J. Differential Equations 252 (2012) 2370–2387

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On a class of singular boundary value problems with singular perturbation

Feng Xie

Department of Applied Mathematics, Donghua University, Shanghai 201620, PR China

ARTICLE INFO

Article history:
Received 12 May 2011
Revised 28 September 2011
Available online 26 October 2011

MSC:
34E15
34B16
34E05
34B08

Keywords:
Singular boundary value problem
Singular perturbations
Lower and upper solutions
Existence and uniqueness
Asymptotic estimate

ABSTRACT

In this paper we investigate a class of singular second order differential equations with singular perturbation subject to three-point boundary value conditions, whose solution exhibits a couple of boundary layers at two endpoints. We first establish a lower–upper solutions theorem by using the Schauder fixed point theorem. By the asymptotic expansions and the lower–upper solutions theorem we obtain the existence, asymptotic estimates and uniqueness for the proposed problem. Several examples are given for illustrating our results.

1. Introduction

We consider the singular boundary value problem with singular perturbation of the form

$$\varepsilon \frac{d^2 y}{dx^2} - \frac{1}{x-\lambda} f(x,y) \frac{dy}{dx} + g(x,y) = 0, \quad x \in (0,\lambda) \cup (\lambda,1), \tag{1.1}$$

$$y(0) = A, \qquad y(\lambda) = \mu, \qquad y(1) = B, \tag{1.2}$$

where ε is a small and positive parameter, and A, B, μ and $\lambda \in [0,1]$ are given constants. The functions $f(x,y)$ and $g(x,y)$ are smooth on $[0,1] \times \mathbb{R}$. For $\lambda = 0$ or 1, $\mu = A$ or B, respectively.

E-mail address: fxie@dhu.edu.cn.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.003

A special example with $f(x, y) = -1$, $g(x, y) = -y$ and $\lambda = 0$ which exhibits a right boundary layer was investigated by Bender and Orszag [3], and its asymptotic solution was obtained using the matching procedures. Singular boundary value problems arise in many physical models, such as plasma physics [8], the nonlinear circular membrane [20], Homann flow [21], and so on. Recently, many theories and numerical techniques have been developed and applied to study various singular boundary value problems; see for instance [1,2,8,12,17–19] and references therein.

The singular perturbation theory originating from celestial mechanics has been an important tool for dealing with nonlinear problems. An important character of singularly perturbed problems is that the solution rapidly changes on a very narrow interval, which is called a boundary layer or an interval layer. The study on singularly perturbed problems has been always a hot topic since the last century, and considerable literature has grown up relating to these problems. Singularly perturbed boundary value problems having no singular coefficients have been understood very well; see for instance [4, 6,7,13,15]. However, singular boundary value problems with singular perturbation have received few attention. There are only a few literatures of research on this kind of problem, and they focus mainly on numerical aspect [9,14,16].

Singular boundary value problems with singular perturbation can be traced to the well-known Lagerstrom model for flow at low Reynolds number [11]

$$y'' + \frac{k}{x}y' + cyy' + b(y')^2 = 0,$$
$$y(\varepsilon) = 0, \qquad y(\infty) = 0,$$

for $k \geqslant 1$, $c > 0$, $b \geqslant 0$. In [10], Kelley extended the above equation to more general case but limited on a finite interval $[\varepsilon, 1]$ by using the theory of differential inequalities, and established the sufficient condition ensuring the existence of a solution which asymptotically approximates the solution of the reduced problem on the interval $(0, 1]$.

In the present paper, we are devoted to the asymptotic analysis of (1.1)–(1.2). Due to the singularity at $x = \lambda$, Eq. (1.1) subject to a two-point boundary condition may generally have infinitely many solutions. Therefore we add a condition at the singularity. An interesting phenomenon is that the solution of (1.1)–(1.2) exhibits two boundary layers at two endpoints for an internal singularity, which generally occurs for semi-linear problems (namely, having no first order derivative in the equation) of smooth second order differential equations.

Our analysis proceeds in two steps: constructing a formal approximation of the solution and proving the uniformly validity of this approximation. The first step is almost standard. However we make a modification on the correction of boundary layers, which yields a better approximation. In order to prove the validity of asymptotic solution and obtain an estimate of the remainder term we need to establish the lower–upper solutions theory, which is contained in Section 2. It should be noticed that although some lower–upper solutions theorems for singular boundary value problems have been obtained (see [5], for instance), our case has not been covered.

The remainder of this paper is organized as follows. In Section 2 we establish the lower–upper solutions theory for a class of singular three-point boundary value problems by the Schauder fixed point theorem, which will be used to prove the existence and uniqueness for the corresponding singularly perturbed problems. By the asymptotic expansions and the lower–upper solutions theory established in Section 2, the existence, asymptotic estimates and uniqueness of solutions for the problem (1.1)–(1.2) are obtained in Section 3. Several illustrating examples are given in Section 4.

2. Two basic lemmas

Let us consider the singular three-point boundary value problem

$$\frac{d^2y}{dx^2} = \frac{F(x, y)}{x - \lambda}\frac{dy}{dx} + G(x, y), \quad x \in (a, \lambda) \cup (\lambda, b), \tag{2.1}$$

$$y(a) = A, \qquad y(\lambda) = \mu, \qquad y(b) = B, \tag{2.2}$$

for $\lambda \in [a,b]$, for given constants A, B, μ, and for continuous functions $F(x,y)$ and $G(x,y)$ on the domain $[a,b]\times\mathbb{R}$. For $\lambda = a$ or b, $\mu = A$ or B, respectively.

Definition 2.1. We say that a function $\alpha \in C([a,b]) \cap C^1([a,\lambda)\cup(\lambda,b])$ is a lower solution of the problem (2.1) and (2.2) if

$$\begin{aligned} &\alpha''(x) \geqslant \frac{F(x,\alpha)}{x-\lambda}\alpha'(x) + G(x,\alpha), \quad \text{for } x \in (a,\lambda)\cup(\lambda,b),\\ &\alpha(a) \leqslant A, \qquad \alpha(\lambda) \leqslant \mu, \qquad \alpha(b) \leqslant B. \end{aligned} \tag{2.3}$$

We say that a function $\beta \in C([a,b]) \cap C^1([a,\lambda)\cup(\lambda,b])$ is an upper solution of the problem (2.1) and (2.2) if

$$\begin{aligned} &\beta''(x) \leqslant \frac{F(x,\beta)}{x-\lambda}\beta'(x) + G(x,\beta), \quad \text{for } x \in (a,\lambda)\cup(\lambda,b),\\ &\beta(a) \geqslant A, \qquad \beta(\lambda) \geqslant \mu, \qquad \beta(b) \geqslant B. \end{aligned} \tag{2.4}$$

Lemma 2.1. *Assume that α and β are lower and upper solutions of the problem* (2.1) *and* (2.2) *such that $\alpha \leqslant \beta$, and the functions $F(x,y)$ and $G(x,y)$ are continuous on the set $D_\alpha^\beta = \{(x,y) \in [a,b]\times\mathbb{R} \mid \alpha(x) \leqslant y \leqslant \beta(x)\}$. Then the problem* (2.1) *and* (2.2) *has at least one solution $y \in C([a,b]) \cap C^1([a,\lambda)\cup(\lambda,b])$ such that for all $x \in [a,b]$*

$$\alpha(x) \leqslant y(x) \leqslant \beta(x).$$

Proof. For $a < \lambda < b$, the problem (2.1) and (2.2) can be divided into the following two boundary value problems

$$\frac{d^2y}{dx^2} = \frac{F(x,y)}{x-\lambda}\frac{dy}{dx} + G(x,y), \quad x \in (a,\lambda), \tag{2.5}$$

$$y(a) = A, \qquad y(\lambda) = \mu, \tag{2.6}$$

and

$$\frac{d^2y}{dx^2} = \frac{F(x,y)}{x-\lambda}\frac{dy}{dx} + G(x,y), \quad x \in (\lambda,b), \tag{2.7}$$

$$y(\lambda) = \mu, \qquad y(b) = B, \tag{2.8}$$

with $\alpha(\lambda) \leqslant \mu \leqslant \beta(\lambda)$. Clearly, α and β are lower and upper solutions of the above two problems such that $\alpha \leqslant \beta$. For $\lambda = b$ or $\lambda = a$, the original problem can be reduced to the problem (2.5) and (2.6) or the problem (2.7) and (2.8), respectively. In the following proof, we only consider the problem (2.5) and (2.6) for $a < \lambda < b$.

