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Differentiability of solutions for the non-degenerate p-Laplacian in the Heisenberg group

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Abstract

We propose a direct method to control the first-order fractional difference quotients of solutions to quasilinear subelliptic equations in the Heisenberg group. In this way we implement iteration methods on fractional difference quotients to obtain weak differentiability in the T-direction and then second-order weak differentiability in the horizontal directions.

MSC: Primary 35H20; 35B65; Secondary 35J60; 35J70

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1. Introduction

This paper contains second-order horizontal differentiability results for weak solutions of the non-degenerate p-Laplacian equation

$$-\sum_{i=1}^{2n} X_i((\Lambda + |Xu|^2)^{\frac{p-2}{2}} X_i u) = 0, \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open subset of the Heisenberg group $\mathbb{H}^n$, $Xu = (X_1 u, \ldots, X_{2n} u)$ denotes the horizontal gradient of u, $p > 1$ and $\Lambda > 0$. Our results are an intermediate step to

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show second-order differentiability of solutions to the p-Laplacian equation

$$-\sum_{i=1}^{2n} X_i(|Xu|^{p-2}X_iu) = 0, \quad \text{in } \Omega. \tag{1.2}$$

Toward this goal we will use a method based on difference quotients, considering as test functions in the weak form of Eq. (1.1)

$$\sum_{i=1}^{2n} \int_\Omega (\Lambda + |Xu(x)|^2)^{\frac{p-2}{2}} X_iu(x)X_i\varphi(x)\,dx = 0, \quad \text{for all } \varphi \in HW_0^{1,p}(\Omega) \tag{1.3}$$

fractional difference quotients of the weak solution multiplied by a corresponding cut-off function. In the Euclidean setting, this method was widely used and gave complete answers for regularity problems in the nonlinear setting. For the Euclidean counterpart of Eq. (1.1)

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left((\Lambda + |\nabla u|^2)^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i}\right) = 0, \quad \text{in } \Omega \subset \mathbb{R}^n$$

weak solutions have C^∞ interior regularity (see [9–11,13,15,24] and the references therein).

We should point that the main difference between the Euclidean and the subelliptic cases is that any time we use the difference quotients in the horizontal directions (in the Euclidean case any direction can be considered horizontal) we get extra terms involving difference quotients in the non-horizontal directions, which cannot be absorbed or controlled by using the assumptions on the weak solutions.

In the Heisenberg group there are no complete answers yet. In the case $p = 2$ the left-hand side of the Eq. (1.1) is the real part of the Kohn Laplacian and the C^∞ regularity follows from Hörmander's celebrated theorem [12]. The fractional difference quotients were present already in the paper of Hörmander [12] and used together with the tools useful for linear equations like pseudo-differential operators and Fourier transform. For results connected to this case we also mention the papers by Xu and Zuily [27,28] and by Cutri and Garroni [6].

For the nonlinear $p \neq 2$ case we quote the papers of Capogna [3,4], Capogna and Garofalo [5] and Marchi [17–19]. In the papers [3–5] the a priori assumption on the boundedness of the horizontal gradient allows the use of some aspects of linear theory like L^2 spaces or fractional derivatives defined via Fourier transform to gain control on difference quotients and to prove interior C^∞ regularity for the weak solutions of (1.1) and (1.2). Due to the noncommutativity of the horizontal vector fields in the Heisenberg group, the first thing to be proved is the differentiability in the non-horizontal direction T. Under the boundedness condition of the horizontal gradient it is possible to prove for any $p \geqslant 2$ not just that $Tu \in L^2_{\text{loc}}(\Omega)$ but

$Tu \in HW^{1,2}_{\text{loc}}(\Omega)$. This opens the way to the proof of $u \in HW^{2,2}_{\text{loc}}(\Omega)$ and then differentiating Eq. (1.1) or (1.2) we can prove C^∞-regularity.

The general case is more difficult. In the Euclidean case for the degenerate equation corresponding to $\Lambda = 0$ we have $C^{1,\alpha}$ regularity for $1 < p < \infty$, but second order differentiability is valid just for a bounded interval around 2. For example, the $W^{2,2}$ regularity has been proven for the interval $1 < p < 3 + \frac{2}{n-2}$. As references we quote [1,8,9,13,15,16,21,23,24]. In the Heisenberg group $C^{0,\alpha}$ regularity is valid for $1 < p < \infty$ and was proved by several people during the 90's [2,14,27]. Marchi [17–19] proved that $Tu \in L^p_{\text{loc}}(\Omega)$ for $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$ and that $X^2 u \in L^2_{\text{loc}}(\Omega)$ for $2 \leqslant p < 1 + \sqrt{5}$. She used the fractional difference quotients to show that a weak solution is in some truncated versions of fractional Besov and Bessel-potential spaces. Marchi used the embedding among these spaces (see [20,22,25,26]) to obtain more information on the differentiability of weak solutions.

It is clear that the way we manage the fractional difference quotients constitutes a key point in the further development of this theory. In this paper we propose a direct method to bound the first-order difference quotients. Using the semi-group properties hidden in the second-order difference quotients we will able to control the first-order fractional difference quotients and hence to get a complete nonlinear treatment of the regularity problems. Our main contributions are Theorem 1.1 and the implementation of several iteration schemes on fractional difference quotients. The point here is that using an appropriate test function, and exploiting the geometry of vector fields in the Heisenberg group described by the Baker–Campbell–Hausdorff formula, we get information on the second order difference quotients. Using Theorem 1.1 we transfer these information to the first order difference quotients and do our iterations. In this way first we will extend Marchi's results by proving that $Tu \in L^p_{\text{loc}}(\Omega)$ for $1 < p < 4$. Our method can be used also to give a new proof of $Tu \in HW^{1,2}_{\text{loc}}(\Omega)$ for $1 < p < \infty$ under the boundedness assumption of the papers [3–5].

The following step is to prove second-order differentiability in the horizontal directions. By applying Theorem 1.1 we do modified, detail oriented and at the same time relatively simple versions of Marchi's proofs, that are independent of the embedding properties of Besov and Bessel-potential spaces.

We remark that our $HW^{2,2}$ estimates for $2 \leqslant p < 4$ and the $HW^{2,p}$ estimates for $\frac{\sqrt{17}-1}{2} \leqslant p \leqslant 2$ are essential in [7] to be able to differentiate Eq. (1.1) and use the Cordes conditions in order prove uniform $HW^{2,2}$ bounds, which leads to interior $HW^{2,2}$-regularity of p-harmonic functions in an interval that contains $p = 2$ and depends on n (see [7]).

Here is the plan of the paper. In Section 2 we introduce the first- and second-order difference quotients and prove the next result about their connections.

Theorem 1.1. *Let $u \in L^p(\mathbb{H}^n)$, $0 < \alpha$, $0 < \sigma$, $0 \leqslant M < \infty$ and Z a left-invariant vector field. Suppose that*

$$\sup_{0 < |s| \leqslant \sigma} \frac{\| \triangle^2_{Z,s} u \|_{L^p}}{|s|^\alpha} \leqslant M. \tag{1.4}$$

Define β as follows:

- $\beta = \alpha$ *if* $0 < \alpha < 1$,
- β *any number in* $(0, 1)$ *if* $\alpha = 1$, *and*
- $\beta = 1$ *if* $\alpha > 1$.

Then there exists $c > 0$ *independent of* u *and a possibly different* σ *from that one in* (1.4) *such that*

$$\sup_{0 < |s| \leqslant \sigma} \frac{|| \triangle_{Z,s} u||_{L^p}}{|s|^\beta} \leqslant c(||u||_{L^p} + M). \tag{1.5}$$

In Sections 3 and 4 we develop iteration schemes on fractional difference quotients to prove the following theorem on the differentiability in the T-direction.

Remark 1.1. In the next theorems, the constant $c > 0$ depends on n and p. The constant $\delta > 0$ is the Hölder exponent of the weak solution as found in [2,14,27] and depends on x_0 and the L^p norm of u in a ball around x_0. Both constants are independent of Λ when we consider $\Lambda \to 0$ and the weak solution of (1.1) converging locally in the $HW^{1,p}$ norm to a weak solution of (1.2).

