J. Differential Equations 255 (2013) 3678–3702

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence and multiplicity of solutions for nonlinear periodic problems with the scalar p-Laplacian and double resonance

Evgenia H. Papageorgiou [a], Nikolaos S. Papageorgiou [b,*]

[a] Hellenic Naval Academy, Department of Mathematics, Pireaus 45110, Greece
[b] National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

ARTICLE INFO

Article history:
Received 23 January 2011
Revised 22 February 2013
Available online 16 August 2013

MSC:
34B15
34B18
34C25
58E05

Keywords:
Spectrum of the scalar p-Laplacian
Double resonance
Critical groups reduced homology sequence
Morse theory
Existence and multiplicity theorems

ABSTRACT

We consider nonlinear periodic problems driven by the scalar p–Laplacian and with a Caratheodory reaction which can exhibit double resonance at $\pm\infty$. Combining variational methods based on the critical point theory with Morse theoretic techniques, we show that we have existence when the double resonance occurs at any spectral interval and we have multiplicity with at least three nontrivial solutions, when the double resonance occurs at any spectral interval distinct from the "principal" one $[\hat{\lambda}_0 = 0, \hat{\lambda}_1]$.

1. Introduction

In this paper we study the following nonlinear periodic problem:

$$\begin{cases} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = f\left(t, u(t)\right) & \text{a.e. on } T = [0, b], \\ u(0) = u(b), \quad u'(0) = u'(b), \quad 1 < p < \infty. \end{cases} \tag{1}$$

* Corresponding author.
E-mail addresses: epap@math.ntua.gr (E.H. Papageorgiou), npapg@math.ntua.gr (N.S. Papageorgiou).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.07.053

The aim of this work is to prove existence and multiplicity results when double resonance is possible. To make this situation more transparent, suppose that

$$f(t,x)=\widehat{\lambda}_m|x|^{p-2}x+g(t,x) \quad \text{for all } (t,x)\in T\times\mathbb{R},$$

with $\widehat{\lambda}_m$ $m\geqslant 0$ an eigenvalue of the negative scalar p-Laplacian with periodic boundary conditions and $g(t,x)$ is a Caratheodory function (i.e., for all $x\in\mathbb{R}$, $t\to g(t,x)$ is measurable and for a.a. $t\in T$, $x\to g(t,x)$ is continuous) such that $\lim_{|x|\to\infty}\frac{g(t,x)}{|x|^{p-2}x}=0$ uniformly for a.a. $t\in T$. Then problem (1) is said to be "resonant at infinity" with respect to the eigenvalue $\widehat{\lambda}_m$. If this resonance phenomenon can occur with respect to two successive eigenvalues $\widehat{\lambda}_m<\widehat{\lambda}_{m+1}$, then we have "double resonance", a term coined by Berestycki and de Figueiredo [4] who examined such problems in the context of semilinear (i.e., $p=2$) Dirichlet elliptic equations.

