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Invariant manifolds for stochastic wave equations ☆

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Abstract

In this paper, we consider a class of stochastic wave equations with nonlinear multiplicative noise. We first show that these stochastic wave equations generate random dynamical systems (or stochastic flows) by transforming the stochastic wave equations to random wave equations through a stationary random homeomorphism. Then, we establish the existence of random invariant manifolds for the random wave equations. Due to the temperedness of the nonlinearity, we obtain only local invariant manifolds no matter how large the spectral gap is unlike the deterministic cases. Based on these random dynamical systems, we prove the existence of random invariant manifolds in a tempered neighborhood of an equilibrium. Finally, we show that the images of these invariant manifolds under the inverse stationary transformation give invariant manifolds for the stochastic wave equations.
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1. Introduction

In this paper, we study the dynamical properties of the wave equations under random influences

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \mathcal{A}u + b(u) = c(u)\dot{\omega}(t), \quad t \in \mathbb{R},$$

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$$u(0,x)=u_{01}(x), \qquad \frac{\partial u(0,x)}{\partial t}=u_{02}(x), \quad x\in D,\ u(\cdot,x)|_{x\in\partial D}=0, \tag{1}$$

where D is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, $\mathcal{A}$ is a linear elliptic differential operator, b and c are nonlinear functions satisfying particular regularity assumptions and $\dot{\omega}$ presents a white noise.

When the noise is not taken into account, the dynamics of Eq. (1) have been extensively studied. See for example, Hale [15], Temam [27], and their references therein.

The theory of invariant manifolds plays a very important role in the study of dynamics of dynamical systems generated by differential equations arising in applications. This theory has been well developed for deterministic systems. For stochastic partial differential equations, due to their nonclassical fluctuation of driving noise and infinite dimensionality, the theory of invariant manifold is still in its infancy. Recently, there have been some work on invariant manifolds for stochastic partial differential equations with either additive noise or multiplicative noise, see [2, 3,12,13,20] for pathwise random invariant manifolds and see [8,11] for almost surely invariant manifolds.

The way to get pathwise invariant manifolds for a stochastic partial differential equation is to show that the equation generates a random dynamical system. The need for studying random dynamical systems was pointed out by Ulam and von Neumann [28] in 1945. It has flourished since 1980s due to the discovery that stochastic ordinary differential equations generate random dynamical systems through the efforts of Arnold, Harris, Elworthy, Baxendale, Bismut, Ikeda, Kunita, Watanabe, and others. For the theory of invariant manifolds for random dynamical systems, see Ruelle [26], Wanner [29], Liu and Qian [19], Arnold [1], Mohammed and Scheutzow [23], and their references therein.

However, the problem on the existence of random dynamical systems generated by *general* stochastic partial differential equations is still open. One reason is that the stochastic integral is only defined almost surely where the exceptional set may depend on the initial state and for infinite-dimensional systems one cannot apply Kolmogorov's theorem on the existence of a solution version which depends continuously on the initial conditions to ensure the stochastic flow property (see Kunita [22, Theorem 1.4.1]) and the perfection property (see Arnold [1]). These properties allow one to prove under the standard regularity conditions the existence of a random dynamical system generated by a stochastic ordinary differential equation. But, for stochastic partial differential equations, there are results on the existence of random dynamical systems only for equations with special diffusion terms such as pure white noise, see, for example, Flandoli and Lisei [14], Caraballo, Kloeden, and Schmalfuss [4]. For finite-dimensional systems with more general diffusions see Keller [17] and Keller and Schmalfuß [18].

A way to show the existence of a random dynamical system is to transform the stochastic differential equation to a random differential equation without white noise but with random coefficients since it is simpler to work with a random differential equation in the study of its dynamical behavior.

In this paper, we consider a stochastic wave equation with a relative general diffusion term (white noise) in a Hilbert space, which includes Eq. (1) as its special case. We first construct a stationary random transformation based on the Ornstein–Uhlenbeck process and the feature of the stochastic wave equation, which transforms the stochastic wave equation to a system of hyperbolic equations with random coefficients which are tempered functions. The solutions of this random hyperbolic system generate a random dynamical system consisting of the state motion in the state space and the measure preserving random flow on the random basis, which form a cocycle under their interaction. Furthermore, the inverse of this transformation carries a mild solution

of the random system to a version of mild solution of the stochastic wave equation for either $t \geqslant 0$ or $t \leqslant 0$, depending on the choice of the Ornstein–Uhlenbeck process, see (4) and (6). We then establish the existence of local random invariant manifolds for the random hyperbolic system by the Lyapunov–Perron's approach and a cut-off procedure. The images of these invariant manifolds under the inverse stationary transformation give invariant manifolds for the stochastic wave equations, which actually are adapted local random invariant manifolds.

It is known that the existence of invariant manifolds depends on the size of perturbations and the spectral gap of the linear part of equations. The size of perturbations we have here may grow at a subexponential rate along the random base flow. Thus, we will be able to obtain only invariant manifolds in a tempered neighborhood of an equilibrium. Since the size of a tempered neighborhood can shrink only at a subexponential rate, the solutions on invariant manifolds will stay in the tempered neighborhood for a relatively long period of time, especially for the unstable manifold the backward orbits will stay in the neighborhood for all past time.

We organize this paper as follows. In Section 2, we introduce basic concepts on random dynamical systems and the properties of the Ornstein–Uhlenbeck process. In Section 3, we establish the existence of random dynamical systems for the stochastic wave equations. We introduce the modified equations in Section 4 and discuss the properties of the modified nonlinearities. In Section 5, we first establish the existence of global invariant manifolds for the modified equations, then we prove the existence of local invariant manifolds for the original random system and the stochastic wave equations. As an application, we consider a scalar stochastic wave equation in Section 6.

2. Random dynamical systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. On Ω we define a measurable flow $\theta = (\theta_t)_{t\in\mathbb{R}}$

$$\theta : \big(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R})\big) \to (\Omega, \mathcal{F}),$$
$$\theta_t \circ \theta_\tau = \theta_{t+\tau}, \qquad \theta_0 = \mathrm{id}_\Omega.$$

The measure $\mathbb{P}$ is supposed to be ergodic with respect to θ.

Definition 2.1. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

For our particular applications we introduce the metric dynamical system generated by a *Wiener process*. Let W be a twosided[1] Wiener process with values in some separable Hilbert space U defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that the covariance Q of this Wiener process is finite: $\mathrm{tr}_U\, Q < \infty$. Then we have that W has continuous trajectories in U. Hence, we can assume that Ω is the set of continuous functions on $\mathbb{R}$ which are zero at zero with values in U denoted by $C_0(\mathbb{R}, U)$. The σ-algebra $\mathcal{F}$ is generated by the compact open topology for this set and $\mathbb{P}$ is supposed to be the Wiener measure on $\mathcal{F}$. We set

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t).$$

[1] Twosided means that we take two independent Wiener processes with same covariance, reflect one of these processes to the negative axis and glue them together at zero.

Then $\mathbb{P}$ is ergodic with respect to this flow, see Arnold [1].

To introduce a filtration on Ω we consider

$$\mathcal{F}_s^t = \sigma\big\{\omega(u) - \omega(v)\colon\ s \leqslant u \leqslant v \leqslant t\big\}.$$

These σ-algebras satisfy $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$.

To deal with stochastic differential equations in the *classical* sense we have to consider the completion of $(\Omega, \mathcal{F}, \mathbb{P})$ denoted by $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. The augmention of all zero sets of $\bar{\mathcal{F}}$ to $\mathcal{F}_s^t$ is denoted by $\bar{\mathcal{F}}_s^t$. In addition, we introduce the forward filtration

$$\big(\bar{\mathcal{F}}_s^t\big)_{t\geqslant s}, \quad s \text{ fixed}$$

and the backward filtration

$$\big(\bar{\mathcal{F}}_s^t\big)_{t\geqslant s}, \quad t \text{ fixed}.$$

For our considerations we need random variables with a particular subexponential growing property.

Definition 2.2. Let X be a positive random variable on the ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Then $X > 0$ is called tempered from above if there exists a set $\bar{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \bar{\Omega}$

$$\lim_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = 0.^2 \tag{2}$$

X is called tempered from below if $1/X$ is tempered from above.

It is easily seen that $\bar{\Omega}$ is $(\theta_t)_{t\in\mathbb{R}}$-invariant. Note that under our assumption of ergodicity there is only one alternative for temperedness from above:

$$\limsup_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = +\infty.$$

A sufficient condition for temperedness from above can be found in Arnold [1, p. 167]:

$$\mathbb{E} \sup_{t\in[0,1]} \log^+ X(\theta_t\omega) < \infty. \tag{3}$$

Let E be a Polish space. A set-valued mapping

$$\omega \to M(\omega) \subset E, \quad M(\omega) \neq \emptyset$$

is called a *random set* if $M(\omega)$ is closed and

$$\omega \to \operatorname{dist}_E\big(x, M(\omega)\big)$$

[2] $\log^+ x = \max(\log x, 0)$.

is a random variable for every $x \in E$.

We now introduce an Ornstein–Uhlenbeck process on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ given by the Brownian motion. Set

$$t \to z(\theta_t\omega) := -\mu \int_{-\infty}^{0} e^{\mu s}\theta_t\omega(s)\,ds, \quad t \in \mathbb{R}, \tag{4}$$

where μ is positive. Indeed the above Bochner integral makes sense for any path ω with a subexponential growth. By the integration by parts technique for stochastic integrals (see Øksendale [21, Chapter 4]) we know that z solves the Itó equation

$$dz + \mu z\,dt = d\omega(t) \Longleftrightarrow z(t) + \mu \int_{0}^{t} z(\tau)\,d\tau = z(0) + \omega(t) \tag{5}$$

for $t \geqslant 0$, see also [10, Da Prato and Zabczyk, Theorem 5.2]. Indeed $t \mapsto z(\theta_t\omega)$ is a $(\theta_t)_{t\in\mathbb{R}}$-invariant stationary process which is called the (stationary) Ornstein–Uhlenbeck process. This process is adapted with respect to $\mathcal{F}^t_{-\infty}$ where

$$\mathcal{F}^t_{-\infty} := \bigvee_{s\leqslant t} \mathcal{F}^t_s.$$

Similarly, we can define for $\mu < 0$ the Ornstein–Uhlenbeck process

$$s \to z(\theta_s\omega) := \mu \int_{0}^{\infty} e^{\mu\tau}\theta_s\omega(\tau)\,d\tau, \quad s \in \mathbb{R}, \tag{6}$$

satisfying (5) which is adapted with respect to

$$\mathcal{F}^\infty_s = \bigvee_{s\leqslant t} \mathcal{F}^t_s.$$

Lemma 2.3. *Assume that* $\operatorname{tr}_U Q < +\infty$.

(i) *There exists a* $(\theta_t)_{t\in\mathbb{R}}$*-invariant measurable set* $\Omega' \subset C_0(\mathbb{R}, U)$ *of full* $\mathbb{P}$*-measure such that the mapping* $s \to z(\theta_s\omega)$ *is continuous on* U *for any* $\omega \in \Omega'$,
(ii) *The random variable* $|z|_U$ *is tempered from above.*

For a subexponentially growing $\omega \in \Omega = C_0(\mathbb{R}, U)$ the expression

$$\tau \to \sup_{s\in[s_0-1,s_0+1]} \left|\theta_s\omega(\tau)\right|$$

is still subexponentially growing for every $s_0 \in \mathbb{R}$. Then this mapping is a majorant for the integrand in (4). Lebesgue's theorem gives us the continuity of (4) at $s_0 \in \mathbb{R}$.

It follows from Itó's formula and (5) that $\mathbb{E}|z|_U^2 = \frac{\operatorname{tr}_U Q}{2|\mu|} < +\infty$. Thus the second assertion follows by the Burkholder inequality

$$\mathbb{E} \sup_{t\in[0,1]} \left|z(\theta_t\omega)\right|_U^2 < \infty.$$

We then can apply (3) to get the temperedness of $|z|_U$. Let us redefine z on the $(\theta_t)_{t\in\mathbb{R}}$-invariant set $(\Omega' \cap \bar{\Omega})^c$ of measure zero by zero, where $\bar{\Omega}$ is the $(\theta_t)_{t\in\mathbb{R}}$-invariant set such that (2) holds.

Let E be a phase space which is supposed to be Polish. The states of a system under the influence of random perturbations will be given by a measurable mapping

$$\phi : \mathbb{R} \times \Omega \times E \to E$$

having the cocycle property

$$\begin{aligned} &\phi(0,\omega,x) = x,\\ &\phi(t,\theta_\tau\omega,\cdot)\circ\phi(\tau,\omega,x) = \phi(t+\tau,\omega,x), \end{aligned}$$

for $t, \tau \in \mathbb{R}$, $x \in E$ and $\omega \in \Omega$. Such a mapping is called a *random dynamical system.* For other applications it is also possible to consider instead of the time set $\mathbb{R}$ the time set $\mathbb{R}^+$. Examples for random dynamical systems are given by the solution operator of differential equations under random influences. For motivations and general examples see Arnold [1].