Let us define the following modifications of the functions in the right-hand side of (2.5)

$$R(x,y,v) = \begin{cases} \frac{F(x,\alpha(x))}{x-\lambda}v + G(x,\alpha(x)) + \frac{y-\alpha(x)}{1+|y-\alpha(x)|}, & \text{if } y < \alpha(x),\\ \frac{F(x,y)}{x-\lambda}v + G(x,y), & \text{if } \alpha(x) \leqslant y \leqslant \beta(x),\\ \frac{F(x,\beta(x))}{x-\lambda}v + G(x,\beta(x)) + \frac{y-\beta(x)}{1+|y-\beta(x)|}, & \text{if } y > \beta(x), \end{cases}$$

and

$$H(x,y,z)=\begin{cases} R(x,y,\frac{-N}{x-\lambda}), & \text{if } z<-N,\\ R(x,y,\frac{z}{x-\lambda}), & \text{if } -N\leqslant z\leqslant N,\\ R(x,y,\frac{N}{x-\lambda}), & \text{if } z>N, \end{cases}$$

where $N>0$ is a large enough number such that

$$N>\max\Big\{\sup_{x\in[a,\lambda]}\big|(x-\lambda)\alpha'(x)\big|,\sup_{x\in[a,\lambda]}\big|(x-\lambda)\beta'(x)\big|\Big\}+|A-\mu|,$$

and

$$\int_{|A-\mu|}^{N}\frac{1}{s+1}\,\mathrm{d}s>\frac{M_1+1}{|A-\mu|+1}\max_{x\in[a,\lambda]}\big|\beta(x)-\alpha(x)\big|+\frac{M_2}{|A-\mu|+1}, \tag{2.9}$$

while

$$M_1=\max_{(x,y)\in[a,\lambda]\times D_\alpha^\beta}\big|F(x,y)\big|,\qquad M_2=\max_{(x,y)\in[a,\lambda]\times D_\alpha^\beta}\big|(x-\lambda)G(x,y)\big|(\lambda-a).$$

Consider the modified problem

$$y''(x)+H\big(x,y,(x-\lambda)y'\big)=0, \tag{2.10}$$

$$y(a)=A,\qquad y(\lambda)=\mu. \tag{2.11}$$

Let us write the boundary value problem (2.10) and (2.11) as an integral equation

$$y(x)=\int_a^\lambda G(x,s)H\big(s,y(s),(s-\lambda)y'(s)\big)\,\mathrm{d}s+w(x),$$

where

$$G(x,s)=\begin{cases}\frac{(\lambda-s)(x-a)}{\lambda-a}, & a\leqslant x\leqslant s\leqslant\lambda,\\ \frac{(\lambda-x)(s-a)}{\lambda-a}, & a\leqslant s\leqslant x\leqslant\lambda,\end{cases}$$

and

$$w(x)=\frac{\lambda A-\mu a}{\lambda-a}+\frac{\mu-A}{\lambda-a}x.$$

Define an operator $T:X\to X$ as follows

$$(Ty)(x)=\int_a^\lambda G(x,s)H\big(s,y(s),(s-\lambda)y'(s)\big)\,\mathrm{d}s+w(x),$$

where X is a Banach space defined by $X = \{y \in C([a,\lambda]) \cap C^1([a,\lambda)),\ (x-\lambda)y'(x) \in C([a,\lambda])\}$ endowed with the norm

$$\|y\| = \max\Big\{\sup_{x\in[a,\lambda]}\big|y(x)\big|,\ \sup_{x\in[a,\lambda]}\big|(x-\lambda)y'(x)\big|\Big\}.$$

It can be shown that T is completely continuous and bounded by the Arzelà–Ascoli theorem. It follows from the Schauder fixed point theorem that T has at least one fixed point $y(x) \in X$.

We are now ready to prove that any solution $y(x)$ of the problem (2.10) and (2.11) satisfies $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ and $|(x-\lambda)y'(x)| \leqslant N$ for $x \in [a,\lambda]$. Let us first prove that $\alpha(x) \leqslant y(x)$. Suppose, on the contrary, that the function $h(x) = \alpha(x) - y(x)$ has a positive maximum at some $x_0 \in [a,\lambda]$. Considering the fact $h(a) = \alpha(a) - y(a) \leqslant A - A = 0$ and $h(\lambda) = \alpha(\lambda) - y(\lambda) \leqslant \mu - \mu = 0$, we have $x_0 \in (a,\lambda)$, which implies that $h(x_0) > 0, h'(x_0) = 0, h''(x_0) \leqslant 0$. On the other hand, in view of $|(x_0-\lambda)y'(x_0)| = |(x_0-\lambda)\alpha'(x_0)| < N$, we have

$$\begin{aligned} h''(x_0) &= \alpha''(x_0) - y''(x_0) \\ &\geqslant \frac{F(x_0,\alpha(x_0))}{x_0-\lambda}\alpha'(x_0) + G\big(x_0,\alpha(x_0)\big) \\ &\quad - \left[\frac{F(x_0,\alpha(x_0))}{x_0-\lambda}y'(x_0) + G\big(x_0,\alpha(x_0)\big) + \frac{y(x_0)-\alpha(x_0)}{1+|y(x_0)-\alpha(x_0)|}\right] \\ &> 0, \end{aligned}$$

which yields a contradiction. This shows that $\alpha(x) \leqslant y(x)$ for $x \in [a,\lambda]$. In a similar way, we can prove that $y(x) \leqslant \beta(x)$ for $x \in [a,\lambda]$.

We turn to prove $|(x-\lambda)y'(x)| \leqslant N$ for $x \in [a,\lambda]$. Thanks to $|y(a) - y(\lambda)| = |A-\mu|$, there exists a $\tau \in (a,\lambda)$ such that $(\lambda-a)|y'(\tau)| \leqslant |A-\mu|$. So, $|(\tau-\lambda)y'(\tau)| \leqslant |A-\mu|$. Suppose that $|(x-\lambda)y'(x)| \leqslant N$ is not satisfied. Without loss of generality, we assume that there exist $x_1 \in [a,\tau)$ and $x_2 \in (x_1,\tau)$ such that

$$(x_1-\lambda)y'(x_1) = |A-\mu|, \qquad (x_2-\lambda)y'(x_2) = N, \qquad |A-\mu| \leqslant (x-\lambda)y'(x) \leqslant N, \quad x \in [x_1,x_2].$$

Then we have

$$\begin{aligned} \int_{|A-\mu|}^{N} \frac{1}{s+1}\,\mathrm{d}s &= \int_{x_1}^{x_2} \frac{\mathrm{d}(x-\lambda)\,y'(x)}{(x-\lambda)y'(x)+1} \\ &= \int_{x_1}^{x_2} \frac{(x-\lambda)y''(x)+y'(x)}{(x-\lambda)y'(x)+1}\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} \frac{(x-\lambda)[\frac{F(x,y(x))}{x-\lambda}y'(x) + G(x,y(x))] + y'(x)}{(x-\lambda)y'(x)+1}\,\mathrm{d}x \\ &\leqslant \frac{(M_1+1)}{|A-\mu|+1}\int_{x_1}^{x_2} y'(x)\,\mathrm{d}x + \int_a^{\lambda} \frac{\max_{(x,y)\in D_\alpha^\beta}|(x-\lambda)G(x,y(x))|}{|A-\mu|+1}\,\mathrm{d}x \\ &\leqslant \frac{M_1+1}{|A-\mu|+1}\max_{x\in[a,\lambda]}\big|\beta(x)-\alpha(x)\big| + \frac{M_2}{|A-\mu|+1}, \end{aligned}$$

which is contradictory to (2.9).