Theorem 1.2. *Let* $1 < p < 4$ *and* $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1.1). *Consider* $x_0 \in \Omega$ *and* $r > 0$ *such that* $B(x_0, 3r) \subset \Omega$. *Then there exist a number* $k \in \mathbb{N}$ *depending only on* p *and a constant* $c > 0$ *such that we have*

$$\int_{B(x_0, \frac{r}{2^{k+1}})} |Tu(x)|^p \, dx \leqslant c \int_{B(x_0, 2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx \tag{1.6}$$

and hence $Tu \in L^p_{loc}(\Omega)$.

In Section 5 we prove second-order differentiability in the case $2 \leqslant p < 4$.

Theorem 1.3. *Let* $2 \leqslant p < 4$ *and* $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1.1). *Consider* $x_0 \in \Omega$, $r > 0$ *such that* $B(x_0, 3r) \in \Omega$. *Then there exist a number* $k \in \mathbb{N}$ *depending only on* p *and a constant* $c > 0$ *such that we have*

$$\begin{aligned} &\int_{B(x_0, \frac{r}{2^{k+2}})} (\Lambda + |Xu(x)|^2)^{\frac{p-2}{2}} |X^2 u(x)|^2 \, dx \\ &\quad \leqslant c \int_{B(x_0, 2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p \, dx, \end{aligned} \tag{1.7}$$

and hence $u \in HW^{2,2}_{\mathrm{loc}}(\Omega)$.

To prove the second-order differentiability for $p < 2$ we need first to show that $Tu \in L^2_{\mathrm{loc}}(\Omega)$.

Theorem 1.4. *Let* $\frac{\sqrt{17}-1}{2} \leqslant p \leqslant 2$ *and* $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1.1). *Consider* $x_0 \in \Omega$, $r > 0$ *such that* $B(x_0, 3r) \in \Omega$. *Then there exist* $k \in \mathbb{N}$ *depending only on* p *and constants* $c > 0$ *and* $\delta > 0$ *such that we have*

$$\begin{aligned}
&\int_{B(x_0, \frac{r}{2^{k+2}})} |Tu(x)|^2 \, dx \\
&\quad \leqslant c \Bigg(||u||^{2-p}_{C^\delta(B(x_0, \frac{r}{2^{k+1}}))} \int_{B(x_0, 2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx \\
&\qquad + \ ||u||^2_{L^2(B(x_0, \frac{r}{2^{k+1}}))} \Bigg). \qquad (1.8)
\end{aligned}$$

For the interval given by Theorem 1.4 we can prove the following theorem.

Theorem 1.5. *Let* $\frac{\sqrt{17}-1}{2} \leqslant p \leqslant 2$ *and* $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1.1). *Consider* $x_0 \in \Omega$, $r > 0$ *such that* $B(x_0, 3r) \in \Omega$. *Then there exist* $k \in \mathbb{N}$ *depending only on* p *and constants* $c > 0$ *and* $\delta > 0$ *such that we have*

$$\begin{aligned}
&\int_{B(x_0, \frac{r}{2^{k+3}})} |X^2 u(x)|^p \, dx \\
&\quad \leqslant c \Bigg(\Lambda^{\frac{p-2}{2}} ||u||^{2-p}_{C^\delta(B(x_0, \frac{r}{2^{k+1}}))} \int_{B(x_0, 2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx \\
&\qquad + \Lambda^{\frac{p-2}{2}} ||u||^2_{L^2(B(x_0, \frac{r}{2^{k+1}}))} + \int_{B(x_0, 2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx \Bigg) \qquad (1.9)
\end{aligned}$$

and hence $u \in HW^{2,p}_{\mathrm{loc}}(\Omega)$.

2. Fractional difference quotients in the Heisenberg group

Let us consider the Heisenberg group $\mathbb{H}^n$ as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group multiplication

$$\begin{aligned}
&(x_1, \ldots, x_{2n}, t) \cdot (y_1, \ldots, y_{2n}, u) \\
&\quad = \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, t + u - \frac{1}{2} \sum_{i=1}^{n} (x_{n+i} y_i - x_i y_{n+i}) \right).
\end{aligned}$$

Throughout this paper $x \cdot y$ will denote the group multiplication in $\mathbb{H}^n$.

The left invariant vector fields corresponding to the canonical basis of the Lie algebra are

$$X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t},$$
$$X_{n+i} = \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial t}$$

and they are called horizontal vector fields. Denote by

$$T = \frac{\partial}{\partial t}$$

and observe that $[X_i, X_{n+i}] = T$, otherwise $[X_i, X_j] = 0$.

Let Ω be a domain in $\mathbb{H}^n$ and let $p > 1$. Recall that the Haar measure in $\mathbb{H}^n$ is the Lebesque measure of $\mathbb{R}^{2n+1}$, therefore the space $L^p(\Omega)$ is defined in the usual way. Consider the following Sobolev space with respect to the horizontal vector fields X_i

$$HW^{1,p}(\Omega) = \{u \in L^p(\Omega):\ X_i u \in L^p(\Omega),\ \text{for all } i \in \{1, \dots, 2n\}\}.$$

$HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$||u||_{HW^{1,p}} = ||u||_{L^p} + \sum_{i=1}^{2n} ||X_i u||_{L^p}.$$

We denote by $HW_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$.

If Z is a left invariant vector field then for some

$$z = (z_H, z_T) = (z_1, \dots, z_{2n}, z_T)$$

we can write

$$Z = \sum_{i=1}^{2n} z_i X_i + z_T T.$$

The exponential mapping in canonical coordinates is defined by

$$e^Z = z.$$

Recall that in the Heisenberg group the Baker–Campbell–Hausdorff formula for two left invariant vector fields Z and V is

$$e^Z e^V = e^{Z+V+\frac{1}{2}[Z,V]}.$$

Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. For $x \in \Omega$, a left invariant vector field Z, $s \in \mathbb{R}$ sufficiently small, $0 < \alpha, \theta \leqslant 1$, and $u : \Omega \to \mathbb{R}$ let us define:

$$\triangle_{Z,s} u(x) = u(x \cdot e^{sZ}) - u(x),$$

$$\triangle^2_{Z,s} u(x) = u(x \cdot e^{sZ}) + u(x \cdot e^{-sZ}) - 2u(x),$$

$$D_{Z,s,\theta} u(x) = \frac{u(x \cdot e^{sZ}) - u(x)}{|s|^\theta},$$

$$D_{Z,-s,\theta} u(x) = \frac{u(x \cdot e^{-sZ}) - u(x)}{-|s|^\theta}.$$

Then

$$D_{Z,-s,\alpha} D_{Z,s,\theta} u(x) = D_{Z,s,\theta} D_{Z,-s,\alpha} u(x)$$
$$= \frac{u(x \cdot e^{sZ}) + u(x \cdot e^{-sZ}) - 2u(x)}{|s|^{\alpha+\theta}} = \frac{\triangle^2_{Z,s} u(x)}{|s|^{\alpha+\theta}}.$$

We will use the following result [3,12]:

Proposition 2.1. *Let $\Omega \subset \mathbb{H}^n$ be an open set, K a compact set included in Ω, Z a left invariant vector field and $u \in L^p_{\mathrm{loc}}(\Omega)$. If there exist σ and C two positive constants such that*

$$\sup_{0<|s|<\sigma} \int_K |D_{Z,s,1} u(x)|^p \, dx \leqslant C^p$$

then $Zu \in L^p(K)$ and $||Zu||_{L^p(K)} \leqslant C$.

Conversely, if $Zu \in L^p(K)$ then for some $\sigma > 0$

$$\sup_{0<|s|<\sigma} \int_K |D_{Z,s,1} u(x)|^p \, dx \leqslant (2||Zu||_{L^p(K)})^p.$$

The following result is a direct consequence of the Baker–Campbell–Hausdorff formula (see [3,12]).

Proposition 2.2. *Let $\Omega \in \mathbb{H}^n$ be an open set, $1 \leqslant p < \infty$, $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$, $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 3r) \subset \Omega$. Then there exists a positive constant c independent of u such that*

$$\int_{B(x_0,r)} |D_{T,s,\frac{1}{2}} u(x)|^p \, dx \leqslant c \int_{B(x_0,2r)} (|u|^p + |Xu|^p) \, dx. \tag{2.1}$$

Remark 2.1. Let us observe that if g is a cut-off function between $B(x_0, r)$ and $B(x_0, 2r)$ then

$$\int_{B(x_0,r)} |D_{T,s,\frac{1}{2}} u(x)|^p \, dx \leqslant \int_{B(x_0,2r)} |D_{T,s,\frac{1}{2}} (g^2 u)(x)|^p \, dx \leqslant c \int_{B(x_0,2r)} (|u|^p + |Xu|^p) \, dx. \tag{2.2}$$

We will prove now Theorem 1.1 which constitutes our main result on fractional difference quotients. The proof is based on a classical argument of Zygmund [29, Theorem 3.4]. Let us observe that a similar proof can be carried out in a nilpotent stratified Lie group.

