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Evolution inclusions governed by the difference of two subdifferentials in reflexive Banach spaces

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Abstract

The existence of strong solutions of Cauchy problem for the following evolution equation $du(t)/dt + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t)$ is considered in a real reflexive Banach space V, where $\partial\varphi^1$ and $\partial\varphi^2$ are subdifferential operators from V into its dual V^*. The study for this type of problems has been done by several authors in the Hilbert space setting.

The scope of our study is extended to the V-V^* setting. The main tool employed here is a certain approximation argument in a Hilbert space and for this purpose we need to assume that there exists a Hilbert space H such that $V \subset H \equiv H^* \subset V^*$ with densely defined continuous injections.

The applicability of our abstract framework will be exemplified in discussing the existence of solutions for the nonlinear heat equation: $u_t(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t)$, $x \in \Omega$, $t > 0$, $u|_{\partial\Omega} = 0$, where Ω is a bounded domain in $\mathbb{R}^N$. In particular, the existence of local (in time) weak solution is shown under the subcritical growth condition $q < p^*$ (Sobolev's critical exponent) for all initial data $u_0 \in W_0^{1,p}(\Omega)$. This fact has been conjectured but left as an open problem through many years.

Keywords: Evolution equation; Subdifferential; Reflexive Banach space; p-Laplacian; Subcritical; Local existence; Degenerate parabolic equation

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1. Introduction

Let V be a real reflexive Banach space and let V^* be its dual. The main purpose of this paper is to investigate the solvability of the following Cauchy problem in the V-V^* setting, i.e., to find a solution $u(t)$ in V satisfying the equation in V^*:

$$\text{(CP)} \begin{cases} \dfrac{du}{dt}(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t) \quad \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\partial\varphi^1,\ \partial\varphi^2 : V \to 2^{V^*}$ are the subdifferential operators of proper lower semicontinuous convex functions $\varphi^1,\ \varphi^2 : V \to (-\infty, +\infty]$.

The existence and the asymptotic behavior of strong solutions are already studied by Koi-Watanabe [8], Ishii [6] and Ôtani [10–12] in the Hilbert space framework. In particular, the following initial-boundary value problem falls within the scope of the nonmonotone perturbation theory developed in [10,12]:

$$\text{(NHE)} \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Delta_p u(x) := \operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x))$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

On the other hand, Faedo–Galerkin's method gives another useful tool to study (NHE) such as in Lions [9] and Tsutsumi [13].

The theory of perturbation for subdifferential operators in the Hilbert space setting has an advantage over Faedo–Galerkin's method in that it can assure a better regularity of solutions such as $u_t, \Delta_p u \in L^2(0,T; L^2(\Omega))$.

For the quasilinear case where $p \neq 2$, however, it requires a strong restriction on the growth order q of the perturbed term $|u|^{q-2}u$, which is caused by the loss of the elliptic estimate for Δ_p.

As is well known, the theory of elliptic equations bears close relations with the theory of evolution equations, and in the theory of elliptic equations, the Fréchet derivative $d\phi$ of a C^1-function ϕ from V into $\mathbb{R}$ is usually regarded as the operator from V into V^*. We also recall that the statement of "Palais–Smale" condition or Mountain Pass lemmas is formulated in the V-V^* setting; this setting plays a natural and essential role to derive the well-known fact that the equation $-\Delta_p u(x) = |u|^{q-2}u(x),\ x \in \Omega,\ u|_{\partial\Omega} = 0$ admits a nontrivial positive solution if and only if q is subcritical, i.e., $1 < q < p^*$, where p^* denotes the so-called Sobolev's critical exponent, provided that Ω is a bounded star-shaped domain. From this point of view, it would be very natural and important to investigate the solvability of (CP) in the V-V^* setting. However the study in this direction is not fully pursued yet even for the nonperturbed case where $\partial\varphi^2 \equiv 0$, except in [2] and [7].

Moreover it is readily suggested from the study of nonlinear elliptic equations that the perturbation theory for subdifferentials in the V-V^* setting should remedy the deficiency in the Hilbert space setting mentioned above. In fact, as an application of our abstract results, it is shown that (NHE) admits a local (in time) weak solution under the subcritical growth condition $q < p^*$ for all $u_0 \in W_0^{1,p}(\Omega)$. This fact is already known for the semilinear case $p = 2$. For the general case, however, this fact has been conjectured but left as an open problem through many years.

The content of this paper is as follows. In the next section, our main results on the existence of local or global (in time) solutions are formulated and some related materials to be used later are prepared. The proofs for main results are given in Section 3, and the applicability of our abstract results are exemplified in Section 4.

2. Main results

Let V be a real reflexive Banach space and let V^* be its dual. Throughout this paper, we assume that there exists a real Hilbert space H whose dual space H^* is identified with H such that

$$V \subset H \equiv H^* \subset V^*, \tag{1}$$

where the natural injection from V into H as well as that from H^* into V^* are densely defined and continuous.

To formulate our results, we need the notion of subdifferential operators from a Banach space X into its dual X^* defined below. Let $\Phi(X)$ be the set of all *proper lower semicontinuous convex functions* from X into $(-\infty, +\infty]$, where "*proper*" means that the *effective domain* $D(\varphi)$ of φ defined by $D(\varphi) := \{u \in X; \varphi(u) < +\infty\}$ is not empty. The *subdifferential* $\partial_X\varphi(u)$ of φ at u in X is defined by

$$\partial_X\varphi(u) := \left\{ f \in X^*; \varphi(v) - \varphi(u) \geqslant {}_{X^*}\langle f, v-u\rangle_X \text{ for all } v \in D(\varphi)\right\}$$

with domain $D(\partial_X\varphi) := \{u \in D(\varphi); \partial_X\varphi(u) \neq \emptyset\}$, where ${}_{X^*}\langle\cdot,\cdot\rangle_X$ denotes the duality pairing between X and X^*. For simplicity of notation, we write $\partial\varphi$ and $\langle\cdot,\cdot\rangle$ instead of $\partial_X\varphi$ and $\langle\cdot,\cdot\rangle_X$, respectively, if no confusion arises.

In particular, when X is a Hilbert space H and $\varphi \in \Phi(H)$, then

$$\partial_H\varphi(u) := \{f \in H; \varphi(v) - \varphi(u) \geqslant (f, v-u)_H \text{ for all } v \in D(\varphi)\},$$

where $(\cdot,\cdot)_H$ denotes the inner product of H. It is well known that the subdifferential operator $\partial_X\varphi$ becomes a (possibly multi-valued) maximal monotone operator from X into X^* (see [3–5]). Especially, in the Hilbert space setting, various nice properties are known. We summarize some of them without their proofs for later use.

Let $\varphi \in \Phi(H)$. Then the Yosida approximation $(\partial_H \varphi)_\lambda$ of $\partial_H \varphi$ coincides with the subdifferential of the *Moreau–Yosida regularization* φ_λ of φ given by

$$\varphi_\lambda(u) := \inf_{v \in H} \left\{ \frac{1}{2\lambda} |u - v|_H^2 + \varphi(v) \right\}.$$

More precisely, the following lemma holds.

Proposition 1. *Let $\varphi \in \Phi(H)$. Then φ_λ becomes a Fréchet differentiable convex function from H into $\mathbb{R}$ and is characterized by*

$$\varphi_\lambda(u) = \frac{1}{2\lambda} |u - J_\lambda u|_H^2 + \varphi(J_\lambda u) = \frac{\lambda}{2} |(\partial_H \varphi)_\lambda(u)|_H^2 + \varphi(J_\lambda u),$$

where $(\partial_H \varphi)_\lambda$ and J_λ are the Yosida approximation and the resolvent of $\partial_H \varphi$, respectively, i.e., $J_\lambda = (I + \lambda \partial \varphi_H)^{-1}$ and $(\partial_H \varphi)_\lambda = (I - J_\lambda)/\lambda$. Moreover $\partial_H(\varphi_\lambda) = (\partial_H \varphi)_\lambda$, where $\partial_H(\varphi_\lambda)$ denotes the subdifferential (Fréchet derivative) of φ_λ, $\varphi(J_\lambda u) \leqslant \varphi_\lambda(u) \leqslant \varphi(u)$ for all $u \in H$, $\lambda > 0$ and $\varphi_\lambda(u) \to \varphi(u)$ as $\lambda \to 0$ for all $u \in H$.

The following proposition yields an information on the chain rule for φ.

Proposition 2. *Let $\varphi \in \Phi(H)$ and suppose that $u \in W^{1,2}(0, T; H)$, $u(t) \in D(\partial_H \varphi)$ for a.e. $t \in (0, T)$ and that there exists $g \in L^2(0, T; H)$ such that $g(t) \in \partial_H \varphi(u(t))$ for a.e. $t \in (0, T)$. Then the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0, T]$ and the following holds*:

$$\frac{d}{dt} \varphi(u(t)) = \left(h(t), \frac{du}{dt}(t) \right)_H \quad \forall\, h(t) \in \partial_H \varphi(u(t)) \quad for\ a.e.\ t \in (0, T).$$

In the present paper, we are concerned with *strong solutions* of (CP) in the following sense.