Therefore, the solution of (2.10) and (2.11) is also that of (2.5) and (2.6) and satisfies $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ for $x \in [a, \lambda]$. In a similar fashion, we can show that (2.7) and (2.8) has at least a solution $y \in C([\lambda, b]) \cap C^1((\lambda, b])$ such that $\alpha(x) \leqslant y(x) \leqslant \beta(x)$ for all $x \in [\lambda, b]$. The proof of the lemma is completed. □

Lemma 2.2. *Assume that α and β are lower and upper solutions of the problem* (2.1) *and* (2.2)*, respectively, and one of the inequalities* (2.3) *and* (2.4) *is strict. If the functions $F(x, y)$ and $G(x, y)$ are continuous on $(x, y) \in [a, b] \times \mathbb{R}$ and satisfy*

$$(x-\lambda)\frac{\partial F}{\partial y}(x, y) \geqslant 0, \qquad \frac{\partial G}{\partial y}(x, y) \geqslant 0, \quad \textit{for } (x, y) \in [a, b] \times \mathbb{R}, \tag{2.12}$$

then the solution of the problem (2.1) *and* (2.2) *is unique.*

Proof. To establish the uniqueness, it suffices to show that $\alpha(x) \leqslant \beta(x)$ for $x \in [a, b]$. Let $w(x) = \alpha(x) - \beta(x)$, for $x \in [a, b]$. Suppose that $w(x) \leqslant 0$ is not true for $x \in [a, b]$. Then, noting that $w(a) = \alpha(a) - \beta(a) \leqslant 0$, $w(\lambda) = \alpha(\lambda) - \beta(\lambda) \leqslant 0$, and $w(b) = \alpha(b) - \beta(b) \leqslant 0$, $w(x)$ has a positive maximum at some $x_0 \in (a, \lambda) \cup (\lambda, b)$, which implies $w(x_0) > 0$, $w'(x_0) = 0$ and $w''(x_0) \leqslant 0$. Without loss of generality, we assume

$$\alpha''(x_0) > \frac{F(x_0, \alpha(x_0))}{x_0 - \lambda}\alpha'(x_0) + G\big(x_0, \alpha(x_0)\big).$$

On the other hand, with (2.12) we have

$$\alpha''(x_0) \leqslant \beta''(x_0) \leqslant \frac{F(x_0, \beta(x_0))}{x_0 - \lambda}\beta'(x_0) + G\big(x_0, \beta(x_0)\big) \leqslant \frac{F(x_0, \alpha(x_0))}{x_0 - \lambda}\alpha'(x_0) + G\big(x_0, \alpha(x_0)\big),$$

which yields a contradiction. Therefore, $\alpha(x) \leqslant \beta(x)$ for $x \in [a, b]$. This ends the proof. □

3. Main results

In this section, we are interested in the asymptotic behavior of solution with respect to the small parameter ε, as well as the existence and uniqueness for the singular boundary value problem (1.1) and (1.2). To avoid tedious bookkeeping we only consider the approximation of zero order.

We first make two basic assumptions.

(H1) The functions $f(x, y)$ and $g(x, y)$ are C^1-smooth on $[0, 1] \times \mathbb{R}$.
(H2) There exists a positive constant σ_0 such that $f(x, y) \geqslant \sigma_0 > 0$ for $(x, y) \in [0, 1] \times \mathbb{R}$.

Because the case $\lambda = 1$ reduces to that $\lambda = 0$ by the transformation $\bar{x} = 1 - x$, in what follows, we distinguish two cases: $0 < \lambda < 1$ and $\lambda = 0$.

3.1. *Case* $0 < \lambda < 1$

For $0 < \lambda < 1$, the boundary layers occur at the two endpoints $x = 0$ and $x = 1$. We further assume that:

(H3) The reduced problem

$$f(x, y)\frac{dy}{dx} = (x - \lambda)g(x, y), \qquad y(\lambda) = \mu$$

has a solution $y = \varphi(x) \in C^2([0, 1])$.

Noting that the solution $\varphi(x)$ of the reduced problem do not generally meet the boundary conditions $y(0) = A$ and $y(1) = B$, we need introduce the boundary layer correcting terms $u(\tau)$ and $v(t)$ with $\tau = x/\varepsilon$ and $t = (1-x)/\varepsilon$. Setting $y(x) = \varphi(x) + u(\tau) + v(t)$ in (1.1), we obtain

$$\varepsilon\varphi''(x) + \frac{1}{\varepsilon}\frac{d^2u}{d\tau^2} + \frac{1}{\varepsilon}\frac{d^2v}{dt^2} - \frac{f(x,\varphi(x)+u(\tau)+v(t))}{x-\lambda}\left(\varphi'(x) + \frac{1}{\varepsilon}\frac{du}{d\tau} - \frac{1}{\varepsilon}\frac{dv}{dt}\right)$$
$$+ g\bigl(x, \varphi(x) + u(\tau) + v(t)\bigr) = 0.$$

Considering the approximation of zero order for the boundary layer terms, we have

$$\frac{d^2u_0}{d\tau^2} - \frac{f(0,\varphi(0)+u_0(\tau))}{\varepsilon\tau-\lambda}\frac{du_0}{d\tau} = 0, \tag{3.1}$$

$$\frac{d^2v_0}{dt^2} + \frac{f(1,\varphi(1)+v_0(t))}{1-\varepsilon t-\lambda}\frac{dv_0}{dt} = 0. \tag{3.2}$$

Here we have retained ε-term in the denominators. As we will see, in doing so, a better approximation for the boundary layer terms $u_0(\tau)$ and $v_0(t)$ can be obtained. The corresponding boundary conditions become

$$u_0(0) = A - \varphi(0), \qquad u_0\left(\frac{\lambda}{\varepsilon}\right) = 0, \tag{3.3}$$

$$v_0(0) = B - \varphi(1), \qquad v_0\left(\frac{1-\lambda}{\varepsilon}\right) = 0. \tag{3.4}$$

Due to the continuity of $f(x, y)$ we denote by

$$\sigma_1 = \max\bigl\{f\bigl(0, \varphi(0) + u\bigr)\colon -\bigl|A - \varphi(0)\bigr| \leqslant u \leqslant \bigl|A - \varphi(0)\bigr|\bigr\},$$
$$\sigma_2 = \max\bigl\{f\bigl(1, \varphi(1) + v\bigr)\colon -\bigl|B - \varphi(1)\bigr| \leqslant v \leqslant \bigl|B - \varphi(1)\bigr|\bigr\}.$$

The following two lemmas are concerned with the asymptotic behavior of the boundary layer terms.

Lemma 3.1. *Under the assumptions* (H1) *and* (H2), *for sufficiently small* $\varepsilon > 0$, *the boundary value problem* (3.1) *and* (3.3) *has a solution* $u_0(\tau)$ *satisfying the following estimates*:

$$\bigl(A - \varphi(0)\bigr)\left(\frac{\lambda - \varepsilon\tau}{\lambda}\right)^{\frac{\underline{\sigma}+\varepsilon}{\varepsilon}} \leqslant u_0(\tau) \leqslant \bigl(A - \varphi(0)\bigr)\left(\frac{\lambda - \varepsilon\tau}{\lambda}\right)^{\frac{\overline{\sigma}+\varepsilon}{\varepsilon}} \tag{3.5}$$

and

$$\left|\frac{du_0}{d\tau}\right| \leqslant 2\bigl|A - \varphi(0)\bigr|(\sigma_1 + \varepsilon)\left(\frac{\lambda - \varepsilon\tau}{\lambda}\right)^{\frac{\sigma_0}{\varepsilon}}, \tag{3.6}$$

where

$$\underline{\sigma} = \begin{cases} \sigma_1, & \textit{if } A - \varphi(0) > 0, \\ \sigma_0, & \textit{if } A - \varphi(0) < 0, \end{cases} \qquad \overline{\sigma} = \begin{cases} \sigma_0, & \textit{if } A - \varphi(0) > 0, \\ \sigma_1, & \textit{if } A - \varphi(0) < 0. \end{cases}$$

Lemma 3.2. *Under the assumptions* (H1) *and* (H2), *for sufficiently small* $\varepsilon > 0$, *the boundary value problem* (3.2) *and* (3.4) *has a solution* $v_0(t)$ *satisfying the following estimates*:

$$\bigl(B-\varphi(1)\bigr)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\underline{\underline{\sigma}}+\varepsilon}{\varepsilon}} \leqslant v_0(t) \leqslant \bigl(B-\varphi(1)\bigr)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\overline{\overline{\sigma}}+\varepsilon}{\varepsilon}} \tag{3.7}$$

and

$$\left|\frac{dv_0}{dt}\right| \leqslant 2\bigl|B-\varphi(1)\bigr|(\sigma_2+\varepsilon)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\sigma_0}{\varepsilon}}, \tag{3.8}$$

where

$$\underline{\underline{\sigma}} = \begin{cases} \sigma_2, & \text{if } B-\varphi(1)>0,\\ \sigma_0, & \text{if } B-\varphi(1)<0,\end{cases} \qquad \overline{\overline{\sigma}} = \begin{cases} \sigma_0, & \text{if } B-\varphi(1)>0,\\ \sigma_2, & \text{if } B-\varphi(1)<0.\end{cases}$$

The proofs of the above lemmas are essentially similar, so we only present the proof of Lemma 3.2.