Proof of Theorem 1.1. Using $u \in L^p(\mathbb{H}^n)$ we have that $\triangle_{Z,s} u \in L^p(\mathbb{H}^n)$ and $||\triangle_{Z,s} u||_{L^p} \leqslant 2||u||_{L^p}$ for all $0 < |s| \leqslant \sigma$. Let us denote $g(s)(x) = u(x \cdot e^{sZ}) - u(x)$. Condition (1.4) implies that

$$||u(x \cdot e^{sZ}) + u(x \cdot e^{-sZ}) - 2u(x)||_{L^p} \leqslant M \, |s|^\alpha.$$

Without loss of generality we can work just with $s > 0$. Replacing s by $\frac{s}{2}$ and then changing the variables $x \to x \cdot e^{\frac{s}{2}Z}$ in the integral gives

$$||u(x \cdot e^{sZ}) + u(x) - 2u(x \cdot e^{\frac{s}{2}Z})||_{L^p} \leqslant \frac{M}{2^\alpha} s^\alpha.$$

Denoting $M' = \frac{M}{2^\alpha}$ we get

$$\left|\left| g(s) - 2g\left(\frac{s}{2}\right) \right|\right|_{L^p} \leqslant M' s^\alpha. \tag{2.3}$$

Replacing s by $\frac{s}{2}$ in formula (2.3) we get

$$\left|\left| g\left(\frac{s}{2}\right) - 2g\left(\frac{s}{2^2}\right) \right|\right|_{L^p} \leqslant M' \frac{s^\alpha}{2^\alpha}$$

and hence

$$\left|\left| 2g\left(\frac{s}{2}\right) - 2^2 g\left(\frac{s}{2^2}\right) \right|\right|_{L^p} \leqslant M' s^\alpha 2^{1-\alpha}. \tag{2.4}$$

Repeating this procedure we obtain

$$\left|\left|2^{n-1}g\left(\frac{s}{2^{n-1}}\right) - 2^n g\left(\frac{s}{2^n}\right)\right|\right|_{L^p} \leqslant M' s^\alpha 2^{(1-\alpha)(n-1)}. \tag{2.5}$$

Adding the above inequalities we get

$$\left|\left|g(s) - 2^n g\left(\frac{s}{2^n}\right)\right|\right|_{L^p} \leqslant M' s^\alpha \sum_{k=0}^{n-1} 2^{(1-\alpha)k}. \tag{2.6}$$

If $0<\alpha<1$ then

$$\left|\left|g(s) - 2^n g\left(\frac{s}{2^n}\right)\right|\right|_{L^p} \leqslant M' s^\alpha \frac{2^{(1-\alpha)n} - 1}{2^{1-\alpha} - 1} \leqslant M' s^\alpha \frac{2^{(1-\alpha)n}}{2^{1-\alpha} - 1}$$

and hence

$$\left|\left|g\left(\frac{s}{2^n}\right)\right|\right| \leqslant \frac{1}{2^n} 2||u||_{L^p} + cM' s^\alpha 2^{-\alpha n}.$$

Consider now $0<a<\frac{\sigma}{2}$ fixed. For all $h>0$ sufficiently small there exist $n \in \mathbb{N}$ and $s \in [\frac{a}{2}, a]$ such that $h = \frac{s}{2^n}$. Then

$$||g(h)||_{L^p} \leqslant \frac{4h}{a} ||u||_{L^p} + cM' h^\alpha.$$

Dividing this last inequality by h^α we get (1.5).

If $\alpha = 1$, then inequality (2.6) implies that

$$\left|\left|g(s) - 2^n g\left(\frac{s}{2^n}\right)\right|\right|_{L^p} \leqslant M' sn. \tag{2.7}$$

Consider now $h = \frac{s}{2^n}$ in a similar way as for the previous case and observe that $n = O(\log h)$ to get

$$||g(h)||_{L^p} \leqslant \frac{4}{a} h ||u||_{L^p} + hO(\log h) \tag{2.8}$$

and hence we can use any $\beta<1$ to get (1.5).

If $\alpha>1$ then inequality (2.6) implies that

$$\left|\left|g(s) - 2^n g\left(\frac{s}{2^n}\right)\right|\right|_{L^p} \leqslant M' s^\alpha \frac{1}{1 - 2^{(1-\alpha)}}. \tag{2.9}$$

Therefore, we have

$$\left|\left|g\left(\frac{s}{2^n}\right)\right|\right| \leqslant \frac{1}{2^n} 2||u||_{L^p} + \frac{1}{2^n} M' s^\alpha \frac{1}{1-2^{(1-\alpha)}}$$

and hence for $h = \frac{s}{2^n}$ and $s \in [\frac{a}{2}, a]$ we obtain

$$||g(h)||_{L^p} \leqslant \frac{4h}{a} ||u||_{L^p} + \frac{2h}{a} M' \frac{1}{1-2^{(1-\alpha)}} a^\alpha. \tag{2.10}$$

Now we can use $\beta = 1$ to get (1.5). $\square$

Remark 2.2. Proposition 2.1 together with Theorem 1.1 implies that if u has compact support K and (1.4) is satisfied with $\alpha > 1$, then $Zu \in L^p(K)$.

3. Iterations in the T-direction, for $p \geqslant 2$

Let us rewrite Eq. (1.1) in the following way:

$$\sum_{i=1}^{2n} X_i(a_i(Xu)) = 0, \quad \text{in } \Omega \tag{3.1}$$

where

$$a_i(\xi) = (\Lambda + |\xi|^2)^{\frac{p-2}{2}} \xi_i, \quad \text{for all } \xi \in \mathbb{R}^{2n}.$$

We will use the following properties of the functions a_i:

(i) There exists a constant $c > 0$ such that

$$c(\Lambda + |\xi|^2)^{\frac{p-2}{2}} |q|^2 \leqslant \sum_{i,j=1}^{2n} \frac{\partial a_i(\xi)}{\partial \xi_j} q_i q_j, \quad \text{for all } \xi, q \in \mathbb{R}^{2n}. \tag{3.2}$$

(ii) there exists a constant $c > 0$ such that

$$\left|\frac{\partial a_i(\xi)}{\partial \xi_j}\right| \leqslant c(\Lambda + |\xi|^2)^{\frac{p-2}{2}}, \quad \text{for all } \xi \in \mathbb{R}^{2n}. \tag{3.3}$$

We prove a general lemma, that constitutes the key step in our iteration. In an informal way, we can say that if u has locally $\frac{1}{2} + \alpha$ derivatives in the L^p-sense in the T direction, then it also has $\frac{1}{2} + \frac{1}{p} + \frac{2}{p}\alpha$ derivatives in the L^p-sense in the same direction.

Lemma 3.1. *Let* $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1.1), $x_0 \in \Omega$, $r > 0$ *such that* $B(x_0, 3r) \in \Omega$. *Let us suppose that there exists a constant* $c > 0$, $\sigma > 0$ *and* $\alpha \in [0, \frac{1}{2})$ *such that*

$$\sup_{0 \neq |s| \leqslant \sigma} \int_{B(x_0,r)} |D_{T,s,\frac{1}{2}+\alpha} u(x)|^p \, dx \leqslant c \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx. \tag{3.4}$$

If we have

$$\frac{1 + 2\alpha}{p} < \frac{1}{2}$$

then for possibly different $c > 0$, $\sigma > 0$ *holds*

$$\sup_{0 \neq |s| \leqslant \sigma} \int_{B(x_0,\frac{r}{2})} |D_{T,s,\frac{1}{2}+\frac{1}{p}+\frac{2}{p}\alpha} u(x)|^p \, dx \leqslant c \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx. \tag{3.5}$$

In the case

$$\frac{1 + 2\alpha}{p} \geqslant \frac{1}{2}$$

we have that

$$\int_{B(x_0,\frac{r}{4})} |Tu(x)|^p \, dx \leqslant c \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx. \tag{3.6}$$