3. The abstract stochastic evolution equation

Let H be a separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_H$ and A be a closed self-adjoint linear operator with dense domain $D(A)$ in H. We assume that

Hypothesis A1. *There exists a constant $a \geqslant 0$ such that $(A + a\,\mathrm{id})$ is positive and $(A + a\,\mathrm{id})^{-1}$ is compact.*

This assumption implies that the spectrum of A consists of only eigenvalues with finite multiplicities,

$$-a < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty, \tag{7}$$

and the associated eigenvectors $\{e_n\}_{n\in\mathbb{N}}$, $e_i \in D(A) \subset H$ form an orthonormal basis of H. The positivity of $(A + a\,\mathrm{id})$ allows one to define the fractional powers of $(A + a\,\mathrm{id})$, which we denote by $(A + a\,\mathrm{id})^\sigma$ for all $\sigma \in \mathbb{R}$, see Henry [16] and Temam [27]. The domain of $(A + a\,\mathrm{id})^\sigma$, which we denote by H^σ, is a Hilbert space under the product $((u,v))_\sigma = ((A + a\,\mathrm{id})^\sigma u, (A + a\,\mathrm{id})^\sigma v)$. For the reminder of this paper, we let $V = H^{1/2}$ and denote its product and norm by $((\cdot,\cdot))$ and $\|\cdot\|_V$ respectively. In addition, we use E to denote the space $V \times H$ with norm $\|\cdot\|_E$.

We consider the following stochastic wave equation

$$\frac{d^2u}{dt^2} + \alpha\frac{du}{dt} + Au + B(u) = C(u)[\dot{W}], \tag{8}$$

where $u \in H$, A is the operator given by the above, B and C are nonlinear operators on V, and $\dot{W}$ is a white noise, given by the generalized time derivative of a twosided Wiener process W.

Let $\phi_1 = u$, $\phi_2 = \phi_1' + \eta\phi_1$ for $\eta \in \mathbb{R}$. Then, we can write Eq. (8) as

$$\begin{aligned} d\phi_1 &= (\phi_2 - \eta\phi_1)\,dt, \\ d\phi_2 &= \big((\eta-\alpha)\phi_2 + \eta(\alpha-\eta)\phi_1 - A\phi_1 - B(\phi_1)\big)\,dt + C(\phi_1)[dW]. \end{aligned} \tag{9}$$

We will study this problem in space E.

For the operator B, we assume

Hypothesis A2.

(i) *$B : V \to H$ is Lipschitz continuous and $B(0) = 0$.*
(ii) *There exists a constant $c_B > 0$ such that*

$$\big\|B(\phi_1) - B(\phi_2)\big\|_H \leqslant c_B\big(\|\phi_1\|_V + \|\phi_2\|_V\big)\|\phi_1 - \phi_2\|_H \quad \text{for } \phi_1, \phi_2 \in V.$$

Let W be a Wiener process with values in the separable Hilbert space U with norm $|\cdot|_U$ defined in Section 2. We will consider W as a *canonical* Wiener process $W(\cdot\,, \omega) = \omega(\cdot)$. Assume that the covariance operator Q of W has a finite trace in U. We note that this Wiener process generates a metric dynamical system described in Section 2.

The operator C is assumed to satisfy

Hypothesis A3.

(i) *$C : V \to \mathcal{L}(U, H)$, the space of all linear bounded operators from U to H, is Lipschitz continuous.*
(ii) *C is a continuously differentiable nonlinear operator from V into $\mathcal{L}(U, V)$ with $C(0) = 0$ and $C'(0) = 0$. In addition, there exists a constant $c_C > 0$ such that*

$$\begin{aligned} &\big\|C(\phi_1)[\cdot] - C(\phi_2)[\cdot]\big\|_{\mathcal{L}(U,V)} \leqslant c_C\big(\|\phi_1\|_V + \|\phi_2\|_V\big)\|\phi_1 - \phi_2\|_V, \\ &\big\|C(\phi_1)[\cdot]\big\|_{\mathcal{L}(U,V)} \leqslant c_C\|\phi_1\|_V \quad \text{for } \phi_1, \phi_2 \in V. \end{aligned} \tag{10}$$

(iii) *$C'(\phi)$ can be extended to a bounded linear operator from H to $\mathcal{L}(U, H) =: \mathcal{L}_2$ and there exists $c_C' > 0$ such that*

$$\big\|C'(\phi_1)[\cdot\,,\cdot] - C'(\phi_2)[\cdot\,,\cdot]\big\|_{\mathcal{L}_2} \leqslant c_C'\|\phi_1 - \phi_2\|_V, \quad \text{for } \phi_1, \phi_2 \in V. \tag{11}$$

Remark 3.1.

(a) Condition (i) implies that the operator $C(\phi)$ is a Hilbert–Schmidt operator from $U_0 = Q^{\frac{1}{2}}U$ to H and is Lipschitz continuous in ϕ, which is important for the existence of solutions for the stochastic wave equation. Thus, the stochastic wave equation has a unique global mild solution.
(b) Conditions (ii) and (iii) yield that the stochastic wave equation generates a random dynamical system which is given by Theorem 3.6.

(c) Conditions $B(0) = 0$ and $C(0) = 0$ simply mean that 0 is an equilibrium point. To get a local invariant manifold, we also need $C'(O) = 0$.

We now give some basic properties of Eq. (9). Let Λ_η denote the linear part of (9). Then,

$$\Lambda_\eta = \begin{pmatrix} -\eta\,\mathrm{id} & \mathrm{id} \\ -A + \eta(\alpha - \eta)\,\mathrm{id} & -(\alpha - \eta)\,\mathrm{id} \end{pmatrix} \tag{12}$$

with domain $D(\Lambda_\eta) = D(A) \times V$.

Lemma 3.2. *The operator Λ_η generates a strongly continuous group $G(t)$ on E.*

For the proof we refer to Pazy [24], Renardy and Rogers [25, p. 409], and Temam [27, p. 185]. The operator Λ_η has a discrete spectrum. More precisely we have

Lemma 3.3. *The spectrum of Λ_η consists of only eigenvalues $\mu_k^\pm$ given by*

$$\mu_k^\pm = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \lambda_k} \in \mathbb{C}$$

with eigenvectors $f_k^\pm = (f_{1k}^\pm, f_{2k}^\pm) \in E + \mathbb{C}E$, where

$$f_{1k}^\pm = e_k, \qquad f_{2k}^\pm = \big(\mu_k^\pm + \eta\big)e_k.$$

Here λ_k is given by (7). *Note that the eigenvalues are independent of η.*

Proof. The proof of this lemma follows from a straightforward computation and the fact that A has a compact resolvent. □

We now are in a position to formulate a *classical* existence and uniqueness result for Eq. (9).

Theorem 3.4. *Let $\phi^0 = (\phi_1^0, \phi_2^0)$ be an $\bar{\mathcal{F}}_{-\infty}^0$ measurable random variable with values in E. Then on every interval $[0, T]$, $T > 0$ (9) has a unique $(\bar{\mathcal{F}}_{-\infty}^t)_{t\in[0,T]}$-adapted mild solution which is continuous on E. Similarly, let $\phi^0 = (\phi_1^0, \phi_2^0)$ be an $\bar{\mathcal{F}}_0^\infty$ measurable random variable with values in E. Then on every interval $[T, 0]$, $T < 0$ (9) has a unique $(\bar{\mathcal{F}}_t^\infty)_{t\in[T,0]}$-adapted mild solution which is continuous on E.*

This theorem follows directly from Da Prato and Zabczyk [10, Theorem 7.2]. We emphasize that Λ_η generates a group.

However, our purpose is to deal with invariant manifolds for (9). We will not be able to obtain a stable/unstable invariant manifold by working directly with Eq. (9) because of the strong adaptedness we have in the last theorem. Therefore for this intention we introduce an extension of the above solution that allows us to consider stable/unstable invariant manifolds. This solution will be defined for *every* ω contained in a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full $\mathbb{P}$ measure independent of the initial condition. More precisely, this solution generates a random dynamical system. To define

a random dynamical system we introduce another equation with random coefficients but without white noise, which will be transformed into the original equation by a stationary transformation.

We consider the equation

$$\begin{aligned}
\frac{d\hat{\phi}_1}{dt} &= \hat{\phi}_2 - \eta\hat{\phi}_1 + C(\hat{\phi}_1)\big[z(\theta_t\omega)\big],\\
\frac{d\hat{\phi}_2}{dt} &= (\eta-\alpha)\hat{\phi}_2 + \eta(\alpha-\eta)\hat{\phi}_1 - A\hat{\phi}_1 - B(\hat{\phi}_1) - (\alpha-\mu-\eta)C(\hat{\phi}_1)\big[z(\theta_t\omega)\big]\\
&\quad - C'(\hat{\phi}_1)\big[z(\theta_t\omega), \hat{\phi}_2 - \eta\hat{\phi}_1 + C(\hat{\phi}_1)\big[z(\theta_t\omega)\big]\big],
\end{aligned} \tag{13}$$

where z is the Ornstein–Uhlenbeck process introduced in Section 2 and μ is given in (4) and (6).

Theorem 3.5. *For every $\omega \in \Omega$ and $\phi^0 := (\phi_1^0, \phi_2^0) \in E$ Eq.* (13) *has a unique global mild solution $\hat{\phi}(\cdot\,, \omega, \phi^0)$ with values in E. Furthermore, $\hat{\phi}(t, \omega, \phi^0)$ is Lipschitz continuous in ϕ^0 and generates a random dynamical system.*

The proof of this theorem is based on the standard argument and Lipschitz continuity of B, and the local Lipschitz continuity of C together the linear bound of C, see Pazy [24].

The following theorem allows us to interpret (9) as a random dynamical system. In particular, the solution version we are going to define is independent of the initial state and hence allows to plug in non-adapted initial conditions.

Theorem 3.6. *Let $z(\omega)$ be the random variable defined in Section* 3 *and let $T : E \times \Omega \to E$ be given by*

$$\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2) \to \big(\hat{\phi}_1, \hat{\phi}_2 + C(\hat{\phi}_1)\big[z(\omega)\big]\big) \quad \text{for } \omega \in \Omega. \tag{14}$$

Then T is a random homeomorphism with the inverse

$$T^{-1}(\cdot\,, \omega) : \phi = (\phi_1, \phi_2) \to \big(\phi_1, \phi_2 - C(\phi_1)\big[z(\omega)\big]\big) \quad \text{for } \omega \in \Omega.$$

Furthermore, the mapping

$$t \to T\big(\hat{\phi}\big(t, \omega, T^{-1}\big(\phi^0, \omega\big)\big), \theta_t\omega\big) =: \phi\big(t, \omega, \phi^0\big) \in E \tag{15}$$

defines a random dynamical system on $\mathbb{R}$. For fixed $\phi^0 \in E$ and z defined by (4)*, the process*

$$(t, \omega) \to \phi\big(t, \omega, \phi^0\big), \quad t \geqslant 0,$$

is a $\bar{\mathcal{F}}^t_{-\infty}$-measurable version of the mild solution of the stochastic differential equation (9)*. Similarly, considering z given by* (6)*, then for fixed $\phi^0 \in E$*

$$(t, \omega) \to \phi\big(t, \omega, \phi^0\big), \quad t \leqslant 0,$$

is a $\bar{\mathcal{F}}^\infty_t$-measurable version of the mild solution of (9)*.*

Proof. We only consider $t \geqslant 0$ since the case $t \leqslant 0$ is similar. A simple calculation gives that T is a homeomorphism on E with the inverse T^{-1} and $T(\hat{\phi}, \cdot)$ and $T^{-1}(\phi, \cdot)$ are measurable. From Castaing and Valadier [5, Chapter III] it follows that

$$(\phi, \omega) \to T^{-1}(\phi, \omega), \qquad (\hat{\phi}, \omega) \to T(\hat{\phi}, \omega)$$

are measurable, here the Lipschitz continuity of C is used.