Proof of Lemma 3.2. We choose the auxiliary functions

$$\alpha(t) = \bigl(B-\varphi(1)\bigr)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\underline{\underline{\sigma}}+\varepsilon}{\varepsilon}},$$

$$\beta(t) = \bigl(B-\varphi(1)\bigr)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\overline{\overline{\sigma}}+\varepsilon}{\varepsilon}}.$$

It is easy to see that

$$\alpha(t) \leqslant \beta(t), \qquad 0 \leqslant t \leqslant \frac{1-\lambda}{\varepsilon},$$

$$\alpha(0)=\beta(0)=B-\varphi(1), \qquad \alpha\left(\frac{1-\lambda}{\varepsilon}\right)=\beta\left(\frac{1-\lambda}{\varepsilon}\right)=0.$$

Moreover, we have

$$\begin{aligned} &\alpha''(t)+\frac{f(1,\varphi(1)+\alpha(t))}{1-\varepsilon t-\lambda}\alpha'(t)\\ &= \bigl(B-\varphi(1)\bigr)\frac{(\underline{\underline{\sigma}}+\varepsilon)}{(1-\lambda)^2}\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\underline{\underline{\sigma}}-\varepsilon}{\varepsilon}}\bigl(\underline{\underline{\sigma}}-f\bigl(1,\varphi(1)+\alpha(t)\bigr)\bigr)\\ &\geqslant 0 \end{aligned}$$

and

$$\begin{aligned} &\beta''(t)+\frac{f(1,\varphi(1)+\beta(t))}{1-\varepsilon t-\lambda}\alpha'(t)\\ &= \bigl(B-\varphi(1)\bigr)\frac{(\overline{\overline{\sigma}}+\varepsilon)}{(1-\lambda)^2}\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\overline{\overline{\sigma}}-\varepsilon}{\varepsilon}}\bigl(\overline{\overline{\sigma}}-f\bigl(1,\varphi(1)+\alpha(t)\bigr)\bigr)\\ &\leqslant 0. \end{aligned}$$

It follows from Lemma 2.1 that the problem (3.2) and (3.4) has a solution $v_0(t)$ such that (3.5) holds.

Now we turn to (3.8). Using the definition of limits and (3.7) we obtain

$$\frac{(\underline{\underline{\sigma}}+\varepsilon)(B-\varphi(1))}{1-\lambda} \leqslant \frac{dv_0}{dt}(0) \leqslant \frac{(\overline{\overline{\sigma}}+\varepsilon)(B-\varphi(1))}{1-\lambda}. \tag{3.9}$$

Let $z = v_0'(t)$. With (H2), from (3.2) we have

$$\frac{dz}{z\,dt} = -\frac{f(1, \varphi(1)+v_0(t))}{1-\varepsilon t-\lambda} \leqslant -\frac{\sigma_0}{1-\varepsilon t-\lambda}.$$

With (3.9) and integrating the above inequality from 0 to t we have the desired bound

$$\left|z(t)\right| \leqslant \frac{|B-\varphi(1)|(\sigma_2+\varepsilon)}{1-\lambda}\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\sigma_0}{\varepsilon}}.$$

This ends the proof of Lemma 3.1. □

Remark. From the estimates (3.5) and (3.7) we see that the boundary layer terms $u_0(x/\varepsilon)$ and $v_0((1-x)/\varepsilon)$ decay exponentially within the right neighborhood of $x=0$ and within the left neighborhood of $x=1$, respectively. As a matter of fact, here $u_0(x/\varepsilon)$ and $v_0((1-x)/\varepsilon)$ decay faster than the usual boundary layer terms in the problem having no singular coefficients. This can be seen from the following facts

$$\left(\frac{\lambda-x}{\lambda}\right)^{\frac{\sigma}{\varepsilon}} = \exp\left(\frac{\sigma}{\varepsilon}\ln\frac{\lambda-x}{\lambda}\right) \leqslant \exp\left(\frac{-\sigma}{\lambda}\frac{x}{\varepsilon}\right)$$

for $0 \leqslant x < \lambda$, and

$$\left(\frac{x-\lambda}{1-\lambda}\right)^{\frac{\sigma}{\varepsilon}} = \exp\left(\frac{\sigma}{\varepsilon}\ln\frac{x-\lambda}{1-\lambda}\right) \leqslant \exp\left(\frac{-\sigma}{1-\lambda}\frac{1-x}{\varepsilon}\right)$$

for $\lambda < x \leqslant 1$ and $\sigma > 0$.

Theorem 3.1. *Let the conditions* (H1), (H2) *and* (H3) *hold. Moreover, we assume that*

$$\frac{\partial f}{\partial y}\big(x, \varphi(x)+u_0(x/\varepsilon)\big)\big(A-\varphi(0)\big) \geqslant 0, \quad \textit{for } x \in (0, \lambda), \tag{3.10}$$

$$\frac{\partial f}{\partial y}\big(x, \varphi(x)+v_0\big((1-x)/\varepsilon\big)\big)\big(B-\varphi(1)\big) \geqslant 0, \quad \textit{for } x \in (\lambda, 1). \tag{3.11}$$

Then for sufficiently small $\varepsilon > 0$ *the boundary value problem* (1.1) *and* (1.2) *has a solution* $y(x, \varepsilon)$ *satisfying*:

$$y(x, \varepsilon) = \varphi(x) + u_0\left(\frac{x}{\varepsilon}\right) + v_0\left(\frac{1-x}{\varepsilon}\right) + \mathcal{O}(\varepsilon), \tag{3.12}$$

where $u_0(x/\varepsilon)$ *and* $v_0((1-x)/\varepsilon)$ *are defined in Lemmas* 3.1 *and* 3.2.

Proof. It follows from the assumptions (H1) and (H3) that there exists a positive constant K such that for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
&\left|\varphi''(x)\right| \leqslant K, \quad x \in [0,1],\\
&\left|f\big(\varepsilon\tau, \varphi(\varepsilon\tau) + u_0\big) - f\big(0, \varphi(0) + u_0\big)\right| \leqslant \frac{K\tau\varepsilon}{2(\sigma_1 + \varepsilon)}, \quad \tau \in \left[0, \frac{\lambda}{\varepsilon}\right],\\
&\left|f\big(1 - \varepsilon t, \varphi(1 - \varepsilon t) + v_0\big) - f\big(1, \varphi(1) + v_0\big)\right| \leqslant \frac{K(1-\lambda)t\varepsilon}{2(\sigma_2 + \varepsilon)}, \quad t \in \left[0, \frac{1-\lambda}{\varepsilon}\right],\\
&\left|\frac{\partial g}{\partial y}(x, y) - \frac{g(x, \varphi(x))}{f(x, \varphi(x))}\frac{\partial f}{\partial y}(x, y)\right| \leqslant K, \quad (x, y) \in [0,1] \times [-\vartheta - K\varepsilon, \vartheta + K\varepsilon],
\end{aligned}$$

where $\vartheta = \max\{|\varphi(x)| + |A - \varphi(0)| + |B - \varphi(1)|,\ 0 \leqslant x \leqslant 1\}$.