Scalar periodic problems with double resonance were investigated primarily for semilinear equations. In this direction we mention the works of Fabry and Fonda [11], Gasiński and Papageorgiou [15], Omari and Zanolin [21] and Su and Zhao [26]. Fabry and Fonda [11], Gasiński and Papageorgiou [15] and Su and Zhao [26], use Landesman–Lazer type conditions, while Omari and Zanolin [21] use certain nonresonance conditions involving the quotient $\frac{2F(t,x)}{x^2}$ where $F(t,x)=\int_0^x f(t,s)\,ds$ is the potential function corresponding to $f(t,x)$. In Fabry and Fonda [11] and Omari and Zanolin [21] the approach is degree theoretic, while in Gasiński and Papageorgiou [15] and Su and Zhao [26] the authors use variational methods coupled with techniques from Morse theory. From the aforementioned works Fabry and Fonda [11] and Omari and Zanolin [21] prove only existence theorems, while Gasiński and Papageorgiou [15] and Su and Zhao [26] have multiplicity results. Gasiński and Papageorgiou [15] produce four solutions, while Su and Zhao [26] obtain two solutions. We also mention the recent works on semilinear (i.e., $p=2$) partial differential equations of Gasiński and Papageorgiou [16] (Dirichlet problems) and O'Regan, Papageorgiou and Smyrlis [22] (Neumann problems), where similar multiplicity results are proved. The orthogonal direct sum decomposition of the ambient Sobolev Hilbert space in terms of the eigenspaces, makes the analysis of the semilinear problem easier. For equations driven by the periodic scalar p-Laplacian, to the best of our knowledge, there is only the works of Fabry and Fayyad [12] and Kyritsi–Papageorgiou [18]. In Fabry and Fayyad [12] the authors prove an existence theorem for problems with asymmetric nonlinearities using an alternative interesting approach based on degree theory. The asymptotic conditions at $\pm\infty$ are similar using this time the Fucik spectrum and the a priori bounds for the solutions are obtained by means of a count of the number of revolutions in the phase plane. In Kyritsi and Papageorgiou [18] the authors prove an existence theorem using asymptotic conditions similar to those employed by Omari and Zanolin [21].

Our approach here combines minimax arguments based on the critical point theory, with Morse theoretic techniques. We prove a multiplicity theorem and an existence theorem. In the existence theorem the asymptotic double resonance condition is essentially complementary to the corresponding one used in the multiplicity theorem.

2. Mathematical background

In this section, for the convenience of the reader, we recall some of the main mathematical tools which we will use in the sequel.

We start with critical point theory. So, let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi\in C^1(X)$, we say that φ satisfies the "C-condition", if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1}\subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1}\subseteq\mathbb{R}$ is bounded and $(1+\|u_n\|)\varphi'(u_n)\to 0\in X^*$, admits a strongly convergent subsequence".

Using this compactness-type condition on φ, we can have the following minimax characterization of certain critical values of φ. The result is known in the literature as the "mountain pass theorem".

Theorem 2.1. **If** *X is Banach space, $\varphi \in C^1(X)$ and satisfies the C-condition, $u_0, u_1 \in X$, $r > 0$, $\|u_1 - u_0\| > r$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\bigl[\varphi(u)\colon \|u - u_0\| = r\bigr] = \eta_r,$$
$$c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \varphi\bigl(\gamma(t)\bigr) \quad \text{where } \Gamma = \bigl\{\gamma \in C\bigl([0,1], X\bigr)\colon \gamma(0) = u_0,\ \gamma(1) = u_1\bigr\},$$

then *$c \geqslant \eta_r$ and c is a critical value of φ.*

Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \bigl\{u \in X\colon \varphi(u) \leqslant c\bigr\}, \qquad K_\varphi = \bigl\{u \in X\colon \varphi'(u) = 0\bigr\}, \qquad K_\varphi^c = \bigl\{u \in K_\varphi\colon \varphi(u) = c\bigr\}.$$

If $Y_2 \subseteq Y_1 \subseteq X$ and $k \geqslant 0$ is an integer, then by $H_k(Y_1, Y_2)$ we denote the k*th* relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Then the critical groups of φ at an isolated critical point u_0 with $c = \varphi(u_0)$ (i.e., $u_0 \in K_\varphi^c$), are defined by

$$C_k(\varphi, u_0) = H_k\bigl(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}\bigr), \quad k \geqslant 0.$$

Here U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology theory, implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $a < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k\bigl(X, \varphi^a\bigr) \quad \text{for all } k \geqslant 0.$$

We know that the deformation theorem is still valid under the C-condition (see, for example, Papageorgiou and Kyritsi [23]). So, using the deformation theorem, we see that the above definition of critical groups at infinity, is independent of the particular choice of the level $a < \inf \varphi(K_\varphi)$. If for some integer $m \geqslant 0$, $C_m(\varphi, \infty) \neq 0$, then exists $u \in K_\varphi$ such that $C_m(\varphi, u) \neq 0$.