Next, we show that T carries a solution of Eq. (13) to a version of the mild solution of Eq. (9). For simplicity, we write Eq. (13) as

$$\frac{d\hat{\phi}}{dt} = \Lambda_\eta \hat{\phi} + F(\theta_t \omega, \hat{\phi}), \tag{16}$$

where

$$F(\omega, \hat{\phi}) = \begin{pmatrix} C(\hat{\phi}_1)[z(\omega)] \\ -B(\hat{\phi}_1) - (\alpha - \mu - \eta)C(\hat{\phi}_1)[z(\omega)] - C'(\hat{\phi}_1)[z(\omega), \hat{\phi}_2 - \eta\hat{\phi}_1 + C(\hat{\phi}_1)[z(\omega)]] \end{pmatrix}.$$

Let $\hat{\phi}(t)$ be the mild solution of Eq. (16) with initial condition $\hat{\phi}(0) = \hat{\phi}^0$. Recall that $G(t)$ is the group generated by the linear operator Λ_η. Then, $\hat{\phi}(t)$ satisfies

$$\hat{\phi}(t) = G(t)\hat{\phi}^0 + \int_0^t G(t-\tau) F\big(\theta_\tau \omega, \hat{\phi}(\tau)\big)\, d\tau$$

and $\phi(t) = T(\hat{\phi}(t), \theta_t \omega)$ satisfies

$$\begin{aligned} \phi(t) = {} & G(t)\phi^0 + \int_0^t G(t-\tau) \begin{pmatrix} 0 \\ -B(\phi_1(\tau)) \end{pmatrix} d\tau \\ & + G(t) \begin{pmatrix} 0 \\ -C(\phi_1^0)[z(\omega)] \end{pmatrix} + \begin{pmatrix} 0 \\ C(\phi_1(t))[z(\theta_t\omega)] \end{pmatrix} \\ & + \int_0^t G(t-\tau) \\ & \times \begin{pmatrix} C(\phi_1(\tau))[z(\theta_\tau\omega)] \\ -(\alpha - \mu - \eta)C(\phi_1(\tau))[z(\theta_\tau\omega)] - C'(\phi_1(\tau))[z(\theta_\tau\omega), \phi_2(\tau) - \eta\phi_1(\tau)] \end{pmatrix} d\tau. \end{aligned} \tag{17}$$

On the other hand, if $\phi(t)$ is a mild solution of Eq. (9), then ϕ is given by

$$\phi(t) = G(t)\phi^0 + \int_0^t G(t-\tau) \begin{pmatrix} 0 \\ -B(\phi_1(\tau)) \end{pmatrix} d\tau$$

$$+\int_0^t G(t-\tau)\begin{pmatrix}0\\ C(\phi_1(\tau))[dW]\end{pmatrix}. \tag{18}$$

Define

$$\phi_1^m(t)=\sum_{i=1}^m \phi_{1,m}(t)e_m \quad \text{and} \quad \phi_2^m(t)=\sum_{i=1}^m \phi_{2,m}(t)e_m$$

where $\phi_{1,m}(t)=(\phi_1(t),e_m)$ and $\phi_{2,m}(t)=(\phi_2(t),e_m)$. Using the condition (i) in Hypothesis A3 and the existence Theorem 3.4, the mild solution $\phi(t)$ satisfies

$$E\int_0^t \big\|C\big(\phi(t)\big)\big\|^2_{\mathcal{L}(U,H)}\,d\tau<\infty.$$

By Theorem 6.5 in Da Prato and Zabczyk [10], the mild solution is also a weak solution. Hence, we have

$$\phi_1^m(t)=\phi_1^{0,m}+\int_0^t \big(\phi_2^m(\tau)-\eta\phi_1^m(\tau)\big)\,d\tau.$$

Since $\phi_1(t)$ and $\phi_2(t)$ are continuous, then ϕ_1^m is continuously differentiable and

$$\frac{d\phi_1^m(t)}{dt}=\phi_2^m(t)-\eta\phi_1^m(t).$$

Since $\phi_1^m(t)\to\phi_1(t)$, using Hypothesis A3(i), we obtain

$$L_2\lim_{m\to\infty}\int_0^t G(t-\tau)\begin{pmatrix}0\\ C(\phi_1^m(\tau))[dW]\end{pmatrix}=\int_0^t G(t-\tau)\begin{pmatrix}0\\ C(\phi_1(\tau))[dW]\end{pmatrix} \tag{19}$$

for $t>0$. Let $0=\tau_0^n<\tau_1^n<\cdots<\tau_n^n=t$ be a sequence of partitions of $[0,t]$ such that the maximal mesh size tends to zero for $n\to\infty$. Then the terms of the left-hand side of (19) are defined as a limit in probability of

$$\sum_{i=0}^{n-1} G\big(t-\tau_i^n\big)\begin{pmatrix}0\\ C(\phi_1^m(\tau_i^n))[W(\tau_{i+1}^n)-W(\tau_i^n)]\end{pmatrix}$$

which is equal to

$$\sum_{i=0}^{n-1} G\big(t-\tau_i^n\big)\begin{pmatrix}0\\ \mu C(\phi_1^m(\tau_i^n))[\int_{\tau_i^n}^{\tau_{i+1}^n} z(\theta_s\omega)\,ds]\end{pmatrix}$$

$$+\sum_{i=0}^{n-1} G\big(t-\tau_i^n\big)\begin{pmatrix}0\\ C(\phi_1^m(\tau_i^n))[z(\theta_{\tau_{i+1}^n}\omega)-z(\theta_{\tau_i^n}\omega)]\end{pmatrix}. \tag{20}$$

The first sum in (20) tends in probability to

$$\int_0^t G(t-\tau)\begin{pmatrix}0\\ \mu C(\phi_1^m(\tau))[z(\theta_\tau\omega)]\end{pmatrix}d\tau. \tag{21}$$

Let us abbreviate the second sum by

$$\sum_{i=0}^{n-1} f\big(\tau_i^n\big)\big[z(\theta_{\tau_{i+1}^n}\omega)-z(\theta_{\tau_i^n}\omega)\big].$$

Rearranging the terms of the sum we can write this expression as

$$\begin{aligned}&-f(0)\big[z(\omega)\big]+f\big(\tau_{n-1}^n\big)\big[z(\theta_{\tau_n^n}\omega)\big]-\sum_{i=1}^{n-1}\big(f\big(\tau_i^n\big)\big[z(\theta_{\tau_i^n}\omega)\big]-f\big(\tau_{i-1}^n\big)\big[z(\theta_{\tau_i^n}\omega)\big]\big)\\ &\quad=-f(0)\big[z(\omega)\big]+f\big(\tau_{n-1}^n\big)\big[z(\theta_t\omega)\big]-\sum_{i=1}^{n-1}\int_{\tau_{i-1}^n}^{\tau_i^n} Df(\tau)\big[z(\theta_{\tau_i^n}\omega)\big]d\tau\end{aligned}$$

since $\tau\to f(\tau)[z]$ is continuously differentiable where Df is in $C([0,t];E)$. Since $\tau\to z(\theta_\tau\omega)$ is continuous this expression tends to

$$-f(0)\big[z(\omega)\big]+f(t)\big[z(\theta_t\omega)\big]-\int_0^t Df(\tau)\big[z(\theta_\tau\omega)\big]d\tau$$

as $n\to\infty$. According to the definition of f, we have that the second sum in (20) approaches to the following as n goes to ∞

$$\begin{aligned}&\begin{pmatrix}0\\ C(\phi_1^m(t))[z(\theta_t\omega)]\end{pmatrix}-G(t)\begin{pmatrix}0\\ C(\phi_1^{0,m})[z(\omega)]\end{pmatrix}+\int_0^t G(t-\tau)\Lambda_\eta\begin{pmatrix}0\\ C(\phi_1^m(\tau))[z(\theta_\tau\omega)]\end{pmatrix}d\tau\\ &-\int_0^t G(t-\tau)\begin{pmatrix}0\\ C'(\phi_1^m(\tau))[z(\theta_\tau\omega),\frac{d}{d\tau}\phi_1^m(\tau)]\end{pmatrix}d\tau.\end{aligned} \tag{22}$$

Here we use the fact that $(0,C(\phi_1^m(\tau))[z(\theta_t\omega)])\in D(\Lambda_\eta)=D(A)\times V$, $C(\cdot)[z]$ is differentiable as an operator from V to V and $\phi_1^m(t)$ is differentiable in V. Thus, we can use the chain rule. Using the form of Λ_η, we obtain that (22) is equal to

$$\begin{pmatrix} 0 \\ C(\phi_1^m(t))[z(\theta_t\omega)] \end{pmatrix} - G(t)\begin{pmatrix} 0 \\ C(\phi_1^{0,m})[z(\omega)] \end{pmatrix} + \int_0^t G(t-\tau)\begin{pmatrix} C(\phi_1^m(\tau))[z(\theta_\tau\omega)] \\ -(\alpha-\eta)C(\phi_1^m(\tau))[z(\theta_\tau\omega)] \end{pmatrix} d\tau - \int_0^t G(t-\tau)\begin{pmatrix} 0 \\ C'(\phi_1^m(\tau))[z(\theta_\tau\omega), \phi_2^m(t)-\eta\phi_1^m(t)] \end{pmatrix} d\tau.$$

Letting $m \to \infty$ together with (21) we can see that (18) can be expressed by (17) where we have used Hypothesis A3. Similarly, we can treat terms containing B. We obtain that the transformed mild solution of (13) is a version of the mild solution of (9). □

4. The cut-off system and its Lipschitz constants

In this section, we introduce a modified equation of (13) by using a cut-off function such that its nonlinear terms has a small Lipschitz constants.

Let $\chi : E \to E$ be a smooth cut-off function such that

$$\chi(x) := \begin{cases} x: & \|x\|_E \leqslant \frac{1}{2}, \\ \frac{x}{\|x\|_E}: & \|x\|_E > 1 \end{cases}$$

with Lipschitz constant L_χ. Set

$$\chi_r(x) := r\chi\left(\frac{1}{r}x\right) \quad \text{for } r > 0.$$

Define

$$\pi_i(\phi_1, \phi_2) = \phi_i \quad \text{for } (\phi_1, \phi_2) \in E,\ i = 1, 2.$$

Note that the norm $\|\pi_i\| = 1$, $i = 1, 2$.

Let $R > 0$ be a random variable. We introduce the modified nonlinear hyperbolic system

$$\begin{aligned} \frac{d\psi_1}{dt} &= \psi_2 - \eta\psi_1 + F_R^1(\theta_t\omega, \psi), \\ \frac{d\psi_2}{dt} &= (\eta-\alpha)\psi_2 + \eta(\alpha-\eta)\psi_1 - A\psi_1 + F_R^2(\theta_t\omega, \psi) \end{aligned} \tag{23}$$

where

$$\begin{aligned} F_R^1(\omega, \psi) &= C\big(\pi_1\chi_{R(\omega)}(\psi)\big)\big[z(\omega)\big], \\ F_R^2(\omega, \psi) &= -B\big(\pi_1\chi_{R(\omega)}(\psi)\big) - (\alpha-\mu-\eta)C\big(\pi_1\chi_{R(\omega)}(\psi)\big)\big[z(\omega)\big] \\ &\quad - C'\big(\pi_1\chi_{R(\omega)}(\psi)\big)\big[z(\omega), \pi_2\chi_{R(\omega)}(\psi) - \eta\pi_1\chi_{R(\omega)}(\psi) \\ &\quad + C\big(\pi_1\chi_{R(\omega)}(\psi)\big)\big[z(\omega)\big]\big], \end{aligned}$$

and $\psi = (\psi_1, \psi_2)$. We denote $F_R = (F_R^1, F_R^2)$.

Clearly, $F_R(\omega, \psi) = F(\omega, \psi)$ for $\|\psi\|_E \leqslant R/2$ and nonlinear operators $F_R^1(\omega, \cdot)$, $F_R^2(\omega, \cdot)$ are Lipschitz continuous. Thus, one has as Theorem 3.5

Lemma 4.1. *Eq.* (23) *generates a random dynamical system $\psi(t, \omega, x)$ which is Lipschitz continuous in x.*

In the following we are going to show that we can find a tempered random variable such that the Lipschitz constants of F_R^1, F_R^2 are arbitrarily small.