From the construction of asymptotic solution we select the barrier functions

$$\alpha(x) = \begin{cases} \varphi(x) + u_0(\frac{x}{\varepsilon}) - \varepsilon u_1(\frac{x}{\varepsilon}) - \gamma(x)\varepsilon, & x \in [0, \lambda],\\ \varphi(x) + v_0(\frac{1-x}{\varepsilon}) - \varepsilon v_1(\frac{1-x}{\varepsilon}) - \gamma(x)\varepsilon, & x \in [\lambda, 1], \end{cases}$$

$$\beta(x) = \begin{cases} \varphi(x) + u_0(\frac{x}{\varepsilon}) + \varepsilon u_1(\frac{x}{\varepsilon}) + \gamma(x)\varepsilon, & x \in [0, \lambda],\\ \varphi(x) + v_0(\frac{1-x}{\varepsilon}) + \varepsilon v_1(\frac{1-x}{\varepsilon}) + \gamma(x)\varepsilon, & x \in [\lambda, 1], \end{cases}$$

where

$$\begin{aligned}
u_1(\tau) &= \frac{K|A - \varphi(0)|p(\tau)}{2(\sigma_0 + \varepsilon)(\sigma_0 + 2\varepsilon)(\sigma_0 + 3\varepsilon)}\left(\frac{\lambda - \varepsilon\tau}{\lambda}\right)^{\frac{\sigma_0}{\varepsilon}}, \quad \tau \in \left[0, \frac{\lambda}{\varepsilon}\right],\ \tau = \frac{x}{\varepsilon},\\
v_1(t) &= \frac{K|B - \varphi(1)|(1-\lambda)q(t)}{2(\sigma_0 + \varepsilon)(\sigma_0 + 2\varepsilon)(\sigma_0 + 3\varepsilon)}\left(\frac{1 - \varepsilon t - \lambda}{1 - \lambda}\right)^{\frac{\sigma_0}{\varepsilon}}, \quad t \in \left[0, \frac{1-\lambda}{\varepsilon}\right],\ t = \frac{1-x}{\varepsilon},
\end{aligned}$$

and

$$\gamma(x) = \begin{cases} \frac{L}{K}[\exp(\frac{K(x-\lambda)^2}{2\sigma_1}) - 1], & x \in [0, \lambda],\\ \frac{L}{K}[\exp(\frac{K(x-\lambda)^2}{2\sigma_0}) - 1], & x \in [\lambda, 1], \end{cases}$$

for $p(\tau) = 2\lambda^2(\lambda + \sigma_0 + 3\varepsilon) + 2\lambda\sigma_0(\lambda + \sigma_0 + 3\varepsilon)\tau + (\sigma_0 + \varepsilon)(\lambda\sigma_0 - 2\sigma_0\varepsilon - 6\varepsilon^2)\tau^2 - \varepsilon(\sigma_0 + \varepsilon)(\sigma_0 + 2\varepsilon)\tau^3$ and $q(t) = 2(1-\lambda)(1 - \lambda + \sigma_0 + 3\varepsilon) + 2(\sigma_0 + \varepsilon)(1 - \lambda + \sigma_0 + 3\varepsilon)t + (\sigma_0 + \varepsilon)(\sigma_0 + 2\varepsilon)t^2$, and for a positive constant L to be determined later.

It can be immediately verified that the following properties hold:

$$\frac{du_1}{d\tau} = \frac{-K|A - \varphi(0)|\tau(\tau + 2)}{2}\left(\frac{\lambda - \varepsilon\tau}{\lambda}\right)^{\frac{\sigma_0}{\varepsilon}} \leqslant 0, \tag{1}$$

$$0 \leqslant u_1(\tau) \leqslant \frac{K|A - \varphi(0)|(2 + \sigma_0)}{\sigma_0^3}, \quad \text{for } \tau \in \left[0, \frac{\lambda}{\varepsilon}\right];$$

(2)
$$\frac{dv_1}{dt} = \frac{-K|B-\varphi(1)|t(t+2)}{2}\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\sigma_0}{\varepsilon}} \leqslant 0,$$

$$0 \leqslant v_1(t) \leqslant \frac{K|B-\varphi(1)|(2+\sigma_0)}{\sigma_0^3}, \quad \text{for } t \in \left[0, \frac{1-\lambda}{\varepsilon}\right];$$

(3)
$$0 \leqslant \gamma(x) \leqslant \frac{L}{K}\left[\exp\left(\frac{K}{2\sigma_0}\right)-1\right], \qquad \left|\gamma''(x)\right| \leqslant \frac{L(\sigma_1+K)}{\sigma_0^2}\exp\left(\frac{K}{2\sigma_0}\right), \quad \text{for } x \in [0,1],$$

$$\gamma'(x) \leqslant 0 \quad \text{for } x \in [0,\lambda] \quad \text{and} \quad \gamma'(x) \geqslant 0 \quad \text{for } x \in [\lambda,1];$$

(4) $u_1(\tau)$ is a solution of the equation

$$\frac{d^2u}{d\tau^2} + \frac{\sigma_0}{\lambda-\varepsilon\tau}\frac{du}{d\tau} + K\left|A-\varphi(0)\right|(\tau+1)\left(\frac{\lambda-\varepsilon\tau}{\lambda}\right)^{\frac{\sigma_0}{\varepsilon}} = 0;$$

(5) $v_1(t)$ is a solution of the equation

$$\frac{d^2v}{dt^2} + \frac{\sigma_0}{1-\varepsilon t-\lambda}\frac{dv}{dt} + K\left|B-\varphi(1)\right|(t+1)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\sigma_0}{\varepsilon}} = 0;$$

(6) $\gamma(x)$ is a solution of the equation

$$\frac{\sigma}{x-\lambda}\frac{d\gamma}{dx} - K\gamma(x) - L = 0,$$

where $\sigma = \sigma_1$ for $0 < x < \lambda$, and $\sigma = \sigma_0$ for $\lambda < x < 1$.

Using the above properties (1)–(6) and the assumptions (H1)–(H3) and with (3.10), we have that for $x \in (0,\lambda)$

$$\begin{aligned}
&\varepsilon\alpha''(x) - \frac{f(x,\alpha(x))}{x-\lambda}\alpha'(x) + g\big(x,\alpha(x)\big) \\
&\quad = \varepsilon\varphi''(x) + \frac{1}{\varepsilon}\frac{d^2u_0}{d\tau^2} - \frac{d^2u_1}{d\tau^2} - \frac{f(x,\alpha(x))}{x-\lambda}\varphi'(x) - \frac{1}{\varepsilon}\frac{f(x,\alpha(x))}{x-\lambda}\frac{du_0}{d\tau} + \frac{f(x,\alpha(x))}{x-\lambda}\frac{du_1}{d\tau} \\
&\qquad + \frac{f(x,\alpha(x))}{x-\lambda}\gamma'(x)\varepsilon + g\big(x,\alpha(x)\big) - \gamma''(x)\varepsilon^2 \\
&\quad = -\frac{d^2u_1}{d\tau^2} + \frac{f(x,\alpha(x))}{x-\lambda}\frac{du_1}{d\tau} - \frac{1}{\varepsilon}\frac{f(x,\varphi(x)+u_0) - f(0,\varphi(0)+u_0)}{x-\lambda}\frac{du_0}{d\tau} \\
&\qquad + \frac{f_y'(x,\varphi(x)+u_0+\theta_1u_1\varepsilon+\theta_1\gamma\varepsilon)(u_1+\gamma)}{x-\lambda}\frac{du_0}{d\tau} + \frac{f(x,\alpha(x))}{x-\lambda}\gamma'(x)\varepsilon - \gamma''(x)\varepsilon^2 \\
&\qquad + \left(\frac{\partial g}{\partial y}(x,\cdot) - \frac{g(x,\varphi(x))}{f(x,\varphi(x))}\frac{\partial f}{\partial y}(x,\cdot)\right)\big(u_0 - u_1\varepsilon - \gamma(x)\varepsilon\big) + \varepsilon\varphi''(x) \\
&\quad \geqslant -\frac{d^2u_1}{d\tau^2} + \frac{\sigma_0}{\varepsilon\tau-\lambda}\frac{du_1}{d\tau} - K\left|A-\varphi(0)\right|(\tau+1)\left(\frac{\lambda-\varepsilon\tau}{\lambda}\right)^{\frac{\sigma_0}{\varepsilon}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma_0}{x-\lambda}\gamma'(x)\varepsilon - K|u_1|\varepsilon - K\gamma(x)\varepsilon - \gamma''(x)\varepsilon^2 + \varepsilon\varphi''(x) \\
&\geqslant \frac{\sigma_0}{x-\lambda}\gamma'(x)\varepsilon - K\gamma(x)\varepsilon - \frac{K^2|A-\varphi(0)|(2+\sigma_0)}{\sigma_0^3}\varepsilon - K\varepsilon - \gamma''(x)\varepsilon^2 \\
&\geqslant \left[L - \frac{K^2|A-\varphi(0)|(2+\sigma_0)}{\sigma_0^3} - K - \frac{L(\sigma_1+K)}{\sigma_0^2}\exp\left(\frac{K}{2\sigma_0}\right)\varepsilon\right]\varepsilon,
\end{aligned}
$$

where $(x,\cdot)$ denotes $(x,\varphi(x)-\theta_2(u_0+u_1\varepsilon+\gamma\varepsilon))$, and $0<\theta_1,\theta_2<1$.