In the study of problem (1) we will use the Sobolev space

$$W^{1,p}_{\mathrm{per}}(0, b) = \bigl\{u \in W^{1,p}(0, b)\colon u(0) = u(b)\bigr\}.$$

Recall that $W^{1,p}(0, b)$ is embedded compactly in $C(T)$ (Sobolev embedding theorem) and so in the above definition, the evaluations at $t = 0$ and $t = b$ make sense. We shall also use the Banach space $\widehat{C}^1(T) = \{u \in C^1(T)\colon u(0) = u(b)\} = C^1(T) \cap W^{1,p}_{\mathrm{per}}(0, b)$. This is an ordered Banach space with positive cone $\widehat{C}_+$ given by

$$\widehat{C}_+ = \bigl\{u \in \widehat{C}^1(T)\colon u(t) \geqslant 0 \text{ for all } t \geqslant 0\bigr\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} \widehat{C}_+ = \bigl\{u \in \widehat{C}_+\colon u(t) > 0 \text{ for all } t \geqslant 0\bigr\}.$$

Let $A : W^{1,p}_{\mathrm{per}}(0, b) \to W^{1,p}_{\mathrm{per}}(0, b)^*$ be the nonlinear map defined by

$$\bigl\langle A(u), y\bigr\rangle = \int_0^b \bigl|u'(t)\bigr|^{p-2} u'(t) y'(t)\, dt \quad \text{for all } u, y \in W^{1,p}_{\mathrm{per}}(0, b). \tag{2}$$

We have (see Gasiński and Papageorgiou [14] and Kyritsi and Papageorgiou [19]):

Proposition 2.2. *The map $A : W^{1,p}_{\mathrm{per}}(0,b) \to W^{1,p}_{\mathrm{per}}(0,b)^*$ defined by (2) is bounded, continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, i.e., if $u_n \xrightarrow{w} u$ in $W^{1,p}_{\mathrm{per}}(0,b)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W^{1,p}_{\mathrm{per}}(0,b)$.*

Next let us recall some basic facts concerning the spectrum of the negative scalar periodic p-Laplacian. So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \widehat{\lambda}\left|u(t)\right|^{p-2}u(t) \quad \text{a.e. on } T=[0,b], \\ u(0)=u(b), \quad u'(0)=u'(b), \quad \widehat{\lambda}\in\mathbb{R},\ 1<p<\infty. \end{cases} \tag{3}$$

A number $\widehat{\lambda}\in\mathbb{R}$ for which problem (3) has a nontrivial solution $u\in C^1(T)$, is said to be an eigenvalue of (3) and u is a corresponding eigenfunction. It is easy to see from (3) that, if $\widehat{\lambda}\in\mathbb{R}$ is an eigenvalue, then $\widehat{\lambda}\geqslant 0$. In fact $\widehat{\lambda}_0 = 0$ is the smallest eigenvalue, with corresponding eigenspace $\mathbb{R}$ (the space of constant functions). Moreover, if $u\in C^1(T)$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda}>0$, then u is necessarily nodal (i.e., sign changing). In addition $u(t)\neq 0$ for a.a. $t\in T$ (in fact $u(\cdot)$ has finitely many zeros).

Let $\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}$. Note that if $p=2$, then $\pi_2=\pi$. The sequence

$$\left\{\widehat{\lambda}_n = \left(\frac{2n\pi_p}{b}\right)^p\right\}_{n\geqslant 0}$$

is the set of all eigenvalues of (3). If $p=2$ (linear eigenvalue problem), then we recover the well-known spectrum of the negative scalar Laplacian with periodic boundary conditions, which is the sequence $\{\widehat{\lambda}_n = (\frac{2n\pi}{b})^2\}_{n\geqslant 0}$.