Lemma 4.2. *For every $l > 0$ and every $\omega \in \Omega$ there exists a random variable R tempered from below such that*

$$\big\|F_R(\omega, \psi_1) - F_R(\omega, \psi_2)\big\|_E \leqslant l\|\psi_1 - \psi_2\|_E$$

and that the Lipschitz constant of the operator T defined in Theorem 3.6 *with respect to $B_E(0, R)$ is less than $1 + l$. In addition F_R is bounded.*

Proof. Let us define the random variable

$$R(\omega) = \min\left(1, \frac{\varepsilon}{2c_C L_\chi |z|_U}, \frac{\varepsilon}{c'_C L_\chi |z|_U}, \frac{\varepsilon}{2c_B L_\chi}\right)$$

where $L_\chi > 0$ is the Lipschitz constant of χ and $\varepsilon > 0$ to be chosen later. Note that $B_E(0, R(\omega))$ is a random set whose radius is tempered from below. According to (10) we have that for $\phi_a, \phi_b \in E$

$$\begin{aligned}
&\big\|C\big(\pi_1\chi_R(\phi_a)\big)[z] - C\big(\pi_1\chi_R(\phi_b)\big)[z]\big\|_V \\
&\quad\leqslant c_C|z|_U\big(\big\|\chi_R(\phi_a)\big\|_E + \big\|\chi_R(\phi_b)\big\|_E\big)\big\|\chi_R(\phi_a) - \chi_R(\phi_b)\big\|_E \\
&\quad\leqslant c_C L_\chi |z|_U 2R\|\phi_a - \phi_b\|_E \leqslant \varepsilon\|\phi_a - \phi_b\|_E. \qquad (24)
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&|-\alpha + \mu + \eta|\big\|C\big(\pi_1\chi_R(\phi_a)\big)[z] - C\big(\pi_1\chi_R(\phi_b)\big)[z]\big\|_H \\
&\quad\leqslant \varepsilon\big(|\alpha| + |\mu| + |\eta|\big)c_e\|\phi_a - \phi_b\|_E
\end{aligned}$$

where c_e is the embedding constant between H and V. We define

$$\begin{aligned}
&\psi_a = \pi_1\chi_R(\phi_a) \in V, \\
&\hat{\psi}_a = \pi_2\chi_R(\phi_a) - \eta\pi_1\chi_R(\phi_a) + C\big(\pi_1\chi_R(\phi_a)\big)[z] \in H,
\end{aligned}$$

and similarly ψ_b, $\hat{\psi}_b$. On account of (11) it follows that

$$\begin{aligned}
&\big\|C'(\psi_a)[z,\hat{\psi}_a]-C'(\psi_b)[z,\hat{\psi}_b]\big\|_H\\
&\quad\leqslant \big\|C'(\psi_a)[z,\hat{\psi}_a]-C'(\psi_b)[z,\hat{\psi}_a]\big\|_H+\big\|C'(\psi_b)[z,\hat{\psi}_a-\hat{\psi}_b]\big\|_H\\
&\quad\leqslant c_C' L_\chi |z|_U \|\hat{\psi}_a\|_H \|\phi_a-\phi_b\|_E + c_C'|z|_U\|\psi_b\|_V\|\hat{\psi}_a-\hat{\psi}_b\|_H,
\end{aligned}$$

where

$$\begin{aligned}
\|\hat{\psi}_a\|_H &\leqslant R+\eta c_e R+\big\|C(\psi_a)[z]\big\|_H\\
&\leqslant R+\eta c_e R+c_C c_e\big\|\chi_R(\phi_a)\big\|_E^2|z|_U \leqslant \frac{\varepsilon(1+\eta c_e)}{2c_C L_\chi|z|_U}+\frac{c_C c_e\varepsilon^2|z|_U}{4c_C^2 L_\chi^2|z|_U^2},\\
\|\psi_b\|_V &\leqslant R\leqslant \frac{\varepsilon}{2c_C L_\chi|z|_U}
\end{aligned}$$

and by (24)

$$\begin{aligned}
\|\hat{\psi}_a-\hat{\psi}_b\|_H &\leqslant \big\|\pi_2\chi_R(\phi_a)-\pi_2\chi_R(\phi_b)\big\|_H+\eta c_e\big\|\pi_1\chi_R(\phi_a)-\pi_1\chi_R(\phi_b)\big\|_V\\
&\quad+\big\|C\big(\chi_R(\psi_a)\big)[z]-C\big(\chi_R(\psi_b)\big)[z]\big\|_H\\
&\leqslant L_\chi\big(\|\phi_a-\phi_b\|_E+\eta c_e\|\phi_a-\phi_b\|_E\big)+\varepsilon c_e\|\phi_a-\phi_b\|_E.
\end{aligned}$$

Similarly, we can treat the B term. Choosing $\varepsilon>0$ sufficiently small we get for F_R a Lipschitz constant less than or equal to some $l>0$. The boundedness of F_R follows from the cut-off. The proof of this lemma is complete. □

5. The existence of random invariant manifolds

In this section, we establish the existence of invariant manifolds for Eqs. (9), (13), and (23). We first write Eq. (23) as

$$\frac{d\psi}{dt}=\Lambda_\eta\psi+F_R(\theta_t\omega,\psi), \tag{25}$$

where Λ_η is given by (12).

For the reminder of this section, we let $\eta=\alpha/2$ and assume that N is a positive integer such that $\frac{\alpha^2}{4}-\lambda_N>0$ and $\lambda_N\neq\lambda_{N+1}$. This assumption implies that not all of the eigenvalues of $\Lambda_{\alpha/2}$ are on the vertical line with $\operatorname{Re}\mu=-\alpha/2$ in the complex plane. Note again that the eigenvalues of Λ_η do not depend on η.

Set

$$\begin{aligned}
E_N&=\operatorname{span}\left\{\begin{pmatrix}e_k\\0\end{pmatrix},\begin{pmatrix}0\\e_k\end{pmatrix}:\ k=1,2,\ldots,N\right\},\\
E_N^\perp&=\operatorname{span}\left\{\begin{pmatrix}e_k\\0\end{pmatrix},\begin{pmatrix}0\\e_k\end{pmatrix}:\ k=N+1,N+2,\ldots\right\}.
\end{aligned}\tag{26}$$

Clearly $E=E_N\oplus E_N^\perp$, E_N is orthogonal to $E_N^\perp$, $\dim(E_N)=2N$, and $E_N^\perp=V_N^\perp\times H_N^\perp$ where $V_N^\perp$ and $H_N^\perp$ are the span of $\{e_{N+1},e_{N+2},\ldots\}$ under the V norm and the H norm respectively.

Furthermore, both E_N and $E_N^\perp$ are invariant subspaces of the linear operator $\Lambda_{\alpha/2}$. We also note that the spectrum of $\Lambda_{\alpha/2}$ consists of only eigenvalues which are given by Lemma 3.3.

Let $0 < \varepsilon < \sqrt{\frac{\alpha^2}{4} - \lambda_N}$. We choose $\delta \in \mathbb{R}$ such that $\sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta} = \varepsilon$. When $\frac{\alpha^2}{4} - \lambda_{N+1} > 0$, we choose $\varepsilon = \sqrt{\frac{\alpha^2}{4} - \lambda_{N+1}}$, thus $\delta = 0$. Then, $(A + \frac{\alpha^2}{4} - 2\lambda_{N+1} + \delta)^{1/2}$ is a positive operator on $V_N^\perp$ which has an equivalent norm given by the product

$$(u, v)_{V_N^\perp} = \left(\left(A + \frac{\alpha^2}{4} - 2\lambda_{N+1} + \delta\right)^{1/2} u, \left(A + \frac{\alpha^2}{4} - 2\lambda_{N+1} + \delta\right)^{1/2} v\right),$$

where $(\cdot, \cdot)$ is the inner product in H. By using the above product in $V_N^\perp$, we define the following product in E, which gives an equivalent norm in E,

$$\big((\psi, \bar{\psi})\big)_E = \big(\psi_{1,N}^\perp, \bar{\psi}_{1,N}^\perp\big)_{V_N^\perp} + \big(\psi_{2,N}^\perp, \bar{\psi}_{2,N}^\perp\big)_H + (\psi_N, \bar{\psi}_N)_E,$$

for $\psi, \bar{\psi} \in E$. Here we decompose ψ and $\bar{\psi}$ into the sum of E_N and $E_N^\perp$ components. Since this new norm is an equivalent norm, for simplicity, we will use the same notation we used before to denote the corresponding norm in E by $\|\cdot\|_E$.

The next lemma is about an exponential dichotomy for the group $G(t)$ generated by $\Lambda_{\alpha/2}$.

Lemma 5.1. *There exists an invariant splitting $E = E_N^- \oplus E_N^+ \oplus E_N^\perp$ with projections π_N^-, π_N^+, and $\pi_N^\perp$ respectively, where $E_N^\perp$ is as in* (26) *and $E_N^\pm \subset E_N$ such that*

$$\begin{aligned}
&\big\|G(t)\pi_N^\perp\big\|_E \leqslant e^{(-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta})t}, \quad t \geqslant 0,\\
&\big\|G(t)\pi_N^-\big\|_E \leqslant \big\|\pi_N^-\big\|_E e^{(-\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \lambda_N})t}, \quad t \geqslant 0,\\
&\big\|G(t)\pi_N^+\big\|_E \leqslant \big\|\pi_N^+\big\|_E e^{(-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \lambda_N})t}, \quad t \leqslant 0.
\end{aligned} \tag{27}$$

Proof. Since $E = E_N \oplus E_N^\perp$ and both E_N and $E_N^\perp$ are invariant subspaces of $\Lambda_{\alpha/2}$, by restricting $\Lambda_{\alpha/2}$ to E_N, according to Lemma 3.3, we obtain the eigenvalues of $\Lambda_{\alpha/2}|_{E_N}$ are

$$\mu_n^\pm = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha}{4} - \lambda_n}, \quad n = 1, 2, \ldots, N, \tag{28}$$

and the corresponding eigenvectors are

$$\begin{pmatrix} e_n \\ (\mu_n^\pm + \frac{\alpha}{2})e_n \end{pmatrix}, \quad n = 1, 2, \ldots, N.$$

Let

$$\begin{aligned}
E_N^- &= \operatorname{span}\left\{\begin{pmatrix} e_n \\ (\mu_n^- + \frac{\alpha}{2})e_n \end{pmatrix} : n = 1, 2, \ldots, N\right\},\\
E_N^+ &= \operatorname{span}\left\{\begin{pmatrix} e_n \\ (\mu_n^+ + \frac{\alpha}{2})e_n \end{pmatrix} : n = 1, 2, \ldots, N\right\}.
\end{aligned}$$

Obviously, $E_N = E_N^- \oplus E_N^+$ and both E_N^- and E_N^+ are invariant subspaces of $\Lambda_{\alpha/2}$ and

$$\left(\left(\begin{pmatrix} e_n \\ (\mu_n^{\pm} + \frac{\alpha}{2})e_n \end{pmatrix}, \begin{pmatrix} e_m \\ (\mu_n^{\pm} + \frac{\alpha}{2})e_m \end{pmatrix}\right)\right)_E = 0 \quad \text{for } m \neq n. \tag{29}$$

Hence, we have an invariant splitting: $E = E_N^{\perp} \oplus E_N^- \oplus E_N^+$. We note that E_N^- is not orthogonal to E_N^+. Let π_N^- and π_N^+ be the corresponding projections. Using (29), we have that the second and third inequality of (27).

We now consider the restriction of $\Lambda_{\alpha/2}$ to $E_N^{\perp}$. For each $\psi \in E_N^{\perp} \cap D(\Lambda_{\alpha/2})$, where $D(\Lambda_{\alpha/2})$ is the domain of $\Lambda_{\alpha/2}$, we have

$$\begin{aligned}
&\left(\left(\left[-\sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta}\,\mathrm{id} + \begin{pmatrix} 0 & \mathrm{id} \\ -A + \frac{\alpha^2}{4}\,\mathrm{id} & 0 \end{pmatrix}\right]\psi, \psi\right)\right)_E \\
&\quad = -\sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta}((\psi, \psi))_E + \left(\frac{\alpha^2}{4} - 2\lambda_{N+1} + \delta\right)(\psi_1, \psi_2) \\
&\quad \leqslant 0,
\end{aligned}$$

which yields that the operator

$$-\sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta}\,\mathrm{id} + \begin{pmatrix} 0 & \mathrm{id} \\ -A + \frac{\alpha^2}{4}\,\mathrm{id} & 0 \end{pmatrix}$$

is dissipative, see Pazy [24]. By the Lumer–Phillips Theorem, the above linear operator generates a contraction semigroup. Thus, we have

$$\left\|\exp\left\{\begin{pmatrix} 0 & \mathrm{id} \\ -A + \frac{\alpha^2}{4}\,\mathrm{id} & 0 \end{pmatrix} t\right\}\right\|_E \leqslant \exp\left\{\left(\sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta}\right)t\right\}, \quad t \geqslant 0.$$

Therefore, we have the first inequality of (27). This completes the proof of this lemma. □

We state the definition of invariant manifolds.

Definition 5.2. Let ϕ be a random dynamical system with state space E.

(1) A random set M is said to be invariant under ϕ if

$$\phi\big(t, \omega, M(\omega)\big) \subset M(\theta_t\omega), \quad \text{for all } t \geqslant 0.$$

(2) A random set M is called a Lipschitz (or C^1) global random pseudo-unstable manifold at equilibrium 0 if M is invariant, $M(\omega)$ is a Lipschitz (or C^1) manifold, and there exists a $\beta \in \mathbb{R}$ such that

$$M(\omega) = \big\{x \in E \mid \lim_{t \to -\infty} e^{-\beta t}\phi(t, \omega, x) = 0\big\}.$$

Similarly, we can define global random pseudo-stable manifolds.

We first show that the random dynamical system generated by the modified Eq. (23) has a global Lipschitz random pseudo-unstable manifold.

For simplicity, we denote $E_N^{\perp} \oplus E_N^-$ and E_N^+ by E^s and E^u respectively and the corresponding projections by π^u, π^s, and set

$$\mu_{N+1}^* = \begin{cases} -\frac{\alpha}{2}, & \text{if } \frac{\alpha^2}{4} - \lambda_{N+1} \leqslant 0, \\ \mu_{N+1}^+ = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \lambda_{N+1}}, & \text{if } \frac{\alpha^2}{4} - \lambda_{N+1} > 0. \end{cases}$$

Note that $\mu_{N+1}^* < \mu_N^+$. When $\frac{\alpha^2}{4} - \lambda_{N+1} \leqslant 0$, one may choose $\delta > 0$ such that $\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \lambda_{N+1} + \delta} \in \mathbb{R}$ can be arbitrarily small.

We will prove the existence of random invariant manifolds by Lyapunov–Perron's method. This method has been applied to deterministic hyperbolic equations by Chow and Lu [7] and to parabolic equations containing a pure white noise by Chueshov and Girya [8], Chueshov and Scheutzow [9]. To obtain this result we will formulate several lemmas.