Similarly, we have for $x\in(\lambda,1)$

$$
\begin{aligned}
&\varepsilon\alpha''(x) - \frac{f(x,\alpha(x))}{x-\lambda}\alpha'(x) + g\big(x,\alpha(x)\big) \\
&= \varepsilon\varphi''(x) + \frac{1}{\varepsilon}\frac{d^2v_0}{dt^2} - \frac{d^2v_1}{dt^2} - \frac{f(x,\alpha(x))}{x-\lambda}\varphi'(x) + \frac{1}{\varepsilon}\frac{f(x,\alpha(x))}{x-\lambda}\frac{dv_0}{dt} - \frac{f(x,\alpha(x))}{x-\lambda}\frac{dv_1}{dt} \\
&\quad + \frac{f(x,\alpha(x))}{x-\lambda}\gamma'(x)\varepsilon + g\big(x,\alpha(x)\big) - \gamma''(x)\varepsilon^2 \\
&= -\frac{d^2v_1}{dt^2} - \frac{f(x,\alpha(x))}{x-\lambda}\frac{dv_1}{dt} + \frac{1}{\varepsilon}\frac{f(x,\varphi(x)+v_0)-f(1,\varphi(1)+v_0)}{x-\lambda}\frac{dv_0}{dt} \\
&\quad - \frac{f'_y(x,\varphi(x)+v_0-\theta_3v_1\varepsilon-\theta_3\gamma\varepsilon)(v_1+\gamma)}{x-\lambda}\frac{dv_0}{dt} + \frac{f(x,\alpha(x))}{x-\lambda}\gamma'(x)\varepsilon - \gamma''(x)\varepsilon^2 \\
&\quad + \left(\frac{\partial g}{\partial y}(x,\cdot\cdot) - \frac{g(x,\varphi(x))}{f(x,\varphi(x))}\frac{\partial f}{\partial y}(x,\cdot\cdot)\right)\big(v_0 - v_1\varepsilon - \gamma(x)\varepsilon\big) + \varepsilon\varphi''(x) \\
&\geqslant -\frac{d^2v_1}{dt^2} - \frac{\sigma_0}{1-\varepsilon t-\lambda}\frac{dv_1}{dt} - K\big|B-\varphi(1)\big|(t+1)\left(\frac{1-\varepsilon t-\lambda}{1-\lambda}\right)^{\frac{\sigma_0}{\varepsilon}} \\
&\quad + \frac{\sigma_0}{x-\lambda}\gamma'(x)\varepsilon - K|v_1|\varepsilon - K\gamma(x)\varepsilon - \gamma''(x)\varepsilon^2 + \varepsilon\varphi''(x) \\
&\geqslant \frac{\sigma_0}{x-\lambda}\gamma'(x)\varepsilon - K\gamma(x)\varepsilon - \frac{K^2|B-\varphi(1)|(2+\sigma_0)}{\sigma_0^3}\varepsilon - K\varepsilon - \gamma''(x)\varepsilon^2 \\
&\geqslant \left[L - \frac{K^2|B-\varphi(1)|(2+\sigma_0)}{\sigma_0^3} - K - \frac{L(\sigma_1+K)}{\sigma_0^2}\exp\left(\frac{K}{2\sigma_0}\right)\varepsilon\right]\varepsilon,
\end{aligned}
$$

where $(x,\cdot\cdot)$ denotes $(x,\varphi(x)-\theta_4(v_0+v_1\varepsilon+\gamma\varepsilon))$, and $0<\theta_3,\theta_4<1$. Hence, we can choose

$$
L > K + \frac{K^2(2+\sigma_0)}{\sigma_0^3}\max\big\{\big|A-\varphi(0)\big|,\big|B-\varphi(1)\big|\big\}
$$

such that for small enough $\varepsilon>0$

$$
\varepsilon\alpha''(x) - \frac{f(x,\alpha(x))}{x-\lambda}\alpha'(x) + g\big(x,\alpha(x)\big) \geqslant 0, \quad x\in(0,\lambda)\cup(\lambda,1).
$$

We can prove in a similar fashion that for sufficiently small $\varepsilon>0$

$$
\varepsilon\beta''(x) - \frac{f(x,\beta(x))}{x-\lambda}\beta'(x) + g\big(x,\beta(x)\big) \leqslant 0, \quad x\in(0,\lambda)\cup(\lambda,1).
$$

Moreover, it is evident that

$$\alpha(x) \leqslant \beta(x), \quad 0 \leqslant x \leqslant 1,$$
$$\alpha, \beta \in C\big([0,1]\big) \cap C^1\big([0,\lambda) \cup (\lambda, 1]\big),$$
$$\alpha(0) \leqslant A \leqslant \beta(0), \qquad \alpha(\lambda) = \beta(\lambda), \qquad \alpha(1) \leqslant B \leqslant \beta(1).$$

It follows from Lemma 2.1 that the boundary value problem (1.1) and (1.2) has a solution $y(x,\varepsilon)$ such that $\alpha(x) \leqslant y(x,\varepsilon) \leqslant \beta(x)$ for $x \in [0,1]$. The proof is completed. □

We remark that the conditions (3.10) and (3.11) are not essential to the existence of asymptotic solution for the singular boundary value problem (1.1) and (1.2). If we relax the estimate of asymptotic solution we can remove the conditions (3.10) and (3.11). We state the following theorem.

Theorem 3.2. *Under the assumptions* (H1), (H2) *and* (H3), *for sufficiently small* $\varepsilon > 0$ *the boundary value problem* (1.1) *and* (1.2) *has a solution* $y(x)$ *such that for all* $x \in [0,\lambda]$

$$\big(A - \varphi(0)\big)\left(\frac{\lambda - x}{\lambda}\right)^{\frac{\underline{\sigma}+\varepsilon}{\varepsilon}} - \Delta_1\varepsilon \leqslant y(x,\varepsilon) - \varphi(x) \leqslant \big(A - \varphi(0)\big)\left(\frac{\lambda - x}{\lambda}\right)^{\frac{\overline{\sigma}+\varepsilon}{\varepsilon}} + \Delta_1\varepsilon,$$

and for all $x \in [\lambda, 1]$

$$\big(B - \varphi(1)\big)\left(\frac{x - \lambda}{1 - \lambda}\right)^{\frac{\underline{\underline{\sigma}}+\varepsilon}{\varepsilon}} - \Delta_1\varepsilon \leqslant y(x,\varepsilon) - \varphi(x) \leqslant \big(B - \varphi(1)\big)\left(\frac{x - \lambda}{1 - \lambda}\right)^{\frac{\overline{\overline{\sigma}}+\varepsilon}{\varepsilon}} + \Delta_1\varepsilon,$$

where $\underline{\sigma}$, $\overline{\sigma}$, $\underline{\underline{\sigma}}$ *and* $\overline{\overline{\sigma}}$ *are defined in Lemmas* 3.1 *and* 3.2, *and* Δ_1 *is a positive constant.*

Proof. Define

$$\alpha(x) = \begin{cases} \varphi(x) + (A - \varphi(0))(\frac{\lambda - x}{\lambda})^{\frac{\underline{\sigma}+\varepsilon}{\varepsilon}} - \gamma(x)\varepsilon, & x \in [0,\lambda], \\ \varphi(x) + (B - \varphi(1))(\frac{x-\lambda}{1-\lambda})^{\frac{\underline{\underline{\sigma}}+\varepsilon}{\varepsilon}} - \gamma(x)\varepsilon, & x \in [\lambda,1], \end{cases}$$

$$\beta(x) = \begin{cases} \varphi(x) + (A - \varphi(0))(\frac{\lambda - x}{\lambda})^{\frac{\overline{\sigma}+\varepsilon}{\varepsilon}} + \gamma(x)\varepsilon, & x \in [0,\lambda], \\ \varphi(x) + (B - \varphi(1))(\frac{x-\lambda}{1-\lambda})^{\frac{\overline{\overline{\sigma}}+\varepsilon}{\varepsilon}} + \gamma(x)\varepsilon, & x \in [\lambda,1], \end{cases}$$

where $\gamma(x)$ is defined as in the proof of Theorem 3.1. To apply Lemma 2.1, it remains only to show that $\alpha(x)$ and $\beta(x)$ are lower and upper solutions of the problem (1.1) and (1.2), which follows the similar lines as the proof of Theorem 3.1, and is omitted here. □

Remark. If the functions $f(x,y)$ and $g(x,y)$ are sufficiently smooth on $[0,1] \times \mathbb{R}$, asymptotic expansions of arbitrary order for the solution $y(x,\varepsilon)$ can be derived:

$$y(x,\varepsilon) = \sum_{i=0}^{\infty}\left[y_i(x) + u_i\left(\frac{x}{\varepsilon}\right) + v_i\left(\frac{1-x}{\varepsilon}\right)\right]\varepsilon^i.$$

Under some additional conditions, we have the following uniqueness result.