Proof. Let us consider

$$\gamma = \frac{1}{2} + \alpha,$$

and let g be a cut-off function between $B(x_0, \frac{r}{2})$ and $B(x_0, r)$. We use now the test function

$$\varphi = D_{T,-s,\gamma}(g^2 D_{T,s,\gamma} u) \tag{3.7}$$

to get

$$\sum_{i=1}^{2n} \int_{\Omega} a_i(Xu(x)) X_i(D_{T,-s,\gamma}(g^2 D_{T,s,\gamma} u(x))) \, dx = 0 \tag{3.8}$$

and from here, by the fact that X_i commutes with $D_{T,s,\gamma}$ and $D_{T,-s,\gamma}$, we obtain

$$\sum_{i=1}^{2n} \int_\Omega D_{T,s,\gamma}\, a_i(Xu(x))g^2(x)D_{T,s,\gamma}(X_iu(x))\,dx \\ + \sum_{i=1}^{2n} \int_\Omega D_{T,s,\gamma}\, a_i(Xu(x))D_{T,s,\gamma}\, u(x)2g(x)X_ig(x)\,dx = 0. \tag{3.9}$$

We can use now the properties of the functions a_i and [11, Lemma 8.3] to get

$$\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}Xu(x)|^2\,dx \\ \leqslant c\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}Xu(x)| \\ \times |D_{T,s,\gamma}u(x)||g(x)||Xg(x)|\,dx.$$

Using the fact that $p \geqslant 2$ we get

$$\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}Xu(x)|^2\,dx \\ \leqslant c\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}\, u(x)|^2|Xg(x)|^2\,dx. \tag{3.10}$$

Denoting by RHS the right-hand side of (3.10) we have that

$$RHS \leqslant c\int_{B(x_0,r)} ((\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}} + |D_{T,s,\gamma}\, u(x)|^p)\,dx.$$

Using (3.4) we get that

$$RHS \leqslant c\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx$$

and therefore

$$\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}Xu(x)|^2\,dx \\ \leqslant c\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx. \tag{3.11}$$

Using

$$|D_{T,s,\gamma}Xu|^p = |D_{T,s,\gamma}Xu|^{p-2} \cdot |D_{T,s,\gamma}Xu|^2$$

and the inequality

$$|s^\gamma D_{T,s,\gamma}Xu(x)| \leqslant \sqrt{2}(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sT})|)^{\frac{1}{2}},$$

formula (3.11) gives

$$\int_{B(x_0,r)} g^2(x)s^{(p-2)\gamma}|D_{T,s,\gamma}Xu(x)|^p\,dx \leqslant c\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx.$$

Since

$$D_{T,s,\gamma}X(g^2u)(x) = D_{T,s,\gamma}X(g^2)(x)u(x \cdot e^{sT}) + X(g^2)(x)D_{T,s,\gamma}u(x) + D_{T,s,\gamma}g^2(x)Xu(x \cdot e^{sT}) + g^2(x)D_{T,s,\gamma}Xu(x)$$

it follows that

$$\int_{B(x_0,r)} |D_{T,s,\frac{2\gamma}{p}}X(g^2u)(x)|^p\,dx \leqslant c\int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p)\,dx. \tag{3.12}$$

Let us denote the right-hand side of (3.12) by M^p. Using Proposition 2.2 we get

$$\int_{B(x_0,r)} |D_{T,-s,\frac{1}{2}}D_{T,s,\frac{2\gamma}{p}}(g^2u)(x)|^p\,dx \leqslant M^p. \tag{3.13}$$

Therefore, for all s sufficiently small we have

$$\frac{||\triangle^2_{T,s}(g^2u)||_{L^p(\mathbb{H}^n)}}{s^{\frac{1}{2}+\frac{1+2\alpha}{p}}} \leqslant M,$$

so there exists $\sigma > 0$ such that

$$\sup_{0<|s|\leqslant\sigma} \frac{||\triangle^2_{T,s}(g^2u)||_{L^p(\mathbb{H}^n)}}{s^{\frac{1}{2}+\frac{1+2\alpha}{p}}} \leqslant M. \tag{3.14}$$

If it happens that

$$\frac{1+2\alpha}{p}<\frac{1}{2}$$

then by Theorem 1.1 we get (3.5).

If we have

$$\frac{1+2\alpha}{p}>\frac{1}{2}$$

then, by Remark 2.2 we have $Tu \in L^p_{\mathrm{loc}}(\Omega)$ and estimate (3.6) is valid.

In the remaining case

$$\frac{1+2\alpha}{p}=\frac{1}{2}$$

and then using that $\alpha \in [0, \frac{1}{2})$ we get

$$0 \leqslant \frac{p-2}{4} < \frac{1}{2}$$

which gives $2 \leqslant p < 4$. Theorem 1.1 implies that we can use α' arbitrarily close to $\frac{1}{2}$, in particular $\alpha' > \frac{p-2}{4}$, and the following form of (3.4)

$$\begin{aligned}
&\sup_{0 \neq |s| \leqslant \sigma} \int_{B(x_0,\frac{r}{2})} |D_{T,s,\frac{1}{2}+\alpha'} u|^p \, dx \\
&\quad \leqslant c \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx.
\end{aligned}$$

Using a cut-off function g between $B(x_0, \frac{r}{4})$ and $B(x_0, \frac{r}{2})$ we get back (3.14) with

$$\frac{1+2\alpha'}{p}>\frac{1}{2}$$

and then use the previous case. $\square$

Remark 3.1. The proof shows that in the case

$$\frac{1+2\alpha}{p}>\frac{1}{2}$$

we can have a larger radius on the left hand side of (3.6), namely

$$\int_{B(x_0,\frac{r}{2})} |Tu(x)|^p \, dx \leqslant c \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx. \tag{3.15}$$

Proof of Theorem 1.2 for $2 \leqslant p < 4$. Proposition 2.2 implies that we can start with $\alpha_0 = 0$ in the assumption (3.4) to get $\alpha_1 = \frac{1}{p}$ in (3.5). Now we can use α_1 in (3.4) to get

$$\alpha_2 = \frac{1}{p} + \frac{2}{p}\alpha_1$$

such that estimate (3.5) is true. In general, if we already found $\alpha_1, \dots, \alpha_k$, then we get

$$\alpha_{k+1} = \frac{1}{p} + \frac{2}{p}\alpha_k = \frac{1}{p} + \cdots + \frac{2^{k-2}}{p^{k-1}} + \frac{2^{k-1}}{p^{k-1}}\alpha_1 = \frac{1}{p}\sum_{i=0}^{k-1}\left(\frac{2}{p}\right)^i = \frac{1}{p}\frac{1-\left(\frac{2}{p}\right)^k}{1-\frac{2}{p}}.$$

Therefore, for a given $p > 2$ the upper bound for α_k is given by

$$\frac{1}{p-2}.$$

Hence, for $p \in [2, 4)$, after a number sufficiently large of k iterations, we get that $\alpha_k > \frac{1}{2}$ and this means that $Tu \in L^p_{\mathrm{loc}}(\Omega)$. $\square$

Remark 3.2. If we ask for $\alpha_2 > \frac{1}{2}$ then we get the inequality

$$p^2 - 2p - 4 < 0$$

that leads to Marchi's result $p \in [2, 1 + \sqrt{5})$.

In the case $p \geqslant 4$ our iterations give the following result.

Proposition 3.1. *For $p \geqslant 4$ and weak solutions u of* (1.1) *we have*

$$\begin{aligned}\sup_{0 \neq |s| \leqslant \sigma} \int_{B(x_0, \frac{r}{2^k})} |D_{T,s,\frac{1}{2}+\alpha'} u(x)|^p \, dx \\ \leqslant c \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p)\, dx. \end{aligned} \tag{3.16}$$

for α' less then, but arbitrarily close to $\frac{1}{p-2}$, and a corresponding number k of iterations.