In the following we are going to choose the abbreviation $F_R^{u/s}$, $\psi^{u/s}$, etc. for $\pi^{u/s} F_R$, $\pi^{u/s}\psi$, etc.

For $\beta \in \mathbb{R}$, we define the following Banach space

$$\mathcal{E}^\beta = \Big\{ \psi(\cdot) \in C\big((-\infty, 0]; E\big) \colon \sup_{t \leqslant 0} e^{-\beta t} \big\| \psi(t) \big\|_E < \infty \Big\}$$

with norm

$$\|\psi\|_\beta = \sup_{t \leqslant 0} e^{-\beta t} \big\| \psi(t) \big\|_E.$$

Lemma 5.3. *Let R be a random variable tempered from below such that the Lipschitz constant l of F_R^u, F_R^s satisfies the following gap condition*

$$\mu_N^+ - \mu_{N+1}^* - 2l\big(\|\pi^u\| + \|\pi^s\|\big) > 0. \tag{30}$$

Then the following operator for $\omega \in \Omega$, $\phi_0^u \in E^u$

$$\begin{aligned} T_{\phi_0^u}^\omega(\phi)[t] := {} & G^u(t)\phi_0^u + \int_0^t G^u(t-\tau) F_R^u\big(\theta_\tau\omega, \phi(\tau)\big)\, d\tau \\ & + \int_{-\infty}^t G^s(t-\tau) F_R^s\big(\theta_\tau\omega, \omega, \phi(t)\big)\, d\tau \end{aligned} \tag{31}$$

is well defined from $\mathcal{E}^\beta$ to $\mathcal{E}^\beta$ with $\beta := \frac{\mu_N^+ + \mu_{N+1}^*}{2}$ and has a unique fixed point in $\mathcal{E}^\beta$.

Proof. By a straightforward computation, we have that $T^{\omega}_{\phi_0^u}$ is well defined from $\mathcal{E}^{\beta}$ to itself. For the existence of a random fixed point we use the Banach fixed point theorem. We show that $T^{\omega}_{\phi_0^u}$ is a contraction for $\omega \in \Omega$. For $\phi^1, \phi^2 \in \mathcal{E}^{\beta}$, we have

$$\begin{aligned}
&\big\|T^{\omega}_{\phi_0^u}(\phi^1) - T^{\omega}_{\phi_0^u}(\phi^2)\big\|_{\beta} \\
&\leqslant \sup_{t\leqslant 0} e^{-\beta t}\Bigg(\int_t^0 \big\|\pi^u\big\| e^{\mu_N^+(t-\tau)} e^{\beta\tau} e^{-\beta\tau} l \big\|\phi^1(\tau) - \phi^2(\tau)\big\|_E \, d\tau \\
&\quad + \int_{-\infty}^t \big\|\pi^s\big\| e^{\mu_{N+1}^{\delta}(t-\tau)} e^{\beta\tau} e^{-\beta\tau} l \big\|\phi^1(\tau) - \phi^2(\tau)\big\|_E \, d\tau\Bigg) \\
&\leqslant l \sup_{t\leqslant 0}\Bigg(\big\|\pi^u\big\| \int_t^0 e^{(\mu_N^+ - \beta)(t-\tau)}\, d\tau + \big\|\pi^s\big\| \int_{-\infty}^t e^{(\beta - \mu_{N+1}^{\delta})(t-\tau)}\, d\tau\Bigg)\big\|\phi^1 - \phi^2\big\|_{\beta} \\
&\leqslant l\left(\frac{\|\pi^u\|}{\mu_N^+ - \beta} + \frac{\|\pi^s\|}{\beta - \mu_{N+1}^{\delta}}\right)\big\|\phi^1 - \phi^2\big\|_{\beta},
\end{aligned}$$

where $\mu_{N+1}^{\delta} = -\frac{\alpha}{2} + \sqrt{\alpha^2/4 - \lambda_{N+1} + \delta}$. When $\alpha^2/4 - \lambda_{N+1} > 0$, we choose $\delta = 0$ and when $\alpha^2/4 - \lambda_{N+1} \leqslant 0$, we choose $\delta > 0$ such that

$$l\left(\frac{\|\pi^u\|}{\mu_N^+ - \beta} + \frac{\|\pi^s\|}{\beta - \mu_{N+1}^{\delta}}\right) < 1$$

since (30). Thus, by using (30), $T^{\omega}_{\phi_0^u}$ is a contraction from $\mathcal{E}^{\beta}$ into $\mathcal{E}^{\beta}$. Hence, it has a unique fixed point in $\mathcal{E}^{\beta}$. □

Denote the fixed point of the operator $T^{\omega}_{\phi_0^u}$ by $\Gamma(\cdot, \omega, \phi_0^u)$ and set

$$\gamma(\omega, \phi_0^u) := \Gamma^s(0, \omega, \phi_0^u) = \int_{-\infty}^0 G^s(-\tau) F_R^s\big(\theta_\tau\omega, \Gamma\big(\tau, \omega, \phi_0^u\big)\big)\, d\tau. \tag{32}$$

Clearly, $\gamma(\omega, 0) = 0$.

Lemma 5.4. *The mapping $\phi_0^u \to \Gamma(\cdot, \omega, \phi_0^u) \in \mathcal{E}^{\beta}$ is Lipschitz continuous for every $\omega \in \Omega$ and*

$$\operatorname{Lip}\Gamma(\cdot, \omega, \cdot) \leqslant \frac{\|\pi^u\|}{1-k},$$

where

$$k = l\left(\frac{\|\pi^u\|}{\mu_N^+ - \beta} + \frac{\|\pi^s\|}{\beta - \mu_{N+1}^{\delta}}\right).$$

Furthermore,

$$\operatorname{Lip}\gamma(\omega,\cdot)\leqslant\frac{\|\pi^u\|\,\|\pi^s\|}{1-k}.$$

Proof. Let $\phi_{01}^u,\phi_{02}^u\in E^u$. Since $k<1$ is the contraction constant for (31) which is independent of ω. We then have

$$\begin{aligned}
&\big\|\Gamma\big(\cdot,\omega,\phi_{01}^u\big)-\Gamma\big(\cdot,\omega,\phi_{02}^u\big)\big\|\\
&\quad=\big\|T_{\phi_{01}^\omega}\big(\Gamma\big(\cdot,\omega,\phi_{01}^u\big)\big)-T_{\phi_{02}^\omega}\big(\Gamma\big(\cdot,\omega,\phi_{02}^u\big)\big)\big\|_\beta\\
&\quad\leqslant\big\|T_{\phi_{01}^\omega}\big(\Gamma\big(\cdot,\omega,\phi_{01}^u\big)\big)-T_{\phi_{02}^\omega}\big(\Gamma\big(\cdot,\omega,\phi_{01}^u\big)\big)\big\|_\beta\\
&\qquad+\big\|T_{\phi_{02}^\omega}\big(\Gamma\big(\cdot,\omega,\phi_{01}^u\big)\big)-T_{\phi_{02}^\omega}\big(\Gamma\big(\cdot,\omega,\phi_{02}^u\big)\big)\big\|_\beta\\
&\quad\leqslant\big\|G^u(\cdot)\big(\phi_{01}^u-\phi_{02}^u\big)\big\|_\beta+k\big\|\Gamma\big(\cdot,\omega,\phi_{01}^u\big)-\Gamma\big(\cdot,\omega,\phi_{02}^u\big\|_\beta\\
&\quad\leqslant\sup_{t\leqslant0}e^{-\beta t}\big\|G^u(t)\big\|_{\mathcal{L}(E)}\big\|\phi_{01}^u-\phi_{02}^u\big\|_E+k\big\|\Gamma\big(\cdot,\omega,\phi_{01}^u\big)-\Gamma\big(\cdot,\omega,\phi_{02}^u\big\|_\beta.
\end{aligned}$$

Since $\mu_N^+-\beta>0$ the first term on the right of the last inequality is well defined. We obtain that a bound for the Lipschitz constant is given by

$$\sup_{t\leqslant0}e^{-\beta t}\big\|G^u(t)\big\|_{\mathcal{L}(E)}/(1-k)\leqslant\big\|\pi^u\big\|/(1-k).$$

Applying the projection π^s to $\Gamma(0,\omega,\phi_0^u)$, we have that $\operatorname{Lip}\gamma(\omega,\cdot)\leqslant\|\pi^s\|\operatorname{Lip}\Gamma$. This completes the proof of the lemma. □

Note that Γ can be constructed as a pointwise limit of measurable random variables by the fixed point iteration procedure. It follows that $\omega\to\gamma(\omega,\phi_0^u)$ is measurable for every $\phi_0^u\in E^u$.

We define the manifold

$$M(\omega)=\big\{\phi_0^u+\gamma\big(\omega,\phi_0^u\big)\colon\ \phi_0^u\in E^u\big\}.$$

Since $\gamma(\omega,0)=0$ we have $0\in M(\omega)$.

Lemma 5.5. *$M(\omega)$ is invariant, that is,*

$$\psi\big(T,\omega,M(\omega)\big)\subset M(\theta_t\omega)\quad\text{for } T\geqslant0. \tag{33}$$

Proof. To see (33) it suffices to show that the fixed point of $T^{\theta_T\omega}_{\psi^u(T,\omega,\phi_0)}$, which is denoted by $\Gamma(\cdot,\theta_T\omega,\psi^u(T,\omega,\phi_0))$, is equal to

$$\Xi_T(\sigma,\omega)=\begin{cases}\psi(\sigma+T,\omega,\Gamma(0,\omega,\phi_0^u))\colon & \sigma\in[-T,0],\\ \Gamma(\sigma+T,\omega,\phi_0^u)\colon & \sigma<-T.\end{cases} \tag{34}$$

Using the fact that $t\to\psi(t,\omega,\Gamma(0,\omega,\phi_0^u))$ is a solution (23) with initial condition $\Gamma(0,\omega,\phi_0^u)$ (see Lemma 5.6), we have

$$\psi^s\big(t,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big) = G^s(t)\int_{-\infty}^{0} G^s(-\tau)F_R^s\big(\theta_\tau\omega,\Gamma\big(\tau,\omega,\phi_0^u\big)\big)\,d\tau + \int_0^t G^s(t-\tau)F_R^s\big(\theta_\tau\omega,\psi\big(\tau,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\big)\,d\tau.$$

For $\sigma = t - T$, $\sigma \geqslant -T$ we then get

$$\Xi_T^s(\sigma,\omega) = \psi^s\big(\sigma+T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big) = \int_{-\infty}^{\sigma} G^s(\sigma-\tau)F_R^s\big(\theta_{\tau+T}\omega,\Xi_T(\tau,\omega)\big)\,d\tau.$$

For $\sigma < -T$ we can write

$$\int_{-\infty}^{\sigma} G^s(\sigma-\tau)F_R^s\big(\theta_{\tau+T}\omega,\Xi_T(\tau,\omega)\big)\,d\tau = \int_{-\infty}^{\sigma+T} G^s(\sigma+T-\tau)F_R^s\big(\theta_\tau\omega,\Gamma\big(\tau,\omega,\phi_0^u\big)\big)\,d\tau = \pi^s\Gamma\big(\sigma+T,\omega,\phi_0^u\big) = \Xi_T^s(\sigma,\omega).$$

This shows that Eq. (34) gives a fixed point relation for the E^s-component of the operator (31).

On the other hand, for the E^u-component of the operator $T^{\omega}_{\phi_0^u}$ and for $-T \leqslant \sigma := t - T \leqslant 0$, $t \in [0,T]$, we have

$$\begin{aligned}
\Xi_T^u(\sigma,\omega) &= \psi^u\big(\sigma+T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big) = \psi^u\big(t,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\\
&= G^u(t)\phi_0^u + \int_0^t G^u(t-\tau)F_R^u\big(\theta_\tau\omega,\psi\big(\tau,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\big)\,d\tau\\
&= G^u(t-T)\left(G^u(T)\phi_0^u + \int_0^T G^u(T-\tau)F_R^u\big(\theta_\tau\omega,\psi\big(\tau,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\big)\,d\tau\right)\\
&\quad + \int_T^t G^u(t-\tau)F_R^u\big(\theta_\tau\omega,\psi\big(\tau,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\big)\,d\tau\\
&= G^u(\sigma)\psi^u\big(T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\\
&\quad + \int_0^{\sigma} G^u(\sigma-\tau)F_R^u\big(\theta_{\tau+T}\omega,\psi\big(\tau+T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\big)\,d\tau\\
&= G^u(\sigma)\psi^u\big(T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)
\end{aligned}$$

$$+\int_0^\sigma G^u(\sigma-\tau)F_R^u\big(\theta_{\tau+T}\omega,\Xi_T(\tau,\omega)\big)\,d\tau,$$

here it was used that $\Gamma^u(0,\omega,\phi_0^u)=\phi_0^u$. Similarly, for the case $\sigma<-T$,

$$\begin{aligned}\Xi_T^u(\sigma,\omega)&=\Gamma^u\big(\sigma+T,\omega,\phi_0^u\big)=\psi^u\big(\sigma+T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)\\&=G^u(\sigma)\psi^u\big(T,\omega,\Gamma\big(0,\omega,\phi_0^u\big)\big)+\int_0^\sigma G^u(\sigma-\tau)F_R^u\big(\theta_{T+\tau}\omega,\Xi(\tau,\omega)\big)\,d\tau,\end{aligned}$$

by the fact $\Gamma(t,\omega,\phi_0^u)=\psi(t,\omega,\Gamma(0,\omega,\phi_0^u))$, which follows from the uniqueness of the solution. Therefore, $\Xi(0,\omega,\phi_0^u)=\phi(T,\omega,\Gamma(0,\omega,\phi_0^u))\in M(\theta_T\omega)$. This completes the proof of the lemma. □

We note that we can also prove the stronger property

$$\psi\big(T,\omega,M(\omega)\big)=M(\theta_T\omega)\quad\text{for } T\geqslant 0.$$

which is based on the fact that ψ is a random dynamical system defined on the time set $\mathbb{R}$. We omit here the proof which is exactly the same as in Arnold [1, Chapter 7] for finite-dimensional discrete systems.