In addition to the eigenvalue problem (3), we will also consider the following weighted nonlinear eigenvalue problem:

$$\begin{cases} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \left(\widehat{\mu}+\beta(t)\right)\left|u(t)\right|^{p-2}u(t) \quad \text{a.e. on } T=[0,b], \\ u(0)=u(b), \quad u'(0)=u'(b), \quad 1<p<\infty,\ \beta\in L^1(T). \end{cases} \tag{4}$$

As before $\widehat{\mu}\in\mathbb{R}$ is an eigenvalue, if problem (4) admits a nontrivial solution $u\in C^1(T)$. This eigenvalue problem was investigated by Zhang [28] and Binding and Rynne [5,6]. In fact it was shown by Binding and Rynne [6] that problem (4) can have nonvariational eigenvalues, answering this way a question left open by Zhang [28]. Concerning the eigenvalues of (4), we will need the following observation which can be found in Aizicovici, Papageorgiou and Staicu [1].

Proposition 2.3. **If** *$\beta\in L^\infty(T)_+$ satisfy $\widehat{\lambda}_m \leqslant \beta(t) \leqslant \widehat{\lambda}_{m+1}$ a.e. on T for some integer $m\geqslant 0$ and $\widehat{\lambda}_m\neq\beta$, $\widehat{\lambda}_{m+1}\neq\beta$,* **then** *all eigenvalues of (4) are nonzero and do not have zero as a limit point.*

Let $\widehat{u}_0$ denote the positive L^p-normalized eigenfunction corresponding to $\widehat{\lambda}_0=0$. Then $\widehat{u}_0 = \frac{1}{b^{\frac{1}{p}}}\in(0,+\infty)$. Using $\pm\widehat{u}_0$, we can have an alternative variational characterization of $\widehat{\lambda}_1>0$ (the first nonzero eigenvalue of (3)), distinct from the one provided by the Ljusternik–Schnirelmann theory. This alternative characterization can be found in Kyritsi and Papageorgiou [19] and will be used in the proof of the existence theorem (see Section 4).

Proposition 2.4. **If** $\partial B_1^{L^p} = \{u \in L^p(T): \|u\|_p = 1\}$, $M = W_0^{1,p}(0,b) \cap \partial B_1^{L^p}$ *and*

$$\widetilde{\Gamma} = \big\{\widetilde{\gamma} \in C\big([-1,1], M\big): \widetilde{\gamma}(-1) = -\widehat{u}_0,\ \widetilde{\gamma}(1) = \widehat{u}_0\big\},$$

then $\widehat{\lambda}_1 = \inf_{\gamma \in \widetilde{\Gamma}} \max_{-1 \leqslant s \leqslant 1} \|\frac{d}{dt}\widetilde{\gamma}(s)\|_p^p$.

In Kyritsi and Papageorgiou [19] we can also find the following simple lemma:

Lemma 2.5. **If** $\theta \in L^1(T)$, $\theta \leqslant 0$ *a.e. on* T *and* $\theta \neq 0$, **then** *there exists* $\xi_0 > 0$ *s.t.*

$$\|u'\|_p^p - \int_0^b \theta(t)\big|u(t)\big|^p\,dt \geqslant \xi_0\|u\|^p \quad \text{for all } u \in W_{\mathrm{per}}^{1,p}(0,b).$$

Finally in what follows by $\|\cdot\|$ we denote the norm of the Sobolev space $W_{\mathrm{per}}^{1,p}(0,b)$ and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(W_{\mathrm{per}}^{1,p}(0,b)^*, W_{\mathrm{per}}^{1,p}(0,b))$. Also, if $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$.