Definition 1. A function $u \in C([0, T]; V^*)$ is said to be a strong solution of (CP) on $[0, T]$, if the following conditions are satisfied:

(i) $u(t)$ is a V^*-valued absolutely continuous function on $[0, T]$.
(ii) $u(t) \to u_0$ strongly in H as $t \to +0$.
(iii) $u(t) \in D(\partial \varphi^1) \cap D(\partial \varphi^2)$ for a.e. $t \in (0, T)$ and there exist sections $g^i(t) \in \partial \varphi^i(u(t))$ $(i = 1, 2)$ satisfying:

$$\frac{du}{dt}(t) + g^1(t) - g^2(t) = f(t) \text{ in } V^* \text{ for a.e. } t \in (0, T). \tag{2}$$

Throughout the present paper, we denote by C or C_i $(i = 1, 2, \ldots)$ positive constants which do not depend on the elements of the corresponding space or set. Moreover let us denote by $\mathcal{L}$ the set of all monotone nondecreasing functions from $[0, +\infty)$ into itself. For $p \in (1, +\infty)$, p' designates the Hölder conjugate of p, i.e., $p' = p/(p-1)$.

Our basic assumptions are the following:

(A1) $|u|_V^p - C_1|u|_H^2 - C_2 \leqslant C_3\varphi^1(u) \quad \forall u \in D(\varphi^1),\ 1 < p < +\infty.$

(A2) $D(\varphi^1) \subset D(\partial\varphi^2)$. Furthermore if $\{u_n\}$ is a sequence such that $\sup_{t\in[0,T]}\{\varphi^1(u_n(t)) + |u_n(t)|_H\} + \int_0^T |du_n(t)/dt|_H^2\, dt$ is bounded, then for every $g_n(\cdot) \in \partial\varphi^2(u_n(\cdot))$, $\{g_n\}$ forms a precompact subset in $C([0, T]; V^*)$.

(A3) There exists an extension $\tilde{\varphi}^2 \in \Phi(H)$ of φ^2, i.e., $\tilde{\varphi}^2(u) = \varphi^2(u)\ \forall u \in V$, such that $\varphi^1 (J_\lambda u) \leqslant l_1 \left(\varphi^1(u) + l_2(|u|_H)\right)\ \forall \lambda \in (0, 1],\ \forall u \in D(\varphi^1)$, where $l_i \in \mathcal{L}$ $(i = 1, 2)$ and J_λ denotes the resolvent of $\partial_H\tilde{\varphi}^2$, that is, $J_\lambda = (I + \lambda\partial_H\tilde{\varphi}^2)^{-1}$.

(A4) $\varphi^2(u) \leqslant k\varphi^1(u) + C_4|u|_H^2 + C_5 \quad \forall u \in D(\varphi^1),\ 0 \leqslant k < 1.$

Remark 1. (A3) is weaker than the well-known sufficient condition for the maximality of $\partial_H\varphi_H^1 + \partial_H\tilde{\varphi}^2$:

$$\varphi_H^1(J_\lambda u) \leqslant \varphi_H^1(u) + C\lambda \quad \forall \lambda \in (0, 1],\ \forall u \in D(\varphi_H^1),$$

where φ_H^1 denotes the extension of φ^1 onto H which will be given in the proof of Theorem 1.

We note that (A2) assures the continuity of φ^2 in the following sense.

Proposition 3. *Assume that* (A2) *is satisfied. Let* $\{u_n\}$ *be a sequence in* $D(\varphi^1)$ *such that* $u_n \to u$ *weakly in V and* $\varphi^1(u_n)$ *is bounded. Then it follows that* $\varphi^2(u_n) \to \varphi^2(u)$.

Proof of Proposition 3. Let $\{u_n\}$ be a sequence in $D(\varphi^1)$ such that $u_n \to u$ weakly in V as $n \to +\infty$ and $\varphi^1(u_n)$ is bounded. Then from the fact that $\varphi^2 \in \Phi(V)$, it follows that:

$$\varphi^2(u) \leqslant \liminf_{n\to+\infty} \varphi^2(u_n). \tag{3}$$

On the other hand, for each $n \in \mathbb{N}$, let $g_n \in \partial\varphi^2(u_n)$ and set $v_n(t) = u_n$ and $h_n(t) = g_n$ for all $t \in [0, T]$. Then we see that $\sup_{t\in[0,T]}\{\varphi^1(v_n(t)) + |v_n(t)|_H\} = \varphi^1(u_n) + |u_n|_H$ is bounded, $dv_n/dt \equiv 0$ and $h_n(\cdot) \in \partial\varphi^2(v_n(\cdot))$. By (A2), we can extract a subsequence $\{n'\}$ of $\{n\}$ such that $h_{n'} \to h$ strongly in $C([0, T]; V^*)$, which implies $\{g_{n'}\}$ becomes a strongly convergent sequence in V^*.

Hence since $\varphi^2(u_{n'}) \leqslant \varphi^2(u) + \langle g_{n'}, u_{n'} - u \rangle$, we get

$$\limsup_{n' \to +\infty} \varphi^2(u_{n'}) \leqslant \varphi^2(u) + \lim_{n' \to +\infty} \langle g_{n'}, u_{n'} - u \rangle = \varphi^2(u). \tag{4}$$

Therefore it follows from (3) and (4) that $\varphi^2(u_{n'}) \to \varphi^2(u)$. Since the limit is unique, we find that $\varphi^2(u_n) \to \varphi^2(u)$. $\square$

Now our main results are stated as follows.

Theorem 1. *Assume that* (A1)–(A4) *hold. Then for all* $u_0 \in D(\varphi^1)$ *and* $f \in W^{1,p'}(0, T; V^*)$, (CP) *has a strong solution* u *on* $[0, T]$ *satisfying*:

$$\begin{cases} u \in C_w([0, T]; V) \cap W^{1,2}(0, T; H), \\ u(t) \in D(\partial\varphi^1) \cap D(\partial\varphi^2) \quad \textit{for a.e. } t \in (0, T), \\ g^1 \in L^2(0, T; V^*), \quad g^2 \in C([0, T]; V^*), \\ \sup_{t \in [0,T]} \varphi^1(u(t)) < +\infty, \quad \varphi^2(u(\cdot)) \in C([0, T]), \end{cases} \tag{5}$$

where g^i $(i = 1, 2)$ *are the sections of* $\partial\varphi^i$ *satisfying* (2) *and* $C_w([0, T]; V)$ *denotes the set of all V-valued weakly continuous functions on* $[0, T]$.

Moreover the following energy estimate holds true.

$$\int_0^t \left| \frac{du}{d\tau}(\tau) \right|_H^2 d\tau + \varphi^1(u(t)) + \varphi^2(u_0)$$
$$\leqslant \varphi^1(u_0) + \varphi^2(u(t)) + \langle f(t), u(t) \rangle - \langle f(0), u_0 \rangle - \int_0^t \left\langle \frac{df}{d\tau}(\tau), u(\tau) \right\rangle d\tau \tag{6}$$

for all $t \in [0, T]$.

As for the existence of local (in time) strong solutions, we do not need to assume (A4), which might be somewhat restrictive from the view point of applications to P.D.E.

Theorem 2. *Assume that* (A1)–(A3) *hold. Then for all* $u_0 \in D(\varphi^1)$ *and* $f \in W^{1,p'}(0, T; V^*)$, *there exists a number* $T_0 \in (0, T]$ *such that* (CP) *has a strong solution* u *on* $[0, T_0]$ *satisfying* (5) *with* T *replaced by* T_0.

As for the global (in time) existence, we introduce the following assumption:

(A5) $\alpha\varphi^1(u) \leqslant \langle \xi - \eta, u \rangle + l_3(\varphi^2(u)) \cdot \varphi^1(u) \quad \forall [u, \xi] \in \partial\varphi^1, \ \forall [u, \eta] \in \partial\varphi^2,$

where $\alpha > 0$ and l_3 denotes a nondecreasing continuous function from $[0, +\infty)$ to $\mathbb{R}$ satisfying $l_3(0) = 0$.

The following theorem ensures the existence of small global solutions.

Theorem 3. *In addition to all the assumptions in Theorem 2, assume that $C_1 = C_2 = 0$ in* (A1), *$\varphi^2 \geqslant 0$ and* (A5) *is satisfied. Let δ_0 be a positive number such that $l_3(\delta_0) < \alpha$. Then, for all $R > 0$, there exists a positive number δ_R such that for all $T > 0$ and (u_0, f) belonging to*

$$X^T_{\delta_R,R} := \Bigg\{ (u_0, f) \in D(\varphi^1) \times W^{1,p'}(0, T; V^*);$$

$$\varphi^1(u_0) + \int_0^T |f(\tau)|^{p'}_{V^*}\, d\tau + \int_0^T \left|\frac{df}{d\tau}(\tau)\right|^{p'}_{V^*} d\tau \leqslant R,$$

$$\varphi^2(u_0) < \delta_0,$$

$$|u_0|_H + \left\{ \max\left(1, \frac{1}{T}\right) \left\| |f(\cdot)|^{p'}_{V^*} \right\|_{1,T} \right\}^{1/p} < \delta_R \Bigg\},$$

where

$$\left\| |f(\cdot)|^{p'}_{V^*} \right\|_{1,T} := \begin{cases} \displaystyle\int_0^T |f(\tau)|^{p'}_{V^*}\, d\tau & if\ \ T < 1, \\ \displaystyle\sup_{t\in[1,T]} \int_{t-1}^t |f(\tau)|^{p'}_{V^*}\, d\tau & if\ \ T \geqslant 1, \end{cases}$$

(CP) *has a strong solution u on $[0, T]$ satisfying* (5).