4. Iterations in the T-direction for $1 < p < 2$

Proof of Theorem 1.2 for $1 < p < 2$. Let g be a cut-off function between $B(x_0, \frac{r}{2})$ and $B(x_0, r)$. We can follow then the proof of Lemma 3.1 for $\alpha = 0$ and $\gamma = \frac{1}{2}$

until we get

$$\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}Xu(x)|^2\,dx$$
$$\leqslant c\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |D_{T,s,\gamma}Xu(x)|$$
$$\times |D_{T,s,\gamma}u(x)||g(x)||Xg(x)|\,dx. \tag{4.1}$$

Let us denote by RHS the right-hand side of (4.1). We will keep using γ instead of $\frac{1}{2}$ to get a general iteration formula. Then

$$\begin{aligned}
RHS &\leqslant \frac{c}{s^\gamma}\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} \\
&\quad \times |Xu(x\cdot e^{sT}) - Xu(x)||D_{T,s,\gamma}u(x)|\,dx \\
&\leqslant \frac{c}{s^\gamma}\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} \\
&\quad \times (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{1}{2}}|D_{T,s,\gamma}u(x)|\,dx \\
&\leqslant \frac{c}{s^\gamma}\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-1}{2}} |D_{T,s,\gamma}u(x)|\,dx \\
&\leqslant \frac{c}{s^\gamma}\left(\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}}\,dx\right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_{B(x_0,r)} |D_{T,s,\gamma}u(x)|^p\,dx\right)^{\frac{1}{p}} \\
&\leqslant \frac{c}{s^\gamma}\left(\int_{B(x_0,r)} (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}}\,dx\right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_{B(x_0,2r)} (|u(x)|^p + |Xu(x)|^p)\,dx\right)^{\frac{1}{p}} \\
&\leqslant \frac{c}{s^\gamma}\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx.
\end{aligned}$$

Therefore,

$$\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|^2\,dx$$
$$\leqslant cs^\gamma\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx. \tag{4.2}$$

We need the following inequalities used initially in the Euclidean case (see [15]).

$$\begin{aligned}
&(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}} \\
&\quad \leqslant (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}-1}(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2) \\
&\quad \leqslant 3(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}-1} \\
&\qquad \times (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT}) - Xu(x)|^2) \\
&\quad \leqslant 3(\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + 3(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}-1} \\
&\qquad \times |Xu(x\cdot e^{sT}) - Xu(x)|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}}\,dx \\
&\quad \leqslant 3\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2)^{\frac{p}{2}}\,dx \\
&\qquad + c\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx \\
&\quad \leqslant c\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx.
\end{aligned}$$

Also, by Hölder's inequality we get

$$\begin{aligned}
&\int_{B(x_0,r)} g^2(x)|Xu(x\cdot e^{sT}) - Xu(x)|^p\,dx \\
&\quad = \int_{B(x_0,r)} \Big(g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}-1}|Xu(x\cdot e^{sT}) - Xu(x)|^2\Big)^{\frac{p}{2}} \\
&\qquad \times (g^{\frac{4}{p}}(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2))^{(1-\frac{p}{2})\frac{p}{2}}\,dx \\
&\quad \leqslant \left(\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p}{2}-1}|Xu(x\cdot e^{sT}) - Xu(x)|^2\,dx\right)^{\frac{p}{2}} \\
&\qquad \times \left(\int_{B(x_0,r)} (g^{\frac{4}{p}}(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2))^{\frac{p}{2}}\,dx\right)^{1-\frac{p}{2}} \\
&\quad \leqslant \left(cs^{\gamma}\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\,dx\right)^{\frac{p}{2}}
\end{aligned}$$

$$\times \left(\int_{B(x_0,r)} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sT})|^2)^{\frac{p}{2}}\, dx \right)^{1-\frac{p}{2}}$$

$$\leqslant cs^{p\frac{\gamma}{2}} \left(\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\, dx \right)^{\frac{p}{2}}$$

$$\times \left(\int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\, dx \right)^{1-\frac{p}{2}}$$

$$\leqslant cs^{p\frac{\gamma}{2}} \int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\, dx.$$

Therefore,

$$\int_{B(x_0,r)} g^2(x) |D_{T,s,\frac{\gamma}{2}} Xu(x)|^p\, dx \leqslant c \int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\, dx.$$

In the same way as we have obtained inequality (3.12), we get

$$\int_{B(x_0,r)} |D_{T,s,\frac{\gamma}{2}} X(g^2u)(x)|^p\, dx \leqslant c \int_{B(x_0,2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p\, dx. \tag{4.3}$$

Let us denote the right-hand side of (4.3) by M^p. Proposition 2.2 implies that

$$\int_{B(x_0,r)} |D_{T,-s,\frac{1}{2}} D_{T,s,\frac{\gamma}{2}} (g^2u)(x)|^p\, dx \leqslant M^p \tag{4.4}$$

and this means for a sufficiently small σ

$$\sup_{0<|s|\leqslant\sigma} \frac{||\triangle^2_{T,s}(g^2u)||_{L^p(\mathbb{H}^n)}}{|s|^{\frac{1}{2}+\frac{\gamma}{2}}} \leqslant M \tag{4.5}$$

We started with $\gamma = \frac{1}{2}$ and therefore in (4.3) we have a fractional exponent of $\frac{1}{4}$, while in (4.5) we have a power of $\frac{3}{4}$ for s. Using Theorem 1.1 and cut-off functions between $B(x_0, \frac{r}{2^k})$ and $B(x_0, \frac{r}{2^{k-1}})$ we can do iterations to obtain after k steps that

$$\int_{B(x_0,\frac{r}{2^{k-1}})} |D_{T,s,\frac{2^k-1}{2^{k+1}}} X(g^2u)(x)|^p\, dx \leqslant M^p \tag{4.6}$$

and

$$\sup_{0<|s|\leqslant\sigma} \frac{||\triangle_{T,s}(g^2u)||_{L^p(\mathbb{H}^n)}}{|s|^{\frac{2^{k+1}-1}{2^{k+1}}}} \leqslant M. \tag{4.7}$$

Let us consider now $k \in \mathbb{N}$ such that

$$\frac{1}{2^k - 1} < p - 1.$$

Then for

$$a = \frac{2^k - 1}{2^{k+1}} \quad \text{and} \quad b = \frac{2^{k+1} - 1}{2^{k+1}}$$

we have

$$a(p-1) + b > 1.$$

Let us consider

$$\gamma = \frac{a(p-1) + b}{2} > \frac{1}{2}$$

and return to (4.1) with a cut-off function g between $B(x_0, \frac{r}{2^{k+1}})$ and $B(x_0, \frac{r}{2^k})$. Then

$$\begin{aligned} RHS &\leqslant c \int_{B(x_0, \frac{r}{2^k})} (\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x \cdot e^{sT}) - Xu(x)|^{2-p} \\ &\quad \times \frac{|Xu(x \cdot e^{sT}) - Xu(x)|^{p-1}}{s^{a(p-1)}} \frac{|u(x \cdot e^{sT}) - u(x)|}{s^b} \, dx \\ &\leqslant \int_{B(x_0, \frac{r}{2^k})} \frac{|Xu(x \cdot e^{sT}) - Xu(x)|^{p-1}}{s^{a(p-1)}} |D_{T,s,b} u(x)| \, dx \\ &\leqslant \left(\int_{B(x_0, \frac{r}{2^k})} \frac{|Xu(x \cdot e^{sT}) - Xu(x)|^p}{s^{ap}} \, dx \right)^{\frac{p-1}{p}} \left(\int_{B(x_0, \frac{r}{2^k})} |D_{T,s,b} u(x)|^p \, dx \right)^{\frac{1}{p}} \\ &\leqslant c \int_{B(x_0, 2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p \, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{B(x_0, \frac{r}{2^k})} g^2(x) (\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x \cdot e^{sT}) - Xu(x)|^2 \, dx \\ &\quad \leqslant c s^{2\gamma} \int_{B(x_0, 2r)} (\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p \, dx. \end{aligned} \tag{4.8}$$

Doing a similar proof as we did starting from formula (4.2) we get that

$$\sup_{0<|s|\leqslant\sigma} \frac{||\triangle^2_{T,s}(g^2u)||_{L^p(\mathbb{H}^n)}}{|s|^{\frac{1}{2}+\gamma}} \leqslant M \tag{4.9}$$

Using the fact that $\frac{1}{2}+\gamma>1$, Theorem 1.1 implies now that $Tu\in L^p_{\mathrm{loc}}(\Omega)$. $\square$

5. Proof of Theorem 1.3

We present an extension of Marchi's proof [17] taking into consideration our iterations and the extended range $2\leqslant p<4$.

Let $i_0\in\{1,\dots,n\}$, $s>0$ and use the test function

$$\varphi = D_{X_{i_0},-s,1}D_{X_{i_0},s,1}(g^4u),$$

where g is a cut-off function between $B(x_0,\frac{r}{2^{k+2}})$ and that $B(x_0,\frac{r}{2^{k+1}})$.

