. To facilitate the work, we introduce some properties of the operator b, but we skip the proof (cf. [7, Lemma 7.2]).

Lemma 8.3. *The trilinear form b is defined and continuous on $\boldsymbol{V}^2_{\sigma,\tau}(\Omega) \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega) \times \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Also we have, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$,*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{0} \tag{8.2}$$

and

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad .$$

Moreover,

$$b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\beta}) = 0 \quad \textit{and} \quad b(\boldsymbol{\beta}, \boldsymbol{\beta}, \boldsymbol{u}) = 0, \quad \textit{where } \boldsymbol{\beta} \textit{ is defined in (2.4).}$$

8.1. Existence and regularity

Now, we can prove the existence of weak solutions of the Navier-Stokes problem (NS). First, we study the Hilbertian case.

Proof of Theorem 2.3 for $p = 2$. (i) Existence: The existence of solution of (8.1) can be proved by using standard arguments, *i.e.*, we construct an approximate solution by using the Galerkin method and then, we pass to the limit. Nonetheless, we state it briefly for completeness.

For each fixed integer $m \geq 1$, define an approximate solution $\boldsymbol{u}_m$ of (8.1) by

$$\begin{aligned} &\boldsymbol{u}_m = \sum_{i=1}^m \xi_{i,m} \boldsymbol{v}_i, \quad \xi_{i,m} \in \mathbb{R} \\ &2\int_\Omega \mathbb{D}\boldsymbol{u}_m : \mathbb{D}\boldsymbol{v}_k + b(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{v}_k) + \int_\Gamma \alpha \boldsymbol{u}_{\boldsymbol{\tau} m} \cdot \boldsymbol{v}_{\boldsymbol{\tau} k} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_k - \int_\Omega \mathbb{F} : \nabla \boldsymbol{v}_k + \langle \boldsymbol{h}, \boldsymbol{v}_k \rangle_\Gamma, \\ &\qquad \text{for } k = 1, \ldots, m \end{aligned} \tag{8.3}$$

and $V_m := \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \rangle$ is the space spanned by the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ and $\{\boldsymbol{v}_i\}_{i \in I\!N}$ is an orthonormal basis of $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Note that V_m is equipped with the scalar product $(\cdot, \cdot)$ induced by $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$. Let the mapping $P_m : V_m \to V_m$ be defined by

$$(P_m(\boldsymbol{w}), \boldsymbol{v}) = 2\int_\Omega \mathbb{D}\boldsymbol{w} : \mathbb{D}\boldsymbol{v} + b(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) + \int_\Gamma \alpha \boldsymbol{w}_{\boldsymbol{\tau}} \cdot \boldsymbol{v}_{\boldsymbol{\tau}} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_\Omega \mathbb{F} : \nabla \boldsymbol{v} - \langle \boldsymbol{h}, \boldsymbol{v} \rangle_\Gamma,$$

for all $\boldsymbol{w}, \boldsymbol{v} \in V_m$. The continuity of the mapping is clear. Also, using (8.2) and Proposition 3.13, we get

$$\begin{aligned} (P_m(\boldsymbol{w}), \boldsymbol{w}) &= 2\|\mathbb{D}\boldsymbol{w}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{w}_{\boldsymbol{\tau}}|^2 - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} + \int_\Omega \mathbb{F} : \nabla \boldsymbol{w} - \langle \boldsymbol{h}, \boldsymbol{w} \rangle_\Gamma \\ &\geq C(\alpha, \Omega) \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \left\{ \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} - C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \right) \right\}. \end{aligned}$$

Hence, $(P_m(\boldsymbol{w}), \boldsymbol{w}) > 0$ for all $\|\boldsymbol{w}\|_{V_m} = k$, where $k > C(\Omega)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\big)$. Therefore, the hypothesis of Brouwer's theorem is satisfied and there exists a solution $\boldsymbol{u}_m$ of (8.3).