Remark 2. All results described above hold true even if $\sup_{t\in[0,T]}\{\varphi^1(u_n(t)) + |u_n(t)|_H\}$ in (A2) and $\varphi^1(J_\lambda u)$ in (A3) are replaced by $\sup_{t\in[0,T]} |u_n(t)|_V$ and $|J_\lambda u|_V$, respectively.

3. Proof of main results

3.1. Proof of Theorem 1

The first step of our proof is to introduce suitable approximation problems for (CP) in the Hilbert space H. To this end, we first define the extension φ^1_H to φ^1 on H by

$$\varphi^1_H(u) = \begin{cases} \varphi^1(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H/V. \end{cases}$$

Then, by virtue of (A1), we can easily verify that $\varphi^1_H \in \Phi(H)$ (see [2]).

Now our approximation problems for (CP) are given by

$$(\mathrm{CP})_\lambda \begin{cases} \dfrac{du_\lambda}{dt}(t) + \partial_H \varphi_H^1(u_\lambda(t)) - \partial_H \tilde{\varphi}_\lambda^2(u_\lambda(t)) \ni f_\lambda(t) \quad \text{in } H, \quad 0 < t < T, \\ u_\lambda(0) = u_0, \end{cases}$$

where f_λ belongs to $C^1([0,T];H)$ such that $f_\lambda \to f$ strongly in $W^{1,p'}(0,T;V^*)$ as $\lambda \to +0$, $\tilde{\varphi}^2$ is the extension of φ^2 on H given in (A3) and $\partial_H \tilde{\varphi}_\lambda^2$ denotes the Yosida approximation of $\partial_H \tilde{\varphi}^2$. We note by Proposition 1 that $\partial_H \tilde{\varphi}_\lambda^2 = \partial_H(\tilde{\varphi}_\lambda^2)$.

Since $\partial_H \tilde{\varphi}_\lambda^2$ is Lipschitz continuous in H, it is well known (see e.g. Proposition 3.12 and Theorem 3.6 of [5]) that there exits a unique strong solution u_λ of $(\mathrm{CP})_\lambda$ on $[0,T]$ satisfying:

$$u_\lambda \in W^{1,2}(0,T;H), \quad u_\lambda(t) \in D(\partial_H \varphi_H^1) \ \text{ for a.e. } t \in (0,T),$$
$$t \mapsto \varphi_H^1(u_\lambda(t)), \ \tilde{\varphi}_\lambda^2(u_\lambda(t)) \text{ are absolutely continuous on } [0,T].$$

Here we can assume that $\varphi^1 \geqslant 0$ without any loss of generality. Indeed, since $\varphi_H^1 \in \Phi(H)$, there exist $v_0 \in H$ and $\mu_0 \in \mathbb{R}$ such that

$$\varphi_H^1(u) \geqslant (v_0, u)_H + \mu_0 \quad \forall u \in H$$

(see [3]). Now set $\hat{\varphi}^1(u) := \varphi^1(u) - (v_0,u)_H - \mu_0$. Then since $\varphi_H^1(u) - (v_0,u) - \mu_0 \geqslant 0$ for all $u \in H$ and $\varphi^1(u) = \varphi_H^1(u)$ for all $u \in V$, it follows that $\hat{\varphi}^1(u) \geqslant 0$ for all $u \in V$. Moreover we can easily get

$$D(\hat{\varphi}^1) = D(\varphi^1), \quad D(\partial\hat{\varphi}^1) = D(\partial\varphi^1), \quad \partial\hat{\varphi}^1(u) = \partial\varphi^1(u) - v_0 \quad \forall u \in D(\partial\varphi^1).$$

Hence (CP) is equivalent to Cauchy problem for the following evolution equation with an initial condition $u(0) = u_0$.

$$\frac{du}{dt}(t) + \partial\hat{\varphi}^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t) - v_0 \quad \text{in } V^*, \quad 0 < t < T.$$

Moreover it is easy to see that if (A1)–(A4) hold, then (A1)–(A4) with φ^1 replaced by $\hat{\varphi}^1$ also hold.

We are going to establish a priori estimates in the following Lemmas 1–3.

Lemma 1. *There exists a constant M_1 such that*

$$\sup_{t\in[0,T]} |u_\lambda(t)|_H \leqslant M_1, \tag{7}$$

$$\sup_{t\in[0,T]} \varphi^1(u_\lambda(t)) \leqslant M_1, \tag{8}$$

$$\int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt \leqslant M_1, \tag{9}$$

$$\sup_{t\in[0,T]} |u_\lambda(t)|_V \leqslant M_1. \tag{10}$$

Proof of Lemma 1. Multiply $(\mathrm{CP})_\lambda$ by $du_\lambda(t)/dt$. Then, by Proposition 2, we obtain

$$\left|\frac{du_\lambda}{dt}(t)\right|_H^2 + \frac{d}{dt}\varphi_H^1(u_\lambda(t)) - \frac{d}{dt}\tilde{\varphi}_\lambda^2(u_\lambda(t)) = \left(f_\lambda(t), \frac{du_\lambda}{dt}(t)\right)_H. \tag{11}$$

Hence, integrating (11) over $(0,t)$, we have by Proposition 1,

$$\begin{aligned}
&\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + \varphi^1(u_\lambda(t)) + \tilde{\varphi}_\lambda^2(u_0)\\
&\quad\leqslant \varphi^1(u_0) + \tilde{\varphi}^2(u_\lambda(t)) + \langle f_\lambda(t), u_\lambda(t)\rangle - \langle f_\lambda(0), u_0\rangle\\
&\qquad - \int_0^t \left\langle \frac{df_\lambda}{d\tau}(\tau), u_\lambda(\tau)\right\rangle d\tau.
\end{aligned} \tag{12}$$

By (A1) and (A4), it follows that

$$\begin{aligned}
&\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + (1-k)\varphi^1(u_\lambda(t))\\
&\quad\leqslant \varphi^1(u_0) - \tilde{\varphi}_\lambda^2(u_0) + C_4|u_\lambda(t)|_H^2 + C_5\\
&\qquad + |f_\lambda(t)|_{V^*}\{C_3\varphi^1(u_\lambda(t)) + C_1|u_\lambda(t)|_H^2 + C_2\}^{1/p} + |f_\lambda(0)|_{V^*}|u_0|_V\\
&\qquad + \int_0^t \left|\frac{df_\lambda}{d\tau}(\tau)\right|_{V^*} \left\{C_3\varphi^1(u_\lambda(\tau)) + C_1|u_\lambda(\tau)|_H^2 + C_2\right\}^{1/p} d\tau.
\end{aligned}$$

Then, by Young's inequality, there exists a constant C depending only on k, p, C_1 and C_3 such that

$$\begin{aligned}
&\int_0^t \left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + \frac{1-k}{2}\varphi^1(u_\lambda(t)) \\
&\leqslant C\left\{|u_0|_V^p + \varphi^1(u_0) + |\tilde{\varphi}_\lambda^2(u_0)| + C_2 + C_5 + \sup_{\tau\in[0,t]} |f_\lambda(\tau)|_{V^*}^{p'}\right. \\
&\left.+ \int_0^t \left|\frac{df_\lambda}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau\right\} + (C_4+1)|u_\lambda(t)|_H^2 \\
&\times \int_0^t \left\{|u_\lambda(\tau)|_H^2 + \varphi_{H,\lambda}^1(u_\lambda(\tau))\right\} d\tau. \qquad (13)
\end{aligned}$$

Here using the fact that $\frac{d}{dt}|u_\lambda(t)|_H \leqslant |\frac{du_\lambda}{dt}(t)|_H$, we get

$$\begin{aligned}
\mu\frac{d}{dt}|u_\lambda(t)|_H^2 &= 2\mu|u_\lambda(t)|_H \frac{d}{dt}|u_\lambda(t)|_H \\
&\leqslant 2\mu|u_\lambda(t)|_H \left|\frac{du_\lambda}{dt}(t)\right|_H \\
&\leqslant \mu^2|u_\lambda(t)|_H^2 + \left|\frac{du_\lambda}{dt}(t)\right|_H^2 \quad \forall\mu > 0. \qquad (14)
\end{aligned}$$