3. Multiplicity theorem

The hypotheses on the reaction term $f(t,x)$ are:

H $f : T \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function s.t. for a.a. $t \in T$, $f(t,0) = 0$ and
(**i**) $|f(t,x)| \leqslant a(t)(1 + |x|^{p-1})$ for a.a. $t \in T$, all $x \in \mathbb{R}$, with $a \in L^1(T)_+$;
(**ii**) there exists an integer $m \geqslant 1$ s.t.

$$\widehat{\lambda}_m \leqslant \liminf_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \limsup_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \widehat{\lambda}_{m+1}$$

uniformly for a.a. $t \in T$, and

$$\lim_{|x|\to\infty}\big[f(t,x)x - pF(t,x)\big] = +\infty \quad \text{uniformly for a.a. } t \in T;$$

(**iii**) there exists a function $\theta \in L^1(T)$, $\theta(t) \leqslant 0$ a.e. on T, $\theta \neq 0$ such that

$$\limsup_{x\to 0} \frac{pF(t,x)}{|x|^p} \leqslant \theta(t) \quad \text{uniformly for a.a. } t \in T,$$

where $F(t,x) = \int_0^x f(t,s)\,ds$;
(**iv**) for every $r > 0$, there exists $\xi_r > 0$ s.t. $f(t,x)x + \xi_r|x|^p \geqslant 0$ for a.a. $t \in T$, all $x \in [-r,r]$.

Remark. Hypothesis H(ii) implies that we have double resonance in the spectral interval $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, $m \geqslant 1$. We emphasize that in contrast to the semilinear works of Gasiński and Papageorgiou [15] and Su and Zhao [26], where as we already mentioned in the Introduction, multiplicity results are proved, we do not require that $f(t,\cdot) \in C^1(\mathbb{R})$. This together with the fact that the ambient space is not Hilbert, create difficulties in the use of the Morse theoretic methods.

Example. The following function $f(x)$ satisfies hypotheses H (for the sake of simplicity we drop the t-dependence):

$$f(x) = \begin{cases} \widehat{\lambda}_m|x|^{r-2}x - |x|^{p-2}x & \text{if } |x| \leqslant 1, \\ \widehat{\lambda}_m|x|^{p-2}x - |x|^{q-2}x & \text{if } |x| > 1, \end{cases}$$

with $m \geqslant 1$ and $1 < q < p < r < \infty$.

First we will produce two constant sign solutions for problem (1). To this end, we choose $\varepsilon \in (0, \widehat{\lambda}_2)$ and consider the following truncations–perturbations of the reaction $f(t,x)$:

$$g_+(t,x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ f(t,x) + \varepsilon x^{p-1} & \text{if } x > 0, \end{cases}$$

and

$$g_-(t,x) = \begin{cases} f(t,x) + \varepsilon|x|^{p-2}x & \text{if } x < 0, \\ 0 & \text{if } x \geqslant 0. \end{cases} \tag{5}$$

Both are Caratheodory functions. We set $G_\pm(t,x) = \int_0^x g_\pm(t,s)\,ds$ and consider the C^1-functionals $\psi_\pm : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\psi_\pm(u) = \frac{1}{p}\|u'\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_0^b G_\pm\big(t,u(t)\big)\,dt \quad \text{for all } u \in W^{1,p}_{\text{per}}(0,b).$$

Also, let $\varphi : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|u'\|_p^p - \int_0^b F\big(t,u(t)\big)\,dt \quad \text{for all } u \in W^{1,p}_{\text{per}}(0,b).$$

We know that $\varphi \in C^1(W^{1,p}_{\text{per}}(0,b))$.

Proposition 3.1. If *hypotheses* H *hold,* **then** $\psi_\pm$ *satisfy the C-condition.*

Proof. We do the proof for ψ_+, the proof for ψ_- being similar.

We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\big|\psi_+(u_n)\big| \leqslant M_1 \quad \text{for some } M_1 > 0,\ \text{all } n \geqslant 1 \tag{6}$$

and

$$\big(1 + \|u_n\|\big)\psi_+'(u_n) \to 0 \quad \text{in } W^{1,p}_{\text{per}}(0,b)^* \text{ as } n \to \infty. \tag{7}$$

From (7) we have

$$\left|\big\langle A(u_n), h\big\rangle + \varepsilon\int_0^b |u_n|^{p-2}u_n h\,dt - \int_0^b g_+(t,u_n)h\,dt\right| \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{8}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$ with $\varepsilon_n \to 0^+$.