For $i\neq i_0$ we have

$$X_i(D_{X_{i_0},-s,1}D_{X_{i_0},s,1}(g^4u)) = D_{X_{i_0},-s,1}D_{X_{i_0},s,1}(X_i(g^4u)),$$

while for $i=i_0+n$ we have

$$\begin{aligned} X_{i_0+n}(D_{X_{i_0},-s,1}D_{X_{i_0},s,1}(g^4u))(x) = D_{X_{i_0},-s,1}D_{X_{i_0},s,1}\bigg(& X_{i_0+n}(g^4u))(x) \\ & - \frac{1}{s}(T(g^4u)(x\cdot e^{sX_{i_0}}) - T(g^4u)(x\cdot e^{-sX_{i_0}})\bigg). \end{aligned} \tag{5.1}$$

To see that formula (5.1) is true it is enough to observe that

$$X_{i_0+n}((g^4u)(x\cdot e^{sX_{i_0}})) = X_{i_0+n}(g^4u)(x\cdot e^{sX_{i_0}}) - sT(g^4u)(x\cdot e^{sX_{i_0}})$$

and

$$X_{i_0+n}((g^4u)(x\cdot e^{-sX_{i_0}})) = X_{i_0+n}(g^4u)(x\cdot e^{-sX_{i_0}}) + sT(g^4u)(x\cdot e^{-sX_{i_0}}).$$

Using the test function φ in equation (1.3) we get

$$\begin{aligned} &\sum_{i=1}^{2n} \int_\Omega a_i(Xu(x))D_{X_{i_0},-s,1}D_{X_{i_0},s,1}X_i(g^4u)(x)\,dx \\ &\quad = \int_\Omega a_{i_0+n}(Xu(x))\frac{1}{s}(T(g^4u)(x\cdot e^{sX_{i_0}}) - T(g^4u)(x\cdot e^{-sX_{i_0}})) \end{aligned}$$

and hence

$$\sum_{i=1}^{2n}\int_\Omega D_{X_{i_0},s,1}a_i(Xu(x))D_{X_{i_0},s,1}X_i(g^4u)(x)\,dx$$
$$=-\int_\Omega a_{i_0+n}(Xu(x))(D_{X_{i_0},s,1}T(g^4u)(x)+D_{X_{i_0},-s,1}T(g^4u)(x)). \tag{5.2}$$

We use that

$$\begin{aligned}
D_{X_{i_0},s,1}X_i(g^4u)(x) &= D_{X_{i_0},s,1}(4g^3(x)X_ig(x)u(x)+g^4(x)X_iu(x))\\
&= 4D_{X_{i_0},s,1}g(x)g^2(x\cdot e^{sX_{i_0}})X_ig(x\cdot e^{sX_{i_0}})u(x\cdot e^{sX_{i_0}})\\
&\quad + 4g(x)D_{X_{i_0},s,1}g(x)g(x\cdot e^{sX_{i_0}})X_ig(x\cdot e^{sX_{i_0}})u(x\cdot e^{sX_{i_0}})\\
&\quad + 4g^2(x)D_{X_{i_0},s,1}g(x)X_ig(x\cdot e^{sX_{i_0}})u(x\cdot e^{sX_{i_0}})\\
&\quad + 4g^3(x)D_{X_{i_0},s,1}X_ig(x)u(x\cdot e^{sX_{i_0}})\\
&\quad + 4g^3(x)X_ig(x)D_{X_{i_0},s,1}u(x)\\
&\quad + D_{X_{i_0},s,1}g(x)g^3(x\cdot e^{sX_{i_0}})X_iu(x\cdot e^{sX_{i_0}})\\
&\quad + g(x)D_{X_{i_0},s,1}g(x)g^2(x\cdot e^{sX_{i_0}})X_iu(x\cdot e^{sX_{i_0}})\\
&\quad + g^2(x)D_{X_{i_0},s,1}g(x)g(x\cdot e^{sX_{i_0}})X_iu(x\cdot e^{sX_{i_0}})\\
&\quad + g^3(x)D_{X_{i_0},s,1}g(x)X_iu(x\cdot e^{sX_{i_0}})\\
&\quad + g^4(x)D_{X_{i_0},s,1}X_iu(x).
\end{aligned}$$

Therefore, Eq. (5.2) has the form

$$\sum_{i=1}^{2n}\int_\Omega D_{X_{i_0},s,1}a_i(Xu(x))D_{X_{i_0},s,1}X_iu(x)g^4(x)\,dx \tag{L1}$$

$$=\int_\Omega D_{X_{i_0},-s,1}a_{i_0+n}(Xu(x))T(g^4u)(x)\,dx$$

$$+\int_\Omega D_{X_{i_0},s,1}a_{i_0+n}(Xu(x))T(g^4u)(x)\,dx \tag{R1}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{X_{i_0},s,1}a_i(Xu(x))4D_{X_{i_0},s,1}g(x)$$

$$\times g^2(x\cdot e^{sX_{i_0}})X_ig(x\cdot e^{sX_{i_0}})u(x\cdot e^{sX_{i_0}})\,dx \tag{R2}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))4g(x)D_{X_{i_0},s,1}g(x)$$
$$\times\, g(x\cdot e^{sX_{i_0}})X_i g(x\cdot e^{sX_{i_0}})u(x\cdot e^{sX_{i_0}})\,dx \tag{R3}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))4g^2(x)D_{X_{i_0},s,1}g(x)$$
$$\times\, X_i g(x\cdot e^{sX_{i_0}})u(x\cdot e^{sX_{i_0}})\,dx \tag{R4}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))4g^3(x)$$
$$\times\, D_{X_{i_0},s,1}X_i g(x)u(x\cdot e^{sX_{i_0}})\,dx \tag{R5}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))4g^3(x)$$
$$\times\, X_i g(x)D_{X_{i_0},s,1}u(x)\,dx \tag{R6}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))D_{X_{i_0},s,1}g(x)$$
$$\times\, g^3(x\cdot e^{sX_{i_0}})X_i u(x\cdot e^{sX_{i_0}})\,dx \tag{R7}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))g(x)D_{X_{i_0},s,1}g(x)$$
$$\times\, g^2(x\cdot e^{sX_{i_0}})X_i u(x\cdot e^{sX_{i_0}})\,dx \tag{R8}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))g^2(x)D_{X_{i_0},s,1}g(x)$$
$$\times\, g(x\cdot e^{sX_{i_0}})X_i u(x\cdot e^{sX_{i_0}})\,dx \tag{R9}$$

$$-\sum_{i=1}^{2n}\int_{\Omega} D_{X_{i_0},s,1}a_i(Xu(x))g^3(x)D_{X_{i_0},s,1}g(x)$$
$$\times\, X_i u(x\cdot e^{sX_{i_0}})\,dx \tag{R10}$$

We estimate now each of the above lines. We will use $\delta > 0$ as a sufficiently small number.

$$(\mathrm{L1}) \geqslant c\int_{\Omega}(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}}|D_{X_{i_0},s,1}Xu(x)|^2 g^4(x)\,dx.$$

$$
\begin{aligned}
(\mathrm{R1}) \leqslant{}& c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{-sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},-s,1}Xu(x)| g^4(x)|Tu(x)|\,dx \\
&+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{-sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},-s,1}Xu(x)| \\
&\times 4|g^3(x)||Tg(x)||u(x)|\,dx \\
&+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} \\
&\times |D_{X_{i_0},s,1}Xu(x)| g^4(x)|Tu(x)|\,\mathrm{d}x \\
&+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)| \\
&\times 4|g^3(x)||Tg(x)||u(x)|\,dx \\
\leqslant{}& \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{-sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},-s,1}Xu(x)|^2 g^4(x)\,dx \\
&+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{-sX_{i_0}})|^2)^{\frac{p-2}{2}} g^4(x)|Tu(x)|^2\,dx \\
&+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{-sX_{i_0}})|^2)^{\frac{p-2}{2}} g^2(x)|Tg(x)|^2|u(x)|^2\,dx \\
&+ \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|^2 g^4(x)\,dx \\
&+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} g^4(x)|Tu(x)|^2\,dx \\
&+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} g^2(x)|Tg(x)|^2|u(x)|^2\,dx.
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{R2}) \leqslant{}& c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)||D_{X_{i_0},s,1}g(x)| \\
&\times g^2(x)|Xg(x\cdot e^{sX_{i_0}})||u(x\cdot e^{sX_{i_0}})|\,\mathrm{d}x \\
&+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)||D_{X_{i_0},s,1}g(x)| \\
&\times s\left|\frac{g^2(x\cdot e^{sX_{i_0}}) - g^2(x)}{s}\right| |Xg(x\cdot e^{sX_{i_0}})||u(x\cdot e^{sX_{i_0}})|\,dx \\
\leqslant{}& \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|^2 g^4(x)\,dx
\end{aligned}
$$

$$+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}g(x)|^2$$

$$\times |Xg(x\cdot e^{sX_{i_0}})|^2 |u(x\cdot e^{sX_{i_0}})|^2\,dx$$

$$+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-1}{2}} |D_{X_{i_0},s,1}g(x)|^2$$

$$\times |Xg(x\cdot e^{sX_{i_0}})||u(x\cdot e^{sX_{i_0}})|\,dx.$$

The estimates of (R3) is similar to that of (R2).