Lemma 5.6. *Suppose that $\psi(\cdot,\omega)\in\mathcal{E}^\beta$. Then $\psi(t,\omega)$ is a solution of Eq.* (25) *with $\psi(0,\omega)=\phi_0$ if and only if $\psi(t,\omega)$ satisfies*

$$\begin{aligned}\psi(t,\omega)&=G(t)\phi_0^u+\int_0^t G^u(t-\tau)F_R^s\big(\theta_\tau\omega,\psi(\tau,\omega)\big)\,d\tau\\&\quad+\int_{-\infty}^t G^s(t-\tau)F_R^u\big(\theta_\tau\omega,\omega,\psi(\tau,\omega)\big)\,d\tau.\end{aligned}\tag{35}$$

Proof. To prove this lemma, we assume that $\psi(\cdot,\omega)\in\mathcal{E}^\beta$ is a solution of Eq. (25) with $\psi(0,\omega)=\phi_0$. By using the variation of constants formula, we get

$$\psi^u(t,\omega)=G^u(t)\phi_0^u+\int_0^t G^u(t-s)F_R^u\big(\theta_\tau\omega,\psi(\tau,\omega)\big)\,d\tau,\tag{36}$$

$$\psi^s(t,\omega)=G^s(t-\tau)\psi^s(\tau,\omega)+\int_\tau^t G(t-s)F_R^s\big(\theta_\tau\omega,\psi(\tau,\omega)\big)\,d\tau.\tag{37}$$

Since $\psi(\cdot,\omega)\in\mathcal{E}^\beta$, we have for $t>\tau$, $\tau<0$ that

$$\begin{aligned}\left\|G(t-\tau)\psi^s(\tau,\omega)\right\|_\beta &\leqslant e^{\mu^\delta_{N+1}(t-\tau)}e^{\beta\tau}\left\|\psi(\cdot,\omega)\right\|_\beta \\ &= e^{\mu^\delta_{N+1}t}e^{(\beta-\mu^\delta_{N+1})\tau}\left\|\psi(\cdot,\omega)\right\|_\beta \to 0 \quad \text{as } \tau\to-\infty.\end{aligned}$$

Then, taking the limit $\tau\to-\infty$ in (37), we obtain

$$\psi^s(t,\omega)=\int_{-\infty}^{t} G^s(t-s)F^s_R\big(\theta_\tau\omega,\psi(\tau,\omega)\big)\,d\tau. \tag{38}$$

Combining (36) and (38), it follows (35). The converse follows from a direct computation. This completes the proof of the lemma. □

In order to have a C^1 invariant manifold, in addition to Hypotheses A1–A3, we assume

Hypothesis A4. *B is C^1 from V to H. $C(\cdot)$ is C^2 from V to $\mathcal{L}(U,V)$ and for each $u\in VC''(\phi_1)$ can be extend to a bilinear form over $H\times H$ with values in $\mathcal{L}(U,H)$.*

Lemma 5.7. *For each $\omega\in\Omega$, $\gamma(\omega,\cdot):E^u\to E^s$ is C^1 smooth and $D\gamma(\omega,\phi^u_0)$ is measurable in ω for each $\phi^u_0\in E^u$.*

Proof. It is sufficient to show that there exists a $\delta_0>0$ such that

$$\Gamma(\cdot,\omega,\cdot):\mathbb{R}^-\times E^u\to\mathcal{E}^{\beta-\delta_0}$$

is C^1. We first notice that the condition (30) implies that there exists a $\delta_0>0$ sufficiently small so that

$$l\left(\frac{\|\pi^u\|}{\mu^+_N-(\beta-\sigma)}+\frac{\|\pi^s\|}{(\beta-\sigma)-\mu^\delta_{N+1}}\right)<1,\quad \text{for all } 0\leqslant\sigma\leqslant\sigma_0.$$

Thus, $T^\omega_{\phi^u_0}$ is a contraction in $\mathcal{E}^{\beta-\sigma}$ for each $0\leqslant\sigma\leqslant\sigma_0$. Since $\mathcal{E}^{\beta_1}\subset\mathcal{E}^{\beta_2}$ for $\beta_2\leqslant\beta_1$, by the uniqueness of fixed points of a contraction, we have that $\Gamma(\cdot,\omega,\phi^u_0)\in\mathcal{E}^{\beta-\sigma}$ for all $\sigma\in[0,\sigma_0]$.

For $\phi^u_0\in E^u$, we set

$$\begin{aligned}Sv=&\int_0^t G^u(t-\tau)D_\phi F^u_R\big(\theta_\tau\omega,\Gamma\big(\tau,\omega,\phi^u_0\big)\big)v\,d\tau\\ &+\int_{-\infty}^t G^s(t-\tau)D_\phi F^s_R\big(\theta_\tau\omega,\Gamma\big(\tau,\omega,\phi^u_0\big)\big)v\,d\tau\end{aligned}$$

for $v \in \mathcal{E}^{\beta-\sigma_0/2}$. From (28), we have that $G(\cdot)$ is a bounded linear operator from E^u to $\mathcal{E}^{\beta-\sigma_0/2}$. Using the same arguments as we proved that $T^{\omega}_{\phi_0^u}$ is a contraction, it follows that S is a bounded linear operator from $\mathcal{E}^{\beta-\sigma/2}$ to itself and

$$\|S\| \leqslant l\frac{\|\pi^u\|}{\mu_N^+ - (\beta-\sigma_0/2)} + \frac{\|\pi^s\|}{(\beta-\sigma_0/2) - \mu_{N+1}^{\delta}} < 1$$

which implies that $\mathrm{id} - S$ is invertible in $\mathcal{E}^{\beta-\sigma_0/2}$. For $\bar{\phi}_0^u, \phi_0^u \in E^u$, we set

$$\begin{aligned} I = \int_0^t & G^u(t-\tau)\big[F_R^u\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\bar{\phi}_0^u\big)\big) - F_R^u\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\phi_0^u\big)\big) \\ & - D_\phi F_R^u\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\phi_0^u\big)\big)\big(\Gamma\big(\tau,\omega,\bar{\phi}_0^u\big) - \Gamma\big(\tau,\omega,\phi_0^u\big)\big)\big]\,d\tau \\ & + \int_{-\infty}^t G^s(t-\tau)\big[F_R^s\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\bar{\phi}_0^u\big)\big) - F_R^s\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\phi_0^u\big)\big) \\ & - D_\phi F_R^s\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\phi_0^u\big)\big)\big(\Gamma\big(\tau,\omega,\bar{\phi}_0^u\big) - \Gamma\big(\tau,\omega,\phi_0^u\big)\big)\big]\,d\tau. \end{aligned}$$

Using the same arguments as in Chow, Lin, and Lu [6], we have $|I|_{\beta-\sigma_0/2} = o(|\bar{\phi}_0^u - \phi_0^u|)$ as $\bar{\phi}_0^u \to \phi_0^u$. Thus,

$$\begin{aligned} &\Gamma\big(\cdot,\omega,\bar{\phi}_0^u\big) - \Gamma\big(\cdot,\omega,\phi_0^u\big) - S\big(\Gamma\big(\cdot,\omega,\bar{\phi}_0^u\big) - \Gamma\big(\cdot,\omega,\phi_0^u\big)\big) \\ &\quad = G(\cdot)\big(\bar{\phi}_0^u - \phi_0^u\big) + I \\ &\quad = G(\cdot)\big(\bar{\phi}_0^u - \phi_0^u\big) + o\big(\big|\bar{\phi}_0^u - \phi_0^u\big|\big), \quad \text{as } \bar{\phi}_0^u \to \phi_0^u \end{aligned}$$

which yields

$$\Gamma\big(\cdot,\omega,\bar{\phi}_0^u\big) - \Gamma\big(\cdot,\omega,\phi_0^u\big) = (\mathrm{id}-S)^{-1}S\big(\bar{\phi}_0^u - \phi_0^u\big) + o\big(\big|\bar{\phi}_0^u - \phi_0^u\big|\big).$$

Hence, $\Gamma(\cdot,\omega,\phi_0^u)$ is differentiable in ϕ_0^u and its derivative satisfies $D_{\phi_0^u}\Gamma(\cdot,\omega,\phi_0^u) \in \mathcal{L}(E^u, \mathcal{E}^{\beta-\sigma_0/2})$, where $\mathcal{L}(E^u, \mathcal{E}^{\beta-\sigma_0/2})$ is the usual space of bounded linear operators and

$$\begin{aligned} D_{\phi_0^u}\Gamma\big(t,\omega,\phi_0^u\big) = G^u(t) &+ \int_0^t G^u(t-\tau)D_\phi F_R^u\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\phi_0^u\big)\big)D_{\phi_0^u}\Gamma\big(\tau,\omega,\phi_0^u\big)v\,d\tau \\ &+ \int_{-\infty}^t G^s(t-\tau)D_\phi F_R^s\big(\theta_\tau\omega, \Gamma\big(\tau,\omega,\phi_0^u\big)\big)D_{\phi_0^u}\Gamma\big(\tau,\omega,\phi_0^u\big)\,d\tau. \end{aligned}$$

Similarly, we have that $D_{\phi_0^u}\Gamma(\cdot,\omega,\cdot)$ is continuous from E^u to $\mathcal{L}(E^u, \mathcal{E}^{\beta-\sigma_0})$. Therefore, $\Gamma(\cdot,\omega,\cdot)$ is C^1 from E^u to $\mathcal{E}^{\beta-\sigma_0}$. We point out that the measurability of $D_{\phi_0^u}\Gamma(\cdot,\omega,\phi_0^u)$ follows from ω-wise limits of the iteration of measurable contraction mappings starting at 0. This completes the proof of the lemma. □

Lemma 5.8. *For each $\phi_0^u \in E^u$, there exists a deterministic constant C such that*

$$\left\|\psi\left(t,\omega,\phi_0^u+\gamma\left(\omega,\phi_0^u\right)\right)\right\|_E \leqslant C\left\|\phi_0^u\right\|_E e^{\beta t}, \quad t \leqslant 0. \tag{39}$$

Proof. Since $\phi_0^u + \gamma(\omega,\phi_0^u) \in M(\omega)$, we have $\psi(\cdot,\omega,\phi_0^u+\gamma(\omega,\phi_0^u)) \in \mathcal{E}^\beta$. By Lemma 5.6, $\psi(\cdot,\omega,\phi_0^u+\gamma(\omega,\phi_0^u))$ satisfies Eq. (35). By a similar computation in Lemma 5.3, we have

$$\left\|\psi\left(\cdot,\omega,\phi_0^u+\gamma\left(\omega,\phi_0^u\right)\right)\right\|_\beta \leqslant \frac{\|\pi^u\|}{1-k}\left\|\phi_0^u\right\|,$$

which implies (39). □

Summarizing the above results we have

Theorem 5.9. *There exists a random variable $R(\omega) > 0$ tempered from below, which is introduced in Lemma* 4.2*, such that*

(i) *If Hypotheses* A1–A3 *hold, then the random dynamical system generated by* (23) *has a Lipschitz global random pseudo-unstable manifold given by*

$$M(\omega) = \left\{\phi^u + \gamma\left(\omega,\phi^u\right)\colon\ \phi^u \in E^u\right\}$$

where $\gamma(\cdot,\cdot)\colon \Omega \times E^u \to E^s$ is Lipschitz in ϕ^u and is measurable in ω, and $\gamma(\omega,0) = 0$.

(ii) *In addition, if Hypothesis* A4 *holds, then $\gamma(\omega,\phi^u)$ is C^1 in ϕ^u and γ and $D_{\phi_0^u}\gamma$ are measurable in ω.*

(iii) *Furthermore, if $\mu_N^+ + \mu_{N+1}^* > 0$ then this manifold is a unstable manifold in the usual sense.*

We introduce *local* random invariant manifolds.