Hence, putting $\mu = C_4 + 2$ and combining (13) with (14), we obtain by Gronwall's inequality,

$$\begin{aligned}
&|u_\lambda(t)|_H^2 + \varphi^1(u_\lambda(t)) \\
&\leqslant C\left\{|u_0|_H^2 + \varphi^1(u_0) + |\tilde{\varphi}_\lambda^2(u_0)| + |u_0|_V^p + C_2 + C_5\right. \\
&\left.+ \sup_{\tau\in[0,T]} |f_\lambda(\tau)|_{V^*}^{p'} + \int_0^T \left|\frac{df_\lambda}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau\right\},
\end{aligned}$$

where C depends on k, p, C_1, C_3, C_4 and T. Therefore since f_λ is bounded in $W^{1,p'}(0,T;V^*)$ and $\tilde{\varphi}_\lambda^2(u_0)$ is bounded, it follows that (7) and (8) hold. Moreover, (7) and (13) imply (9). Furthermore, by (A1), we get

$$|u_\lambda(t)|_V^p \leqslant C_1|u_\lambda(t)|_H^2 + C_2 + C_3\varphi^1(u_\lambda(t)).$$

Hence, (7) and (8) imply (10). $\square$

Lemma 2. *There exists a constant* M_2 *such that*

$$\sup_{t\in[0,T]} |J_\lambda u_\lambda(t)|_H \leqslant M_2, \tag{15}$$

$$\sup_{t\in[0,T]} \varphi^1(J_\lambda u_\lambda(t)) \leqslant M_2, \tag{16}$$

$$\sup_{t\in[0,T]} |J_\lambda u_\lambda(t)|_V \leqslant M_2, \tag{17}$$

$$\int_0^T \left|\frac{d}{dt} J_\lambda u_\lambda(t)\right|_H^2 dt \leqslant M_2. \tag{18}$$

Proof of Lemma 2. Since J_λ is nonexpansive in H (see [4, p. 102]), we can derive (15) from (7). By (A3), (7) and (8) yield (16), which together with (A1) and (15) implies (17). Moreover since $|J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t)|_H/h \leqslant |u_\lambda(t+h) - u_\lambda(t)|_H/h$ for all $h \in \mathbb{R}$ with $t+h \in [0,T]$, we have

$$\int_0^T \left|\frac{d}{dt} J_\lambda u_\lambda(t)\right|_H^2 dt \leqslant \int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt,$$

which together with (9) implies (18). □

Lemma 3. *There exists a constant* M_3 *such that*

$$\sup_{t\in[0,T]} \left|\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t))\right|_{V^*} \leqslant M_3, \tag{19}$$

$$\int_0^T \left|g^1_\lambda(t)\right|_{V^*}^2 dt \leqslant M_3, \tag{20}$$

where $g^1_\lambda(t) = f_\lambda(t) - du_\lambda(t)/dt + \partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t)) \in \partial_H \varphi^1_H(u_\lambda(t))$.

Proof of Lemma 3. Since $J_\lambda u_\lambda(t) \in D(\partial_H \tilde{\varphi}^2) \cap V$ for all $t \in [0,T]$, we get $\partial_H \tilde{\varphi}^2(J_\lambda u_\lambda(t)) \subset \partial \varphi^2(J_\lambda u_\lambda(t))$ for all $t \in [0,T]$. Furthermore, since $\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(\cdot)) \in \partial_H \tilde{\varphi}^2(J_\lambda u_\lambda(\cdot))$ (see [4, p. 104]), it follows from (A2), (15), (16) and (18) that

$$\{\partial_H \tilde{\varphi}^2_\lambda(u_\lambda(\cdot))\} \text{ forms a precompact subset of } C([0,T]; V^*), \tag{21}$$

which yields (19).

Since f_λ is bounded in $W^{1,p'}(0,T;V^*)$ and $g^1_\lambda(t) = f_\lambda(t) - du_\lambda(t)/dt + \partial_H \tilde{\varphi}^2_\lambda(u_\lambda(t))$ for a.e. $t \in (0,T)$, (9) and (19) imply (20). □

From Lemmas 1–3, we can extract a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and the following Lemmas 4–6 hold.

Lemma 4. *There exists $u \in C_w([0,T];V) \cap W^{1,2}(0,T;H)$ such that*

$$u_{\lambda_n} \to u \quad \text{weakly in } L^2(0,T;V) \cap W^{1,2}(0,T;H), \tag{22}$$

$$u_{\lambda_n}(t) \to u(t) \quad \text{weakly in } H \text{ for all } t \in [0,T], \tag{23}$$

$$J_{\lambda_n} u_{\lambda_n} \to u \quad \text{weakly in } L^2(0,T;V) \cap W^{1,2}(0,T;H). \tag{24}$$

Moreover $u(t) \to u_0$ strongly in H as $t \to +0$.

Proof of Lemma 4. Since H and V are reflexive, (7), (9) and (10) imply (22), which also yields $u \in C([0,T];H)$. Moreover, let $q \in [1,+\infty)$ be fixed. Then by (7), we can extract a subsequence $\{\lambda^q_n\}$ of $\{\lambda_n\}$ depending on q such that $u_{\lambda^q_n} - u_0 \to u - u_0$ weakly in $L^q(0,T;H)$. Hence it is obvious that $u_{\lambda^q_n} - u_0 \to u - u_0$ weakly in $L^q(0,t;H)$ for any $t \in [0,T]$. Therefore since $u_{\lambda^q_n}(0) = u_0$, it follows from (9) that

$$\begin{aligned} \|u-u_0\|_{L^q(0,t;H)} &\leqslant \liminf_{\lambda^q_n \to 0} \|u_{\lambda^q_n} - u_0\|_{L^q(0,t;H)} \\ &\leqslant \liminf_{\lambda^q_n \to 0} \left\{ \int_0^t \left(\int_0^\tau \left| \frac{du_{\lambda^q_n}}{ds}(s) \right|_H^2 ds \right)^{q/2} \tau^{q/2}\, d\tau \right\}^{1/q} \\ &\leqslant M_1^{1/2} \left(\frac{2}{q+2} \right)^{1/q} t^{(1/2+1/q)}. \end{aligned}$$

Thus we have

$$\begin{aligned} |u(t)-u_0|_H &\leqslant \sup_{\tau \in [0,t]} |u(\tau) - u_0|_H \\ &= \lim_{q \to +\infty} \|u-u_0\|_{L^q(0,t;H)} \leqslant M_1^{1/2} t^{1/2} \quad \text{for all } t \in [0,T], \end{aligned}$$

which implies $u(t) \to u_0$ strongly in H as $t \to +0$.

Now let $t \in [0,T]$ be fixed. Since $u_{\lambda_n}(0) = u(0) = u_0$, (22) shows that

$$\begin{aligned} \big(u_{\lambda_n}(t) - u(t), \phi\big)_H &= \int_0^t \left(\frac{du_{\lambda_n}}{d\tau}(\tau) - \frac{du}{d\tau}(\tau), \phi \right)_H d\tau \\ &\to 0 \quad \forall \phi \in H, \ \forall t \in [0,T], \end{aligned}$$

which yields (23).

By (10) and (23), for any $t \in [0, T]$, we can take a subsequence $\{\lambda_n^t\}$ of $\{\lambda_n\}$ depending on t such that

$$u_{\lambda_n^t}(t) \to u(t) \quad \text{weakly in } V. \tag{25}$$

It then follows from (10) that $|u(t)|_V \leqslant \liminf_{\lambda_n^t \to 0} |u_{\lambda_n^t}(t)|_V \leqslant M_1$, where M_1 is independent of t. Therefore we conclude that $u(t) \in V$ for all $t \in [0, T]$ and $\sup_{t\in[0,T]} |u(t)|_V \leqslant M_1 < +\infty$. Hence, for all $t \in [0, T]$ and $\{t_n\}$ with $t_n \to t$ as $n \to +\infty$, there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and $w \in V$ such that $u(t_{n_k}) \to w$ weakly in V as $n_k \to +\infty$. On the other hand, $u(t_{n_k}) \to u(t)$ strongly in H as $n_k \to +\infty$, since $u \in C([0, T]; H)$. Then, by virtue of (1), we find $w = u(t)$, whence follows $u \in C_w([0, T]; V)$.