Next, since $\boldsymbol{u}_m$ is a solution of (8.3), we have

$$2\|\mathbb{D}\boldsymbol{u}_m\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}m}|^2 = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_m - \int_\Omega \mathbb{F} : \nabla\boldsymbol{u}_m + \langle\boldsymbol{h}, \boldsymbol{u}_m\rangle_\Gamma\,,$$

which yields the a priori estimate

$$\|\boldsymbol{u}_m\|_{\boldsymbol{H}^1(\Omega)} \le C(\alpha)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

Since the sequence $\boldsymbol{u}_m$ remains bounded in $\boldsymbol{V}^2_{\sigma,\tau}(\Omega)$, there exists some $\boldsymbol{u} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ and a subsequence, which we still call $\boldsymbol{u}_m$, such that

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{V}^2_{\sigma,\tau}(\Omega).$$

Due to the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ and by using Lemma 3.9 to handle the boundary integral $\int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}m}\cdot\boldsymbol{v}_{\boldsymbol{\tau}k}$, we can pass to the limit in (8.3) and we obtain for any $\boldsymbol{v} \in \boldsymbol{V}^2_{\sigma,\tau}(\Omega)$ that

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{v}_{\boldsymbol{\tau}} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{v} + \langle\boldsymbol{h}, \boldsymbol{v}\rangle_\Gamma\,,$$

and thus, $\boldsymbol{u}$ is a solution of (8.1).

(ii) Estimates: The estimates can be proved in a similar way as in the linear case given in Theorem 4.3. □

Proposition 8.4. *The solution of the problem (NS), given by Theorem 2.3 is unique provided*

$$\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} < \frac{1}{C(\alpha, \Omega)}, \tag{8.4}$$

where the constant $C(\alpha, \Omega)$ depends on the continuity constant of the trilinear form b and the equivalence constant of H^1-norm, which will be shown in the proof.

Remark 8.5. Interestingly, in the case of $\alpha \equiv 0$, there is no uniqueness of the solution of the system (NS) even for small data. But in our case, when $\alpha \neq 0$ on some $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| > 0$, there is indeed uniqueness of the solution under the assumption of small data as in the case of Dirichlet boundary condition. The reason of this behavior is the presence of a nontrivial kernel of the Stokes operator for $\alpha \equiv 0$.

Proof. Choosing $\boldsymbol{\varphi} = \boldsymbol{u}$ in the weak formulation (8.1) and using the relation (8.2) and the Proposition 3.13, we obtain that any solution of (8.1) satisfies the estimate

$$C(\Omega,\alpha)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \left(2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2\right)$$
$$\leq C(\Omega)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$$

which gives,

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega,\alpha)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right). \tag{8.5}$$

Now, if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two different solutions of (8.1), let us define $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and subtracting the equations (8.1) corresponding to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we get

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}_1,\boldsymbol{u},\boldsymbol{\varphi}) + b(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{\varphi}) + \int_\Gamma \alpha\boldsymbol{u}_{\boldsymbol{\tau}}\cdot\boldsymbol{\varphi}_{\boldsymbol{\tau}} = 0, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{V}^2_{\sigma,\tau}(\Omega). \tag{8.6}$$

Taking $\boldsymbol{\varphi} = \boldsymbol{u}$ in (8.6) and using again (8.2), we have

$$2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2 = -b(\boldsymbol{u},\boldsymbol{u}_2,\boldsymbol{u})$$

which implies, by using the Proposition 3.13, the continuity of b and the estimate (8.5) for $\boldsymbol{u}_2$,

$$C(\Omega,\alpha)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \left(2\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2\right)$$
$$\leq C(\Omega)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{H}^1(\Omega)}$$
$$\leq C(\Omega,\alpha)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}\right).$$

Thus, considering the condition (8.4), the above inequality implies that $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 0$ that is $\boldsymbol{u}_1 = \boldsymbol{u}_2$. $\square$

Next, we prove the existence of solution of the system (NS) in $\boldsymbol{W}^{1,p}(\Omega)$ by using the Hilbertian case and the Stokes regularity result.

Proof of Theorem 2.3 for $p \neq 2$**.** **(i)** First, let us consider $p > 2$. We have the existence of a weak solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$. Since $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^{r(p)}(\Omega)$ if $p \leq 3$. Hence, the regularity result for Stokes problem in Corollary 5.7 implies that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. For $p > 3$, repeating the same argument with $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega)$, we deduce the required regularity.