Theorem 3.3. *Assume the conditions in Theorem* 3.2 *hold, and*

$$(x-\lambda)\frac{\partial f}{\partial y}(x,y) \geqslant 0, \qquad \frac{\partial g}{\partial y}(x,y) \leqslant 0, \quad \textit{for } (x,y) \in [0,1]\times\mathbb{R}.$$

Then for sufficiently small $\varepsilon > 0$ the boundary value problem (1.1) *and* (1.2) *has a unique solution $y(x) \in C([0,1]) \cap C^1([0,\lambda) \cup (\lambda,1])$.*

Proof. From the construction of lower and upper solutions $\alpha(x)$ and $\beta(x)$ we know that for sufficiently small $\varepsilon > 0$ the inequalities (2.3) and (2.4) are strict. The conclusion follows directly from Lemma 2.2. □

3.2. Case $\lambda = 0$

For the case $\lambda = 0$, (1.1) and (1.2) reduces to the following singular two-point boundary value problem

$$\varepsilon\frac{d^2y}{dx^2} - \frac{1}{x} f(x,y)\frac{dy}{dx} + g(x,y) = 0, \quad x \in (0,1), \tag{3.13}$$

$$y(0) = A, \qquad y(1) = B. \tag{3.14}$$

As in nonsingular two-point boundary value problems [6], under the assumption (H2) the boundary layer occurs at $x = 1$. We assume that:

(H3′) The reduced problem

$$f(x,y)\frac{dy}{dx} = xg(x,y), \qquad y(0) = A$$

has a solution $y = \psi(x) \in C^2([0,1])$.

Introducing the boundary layer term $\bar{v}(t)$ with $t = (1-x)/\varepsilon$ we obtain the equation for the approximation of zero order $\bar{v}_0(t)$

$$\frac{d^2\bar{v}_0}{dt^2} + \frac{f(1,\psi(1)+\bar{v}_0(t))}{1-\varepsilon t}\frac{d\bar{v}_0}{dt} = 0, \tag{3.15}$$

subject to the boundary conditions

$$\bar{v}_0(0) = B - \psi(1), \qquad \bar{v}_0\left(\frac{1}{\varepsilon}\right) = 0. \tag{3.16}$$

We state the existence, asymptotic estimates and uniqueness results for the boundary value problem (3.13) and (3.14) as the following theorems whose proofs are completely similar to those in Section 3.1.

Theorem 3.4. *Under the assumptions* (H1), (H2) *and* (H3′)*, for sufficiently small $\varepsilon > 0$ the boundary value problem* (3.13) *and* (3.14) *has a solution $y(x)$ such that for all $x \in [0,1]$*

$$\big(B-\psi(1)\big)x^{\frac{\underline{\underline{\sigma}}+\varepsilon}{\varepsilon}} - \Delta_2\varepsilon \leqslant y(x,\varepsilon) - \psi(x) \leqslant \big(B-\psi(1)\big)x^{\frac{\bar{\bar{\sigma}}+\varepsilon}{\varepsilon}} + \Delta_2\varepsilon,$$

where $\underline{\underline{\sigma}}$ and $\overline{\overline{\sigma}}$ are defined in Lemma 3.2, *and Δ_2 is a positive constant. Further if*

$$\frac{\partial f}{\partial y}\big(x, \psi(x) + \bar{v}_0\big((1-x)/\varepsilon\big)\big)\big(B - \psi(1)\big) \geqslant 0, \quad \textit{for } x \in (0,1),$$

then the solution of (3.13) *and* (3.14) *has the following more precise estimate*

$$y(x,\varepsilon) = \psi(x) + \bar{v}_0\left(\frac{1-x}{\varepsilon}\right) + \mathcal{O}(\varepsilon),$$

where $\bar{v}_0((1-x)/\varepsilon)$ is the solution to (3.15) *and* (3.16).

Theorem 3.5. *Assume the conditions* (H1), (H2) *and* (H3′) *hold, and*

$$\frac{\partial f}{\partial y}(x,y) \geqslant 0, \qquad \frac{\partial g}{\partial y}(x,y) \leqslant 0, \quad \textit{for } (x,y) \in [0,1] \times \mathbb{R}.$$

Then for sufficiently small $\varepsilon > 0$ the boundary value problem (3.13) *and* (3.14) *has a unique solution $y(x) \in C([0,1]) \cap C^1((0,1])$.*

Let us turn to the situation that the condition (H2) is replaced by

(H2′) There exists a positive constant δ_0 such that $f(x,y) \leqslant -\delta_0 < 0$ for $(x,y) \in [0,1] \times \mathbb{R}$.

It is known that the boundary value problem

$$\varepsilon \frac{d^2 y}{dx^2} - f(x,y)\frac{dy}{dx} + g(x,y) = 0, \quad x \in (0,1),$$
$$y(0) = A, \qquad y(1) = B$$

has a left boundary layer at $x = 0$, under the assumption (H2′) (see [4,6], for example). However, it is not expected that this kind of situation appears for the problem (3.13) and (3.14). The following example

$$\varepsilon \frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \qquad y(0) \text{ finite}, \qquad y(1) = e^{-1/2}$$

is checked in [3]. It is shown that the solution does not exhibit a boundary layer at $x = 0$ for the problem above, although a unique solution does exist.

Let us consider a slight modification of Eq. (3.13)

$$\varepsilon \frac{d^2 y}{dx^2} - \frac{1}{x + \rho\varepsilon} f(x,y)\frac{dy}{dx} + g(x,y) = 0, \quad x \in (0,1), \tag{3.17}$$

with $\rho > 0$. (3.17) has the same reduced equation with (3.13). We assume:

(H3″) The reduced problem

$$f(x,y)\frac{dy}{dx} = xg(x,y), \qquad y(1) = B$$

has a solution $y = \phi(x) \in C^2([0,1])$.

Introducing the boundary layer term $\omega(\xi)$ with $\xi = x/\varepsilon^2$ we obtain the equation for the approximation of zero order $\omega_0(\xi)$

$$\frac{d^2\omega_0}{d\xi^2} - \frac{f(0, \phi(0) + \omega_0(\xi))}{\rho + \varepsilon\xi}\frac{d\omega_0}{d\xi} = 0, \tag{3.18}$$

subject to the boundary conditions

$$\omega_0(0) = A - \phi(0), \qquad \omega_0\left(\frac{1}{\varepsilon}\right) = 0. \tag{3.19}$$

The following lemma is concerned with the existence and the decaying estimate for $\omega_0(\xi)$.

Lemma 3.3. *Under the assumptions* (H1) *and* (H2′), *for sufficiently small* $\varepsilon > 0$, *the boundary value problem* (3.18) *and* (3.19) *has a solution* $\omega_0(\xi)$ *satisfying the estimate*:

$$\big(A - \phi(0)\big)(\rho + \varepsilon\xi)^{\frac{-\underline{\delta}+\varepsilon}{\varepsilon}} \leqslant \omega_0(\xi) \leqslant \big(A - \phi(0)\big)(\rho + \varepsilon\xi)^{\frac{-\bar{\delta}+\varepsilon}{\varepsilon}},$$

where

$$\underline{\delta} = \begin{cases} \delta_1, & \text{if } A - \phi(0) > 0, \\ \delta_0, & \text{if } A - \phi(0) < 0, \end{cases} \qquad \bar{\delta} = \begin{cases} \delta_0, & \text{if } A - \phi(0) > 0, \\ \delta_1, & \text{if } A - \phi(0) < 0, \end{cases}$$

and

$$\delta_1 = \max\big\{-f\big(0, \phi(0) + \omega\big)\colon -\big|A - \phi(0)\big| \leqslant \omega \leqslant \big|A - \phi(0)\big|\big\}.$$

Proof. A straightforward computation gives

$$\alpha(\xi) = \big(A - \phi(0)\big)(\rho + \varepsilon\xi)^{\frac{-\underline{\delta}+\varepsilon}{\varepsilon}} \quad \text{and} \quad \beta(\xi) = \big(A - \phi(0)\big)(\rho + \varepsilon\xi)^{\frac{-\bar{\delta}+\varepsilon}{\varepsilon}}$$

are lower and upper solutions of the problem (3.18) and (3.19). □

We close this subsection by stating the following theorem.