By (17) and (18), there exists $v \in L^2(0, T; V) \cap W^{1,2}(0, T; H)$ such that $J_{\lambda_n} u_{\lambda_n} \to v$ weakly in $L^2(0, T; V) \cap W^{1,2}(0, T; H)$. Here, by (19), we notice that

$$|u_{\lambda_n}(t) - J_{\lambda_n} u_{\lambda_n}(t)|_{V^*} = \lambda_n |\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(t))|_{V^*} \leqslant \lambda_n M_3$$

for all $t \in [0, T]$, which implies $u_{\lambda_n} - J_{\lambda_n} u_{\lambda_n} \to 0$ strongly in $C([0, T]; V^*)$ as $\lambda_n \to 0$. Hence it follows from (22) that $v = u$. $\square$

Lemma 5. *There exists $g^2 \in C([0, T]; V^*)$ such that*

$$\begin{aligned} &\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(\cdot)) \to g^2 \ \textit{strongly in } C([0, T]; V^*) \\ &\textit{and} \quad g^2(t) \in \partial\varphi^2(u(t)) \ \textit{for a.e. } t \in (0, T). \end{aligned} \tag{26}$$

Proof of Lemma 5. By (21), there exists $g^2 \in C([0, T]; V^*)$ such that $\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(\cdot)) \to g^2$ strongly in $C([0, T]; V^*)$. Hence since $\partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(t)) \in \partial_H \tilde{\varphi}^2(J_{\lambda_n} u_{\lambda_n}(t)) \subset \partial\varphi^2(J_{\lambda_n} u_{\lambda_n}(t))$, by the demiclosedness of maximal monotone operators (see e.g. [3, Chapter II]) and Proposition 1.1 of [7], it follows from (24) that $g^2(t) \in \partial\varphi^2(u(t))$ for a.e. $t \in (0, T)$. $\square$

Lemma 6. *There exists $g^1 \in L^2(0, T; V^*)$ such that*

$$\begin{aligned} &g^1_{\lambda_n} \to g^1 \ \textit{weakly in } L^2(0, T; V^*) \\ &\textit{and} \quad g^1(t) = f(t) + g^2(t) - \frac{du}{dt}(t) \in \partial\varphi^1(u(t)) \ \textit{for a.e. } t \in (0, T). \end{aligned} \tag{27}$$

Proof of Lemma 6. By (20), it is obvious that there exists $g^1 \in L^2(0,T;V^*)$ such that

$$g^1_{\lambda_n} \to g^1 \text{ weakly in } L^2(0,T;V^*). \tag{28}$$

Moreover, by $(\mathrm{CP})_{\lambda_n}$, it follows from (22) and (26) that $g^1 = f + g^2 - du/dt$.

Hence it remains to prove that $f(t)+g^2(t)-du(t)/dt \in \partial\varphi^1(u(t))$ for a.e. $t \in (0,T)$. To do this, integrating the product of $g^1_{\lambda_n}(t)$ and $u_{\lambda_n}(t)$ over $(0,T)$, we get by $(\mathrm{CP})_{\lambda_n}$,

$$\begin{aligned}\int_0^T \langle g^1_{\lambda_n}(t), u_{\lambda_n}(t)\rangle\, dt &= \int_0^T \langle f_{\lambda_n}(t), u_{\lambda_n}(t)\rangle\, dt + \int_0^T \langle \partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(t)), u_{\lambda_n}(t)\rangle\, dt \\ &\quad -\frac{1}{2}\,|u_{\lambda_n}(T)|^2_H + \frac{1}{2}\,|u_0|^2_H.\end{aligned}$$

Since $f_{\lambda_n} \to f$ strongly in $W^{1,p'}(0,T;V^*)$, it follows from (22), (23) and (26) that

$$\begin{aligned}&\limsup_{\lambda_n\to 0} \int_0^T \langle g^1_{\lambda_n}(t), u_{\lambda_n}(t)\rangle\, dt \\ &\quad= \lim_{\lambda_n\to 0}\int_0^T \langle f_{\lambda_n}(t), u_{\lambda_n}(t)\rangle\, dt + \lim_{\lambda_n\to 0}\int_0^T \langle \partial_H \tilde{\varphi}^2_{\lambda_n}(u_{\lambda_n}(t)), u_{\lambda_n}(t)\rangle\, dt \\ &\qquad -\frac{1}{2}\liminf_{\lambda_n\to 0}\,|u_{\lambda_n}(T)|^2_H + \frac{1}{2}|u_0|^2_H \\ &\quad\leqslant \int_0^T \langle f(t), u(t)\rangle\, dt + \int_0^T \langle g^2(t), u(t)\rangle\, dt - \frac{1}{2}|u(T)|^2_H + \frac{1}{2}\,|u_0|^2_H \\ &\quad= \int_0^T \left\langle f(t) + g^2(t) - \frac{du}{dt}(t), u(t)\right\rangle dt.\end{aligned} \tag{29}$$

By Lemma 1.3 of [3, Chapter II] and Proposition 1.1 of [7], it follows from (22) and (28) that $g^1(t) = f(t) + g^2(t) - du(t)/dt \in \partial\varphi^1(u(t))$ for a.e. $t \in (0,T)$. $\square$

Now, let $t \in [0,T]$ be arbitrarily fixed. Then since $\varphi^1 \in \Phi(V)$, (8) and (25) imply $\varphi^1(u(t)) \leqslant \liminf_{\lambda^t_n\to 0} \varphi^1(u_{\lambda^t_n}(t)) \leqslant M_1$, where M_1 is independent of t. Hence we conclude that $u(t) \in D(\varphi^1)$ for all $t \in [0,T]$ and $\sup_{t\in[0,T]} \varphi^1(u(t)) \leqslant M_1 < +\infty$. Moreover let $\{t_n\}$ be a sequence in $[0,T]$ such that $t_n \to t$. From the fact that $u \in C_w([0,T];V)$, it follows that $u(t_n) \to u(t)$ weakly in V. Since $\varphi^1(u(t_n)) \leqslant \sup_{t\in[0,T]} \varphi^1(u(t)) \leqslant M_1$, where M_1 is independent of n, Proposition 3 assures that $\varphi^2(u(t_n)) \to \varphi^2(u(t))$, whence follows $\varphi^2(u(\cdot)) \in C([0,T])$.

Finally we provide an energy estimate for the strong solution u. To this end, we claim that

$$\varphi^2(u_{\lambda_n^t}(t)) \to \varphi^2(u(t)) \quad \forall t \in [0,T].$$

Indeed, let $t \in [0,T]$ be fixed. Then, by (8), (25) and the fact that $u_{\lambda_n^t}(t) \in D(\varphi^1)$, Proposition 3 assures the assertion above. Hence putting $\lambda = \lambda_n^t$ in (12) and noting that $\tilde{\varphi}^2_{\lambda_n^t}(u_0) \to \varphi^2(u_0)$ as $\lambda_n^t \to 0$, we obtain

$$\begin{aligned}
&\int_0^t \left|\frac{du}{d\tau}(\tau)\right|_H^2 d\tau + \varphi^1(u(t)) + \varphi^2(u_0) \\
&\quad \leqslant \varphi^1(u_0) + \varphi^2(u(t)) + \langle f(t), u(t)\rangle - \langle f(0), u_0\rangle - \int_0^t \left\langle \frac{df}{d\tau}(\tau), u(\tau)\right\rangle d\tau.
\end{aligned}$$

This completes the proof. $\square$

3.2. Proof of Theorem 2

To prove Theorem 2, we need another type of auxiliary problem:

$$(\mathrm{CP})^r \begin{cases} \dfrac{du}{dt}(t) + \partial\varphi^{1,r}(u(t)) - \partial\varphi^2(u(t)) \ni f(t) & \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

Here $r \in \mathbb{R}$ is chosen so that $r > \varphi^2(u_0)$ and $\varphi^{1,r}$ denotes the cut-off function of φ^1 given by

$$\varphi^{1,r}(u) = \begin{cases} \varphi^1(u) \text{ if } \varphi^2(u) \leqslant r, \\ +\infty \quad \text{otherwise.} \end{cases}$$

Then it is easy to see that $\varphi^{1,r} \in \Phi(V)$ and $D(\varphi^{1,r}) = D(\varphi^1) \cap \{u \in V; \varphi^2(u) \leqslant r\}$ and that (A1) and (A2) are satisfied with φ^1 replaced by $\varphi^{1,r}$. Since (A3) assures that $J_\lambda D(\varphi^1) \subset D(\varphi^1)$, we find by Proposition 1 that $\varphi^2(J_\lambda u) = \tilde{\varphi}^2(J_\lambda u) \leqslant \tilde{\varphi}^2(u) = \varphi^2(u) \leqslant r$ for all $u \in D(\varphi^{1,r})$, which implies $J_\lambda u \in D(\varphi^{1,r})$ and $\varphi^{1,r}(J_\lambda u) = \varphi^1(J_\lambda u)$. Hence (A3) is satisfied with φ^1 replaced by $\varphi^{1,r}$. Furthermore, since $\varphi^2(u) \leqslant r$ for all $u \in D(\varphi^{1,r})$, (A4) is satisfied with $k = 0$, $C_4 = 0$, $C_5 = r$ and $\varphi^1 = \varphi^{1,r}$. Noting that $\varphi^2(u_0) < r$ and $u_0 \in D(\varphi^1)$ yield $u_0 \in D(\varphi^{1,r})$, we observe that Theorem 1 assures the existence of strong solution of $(\mathrm{CP})^r$ on $[0,T]$ as follows:

Lemma 7. *Assume that* (A1), (A2) *and* (A3) *are satisfied. Then for all* $u_0 \in D(\varphi^1)$, $f \in W^{1,p'}(0,T;V^*)$ *and* $r \in \mathbb{R}$ *with* $r > \varphi^2(u_0)$, $(\mathrm{CP})^r$ *has a strong solution* u *on*

$[0, T]$ *satisfying* (5) *with* φ^1 *replaced by* $\varphi^{1,r}$ *and the following inequality*:

$$\int_0^t \left| \frac{du}{dt}(\tau) \right|_H^2 d\tau + \varphi^{1,r}(u(t)) + \varphi^2(u_0)$$
$$\leqslant \varphi^1(u_0) + \varphi^2(u(t)) + \langle f(t), u(t) \rangle - \langle f(0), u_0 \rangle - \int_0^t \left\langle \frac{df}{d\tau}(\tau), u(\tau) \right\rangle d\tau \tag{30}$$

for all $t \in [0, T]$.