In order to obtain the existence result for $p \in (\frac{3}{2}, 2)$, we follow a similar argument as it was done in the proof of [45, Theorem 1.1]. Note that we replaced the space $\boldsymbol{W}^{-1,p}(\Omega)$ by $\boldsymbol{L}^{r(p)}(\Omega)$ for the given data in [45]. For example, we use the following lemma instead of the given in [45, Lemma 1.2]: if there exists $(\boldsymbol{v}, \tilde{\pi}) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla\tilde{\pi} - \boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad 2\,[(\mathbb{D}\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{h} \text{ on } \Gamma, \end{cases}$$

for $p \le q \le 2$, then there exists $(\boldsymbol{w}, \bar{\pi}) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + \nabla\bar{\pi} - \boldsymbol{f} \in \boldsymbol{L}^{r(s)}(\Omega), \\ \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad 2\,[(\mathbb{D}\boldsymbol{w})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{w}_{\boldsymbol{\tau}} = \boldsymbol{h} \text{ on } \Gamma, \end{cases}$$

where $\frac{1}{s} = \frac{1}{q} + \frac{1}{p} - \frac{2}{3}$ (thus $s > q$). The rest of the proof follows the same argument as the given in [45] without any further changes.

(ii) Next, to prove the strong regularity result, we consider that the data are more regular. For $p \in (1, \frac{3}{2}]$, since the Sobolev exponent $p^* \in (\frac{3}{2}, 3]$ and thus, $r(p^*) = p$, we have $\boldsymbol{f} \in \boldsymbol{L}^{r(p^*)}(\Omega)$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p^*},p^*}(\Gamma)$. Hence, the above regularity result for the weak solutions of (NS) implies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p^*}(\Omega) \times L^{p^*}(\Omega)$. Now, for $p \in (1, \frac{3}{2})$, $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^s(\Omega)$ with

$$\frac{1}{s} = \frac{2}{p} - 1,$$

which implies that $s > p$ and thus, by using Theorem 5.10, we obtain $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. For $p = \frac{3}{2}$, since $\boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^m(\Omega)$ for any $m \in (1, \infty)$, we have $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^s(\Omega)$, with $\frac{1}{s} = \frac{1}{3} + \frac{1}{m}$. So, choosing $m > 3$, we have that $s > \frac{3}{2}$ and thus, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$.

For $p > \frac{3}{2}$, since $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$, it follows that $\sum_i u_i \partial_i u \in \boldsymbol{L}^{3-\epsilon}(\Omega)$, which yields $\boldsymbol{u} \in \boldsymbol{W}^{2,3-\epsilon}(\Omega)$. Further, repeating the argument, we get $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. $\square$

Finally, we discuss the limiting behavior of the Navier-Stokes system (NS) as α goes to 0 or ∞.

8.2. Limiting cases

Theorem 8.6. *Let $p \ge 2$, Ω be a nonaxisymmetric bounded domain and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be a solution of (NS), with*

$$\boldsymbol{f} \in \boldsymbol{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

If $\|\alpha\|_{L^{t(p)}(\Gamma)} \to 0$, then we have the convergence

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_0, \pi_0) \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega),$$

where $(\boldsymbol{u}_0, \pi_0)$ *is a solution of the following Navier-Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{u}_0 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 + \nabla \pi_0 = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u}_0 = 0 & \textit{in } \Omega, \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_0 + \mathbb{F})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{h} & \textit{on } \Gamma. \end{cases} \tag{8.7}$$