$$\text{(R4)} \leqslant c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|g^2(x)$$

$$\times |D_{X_{i_0},s,1}g(x)||Xg(x\cdot e^{sX_{i_0}})||u(x\cdot e^{sX_{i_0}})|\,dx$$

$$\leqslant \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|^2 g^2(x)\,dx$$

$$+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}g(x)|^2$$

$$\times |u(x\cdot e^{sX_{i_0}})|^2 |Xg(x\cdot e^{sX_{i_0}})|^2\,dx.$$

The estimate of (R5) is similar to that of (R4).

$$\text{(R6)} \leqslant \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|^2 g^4(x)\,dx$$

$$+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}}$$

$$\times g^2(x)|Xg(x)|^2 |D_{X_{i_0},s,1}u|^2\,dx.$$

$$\text{(R7)} \leqslant c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)||D_{X_{i_0},s,1}g(x)|$$

$$\times |g^3(x)||Xu(x\cdot e^{sX_{i_0}})|\,dx$$

$$+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)||D_{X_{i_0},s,1}g(x)|$$

$$\times s\left|\frac{g^3(x\cdot e^{sX_{i_0}}) - g^3(x)}{s}\right| |Xu(x\cdot e^{sX_{i_0}})|\,dx$$

$$\leqslant \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|^2 g^4(x)\,dx$$

$$+ c(\delta)\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sX_{i_0}})^2|$$
$$\times |D_{X_{i_0},s,1}g(x)|^2 g^2(x)\,dx$$
$$+ c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-1}{2}} |D_{X_{i_0},s,1}g(x)|^2 |Xu(x\cdot e^{sX_{i_0}})|\,dx$$

The estimates of (R8)–(R10) are similar to that of (R7). We can go back now to the beginning of the proof and use a test function

$$\varphi = D_{X_{i_0},s,1} D_{X_{i_0},-s,1}(g^4 u)$$

to get similar results with $x\cdot e^{sX_{i_0}}$ changed to $x\cdot e^{-sX_{i_0}}$. Adding the two inequalities, embedding the terms with δ coefficient into the left-hand side and using that u, Xu and Tu are in $L^p_{\mathrm{loc}}(\Omega)$ we get that for all $s>0$ sufficiently small we have

$$\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1}Xu(x)|^2 g^4(x)\,dx$$
$$+ \int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{-sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},-s,1}Xu(x)|^2 g^4(x)\,dx$$
$$\leqslant c\int_{B(x_0,2r)} (\Lambda + |Xu|^2)^{\frac{p}{2}} + |u(x)|^p\,dx.$$

We can repeat the proof for $n<i_0\leqslant 2n$ and then we get that $X^2u\in L^2_{\mathrm{loc}}(\Omega)$ and this leads to (1.7). $\square$

6. Second-order differentiability in the case $\frac{\sqrt{17}-1}{2}\leqslant p\leqslant 2$

Proof of Theorem 1.4. Let us use in Eq. (1.3) a test function

$$\varphi(x) = \triangle_{T,-s}(g^2(x)\triangle_{T,s}u(x)),$$

where g is a cut-off function between $B(x_0,\frac{r}{2^{k+2}})$ and $B(x_0,\frac{r}{2^{k+1}})$, to get

$$\int_\Omega g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|^2\,dx$$
$$\leqslant c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|2|g(x)|$$
$$\times |Xg(x)||u(x\cdot e^{sT}) - u(x)|\,dx. \tag{6.1}$$

Following a method from [11,18] and using Young's inequality we estimate the right-hand side as follows.

$$
\begin{aligned}
RHS &= c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}+\frac{2-p}{2p}} \\
&\quad \times (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2p}} |Xu(x\cdot e^{sT}) - Xu(x)| \\
&\quad \times 2|g(x)||Xg(x)||u(x\cdot e^{sT}) - u(x)|\,dx \\
&\leqslant c\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{(p-2)(p-1)}{2p}} |Xu(x\cdot e^{sT}) - Xu(x)|^{\frac{2(p-1)}{p}} \\
&\quad \times |g(x)||Xg(x)||u(x\cdot e^{sT}) - u(x)|\,dx \\
&\leqslant \delta\int_\Omega (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|^2 g^2(x)\,dx \\
&\quad + c(\delta)\int_\Omega |g(x)|^{2-p}|Xg(x)|^p|u(x\cdot e^{sT}) - u(x)|^p\,dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_\Omega g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}}|Xu(x\cdot e^{sT}) - Xu(x)|^2\,dx \\
&\quad \leqslant c\int_{B(x_0,\frac{r}{2^{k+1}})} |u(x\cdot e^{sT}) - u(x)|^p\,dx.
\end{aligned}
$$

The method used after formula (4.2) for handling the left-hand side while getting the pth power gives

$$
\begin{aligned}
&\int_\Omega g^2(x)|Xu(x\cdot e^{sT}) - Xu(x)|^p\,dx \\
&\quad \leqslant c\left(\int_{B(x_0,\frac{r}{2^{k+1}})} |u(x\cdot e^{sT}) - u(x)|^p\,dx\right)^{\frac{p}{2}} \\
&\qquad \times \left(\int_\Omega (g^{\frac{4}{p}}(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2))^{\frac{p}{2}}\,dx\right)^{1-\frac{p}{2}}.
\end{aligned}
$$

Theorem 1.2 implies that we can control locally $D_{T,s,1}u$, hence we have

$$
\int_\Omega |D_{T,s,\frac{p}{2}}X(g^2u)(x)|^p\,dx \leqslant M^p, \tag{6.2}
$$

where we have denoted again

$$M^p = c\int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p)\, dx.$$

We use Proposition 2.2 to get that for a sufficiently small $\sigma > 0$ we have

$$\sup_{0<s<\sigma} \frac{||\triangle^2_{T,s}(g^2u)||_{L^p}}{s^{\frac{1+p}{2}}} \leqslant M. \tag{6.3}$$

We will use the fact that for a for small $\delta > 0$ we have that u is locally C^δ (see [2,14,27]) and that for $\frac{\sqrt{17}-1}{2} \leqslant p \leqslant 2$ we have

$$2 - \frac{p}{2} - \frac{p^2}{2} \leqslant 0.$$

Therefore, for all $0 < s < \sigma$ and for $\delta' = \delta(2-p)$ we have

$$\begin{aligned}
&\int_\Omega \frac{|\triangle^2_{T,s}(g^2u(x))|^2}{|s|^{2+\delta'}}\, dx \\
&\quad = \int_\Omega \frac{|\triangle^2_{T,s}(g^2u(x))|^p}{|s|^{\frac{p}{2}+\frac{p^2}{2}}} \frac{|\triangle^2_{T,s}(g^2u(x))|^{2-p}}{|s|^{2+\delta'-\frac{p}{2}-\frac{p^2}{2}}}\, dx \\
&\quad \leqslant cM^p ||g^2u||^{2-p}_{C^\delta(\Omega)}.
\end{aligned}$$

Theorem 1.1 gives now (1.8) and that $Tu \in L^2_{\text{loc}}(\Omega)$.

Corollary 6.1. *For $\frac{\sqrt{17}-1}{2} \leqslant p \leqslant 2$ we have that*

$$\begin{aligned}
&\int_{B(x_0,\frac{r}{2^{k+3}})} |TXu(x)|^p\, dx \\
&\quad \leqslant c\Lambda^{\frac{p(p-2)}{4}} \left(||u||^{2-p}_{C^\delta(B(x_0,\frac{r}{2^{k+1}}))} \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p)\, dx \right)^{\frac{p}{2}}.
\end{aligned} \tag{6.4}$$

Therefore, $XTu \in L^p_{\text{loc}}(\Omega)$.