Now we are going to show that $u(t)$ becomes a strong solution of (CP) on $[0, T_0]$ for some $T_0 > 0$. To do this, it is sufficient to prove that there exists a number $T_0 \in (0, T]$ such that $\partial\varphi^{1,r}(u(t)) = \partial\varphi^1(u(t))$ for a.e. $t \in (0, T_0)$. To this end, we prepare a couple of lemmas.

Lemma 8. *If* $u \in D(\partial\varphi^{1,r})$ *and* $\varphi^2(u) < r$, *then* $u \in D(\partial\varphi^1)$ *and* $\partial\varphi^{1,r}(u) = \partial\varphi^1(u)$.

Proof of Lemma 8. Let $[u, \xi] \in \partial\varphi^{1,r}$ be such that $\varphi^2(u) < r$ and take an arbitrary element $v \in D(\varphi^1)$. Then since $u_s := (1-s)u + sv \in D(\varphi^1)$, $\varphi^1(u_s) \leqslant (1-s)\varphi^1(u) + s\varphi^1(v) \leqslant |\varphi^1(u)| + |\varphi^1(v)|$ for all $s \in [0, 1]$ and $u_s \to u$ strongly in V as $s \to 0$, Proposition 3 assures that $\varphi^2(u_s) \to \varphi^2(u)$ as $s \to 0$. Hence from the fact that $\varphi^2(u) < r$, there exists a number $s_0 \in (0, 1)$ such that $\varphi^2(u_{s_0}) \leqslant r$. Since $u_{s_0} \in D(\varphi^{1,r})$, we get $\varphi^1(u_{s_0}) - \varphi^1(u) = \varphi^{1,r}(u_{s_0}) - \varphi^{1,r}(u) \geqslant \langle \xi, u_{s_0} - u \rangle$. Hence, by the convexity of φ^1, we have $s_0(\varphi^1(v) - \varphi^1(u)) \geqslant \langle \xi, s_0(v - u) \rangle$. By dividing both sides by $s_0 > 0$, we deduce $\varphi^1(v) - \varphi^1(u) \geqslant \langle \xi, v - u \rangle$ for all $v \in D(\varphi^1)$, whence follows $u \in D(\partial\varphi^1)$ and $\xi \in \partial\varphi^1(u)$.

On the other hand, it is obvious that $\partial\varphi^1(u) \subset \partial\varphi^{1,r}(u)$ for all $u \in D(\partial\varphi^{1,r})$ with $\varphi^2(u) < r$, which completes the proof. $\square$

Lemma 9. *There exists a number* $T_0 \in (0, T]$ *such that* $\varphi^2(u(t)) < r$ *for all* $t \in [0, T_0)$.

Proof of Lemma 9. For the case where $\max_{t \in [0,T]} \varphi^2(u(t)) < r$, we can take $T_0 = T$. For the case where $\max_{t \in [0,T]} \varphi^2(u(t)) \geqslant r$, since $\varphi^2(u(\cdot)) \in C([0, T])$ and $\varphi^2(u_0) < r$, there exists a number $T_0 \in (0, T]$ such that $\varphi^2(u(t))$ attains r at $t = T_0$ for the first time. $\square$

By Lemmas 8 and 9, there exists a number $T_0 \in (0, T]$ such that $u(t) \in D(\partial\varphi^1)$ and $\partial\varphi^{1,r}(u(t)) = \partial\varphi^1(u(t))$ for a.e. $t \in (0, T_0)$. Consequently we deduce that u becomes a strong solution of (CP) on $[0, T_0]$. Thus the proof of Theorem 2 is completed. $\square$

3.3. Proof of Theorem 3

We first note that $\varphi^i \geqslant 0$ $(i = 1, 2)$ by assumptions of Theorem 3. Moreover from the assumption on l_3 in (A5), we can take a number $\delta_1 > \delta_0$ such that $\max_{x\in[0,\delta_1]} l_3(x) \leqslant (\alpha + \alpha_0)/2 \in (\alpha_0, \alpha)$, where $\alpha_0 := \max_{x\in[0,\delta_0]} l_3(x) < \alpha$. Hence, by (A5), it follows that:

$$\frac{\alpha - \alpha_0}{2}\varphi^1(u) \leqslant \langle \xi - \eta, u\rangle \tag{31}$$

for all $[u, \xi] \in \partial\varphi^1$ and $[u, \eta] \in \partial\varphi^2$ satisfying $u \in D^2_{\delta_1} := \{u \in D(\varphi^2); \varphi^2(u) \leqslant \delta_1\}$.

Put $r = \delta_1$ and recall the auxiliary problem $(\mathrm{CP})^r$. Moreover define

$$X^T_{\delta,R} := \Bigg\{(u_0, f) \in D(\varphi^1) \times W^{1,p'}(0, T; V^*);$$

$$\varphi^1(u_0) + \int_0^T |f(\tau)|^{p'}_{V^*}\, d\tau + \int_0^T \left|\frac{df}{d\tau}(\tau)\right|^{p'}_{V^*} d\tau \leqslant R,$$

$$\varphi^2(u_0) < \delta_0,\ \ |u_0|_H + \left\{\max\left(1, \frac{1}{T}\right) \left\| |f(\cdot)|^{p'}_{V^*} \right\|_{1,T}\right\}^{1/p} < \delta\Bigg\}$$

for all $\delta,\ R,\ T > 0$, where

$$\left\| |f(\cdot)|^{p'}_{V^*} \right\|_{1,T} := \begin{cases} \displaystyle\int_0^T |f(\tau)|^{p'}_{V^*}\, d\tau & \text{if } T < 1, \\ \displaystyle\sup_{t\in[1,T]} \int_{t-1}^t |f(\tau)|^{p'}_{V^*}\, d\tau & \text{if } T \geqslant 1 \end{cases}$$

and $S^T_{\delta,R} := \{u \in C_w([0, T]; V) \cap W^{1,2}(0, T; H); u$ is a strong solution of $(\mathrm{CP})^r$ on $[0, T]$ satisfying (30) and $\varphi^2(u(\cdot)) \in C([0, T])$ with $(u_0, f) \in X^T_{\delta,R}\}$. Here, by Lemma 7, we note that $S^T_{\delta,R} \neq \emptyset$ when $X^T_{\delta,R} \neq \emptyset$. We then define $T_r(u) := \sup\{T_0 \in (0, T]; \varphi^2(u(t)) < r$ for all $t \in [0, T_0]\}$ for all $u \in S^T_{\delta,R}$.

Now by Lemma 8, to complete the proof, it suffices to show that

$$\forall R > 0,\ \exists \delta_R > 0;\ \forall T > 0,\ \forall u \in S^T_{\delta_R,R},\ T_r(u) = T, \tag{32}$$

where we note that δ_R is independent of T. Suppose that the above claim were false, i.e.,

$$\exists R_0 > 0;\ \forall \delta > 0\ \exists T_\delta > 0,\ \exists u_\delta \in S^{T_\delta}_{\delta,R_0};\ \ T_r(u_\delta) < T_\delta,$$

which implies $\varphi^2(u_\delta(T_r(u_\delta))) = r$ and $\varphi^2(u_\delta(t)) < r$ for all $t < T_r(u_\delta)$.

In particular, by taking $\delta = 1/n$ for each $n \in \mathbb{N}$, we put $v_n := u_{1/n} \in S^{T_{1/n}}_{1/n,R_0}$ and $T_{r,n} := T_r(u_{1/n})$. We then find that v_n becomes a strong solution of the following Cauchy problem $(\mathrm{CP})_n$ on $[0, T_{r,n}]$:

$$(\mathrm{CP})_n \begin{cases} \dfrac{dv_n}{dt}(t) + \partial\varphi^1(v_n(t)) - \partial\varphi^2(v_n(t)) \ni f_n(t) \quad \text{in } V^*, \quad 0 < t < T_{r,n}, \\ v_n(0) = u_{0,n}, \end{cases}$$

where $(u_{0,n}, f_n) \in X^{T_{1/n}}_{1/n,R_0}$.