Proof. **(i)** We assume, for ease of calculation, that $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$. Since $\alpha \to 0$ in $L^{t(p)}(\Gamma)$, there does not exist any $\alpha_* > 0$ such that $\alpha \geq \alpha_*$ on $\Gamma_0 \subseteq \Gamma$. Therefore, $(\boldsymbol{u}_\alpha, \pi_\alpha)$ satisfies the estimate (2.5) for $p = 2$. For $2 < p \leq 3$, the Stokes estimate (6.1) yields

$$\begin{aligned} \|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha\|_{L^p(\Omega)} &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha\|^2_{\boldsymbol{H}^1(\Omega)} \right) \\ &\leq C(\Omega) \left(1 + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)}, \end{aligned}$$

and for $p > 3$,

$$\begin{aligned} \|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha\|_{L^p(\Omega)} &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \\ &\leq C(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha\|^2_{\boldsymbol{W}^{1,3}(\Omega)} \right) \\ &\leq C(\Omega) \left[1 + \left(1 + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right)^2 \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right] \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)}. \end{aligned}$$

Then, $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ uniformly with respect to α. So, there exists $(\boldsymbol{u}_0, \pi_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0) \text{ weakly in } \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

Now, like in Theorem 7.1, passing to the limit as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$ in the variational formulation satisfied by $(\boldsymbol{u}_\alpha, \pi_\alpha)$, we get that $\boldsymbol{u}_0$ satisfies

$$2 \int_\Omega \mathbb{D}\boldsymbol{u}_0 : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{\varphi}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_{\sigma,\tau}(\Omega).$$

Indeed, $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_0$ weakly in $\boldsymbol{W}^{1,p}(\Omega)$ implies that $\boldsymbol{u}_\alpha \to \boldsymbol{u}_0$ in $\boldsymbol{L}^s(\Omega)$, where

$$s \in \begin{cases} (1, p^*] & \text{if} \quad p < 3, \\ (1, \infty) & \text{if} \quad p = 3, \\ (1, \infty] & \text{if} \quad p > 3. \end{cases} \tag{8.8}$$

Also, $\nabla \boldsymbol{u}_\alpha \rightharpoonup \nabla \boldsymbol{u}_0$ weakly in $\boldsymbol{L}^p(\Omega)$. Therefore, $\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0$ weakly in $\boldsymbol{L}^q(\Omega)$, where

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{s},$$

and note that, $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{q'}(\Omega)$. Hence, $b(\boldsymbol{u}_\alpha, \boldsymbol{u}_\alpha, \boldsymbol{\varphi}) \to b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{\varphi})$ as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$. Therefore, $(\boldsymbol{u}_0, \pi_0)$ is a solution of the problem (8.7).

(ii) Next, we prove that the convergence $(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0)$ weakly in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ occurs, in fact, in a strong sense. Subtracting the system (8.7) from the system (NS), we get

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) + \nabla(\pi_\alpha - \pi_0) = (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 - (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha & \text{in } \Omega, \\ \operatorname{div}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) = 0 & \text{in } \Omega, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_0) \cdot \boldsymbol{n} = 0, \quad 2\,[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\alpha\boldsymbol{\tau}} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

Note that $\boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 - \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha = \operatorname{div}(\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha - \boldsymbol{u}_0 \otimes \boldsymbol{u}_0)$. Thus, using the Stokes estimate (6.1) for the above system, we have

$$\begin{aligned} &\|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha - \pi_0\|_{L^p(\Omega)} \\ &\leq C \left(\|\boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha - \boldsymbol{u}_0 \otimes \boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} + \|\alpha \boldsymbol{u}_{0\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \\ &= C \left(\|(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) \otimes \boldsymbol{u}_\alpha + \boldsymbol{u}_0 \otimes (\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\|_{\boldsymbol{L}^p(\Omega)} + \|\alpha \boldsymbol{u}_{0\boldsymbol{\tau}}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \right) \\ &\leq C \left[\|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\boldsymbol{L}^s(\Omega)} \left(\|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \right) + \|\alpha\|_{L^{t(p)}(\Gamma)} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \right], \end{aligned}$$

where s is defined in (8.8). Since $\boldsymbol{u}_\alpha$ is bounded in $\boldsymbol{W}^{1,p}(\Omega)$, it follows that $\boldsymbol{u}_\alpha \to \boldsymbol{u}_0$ in $\boldsymbol{L}^s(\Omega)$, by compactness. This proves the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$ as $\alpha \to 0$. □

Remark 8.7. In the same fashion as we did for the Stokes case, we can prove, with the help of the estimate (2.7) and the Remark 6.13, the above theorem for Ω axisymmetric and α constant, provided the compatibility condition (4.5) is satisfied. Indeed, in order to have the limiting system (8.7), we must assume the compatibility condition since this is the necessary condition for the existence of a solution of the system (8.7).