Theorem 3.6. *Under the assumptions* (H1), (H2′) *and* (H3″), *for sufficiently small* $\varepsilon > 0$ *the boundary value problem* (3.17) *and* (3.14) *has a solution* $y(x)$ *such that for all* $x \in [0, 1]$

$$\big(A - \phi(0)\big)\left(\rho + \frac{x}{\varepsilon}\right)^{\frac{-\underline{\delta}+\varepsilon}{\varepsilon}} - \Delta_3\varepsilon \leqslant y(x, \varepsilon) - \phi(x) \leqslant \big(A - \phi(0)\big)\left(\rho + \frac{x}{\varepsilon}\right)^{\frac{-\bar{\delta}+\varepsilon}{\varepsilon}} + \Delta_3\varepsilon,$$

where $\underline{\delta}$ *and* $\bar{\delta}$ *are defined in Lemma* 3.3, *and* Δ_3 *is a positive constant.*

Proof. The proof is similar to that of Theorem 3.2. □

4. Examples

In this section, we present several examples to illustrate our results.

Example 1. Consider a singular three-point boundary value problem

$$\varepsilon\frac{d^2y}{dx^2}-\frac{2(1+x)}{(2x-1)}\frac{dy}{dx}+xe^{-y}=0,\quad x\in(0,1/2)\cup(1/2,1),$$
$$y(0)=1,\qquad y(1/2)=0,\qquad y(1)=\frac{3}{2}\ln 2.$$

The reduced problem

$$2(1+x)\frac{dy}{dx}=x(2x-1)e^{-y},\qquad y(1/2)=0$$

has a unique solution

$$\varphi(x)=\ln\left(\frac{1}{2}x^2-\frac{3}{2}x+\frac{13}{8}+\frac{3}{2}\ln\left(\frac{2(1+x)}{3}\right)\right).$$

The problems (3.1), (3.3) and (3.2), (3.4) for the boundary layer terms $u_0(\tau)$ and $v_0(t)$ have exact solutions

$$u_0(\tau)=\big(1-\varphi(0)\big)(1-2\varepsilon\tau)^{\frac{1+\varepsilon}{\varepsilon}},\qquad v_0(t)=\big(1-\varphi(1)\big)(1-2\varepsilon t)^{\frac{1+\varepsilon}{\varepsilon}}.$$

By Theorem 3.1 the above problem has a solution

$$y(x,\varepsilon)=\varphi(x)+\left(\frac{5}{8}-\frac{3}{2}\ln\frac{2}{3}\right)|1-2x|^{\frac{1+\varepsilon}{\varepsilon}}+\mathcal{O}(\varepsilon),\quad x\in[0,1].$$

In fact, this solution is also unique from Theorem 3.3.

Example 2. Consider a linear singular two-point boundary value problem in [3]

$$\varepsilon\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-y=0,\quad x\in(0,1),$$
$$y(0)=1,\qquad y(1)=1.$$

It follows from Theorems 3.4 and 3.5 that the above problem has a unique solution $y(x,\varepsilon)$ such that

$$y(x,\varepsilon)=\exp\left(\frac{-x^2}{2}\right)+\left[1-\exp\left(\frac{-1}{2}\right)\right]x^{\frac{1+\varepsilon}{\varepsilon}}+\mathcal{O}(\varepsilon),\quad x\in[0,1],$$

which accurate to order ε agrees with the solution obtained using the matching method

$$y_{\text{matching}}=\exp\left(\frac{-x^2}{2}\right)\left[1+\frac{\varepsilon}{4}\left(x^2-1\right)^2\right]+\left[1-\exp\left(\frac{-1}{2}\right)\right]\frac{6\varepsilon-4\varepsilon x-(1-x)^2}{2\varepsilon}\exp\left(\frac{x-1}{\varepsilon}\right).$$

Example 3. Consider a linear singular two-point boundary value problem with boundary perturbation in [3]

$$\varepsilon \frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \quad x \in (\varepsilon, 1),$$
$$y(\varepsilon) = 0, \qquad y(1) = e^{-1/2}.$$

After a transformation $\bar{x} = x - \varepsilon$, the above problem becomes

$$\varepsilon \frac{d^2 y}{d\bar{x}^2} + \frac{1}{\bar{x} + \varepsilon}\frac{dy}{d\bar{x}} + y = 0, \quad x \in (0, 1 - \varepsilon),$$
$$y(0) = 0, \qquad y(1 - \varepsilon) = e^{-1/2}.$$

From Theorem 3.6 we have

$$y(\bar{x}, \varepsilon) = \exp\left(\frac{-\bar{x}^2}{2}\right) - \exp\left(\frac{-1}{2}\right)\left(1 + \frac{\bar{x}}{\varepsilon}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \mathcal{O}(\varepsilon), \quad \bar{x} \in [0, 1 - \varepsilon].$$

Acknowledgments

The author was supported by the National Natural Science Foundation of China (No. 10701023), in part by the Natural Science Foundation of Shanghai (No. 10ZR1400100), and by the Fundamental Research Funds for the Central Universities.

References

[1] R.P. Agarwal, D. O'Regan, Singular Differential and Integral Equations with Applications, Kluwer Academic Publishers, Dordrecht, 2003.
[2] R.P. Agarwal, D. O'Regan, Nonlinear superlinear singular and nonsingular second order boundary value problems, J. Differential Equations 143 (1998) 60–95.
[3] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw–Hill, New York, 1978.
[4] K.W. Chang, F.A. Howes, Nonlinear Singular Perturbation Phenomena: Theory and Application, Springer-Verlag, New York, 1984.
[5] C.D. Coster, P. Habets, Two-Point Boundary Value Problems: Lower and Upper Solutions, Elsevier B.V., New York, 2006.
[6] E.M. de Jager, F. Jiang, The Theory of Singular Perturbations, North-Holland, Amsterdam, 1996.
[7] Z. Du, W. Ge, M. Zhou, Singular perturbations for third-order nonlinear multi-point boundary value problem, J. Differential Equations 218 (2005) 69–90.
[8] M. Gregus, On a special boundary value problem, Acta Math. Univ. Comenian. (N.S.) 40 (1982) 161–168.
[9] M.K. Kadalbajoo, V.K. Aggarwal, Fitted mesh B-spline method for solving a class of singular singularly perturbed boundary value problems, Int. J. Comput. Math. 82 (2005) 67–76.
[10] W. Kelley, Asymptotically singular boundary value problems, Math. Comput. Modelling 32 (2000) 541–548.
[11] P.A. Lagerstrom, Matched Asymptotic Expansions, Springer-Verlag, New York, 1988.
[12] P.M. Lima, L. Morgado, Analytical–numerical investigation of a singular boundary value problem for a generalized Emden–Fowler equation, J. Comput. Appl. Math. 229 (2009) 480–487.
[13] Xiao-Biao Lin, Heteroclinic bifurcation and singularly perturbed boundary value problems, J. Differential Equations 84 (1990) 319–382.
[14] R.K. Mohanty, N. Jha, A class of variable mesh spline in compression method for singularly perturbed two point singular boundary value problems, Appl. Math. Comput. 168 (2005) 704–716.
[15] R.E. O'Malley, Singular Perturbation Methods for Ordinary Differential Equations, Springer-Verlag, New York, 1991.
[16] J. Rashidiniaa, R. Mohammadia, M. Ghasemi, Cubic spline solution of singularly perturbed boundary value problems with significant first derivatives, Appl. Math. Comput. 190 (2007) 1762–1766.
[17] D. O'Regan, Theory of Singular Boundary Value Problems, World Scientific, Singapore, 1994.
[18] R. Precup, Two positive solutions of some singular boundary value problems, Anal. Appl. 8 (2010) 305–314.
[19] R.D. Russell, L.F. Shampine, Numerical methods for singular boundary value problems, SIAM J. Numer. Anal. 12 (1975) 13–36.
[20] J.Y. Shin, A singular nonlinear boundary value problem in the nonlinear circular membrane under normal pressure, J. Korean Math. Soc. 32 (1995) 761–773.
[21] J.Y. Shin, A singular nonlinear differential equations in Homann flow, J. Math. Anal. Appl. 212 (1997) 443–451.