Multiplying $(\mathrm{CP})_n$ by $v_n(t)$ and using (31), we obtain

$$\frac{1}{2}\frac{d}{dt}|v_n(t)|_H^2 + \frac{\alpha - \alpha_0}{2}\varphi^1(v_n(t)) \leqslant \langle f_n(t), v_n(t)\rangle \quad \text{for a.e. } t \in (0, T_{r,n}), \tag{33}$$

since $v_n(t) \in D^2_{\delta_1}$ for all $t \in [0, T_{r,n})$. Hence, by (A1) and (1), it follows that

$$\frac{1}{2}\frac{d}{dt}|v_n(t)|_H^2 + \tilde{\alpha}|v_n(t)|_H^p \leqslant C|f_n(t)|_{V^*}^{p'} \quad \text{for a.e. } t \in (0, T_{r,n}),$$

where $\tilde{\alpha}$ and C denote positive constants independent of n. Then, by Lemma 4.3 of [2], we have

$$|v_n(T_{r,n})|_H \leqslant \sup_{t\in[0,T_{r,n}]} |v_n(t)|_H \leqslant l\left(|u_{0,n}|_H + \left\| |f_n(\cdot)|_{V^*}^{p'} \right\|_{1,T_{r,n}}^{1/p}\right) \leqslant l\left(\frac{1}{n}\right),$$

where $l(\cdot)$ is a monotone increasing function independent of n satisfying $\lim_{x\to 0} l(x) = 0$. Therefore we find

$$v_n(T_{r,n}) \to 0 \quad \text{strongly in } H \text{ as } n \to +\infty. \tag{34}$$

On the other hand, integrating (33) over $(0, T_{r,n})$ and using (A1), we get

$$\begin{aligned} \frac{1}{2}|v_n(T_{r,n})|_H^2 + \frac{\alpha - \alpha_0}{4}\int_0^{T_{r,n}} \varphi^1(v_n(\tau))\,d\tau &\leqslant \frac{1}{2}|u_{0,n}|_H^2 + C\int_0^{T_{r,n}} |f_n(\tau)|_{V^*}^{p'}\,d\tau \\ &\leqslant \frac{1}{2n^2} + CR_0, \end{aligned} \tag{35}$$

whence follows

$$\int_0^{T_{r,n}} \varphi^1(v_n(\tau))\,d\tau \leqslant M_4, \tag{36}$$

where M_4 denotes a constant independent of n.

Since $(u_{0,n}, f_n) \in X_{1/n,R_0}^{T_{1/n}}$, we can show

$$\sup_{t\in[0,T_{1/n}]} |f_n(t)|_{V^*}^{p'} \leqslant M_5, \tag{37}$$

where M_5 denotes a constant independent of n. Indeed, we first note that the Sobolev-type embedding theorem assures that $f_n \in W^{1,p'}(0,T_{1/n};V^*) \subset C([0,T_{1/n}];V^*)$. Hence there exists $t_n \in [0, T_{1/n}]$ such that

$$|f_n(t_n)|_{V^*} = \min_{t\in[0,T_{1/n}]} |f_n(t)|_{V^*},$$

so

$$T_{1/n}|f_n(t_n)|_{V^*}^{p'} \leqslant \int_0^{T_{1/n}} |f_n(\tau)|_{V^*}^{p'}\,d\tau.$$

For the case where $T_{1/n} \geqslant 1$, it then follows that

$$\begin{aligned}
|f_n(t)|_{V^*}^{p'} &= |f_n(t_n)|_{V^*}^{p'} + \int_{t_n}^{t} \frac{d}{dt}|f_n(\tau)|_{V^*}^{p'}\,d\tau \\
&\leqslant \frac{1}{T_{1/n}}\int_0^{T_{1/n}} |f_n(\tau)|_{V^*}^{p'}\,d\tau + p'\int_0^{T_{1/n}} |f_n(\tau)|_{V^*}^{p'-1}\left|\frac{df_n}{d\tau}(\tau)\right|_{V^*} d\tau \\
&\leqslant \int_0^{T_{1/n}} |f_n(\tau)|_{V^*}^{p'}\,d\tau \\
&\quad + p'\left(\int_0^{T_{1/n}} |f_n(\tau)|_{V^*}^{p'}\,d\tau\right)^{1/p}\left(\int_0^{T_{1/n}} \left|\frac{df_n}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau\right)^{1/p'} \\
&\leqslant CR_0 \quad \forall t \in [0, T_{1/n}],
\end{aligned}$$

where C denotes a constant independent of n. For the case where $T_{1/n} < 1$, noticing $(1/T_{1/n})\int_0^{T_{1/n}} |f_n(\tau)|_{V^*}^{p'}\,d\tau < (1/n)^p$, we can verify the same assertion above.

Therefore, by (A1), it follows from (30), (36) and (37) that

$$\begin{aligned}\varphi^1(v_n(T_{r,n})) &\leqslant C\left\{\varphi^1(u_{0,n}) + r + \int_0^{T_{r,n}} \varphi^1(v_n(\tau))\,d\tau\right.\\ &\quad \left. + |f_n(0)|_{V^*}^{p'} + |f_n(T_{r,n})|_{V^*}^{p'} + \int_0^{T_{r,n}} \left|\frac{df_n}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau\right\}\\ &\leqslant C\{R_0 + r + M_4 + M_5\}.\end{aligned}$$

Hence, by (A1), we find

$$\{v_n(T_{r,n})\} \text{ is bounded in } V. \tag{38}$$

Therefore, by (34) and (38), we can extract a subsequence $\{n'\}$ of $\{n\}$ such that $v_{n'}(T_{r,n'}) \to 0$ weakly in V. Moreover, by (A2), it follows from (38) that there exists a subsequence $\{n''\}$ of $\{n'\}$ and $g^2_{n''} \in \partial\varphi^2(v_{n''}(T_{r,n''}))$ such that $g^2_{n''} \to g^2$ strongly in V^* as $n'' \to +\infty$.

Since $u_{0,n''} \to 0$ weakly in V, we find that $\langle g^2_{n''}, v_{n''}(T_{r,n''}) - u_{0,n''}\rangle \to 0$. Hence there exists a number $N_0 \in \mathbb{N}$ such that $|\langle g^2_{N_0}, v_{N_0}(T_{r,N_0}) - u_{0,N_0}\rangle| < \delta_1 - \delta_0$. From the fact that $\varphi^2(u_{0,N_0}) < \delta_0$, it follows that

$$\varphi^2(v_{N_0}(T_{r,N_0})) \leqslant \varphi^2(u_{0,N_0}) + \langle g^2_{N_0}, v_{N_0}(T_{r,N_0}) - u_{0,N_0}\rangle < \delta_1 = r,$$

which contradicts the definition of $T_{r,N_0} = T_r(v_{N_0})$. Therefore we conclude that (32) holds true. $\square$

4. Application

In this section, we exemplify the applicability of our abstract results obtained in the present paper by discussing the existence of local or global (in time) solutions of (NHE). Here solutions of (NHE) mean:

Definition 2. A function $u \in C([0,T]; W^{-1,p'}(\Omega))$ is said to be a weak solution of (NHE) on $[0,T]$ if the following conditions are satisfied.

(i) $u(t)$ is a $W^{-1,p'}(\Omega)$-valued absolutely continuous function on [0,T].
(ii) $u(t) \to u_0$ strongly in $L^2(\Omega)$ as $t \to +0$.

(iii) $-\Delta_p u(t)$, $|u|^{q-2}u(t) \in W^{-1,p'}(\Omega)$ for a.e. $t \in (0,T)$ and the following holds true.

$$\left\langle \frac{\partial u}{\partial t}(\cdot,t), \phi \right\rangle_{W_0^{1,p}(\Omega)} + \int_\Omega |\nabla u|^{p-2}\nabla u(x,t)\cdot\nabla\phi(x)\,dx$$
$$-\left\langle |u|^{q-2}u(\cdot,t), \phi \right\rangle_{W_0^{1,p}(\Omega)} = \langle f(\cdot,t), \phi\rangle_{W_0^{1,p}(\Omega)}$$

for a.e. $t \in (0,T)$ and for all $\phi \in W_0^{1,p}(\Omega)$.

The existence of local or global solutions of (NHE) is already studied by Tsutsumi [13] for the case $f(x,t) \equiv 0$ and by Ôtani [10,12] for the case $f \in L^2(0,T;L^2(\Omega))$. The argument in [13] is based on Faedo–Galerkin's method and requires the growth condition $q < 2p/(N+p)$ for the existence of local solutions, and $q < p^*$ for the existence of small global solutions, where $p^* = Np/(N-p)$ if $p < N$; $p^* = +\infty$ if $p \geqslant N$. On the other hand, the method in [10,12] is based on a nonmonotone perturbation theory for subdifferential operators in a real Hilbert space and [10] requires the growth condition $q < p^*/2+1$ for the existence of local and small global solutions. As for the semilinear case $p = 2$, however, it is shown in [12] that (NHE) admits local solution and small global solution under the subcritical growth condition $q < 2^*$.

Since the abstract setting in [12] as well as in [10] is chosen in the Hilbert space and the knowledge of elliptic estimate for Δ_p in $L^2(\Omega)$ is insufficient, [10,12] could not assure the existence of local solutions of (NHE) under the subcritical growth condition $q < p^*$.

Nevertheless, it is quite natural to conjecture that (NHE) should admit local solutions in a suitable space (larger than $L^2(\Omega)$) under the subcritical growth condition $q < p^*$, which has been left as an open problem for long time. It would be noteworthy that our abstract framework enables us to give an affirmative answer to this open problem (see Theorem 4 below).