Proof. We use a cut-off function g between $B(x_0, \frac{r}{2^{k+3}})$ and $B(x_0, \frac{r}{2^{k+2}})$. Theorem 1.4 allows us to estimate the right-hand side of Eq. (6.1) in the following way:

$$RHS \leqslant \delta \int_{\Omega} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|^2\, dx$$
$$+ c(\delta) \int_{\Omega} |Xg(x)|^2 (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |u(x\cdot e^{sT}) - u(x)|^2\, dx.$$

Therefore,

$$\int_{\Omega} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|^2\, dx$$
$$\leqslant c \int_{\Omega} |Xg(x)|^2 (\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |u(x\cdot e^{sT}) - u(x)|^2\, dx$$

and hence

$$\int_{\Omega} g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} |Xu(x\cdot e^{sT}) - Xu(x)|^2\, dx$$
$$\leqslant c\Lambda^{\frac{p-2}{2}} \int_{\Omega} |Xg(x)|^2 |u(x\cdot e^{sT}) - u(x)|^2\, dx.$$

Using again the method that follows formula (4.2) we get

$$\int_{\Omega} g^2(x) |Xu(x\cdot e^{sT}) - Xu(x)|^p\, dx$$
$$\leqslant c\left(\Lambda^{\frac{p-2}{2}} \int_{B(x_0,\frac{r}{2^{k+2}})} |u(x\cdot e^{sT}) - u(x)|^2\, dx\right)^{\frac{p}{2}} \tag{6.5}$$

which gives (6.4) and that $XTu \in L^p_{\mathrm{loc}}(\Omega)$. $\square$

We will prove now Theorem 1.5. A similar theorem was announced in [18].

Proof of Theorem 1.5. Let g be a cut-off functions between $B(x_0, \frac{r}{2^{k+3}})$ and that $B(x_0, \frac{r}{2^{k+2}})$. The proof begins in the same way as the proof of Theorem 1.3, until we get the extended form of our inequality with the lines (L1) and (R1)–(R10). We can remark that although we could use a test function $\varphi = D_{X_{i_0},-s,1}(g^2 D_{X_{i_0},s,1} u)$, but we cannot avoid estimates similar to that of line (R6).

For the line (L1) the estimate is the same as in the proof of Theorem 1.3. For the lines (R1)–(R5) we keep the same estimates and use Theorem 1.4 with the facts that for $p < 2$ we have

$$(\Lambda + |Xu(x)|^2 + |Xu(x\cdot e^{sT})|^2)^{\frac{p-2}{2}} \leqslant \Lambda^{\frac{p-2}{2}}.$$

For (R7) we have

$$
\begin{aligned}
(\mathrm{R7}) \leqslant\, & c\int_{\Omega}(\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-1}{2}}|D_{X_{i_0},s,1}Xu(x)||D_{X_{i_0},s,1}g(x)| \\
& \times |g^3(x\cdot e^{sX_{i_0}})|\,dx \\
=\, & c\int_{\Omega}(\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{4}}|D_{X_{i_0},s,1}Xu(x)|g^2(x) \\
& \times (\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p}{4}}|D_{X_{i_0},s,1}g(x)||g(x)|\,dx \\
& + c\int_{\Omega}(\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-1}{2}}|D_{X_{i_0},s,1}Xu(x)||D_{X_{i_0},s,1}g(x)| \\
& \times s\left|\frac{g^3(x\cdot e^{sX_{i_0}})-g^3(x)}{s}\right|\,dx \\
\leqslant\, & \delta\int_{\Omega}(\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}}\,|D_{X_{i_0},s,1}Xu(x)|^2g^4(x)\,dx \\
& + c(\delta)\int_{\Omega}(\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p}{2}}|D_{X_{i_0},s,1}g(x)|^2g^2(x)\,dx \\
& + c\int_{\Omega}(\Lambda+|Xu(x)|^2+|Xu(x\cdot e^{sX_{i_0}})|^2)^{\frac{p}{2}}|D_{X_{i_0},s,1}g(x)|^3\,dx.
\end{aligned}
$$

The estimates for (R8)–(R10) are similar. It is left the estimate for (R6). Following the methods in [11,18] we consider for small $h>0$ and a.e. $x\in B(x_0,4r)$ the function

$$\alpha_i(x)=\int_0^1 a_i(Xu(x\cdot(te^{sX_{i_0}}))\,dt$$

and

$$Y(x)=\int_0^1(\Lambda+|Xu(x\cdot(te^{sX_{i_0}})|^2))^{\frac{p-1}{2}}\,dt.$$

In the distributional sense we have

$$D_{X_{i_0},s,1}a_i(Xu(x))=X_{i_0}\alpha_i(x).$$

Also,

$$|\alpha_i(x)|\leqslant Y(x),\quad \text{a.e } x\in B(x_0,4r).$$

Therefore, we can estimate (R6) in the following way:

$$
\begin{aligned}
(\mathrm{R6}) &= \sum_{i=1}^{2n} \int_\Omega D_{X_{i_0},s,1} a_i(Xu(x)) 4g^3(x) X_i g(x) D_{X_{i_0},s,1} u(x)\, dx \\
&= 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x) X_{i_0}(g^3(x) X_i g(x) D_{X_{i_0},s,1} u(x))\, dx \\
&= 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x) 3g^2(x) X_{i_0} g(x) X_i g(x) D_{X_{i_0},s,1} u(x)\, dx \\
&\quad + 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x) g^3(x) X_{i_0} X_i g(x) D_{X_{i_0},s,1} u(x)\, dx \\
&\quad + 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x) g^3(x) X_i g(x) D_{X_{i_0},s,1} X_{i_0} u(x)\, dx \\
&\leqslant c \int_\Omega g^2(x) Y(x) |D_{X_{i_0},s,1} u(x)|\, dx \qquad (\mathrm{R6}_1) \\
&\quad + c \int_\Omega |g^3(x)|\, Y(x) |D_{X_{i_0},s,1} Xu(x)|\, dx. \qquad (\mathrm{R6}_2)
\end{aligned}
$$

Because of

$$
Y \in L_{\mathrm{loc}}^{\frac{p}{p-1}}(\Omega)
$$

and $Xu \in L^p_{\mathrm{loc}}(\Omega)$ we get that $(\mathrm{R6}_1)$ is finite. Let us estimate $(\mathrm{R6}_2)$ (see [11] Section 8.2).

$$
\begin{aligned}
(\mathrm{R6}_2) = &\int_\Omega g^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{4}} |D_{X_{i_0},s,1} Xu(x)| \\
&\times |g(x)|\, Y(x)(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{2-p}{4}}\, dx \\
\leqslant\ & \delta \int_\Omega g^4(x)(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1} Xu(x)|^2\, dx \\
&+ c(\delta) \int_\Omega g^2(x) Y^2(x)(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{2-p}{2}}\, dx \\
\leqslant\ & \delta \int_\Omega g^4(x)(\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1} Xu(x)|\, dx \\
&+ c(\delta) \int_\Omega g^2(x)(Y^{\frac{p}{p-1}}(x) + (\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{p}{2}})\, dx.
\end{aligned}
$$

We can now continue the proof in the same way as we did in the case $p \geqslant 2$, going back to the beginning of the proof and use a test function

$$\varphi = D_{X_{i_0},s,1} D_{X_{i_0},-s,1}(g^4 u),$$

then adding the two inequalities and embedding the terms with δ coefficient into the left hand side. In this way we get

$$\begin{aligned} &\int_{B(x_0,\frac{r}{2^{k+3}})} (\Lambda + |Xu(x)|^2 + |Xu(x \cdot e^{sX_{i_0}})|^2)^{\frac{p-2}{2}} |D_{X_{i_0},s,1} Xu(x)|^2 \, dx \\ &\leqslant c \left(\Lambda^{\frac{p-2}{2}} ||u||^{2-p}_{C^\delta(B(x_0,\frac{r}{2^{k+1}}))} \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx \right. \\ &\left. \quad + \Lambda^{\frac{p-2}{2}} ||u||^2_{L^2(B(x_0,\frac{r}{2^{k+1}}))} + \int_{B(x_0,2r)} ((\Lambda + |Xu(x)|^2)^{\frac{p}{2}} + |u(x)|^p) \, dx \right) \end{aligned} \tag{6.6}$$

and quoting again the method used after formula (4.2) we get (1.9) and hence $u \in HW^{2,p}_{\mathrm{loc}}(\Omega)$. $\square$

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