Definition 5.10. A random set $M(\omega)$ called a Lipschitz (or C^1) local random pseudo-unstable manifold at zero for a random dynamical system ϕ if

(i) there exists a random neighborhood $U \subset E^u$ of 0 such that

$$M(\omega) = \left\{\gamma\left(\omega,\phi^u\right) + \phi^u\colon\ \phi^u \in U(\omega)\right\} \quad \text{and} \quad 0 \in M(\omega),$$

where $\gamma(\omega,\cdot)\colon U(\omega) \to E^s$ is a Lipschitz (or C^1) mapping and $\omega \to \gamma(\omega,\phi^u(\omega))$ is measurable for every measurable selector $\phi^u(\omega) \in U(\omega)$.

(ii) there exist $t_-(\phi^u,\omega) > 0$, $t_+(\phi^u,\omega) > 0$ such that

$$\phi^u\left(t,\omega,\phi^u\right) \in U(\theta_\tau\omega)$$

for $\phi^u \in \operatorname{Int}(U(\omega))$ and $t \in (-t_-(\phi^u,\omega), t_+(\phi^u,\omega))$ and $\lim_{\|\phi^u\|\to 0} t_\pm(\phi^u,\omega) = \infty$. In addition,

$$\phi(t,\omega,\phi_0) \in M(\theta_\tau\omega) \quad \text{for } t \in \left(-t_-\left(\phi_0^u,\omega\right), t_+\left(\phi_0^u,\omega\right)\right),\ \phi_0 \in M(\omega).$$

Such a manifold is called local random unstable manifold at zero if there exists a random set V, $V(\omega) \subset U(\omega) \subset E^u$ which is a neighborhood of zero and $t_-(\phi^u, \omega) = \infty$ and for some $\lambda > 0$

$$\lim_{t \to -\infty} \phi\big(t, \omega, \phi^u + \gamma\big(\omega, \phi^u\big)\big)e^{-\lambda t} = 0$$

for $\phi^u \in V(\omega)$.

The next theorem is about the existence of local invariant manifolds for Eq. (13)

Theorem 5.11. *Assume that Hypotheses* A1–A3 *hold. Then,*

(i) *the random dynamical system $\hat{\phi}$ generated by the system* (13) *has a Lipschitz local random pseudo-unstable manifold M_{local} at zero.*
(ii) *In addition, if Hypothesis* A4 *holds, M_{local} is C^1.*
(iii) *If $\mu_N^+ + \mu_{N+1}^* > 0$, then this manifold is unstable with $\lambda = (\mu_N^+ + \mu_{N+1}^*)/2$.*

Proof. Let R be the random variable introduced in Lemma 4.2 and let $L_{\hat{\gamma}}$ be the Lipschitz constant of $\hat{\gamma}$ introduced in (32) which can be chosen independently of ω. Set $R_1 := R/\|\pi^u\|(1 + L_{\hat{\gamma}})$ which is also tempered from below. Then

$$\big(\mathrm{id} \cdot + \hat{\gamma}(\omega, \cdot)\big)\big(\hat{U}(\omega)\big) \subset B_E\big(0, R(\omega)\big)$$

where $\hat{U}(\omega) = \pi^u B_E(0, R_1(\omega))$. Note we have $\hat{\phi}^u(t, \omega, x) = \psi(t, \omega, x)$ as long as $\hat{\phi}^u(t, \omega, x) \in \hat{U}(\theta_t\omega)$. By the Gronwall Lemma and the properties of $G(t)$ and F_R, there exist positive constants $K > 0$ and $L > 0$ such that

$$\big\|\hat{\phi}(t, \omega, x)\big\|_E \leqslant K\|x\|_E e^{L|t|}$$

as long as $\hat{\phi}^u(t, \omega, x) \in \hat{U}(\theta_t\omega)$. Since $t \to R_1(\theta_t\omega) > 0$ is continuous it follows the convergence properties of t_-, t_+ given in Definition 5.10.

It remains to prove the existence of a random set $V = \hat{V}$ given in Definition 5.10. To this end we introduce the function

$$f : (\xi, \omega) \in \mathbb{R}^+ \times \Omega \to \sup_{t \leqslant 0}\big(\xi C e^{-\beta t} - R_1(\theta_t\omega)\big)^+ \in \mathbb{R}^+$$

where C, β is defined in Lemma 5.8. Note that $Ce^{-\beta t} - R_1(\theta_t\omega) < 0$ for large $-t$ since R_1 is tempered from below. Then, there exists some $\hat{\xi} > 0$ such that $f(\hat{\xi}, \omega) = 0$. The mapping f is continuous and monotone in ξ for ω fixed and measurable for ω if ξ is fixed. Let $\{x_1, x_2, \ldots\}$ be a dense set in $\mathbb{R}^+$. Define

$$y_k(\omega) = \begin{cases} 0: & f(x_k, \omega) > 0, \\ x_k: & f(x_k, \omega) = 0. \end{cases}$$

From the properties of f it follows that

$$f^{-1}(\cdot, \omega)\big(\{0\}\big) = \overline{\bigcup_k \big\{y_k(\omega)\big\}} = \big[0, \rho(\omega)\big]$$

where $\rho > 0$ is a random variable. Hence $\hat{V}(\omega) := \pi^u B_E(0, \rho(\omega))$ has the properties from Definition 5.10. Indeed, by the construction of f a trajectory starting in $\hat{V}$ will never leave $\hat{U}$ for $t \leqslant 0$. □

It remains to prove that the original equation (9) has an invariant manifold.

Theorem 5.12. *Assume that Hypotheses* A1–A3 *hold. Then,*

(i) *the random dynamical system ϕ generated by* (9) *has a Lipschitz local random pseudo-unstable manifold at zero*;
(ii) *in addition, if Hypothesis* A4 *holds, then the manifold is C^1*;
(iii) *if $\mu_N^+ + \mu_{N+1}^* > 0$, then this manifold is a unstable in the usual sense.*

Proof. Let $\hat{U}$ be the random set where the graph $\hat{\gamma}$ of $\hat{M}$ is localized. Note that $\hat{\gamma}$ has a Lipschitz constant that can be chosen independently of ω with respect to $\hat{U}$. In addition, $\hat{\gamma}(\omega, 0) = 0$.

First, we assume that there exist a random set $\hat{Z} \subset \hat{U}$ and a random ball $Z = B(0, R_2) \cap E^u$ where R_2 is tempered from below and $t \to R_2(\theta_t\omega)$ is continuous. In addition, the mapping

$$\hat{Z}(\omega) \ni \hat{\phi}^u \to T^u\big(\omega, \hat{\phi}^u + \hat{\gamma}\big(\omega, \hat{\phi}^u\big)\big) \in Z(\omega),$$
$$\big\{T^u\big(\omega, \hat{\phi}^u + \hat{\gamma}\big(\omega, \hat{\phi}^u\big)\big)\colon\ \hat{\phi}^u \in \hat{Z}(\omega)\big\} = Z(\omega)$$

is a random Lipschitz continuous homeomorphism with a Lipschitz continuous inverse $S(\omega, \cdot)$ for which $\omega \to S(\omega, h(\omega))$ is measurable for every measurable selector h of Z. We want to show that there exists a local random invariant manifold for ϕ.

Restrict $\hat{M}$ and $\hat{\gamma}$ to $\hat{Z}$. We then have for $\phi := T(\omega, \hat{\phi})$, $\hat{\phi} = \hat{\phi}^u + \hat{\gamma}(\omega, \hat{\phi}^u)$:

$$\begin{aligned} T\big(\omega, \hat{M}(\omega)\big) &= \big\{T^u\big(\omega, \hat{\phi}^u + \hat{\gamma}\big(\omega, \hat{\phi}^u\big)\big) + T^s\big(\omega, \hat{\phi}^u + \hat{\gamma}\big(\omega, \hat{\phi}^u\big)\big)\colon\ \hat{\phi}^u \in \hat{Z}(\omega)\big\} \\ &= \big\{\phi^u + T^s\big(\omega, S\big(\omega, \phi^u\big) + \hat{\gamma}\big(\omega, S\big(\omega, \phi^u\big)\big)\big)\colon\ S\big(\omega, \phi^u\big) \in \hat{Z}(\omega) \Leftrightarrow \phi^u \in Z(\omega)\big\}. \end{aligned}$$

Define

$$\gamma\big(\omega, \phi^u\big) := T^s\big(\omega, S\big(\omega, \phi^u\big) + \hat{\gamma}\big(\omega, S\big(\omega, \phi^u\big)\big)\big).$$

Since $\hat{\gamma}(\omega, \cdot)$, $T(\omega, \cdot)$ and $S(\omega, \cdot)$ are Lipschitz continuous so is $\gamma(\omega, \cdot)$. We introduce

$$M(\omega) = \big\{\phi^u + \gamma\big(\omega, \phi^u\big)\colon\ \phi^u \in Z(\omega)\big\}.$$

Due to the measurability of $\hat{\gamma}$, T, S, and Z the set M is a random set. We will show that M is a local random manifold for (9). Take some $\phi_0 \in M(\omega)$ with $\hat{\phi}_0^u := \pi^u T^{-1}(\omega, \phi_0) \in \hat{Z}(\omega)$. If $\hat{\phi}^u(t, \omega, \hat{\phi}_0) \in \hat{Z}(\theta_t\omega)$ then $T^u(\theta_t\omega, \hat{\phi}(t, \omega, \hat{\phi}_0)) \in Z(\theta_t\omega)$. By the definition of Z, $\hat{Z}$ we have that

$$\begin{aligned} \phi(t, \omega, \phi_0) &= T\big(\theta_t\omega, \hat{\phi}(t, \omega, \hat{\phi}_0)\big) \\ &= T^u\big(\theta_t\omega, \hat{\phi}^u(t, \omega, \hat{\phi}_0) + \hat{\gamma}\big(\theta_t\omega, \hat{\phi}^u(t, \omega, \hat{\phi}_0)\big)\big) \end{aligned}$$

$$
\begin{aligned}
&+ T^s\big(\theta_t\omega, \hat{\phi}^u(t,\omega,\hat{\phi}_0) + \hat{\gamma}\big(\theta_t\omega, \hat{\phi}^u(t,\omega,\hat{\phi}_0)\big)\big)\\
&= \phi^u(t,\omega,\phi_0) + T^s\big(\theta_t\omega, S\big(\theta_t\omega, \phi^u(t,\omega,\phi_0)\big) + \hat{\gamma}\big(\theta_t\omega, S\big(\theta_t\omega, \phi^u(t,\omega,\phi_0)\big)\big)\big)\\
&= \phi^u(t,\omega,\phi_0) + \gamma\big(\theta_t\omega, \phi^u(t,\omega,\phi_0)\big) \in M(\theta_t\omega).
\end{aligned}
$$

Thus M is an invariant random set that can be presented by a Lipschitz graph γ with respect to Z.

We see later that the Lipschitz constant of γ over Z can be chosen independently of ω. Thus, by Theorem 5.11, we obtain the existence of the times t_+, t_- for the random dynamical system defined in (15). In particular, the Lipschitz constant of T is bounded with respect to $\hat{U}$. Then

$$
\big\|\phi(t,\omega,\phi_0)\big\|_E \leqslant C'\|\phi_0\|_E e^{\beta t}
$$

as long as $\phi(t,\omega,\phi_0) \in Z(\theta_t\omega)$ and $t \leqslant 0$ which allows us to conclude that M is unstable if $\mu_N + \mu^*_{N+1} > 0$.