In order to reduce (NHE) to (CP), we choose $V = W_0^{1,p}(\Omega)$ and $H = L^2(\Omega)$ with norms $|\cdot|_V := |\nabla\cdot|_{L^p(\Omega)}$ and $|\cdot|_H := |\cdot|_{L^2(\Omega)}$, respectively. We further put

$$\varphi_p(u) = \frac{1}{p}\int_\Omega |\nabla u(x)|^p\,dx, \quad \psi_q(u) = \frac{1}{q}\int_\Omega |u(x)|^q\,dx \quad \forall u \in V.$$

Here we assume that

$$(\mathrm{C})_{p,q}\ \frac{2N}{N+2} \leqslant p < +\infty \quad \text{and} \quad 1 < q < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } N > p, \\ +\infty & \text{if } N \leqslant p. \end{cases}$$

Then it is easy to see that (1) is satisfied and V is compactly embedded in $L^q(\Omega)$ (see [1]). Hence φ_p and ψ_q belong to $\Phi(V)$, $\partial\varphi_p(u)$ and $\partial\varphi_q(u)$ coincide with $-\Delta_p u$ and

$|u|^{q-2}u$ respectively in the distribution sense, where $D(\varphi_p) = D(\partial\varphi_p) = D(\psi_q) = D(\partial\psi_q) = V$. Thus (NHE) is reduced to (CP) with $\varphi^1 = \varphi_p$ and $\varphi^2 = \psi_q$. Moreover (A1)–(A3) are all assured by the following lemma.

Lemma 10. *Assume* $(\mathrm{CP})_{p,q}$ *is satisfied, then* (A1), (A2) *and* (A3) *hold true with* $\varphi^1 = \varphi_p$, $\varphi^2 = \psi_q$ *and* $C_1 = C_2 = 0$.

Proof of Lemma 10. Since $\varphi_p(u) = |u|_V^p/p$, (A1) with $C_1 = C_2 = 0$ and $C_3 = p$ follows at once. To check (A2), take any sequence $\{u_n\}$ satisfying $\sup_{t\in[0,T]}\{\varphi_p(u_n(t)) + |u_n(t)|_H\} + \int_0^T |du_n(t)/dt|_H^2\, dt \leqslant C$. Then, since V is compactly embedded in $L^q(\Omega)$ and $|u_n(t) - u_n(s)|_H \leqslant \|du_n/dt\|_{L^2(0,T;H)}|t-s|^{1/2}$, $\{u_n(t)\}$ forms a precompact set in $L^q(\Omega)$ for all $t \in [0,T]$ and an equi-continuous set in $C([0,T];H)$. Moreover, by virtue of Gagliardo–Nirenberg's inequality: $|u|_{L^q(\Omega)} \leqslant C|u|_H^\theta |u|_V^{1-\theta}$, $\theta \in (0,1)$, $\forall u \in V$, we observe that $\{u_n(t)\}$ is also equi-continuous in $C([0,T];L^q(\Omega))$. Therefore by Ascoli's lemma, there exists a subsequence $\{n'\}$ of $\{n\}$ such that $u_{n'} \to u$ strongly in $C([0,T];L^q(\Omega))$, whence easily follows:

$$|u_{n'}|^{q-2}u_{n'}(\cdot) \to |u|^{q-2}u(\cdot) \quad \text{strongly in } C([0,T];L^{q'}(\Omega)).$$

Hence $\partial\psi_q(u_{n'}(\cdot)) \to \partial\psi_q(u(\cdot))$ strongly in $C([0,T];V^*)$.

As for (A3), we put $\tilde{\varphi}^2(u) = \varphi^2(u)$ if $u \in V$; $\tilde{\varphi}^2(u) = +\infty$ if $u \in H \setminus V$. Then it is easily seen that $\tilde{\varphi}^2 \in \Phi(H)$ and $\tilde{\varphi}^2|_V = \varphi^2$. Furthermore, since the mapping $r \in \mathbb{R} \mapsto (I + \lambda\partial_H\tilde{\varphi}^2)r = J_\lambda r$ becomes nonexpansive on $\mathbb{R}$, we find that $|\nabla J_\lambda u(x)| \leqslant |\nabla u(x)|$ holds for a.e. $x \in \Omega$. Hence $\varphi^1(J_\lambda u) \leqslant \varphi^1(u)$ which implies (A3) (see the proof for Corollary 16 of [4]). $\square$

4.1. *The case where $p \leqslant q$ and $u_0 \in W_0^{1,p}(\Omega)$*

By applying Theorems 2 and 3, we obtain the following Theorems 4 and 5.

Theorem 4 (*Local existence*). *Assume* $(\mathrm{C})_{p,q}$ *holds and* $p \leqslant q$. *Then, for all* $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in W^{1,p'}(0,T;W^{-1,p'}(\Omega))$, *there exists a number* $T_0 \in (0,T]$ *such that* (NHE) *has a weak solution* u *on* $[0,T_0]$ *satisfying*:

$$u \in C_w([0,T_0];W_0^{1,p}(\Omega)) \cap C([0,T_0];L^q(\Omega)) \cap W^{1,2}(0,T_0;L^2(\Omega)). \tag{39}$$

Proof of Theorem 4. By Lemma 10 and Theorem 2, there exists a number $T_0 \in (0,T]$ such that (NHE) has a solution u on $[0,T_0]$. Moreover since $\psi_q(u(\cdot)) \in C([0,T_0])$, the uniformly convexity of $L^q(\Omega)$ ensures $u \in C([0,T_0];L^q(\Omega))$. $\square$

Theorem 5 (*Global existence*). *Assume* (C)$_{p,q}$ *holds and* $p < q$. *Let* R *be an arbitrary positive number, and let* δ *be a positive number such that* $\delta < C(p,q)^{-p/(q-p)}$, *where* $C(p,q)$ *denotes the best possible constant for the Sobolev–Poincaré-type inequality*: $|u|_{L^q(\Omega)} \leqslant C(p,q)|u|_V$. *Then there exists a positive number* δ_R *independent of* T *such that if* u_0 *and* f *satisfy*

$$\frac{1}{p}|u_0|_V^p + \int_0^T |f(\tau)|_{V^*}^{p'}\, d\tau + \int_0^T \left|\frac{df}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau \leqslant R,$$

$$|u_0|_{L^q(\Omega)} < \delta, \;\; |u_0|_{L^2(\Omega)} + \left\{\max\left(1, \frac{1}{T}\right) \left\| |f(\cdot)|_{V^*}^{p'} \right\|_{1,T}\right\}^{1/p} < \delta_R,$$

then (NHE) *has a weak solution* u *on* $[0,T]$ *satisfying* (39) *with* T_0 *replaced by* T.

Proof of Theorem 5. By the Sobolev–Poincaré-type inequality, it follows that $|u|_{L^q(\Omega)} \leqslant C(p,q)|u|_V$ for all $u \in V$. Hence we find that

$$\begin{aligned}\langle \partial\varphi_p(u) - \partial\psi_q(u), u\rangle &= |u|_V^p - |u|_{L^q(\Omega)}^q \\ &\geqslant |u|_V^p - C(p,q)^p |u|_V^p |u|_{L^q(\Omega)}^{q-p} \\ &= p\varphi_p(u)\left[1 - C(p,q)^p \left\{q\psi_q(u)\right\}^{(q-p)/q}\right],\end{aligned}$$

which implies

$$p\varphi_p(u) \leqslant \langle \partial\varphi_p(u) - \partial\psi_q(u), u\rangle + pC(p,q)^p \left\{q\psi_q(u)\right\}^{(q-p)/q} \varphi_p(u)$$

for all $u \in V$. Therefore (A5) holds with $\alpha = p$, $l_3(r) = pC(p,q)^p (qr)^{(q-p)/q}$ and $\delta_0 = \delta^q/q < C(p,q)^{-pq/(q-p)}/q$. Thus Theorem 3 ensures the existence of weak solutions on $[0,T]$ for (NHE) when u_0 and f satisfy the suitable conditions above. $\square$

4.2. The case where $p > q$ and $u_0 \in W_0^{1,p}(\Omega)$

The case where $p > q$ can be covered by Theorem 1.

Theorem 6 (*Global existence*). *Assume* (C)$_{p,q}$ *holds and* $p > q$. *Then, for all* $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in W^{1,p'}(0,T; W^{-1,p'}(\Omega))$, (NHE) *has a weak solution* u *on* $[0,T]$ *satisfying* (39) *with* T_0 *replaced by* T.

Proof of Theorem 6. Conditions (A1)–(A3) are already assured by Lemma 10. Moreover, since $p > q$, we find

$$\psi_q(u) = \frac{1}{q}|u|^q_{L^q(\Omega)} \leqslant \frac{1}{q}\, C(p,q)^q |u|^q_V \leqslant \frac{1}{2}\,\varphi_p(u) + C \quad \forall u \in V,$$

which implies (A4) with $k = 1/2$. Therefore, by Theorem 1, (NHE) has a global weak solution on $[0, T]$. $\square$

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