Proof of Theorem 2.5. **(i)** Without loss of generality, we assume $\mathbb{F} = 0$ and $\boldsymbol{h} = \boldsymbol{0}$. Since $\alpha \to \infty$, we can consider $\alpha \geq 1$ and then, we have the estimates (2.5) and (2.6). Also, as it was done in Theorem 8.6, the Stokes estimates (6.27) and (6.28) allow us to have for $2 < p \leq 3$,

$$\begin{aligned} \|\boldsymbol{u}_\alpha\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\alpha\|_{L^p(\Omega)} &\leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \\ &\leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} + \|\boldsymbol{u}_\alpha\|^2_{\boldsymbol{H}^1(\Omega)} \right) \\ &\leq C_p(\Omega) \left(1 + \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)} \right) \|\boldsymbol{f}\|_{\boldsymbol{L}^{r(p)}(\Omega)}. \end{aligned}$$

For $p > 3$, a similar estimate, independent of α, can be obtained as in Theorem 8.6. This proves that $(\boldsymbol{u}_\alpha, \pi_\alpha)$ is bounded in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for all $p \geq 2$. Hence, there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{weakly in } \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega).$$

Now, rewriting the system (NS) as

$$\begin{cases} -\Delta \boldsymbol{u}_\alpha + (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha + \nabla \pi_\alpha = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u}_\alpha = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \boldsymbol{u}_\alpha = -\frac{2}{\alpha}[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_{\boldsymbol{\tau}} \quad \text{on } \Gamma, \end{cases} \tag{8.9}$$

and as it was done in Theorem 7.3, letting $\alpha \to \infty$ in the above system, we obtain that $(\boldsymbol{u}_\infty, \pi_\infty)$ satisfies the Navier-Stokes problem (2.9).
(ii) We know that $(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ satisfies the system

$$\begin{cases} -\Delta\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) + \nabla\,(\pi_\alpha - \pi_\infty) = (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty - (\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha & \text{in } \Omega, \\ \operatorname{div}\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) = 0 & \text{in } \Omega, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\cdot \boldsymbol{n} = 0, \quad 2\,[\mathbb{D}\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\,\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}} = -2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} & \text{on } \Gamma. \end{cases}$$

Then the Green formula (3.3) yields, choosing $(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)$ as a test function,

$$\begin{aligned} &2\int_\Omega |\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)|^2 + \alpha \int_\Gamma |(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_{\boldsymbol{\tau}}|^2 \\ =&b\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty, \boldsymbol{u}_\infty, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) - \langle 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

But since $\alpha \to \infty$, $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{L}^4(\Omega)$, by compactness, and thus

$$b\,(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty, \boldsymbol{u}_\infty, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \le \|\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\|^2_{\boldsymbol{L}^4(\Omega)} \|\nabla \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \to 0.$$

Also, since $\boldsymbol{u}_\alpha \rightharpoonup \boldsymbol{u}_\infty$ weakly in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, it implies

$$\langle 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_{\boldsymbol{\tau}}, (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \to 0.$$

Therefore, due to the fact that $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{L}^2(\Omega)$, we obtain the strong convergence $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$. The strong convergence of pressure follows from the estimate (3.12).

Now, in the similar way to the Stokes case, $\boldsymbol{u}_\alpha \to \boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$ implies $\nabla \boldsymbol{u}_\alpha \to \nabla \boldsymbol{u}_\infty$ almost everywhere in Ω. Further, we know that $\nabla \boldsymbol{u}_\alpha$ is a bounded sequence in $\boldsymbol{L}^p(\Omega)$ for all $p \ge 2$. Therefore, the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_\infty$ in $\boldsymbol{W}^{1,q}(\Omega)$ for any $q < p$ follows, cf. [3, Lemma 1.2.3, Chapter 1]. This completes the proof. □

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