We now construct $Z(\omega)$ and $\hat{Z}(\omega)$ together with the mapping S, i.e., we have to find $Z(\omega)$ and $\hat{Z}(\omega)$ such that for each $\phi^u \in Z(\omega)$ the following equation

$$
T^u\big(\omega, \hat{\phi}^u + \hat{\gamma}\big(\omega,\hat{\phi}^u\big)\big) = \big(\hat{\phi}^u_1, \hat{\phi}^u_2 + C^u\big(\hat{\phi}^u_1 + \pi_1\hat{\gamma}\big(\omega,\hat{\phi}^u\big)\big)\big[z(\omega)\big]\big) = \phi^u \tag{40}
$$

has a unique solution $\hat{\phi}^u \in \hat{Z}(\omega)$ and T^u maps $\hat{Z}(\omega)$ into $Z(\omega)$. We first notice that $\phi^u_1 = \hat{\phi}^u_1$. In order to find a solution to the second component of this equation we are going to apply the Banach fixed point theorem to the operator

$$
\mathcal{T}_{\omega,\hat{\phi}^u_1,\phi^u_2}\big(\hat{\phi}^u_2\big) := -C^u\big(\hat{\phi}^u_1 + \pi_1\hat{\gamma}\big(\omega,\hat{\phi}^u_1 + \hat{\phi}^u_2\big)\big)\big[z(\omega)\big] + \phi^u_2 \tag{41}
$$

for parameters ω, $\hat{\phi}^u_1$, ϕ^u_2. Let κ be a positive constant such that

$$
\kappa\big\|\pi^u\big\|(1 + L_{\hat{\gamma}}) < 1, \qquad \big(\big(1 - \kappa\big\|\pi^u\big\|L_{\hat{\gamma}}\big)(1 + L_{\hat{\gamma}}) + L_{\hat{\gamma}}\big) \geqslant 1. \tag{42}
$$

According to Lemma 4.2 we are able to select a random ball $B_E(0, R_3(\omega))$ where R_3 is tempered from below such that

$$
\sup_{\hat{\phi}\in B_E(0,R_3(\omega))} \big\|C'(\pi_1\hat{\phi})\big[z(\omega)\big]\big\|_{\mathcal{L}(H,H)} \leqslant \kappa.
$$

In addition, $t \to R_3(\theta_t\omega)$ is continuous. Set $R_4 = \min(R_3, R_1)$. We want to find an $R_5(\omega)$ such that $\mathcal{T}_{\omega,\hat{\phi}^u_1,\phi^u_2}$ maps $\pi_2(B_E(0, R_5(\omega)) \cap E^u)$ into itself. Let us assume for a while that $\hat{\phi}^u_2$ satisfies

$$
\hat{\phi}^u + \hat{\gamma}\big(\omega, \hat{\phi}^u_1 + \hat{\phi}^u_2\big) \in B_E\big(0, R_4(\omega)\big). \tag{43}
$$

Then the following estimates hold

$$
\begin{aligned}
\big\|\mathcal{T}_{\omega,\hat{\phi}^u_1,\phi^u_2}\big(\hat{\phi}^u_2\big)\big\|_H &\leqslant \big\|C^u\big(\hat{\phi}^u_1 + \pi_1\hat{\gamma}\big(\omega,\hat{\phi}^u_1 + \hat{\phi}^u_2\big)\big)\big[z(\omega)\big]\big\|_H + \big\|\phi^u_2\big\|_H\\
&\leqslant \kappa\big\|\pi^u\big\|\big\|\hat{\phi}^u_1 + \pi_1\hat{\gamma}\big(\omega,\hat{\phi}^u_1 + \hat{\phi}^u_2\big)\big\|_V + \big\|\phi^u_2\big\|_H
\end{aligned}
$$

$$\leqslant \kappa \|\pi^u\| \|\hat{\phi}_1^u\|_V + \kappa \|\pi^u\| L_{\hat{\gamma}} \|\hat{\phi}_1^u\|_V + \kappa \|\pi^u\| L_{\hat{\gamma}} R_5(\omega) + \|\phi_2^u\|_H$$
$$\leqslant R_5(\omega)$$

if

$$2\|\phi^u\|_E \leqslant \big(1 - \kappa \|\pi^u\| L_{\hat{\gamma}}\big) R_5(\omega)$$

for κ satisfying (42). To ensure (43) we determine R_5 by

$$\big(\big(1 - \kappa \|\pi^u\| L_{\hat{\gamma}}\big)(1 + L_{\hat{\gamma}}) + (1 + L_{\hat{\gamma}})\big) R_5 = R_4.$$

Hence, using (42), R_5 is tempered from below and smaller than R_4. Similar we obtain that $\mathcal{T}_{\omega,\hat{\phi}_1^u,\phi_2^u}$ is a contraction such that (41) has a unique fixed point. To see that this fixed point is a random variable it can be constructed point-wise by successive iteration starting from zero. Since the contraction constant is independent of $\hat{\phi}_1^u$, ϕ_2^u the fixed points depend Lipschitz continuously on these parameter contained in

$$Z(\omega) = \left\{ \big(\phi_1^u, \phi_2^u\big) \colon \|\phi_1^u + \phi_2^u\|_H \leqslant \frac{1}{2}\big(1 - \kappa \|\pi^u\| L_{\hat{\gamma}}\big) R_5(\omega) \right\}$$

where the Lipschitz constants are independent of ω, which implies that the Lipschitz constant of γ does not depending on ω. We denote the right-hand side of the last inequality by R_6. Then R_6 is tempered from below and $t \to R_6(\theta_t\omega)$ is continuous. Hence the unique solution of (40) gives us a Lipschitz continuous random homeomorphism

$$S(\omega, \cdot) \colon Z(\omega) \to \hat{Z}(\omega) := S\big(\omega, Z(\omega)\big) \subset \hat{U}(\omega). \qquad \square$$

Remark 5.13. We have proven the existence of invariant manifolds for the random dynamical systems associated with (9). By choosing z as (4) in the random transformation (14) and using Theorems 3.6 and 5.12, the local invariant manifold $M(\theta_t\omega)$ is an $\bar{\mathcal{F}}^t_{-\infty}$-adapted invariant manifold for the stochastic wave equation (9) for $t \geqslant 0$. However, if we choose z as (6), then the manifold $M(\theta_t\omega)$ is neither $\bar{\mathcal{F}}^t_{-\infty}$, $t \geqslant 0$, nor $\bar{\mathcal{F}}^\infty_t$, $t \leqslant 0$-adapted for the stochastic wave equation (9). Similarly, a stable, unstable manifold can never be adapted with respect to $\bar{\mathcal{F}}^t_{-\infty}$, $t \geqslant 0$, $\mathcal{F}^\infty_t$, $t \leqslant 0$.

Remark 5.14. Under the same assumptions as Theorem 5.9, one can show by analog arguments as the pseudo-unstable manifold that there exits a C^1 random pseudo-stable manifold for Eq. (23), which is given by the graph of a C^1 random map from E^s to E^u.

6. An application

Consider a scalar wave equation

$$u_{tt} + \alpha u - u_{xx} + a(x)u + b(u) = c(u)\dot{W}(t, x), \quad 0 < x < 1, \tag{44}$$

with the Dirichlet boundary condition

$$u(t, 0) = u(t, 1) = 0,$$

where a is a continuous function on $[0,1]$, b and c are twice continuously differentiable with $b(0) = b'(0) = c(0) = c'(0) = 0$, and $\dot{W}(t,x)$ is white noise.

Let $H = L^2(0,1)$ and $A = -\frac{\partial^2}{\partial x^2} + a(x)$ with domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$, where $H^2(0,1)$ and $H_0^1(0,1)$ are the usual Sobolev space. Clearly, A is a closed self-adjoint linear operator with dense domain and satisfies the conditions of Hypothesis A1. By the Sturm–Liouville Theorem, all eigenvalues $\{\lambda_n\}_{n\geqslant 1}$ are real and simple, which satisfy $\lim_{n\to\infty} \lambda_n = +\infty$, and the associated eigenvectors form a basis of H. The space V is $H_0^1(0,1)$ with norm

$$\|\phi\|_V^2 = \int_0^1 \big(\phi'(x)\big)^2\,dx.$$

By the Sobolev embedding, V is continuously embedded into the space of continuous functions on $[0,1]$:

$$\sup_{x\in[0,1]} \big|\phi(x)\big| \leqslant \|\phi\|_V \quad \text{for } \phi \in V.$$

Here we assume that $W(t,x)$ yields a twosided Wiener process with values in $U := V$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The covariance Q of the Wiener process Q is a symmetric positive operator with finite trace on V. Its trajectories are in $C(\mathbb{R}, V)$. In particular, we are able to define the Ornstein–Uhlenbeck introduced in Section 2.

The following basic inequality will be frequently used

$$\int_0^1 \big|f(x)g(x)\big|\,dx \leqslant \sup_{(0,1)} \big|f(x)\big| \big\|g(x)\big\|_H \leqslant \|f\|_V \|g\|_H,$$

for $f \in V$ and $g \in H$.

We consider this function b which has bounded first and second derivatives. Let

$$b'_{\max} = \sup_{s\in\mathbb{R}} \big|b'(s)\big| \quad \text{and} \quad b''_{\max} = \sup_{s\in\mathbb{R}} \big|b''(s)\big|.$$

Define

$$B(\phi)(x) := b\big(\phi(x)\big) \quad \text{for } \phi \in V$$

which is contained in V. By the properties of b we then have for $\phi_1, \phi_2 \in V$

$$\big\|B(\phi_2) - B(\phi_1)\big\|_H^2 \leqslant (b'_{\max})^2 \int_0^1 (\phi_2 - \phi_1)^2\,dx = (b'_{\max})^2 \|\phi_2 - \phi_1\|_H^2$$

and

$$\begin{aligned}\big\|B(\phi_2)-B(\phi_1)\big\|_H^2 &\leqslant 2(b'_{\max})^2\int_0^1 \big(|\phi_2(x)|^2+|\phi_1(x)|^2\big)\big(\phi_2(x)-\phi_1(x)\big)^2\,dx\\ &\leqslant 2(b''_{\max})^2\big(\|\phi_2\|_V^2+\|\phi_1\|_V^2\big)\|\phi_2-\phi_1\|_H^2.\end{aligned}$$

Hence, Hypothesis A2 holds for B.

Now, we verify Hypothesis A3. We consider function c which has bounded first and second derivatives. We use the similar notations $c'_{\max}$, $c''_{\max}$ as for b to denote their bounded. We set

$$C(\phi)[z](x)=c\big(\phi(x)\big)z(x)\quad\text{for } \phi,z\in V. \tag{45}$$

Clearly, $c(\phi(x))z(x)\in V$. The following estimates hold

$$\begin{aligned}&\big\|C(\phi_1)[z]-C(\phi_2)[z]\big\|_V^2\\ &\quad=\int_0^1\big(\big(c(\phi_1)z-c(\phi_2)z\big)'\big)^2\,dx\\ &\quad=\int_0^1\big(c'(\phi_1)\phi_1'z-c'(\phi_2)\phi_2'z+c(\phi_1)z'-c(\phi_2)z'\big)^2\,dx\\ &\quad\leqslant 4\int_0^1\big(c'(\phi_1)\phi_1'z-c'(\phi_1)\phi_2'z\big)^2\,dx+4\int_0^1\big(c'(\phi_1)\phi_2'z-c'(\phi_2)\phi_2'z\big)^2\,dx\\ &\qquad+2\int_0^1\big(c(\phi_1)z'-c(\phi_2)z'\big)^2\,dx.\end{aligned}\tag{46}$$

For the first and second term on the right-hand side we have the estimate

$$4(c''_{\max})^2\|\phi_1\|_V^2\|z\|_V^2\|\phi_1-\phi_2\|_V^2+4(c''_{\max})^2\|\phi_1-\phi_2\|_V^2\|z\|_V^2\|\phi_2\|_V^2.$$

Similar to the estimates for b we obtain for the last term in (46) the estimate

$$4(c''_{\max})^2\big(\|\phi_2\|_V^2+\|\phi_1\|_V^2\big)\|\phi_2-\phi_1\|_V^2\|z\|_V^2.$$

These imply that

$$\big\|C(\phi_1)[\cdot]-C(\phi_2)[\cdot]\big\|_{\mathcal{L}(U,V)}\leqslant\sqrt{12}c''_{\max}\|\phi_2-\phi_1\|_V^2.$$

Hence,

$$\big\|C(\phi_1)[\cdot]-C(\phi_2)[\cdot]\big\|_{\mathcal{L}(U,H)}\leqslant\sqrt{12}c''_{\max}\|\phi_2-\phi_1\|_V^2.$$

Thus, condition (i) in Hypothesis A3 holds.

Similarly, we have

$$\big\|C(\phi)[\cdot]\big\|_{\mathcal{L}(U,V)} \leqslant 2c'_{\max}\|\phi\|_V^2.$$

Hence, (10) in Hypothesis A3 holds.

The operator $C : V \to \mathcal{L}(U, H)$ defined in (45) has a Fréchet derivative by the assumptions on c, the Taylor formula and the Sobolev lemma:

$$\sup_{\|z\|_V=1} \big\|c(\phi_1)z - c(\phi_2)z - c'(\phi_2)(\phi_1-\phi_2)z\big\|_H \leqslant \frac{1}{2}c''_{\max}\|\phi_1-\phi_2\|_V^2.$$

Clearly, it makes also sense to consider $C'(\phi)[\cdot,\cdot]$ as a linear bounded operator from H to $\mathcal{L}^2(U, H)$. With respect to the Lipschitz continuity of C' we note

$$\begin{aligned}
&\big\|C'(\phi_1)[\cdot,\cdot] - C'(\phi_2)[\cdot,\cdot]\big\|_{\mathcal{L}^2}^2 \\
&\quad= \sup_{\|z\|_V=1,\ \|\psi\|_H=1} \int_0^1 \big(c'\big(\phi_1(x)\big)z(x)\psi(x) - c'\big(\phi_2(x)\big)z(x)\psi(x)\big)^2\,dx \\
&\quad\leqslant \sup_{\|z\|_V=1,\ \|\psi\|_H=1} \int_0^1 (c''_{\max})^2\big|\phi_1(x)-\phi_2(x)\big|^2\big|z(x)\big|^2\big|\psi(x)\big|^2\,dx \\
&\quad\leqslant (c''_{\max})^2\|\phi_1-\phi_2\|_V^2.
\end{aligned}$$

Therefore, conditions (ii) and (iii) in Hypothesis A3 hold.

Similarly, one can check that Hypothesis A4 is also satisfied. By applying Theorem 5.12, we have

Corollary 6.1. *There is a C^1 local random pseudo-unstable manifold for Eq.* (44) *in the space $H_0^1(0, 1)$.*

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