In (8) we choose $h = -u_n^- \in W^{1,p}_{\mathrm{per}}(0,b)$. Then

$$\begin{aligned}
&\big\|\big(u_n^-\big)'\big\|_p^p + \varepsilon\big\|u_n^-\big\|_p^p \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1 \quad \big(\text{see (5)}\big),\\
&\Rightarrow \quad u_n^- \to 0 \in W^{1,p}_{\mathrm{per}}(0,b).
\end{aligned} \tag{9}$$

Claim. *$\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded.*

We proceed by contradiction. So, suppose that $\|u_n^+\| \to \infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we way assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \quad \text{and} \quad y_n \to y \quad \text{in } C(T). \tag{10}$$

For (8) and (9) we have

$$\left|\big\langle A(y_n), h\big\rangle + \varepsilon\int_0^b |y_n|^{p-2} y_n h\,dt - \int_0^b \frac{g_+(t,u_n^+)}{\|u_n^+\|^{p-1}} h\,dt\right| \leqslant \varepsilon_n'\|h\| \quad \text{with } \varepsilon_n' \to 0^+. \tag{11}$$

Choose $h = y_n - y \in W^{1,p}_{\mathrm{per}}(0,b)$, pass to the limit as $n \to \infty$ and use (10). We obtain

$$\begin{aligned}
&\lim_{n\to\infty}\big\langle A(y_n), y_n - y\big\rangle = 0,\\
&\Rightarrow \quad y_n \to y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b), \ \text{hence } \|y\| = 1,\ y \geqslant 0 \quad (\text{see Proposition 2.2}).
\end{aligned} \tag{12}$$

Note that because of H(i) and (10), we have that $\{\frac{g_+(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\}_{n\geqslant 1} \subseteq L^1(T)$ is uniformly integrable. So, by virtue of the Dunford–Pettis theorem, we may assume that

$$\frac{g_+(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\theta}_+ \quad \text{in } L^1(T). \tag{13}$$

Using hypothesis H(ii) and reasoning as in the proof of Proposition 14 of Aizicovici, Papageorgiou and Staicu [2], we show that

$$\widehat{\theta}_+ = (\xi + \varepsilon)y^{p-1} \quad \text{with } \widehat{\lambda}_m \leqslant \xi(t) \leqslant \widehat{\lambda}_{m+1} \text{ a.e. on } T. \tag{14}$$

So, if we return to (11), pass to the limit as $n \to \infty$ and use (12), (13) and (14), then

$$\begin{aligned}
&\big\langle A(y), h\big\rangle = \int_0^b \xi y^{p-1} h\,dt \quad \text{for all } h \in W^{1,p}_{\mathrm{per}}(0,b),\\
&\Rightarrow \quad A(y) = \xi y^{p-1},\\
&\Rightarrow \quad -\big(\big|y'(t)\big|^{p-2} y'(t)\big)' = \xi(t) y(t)^{p-1} \quad \text{a.e. on T},\\
&y(0) = y(b), \qquad y'(0) = y'(b).
\end{aligned} \tag{15}$$

Recall that $\widehat{\lambda}_m \leqslant \xi(t) \leqslant \widehat{\lambda}_{m+1}$ a.e. on T. If $\xi \neq \widehat{\lambda}_m$ and $\xi \neq \widehat{\lambda}_{m+1}$, then from Proposition 2.3 it follows that $y = 0$, which contradicts (12). If $\xi(t) = \widehat{\lambda}_m$ a.e. on T or $\xi(t) = \widehat{\lambda}_{m+1}$ a.e. on T then from (15) and since $m \geqslant 1$, we infer that y must be nodal, which, contradicts (12). Therefore $\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded. This proves the Claim.

From (9) and the Claim we infer that $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ is bounded. So, we may assume that $u_n \xrightarrow{w} u$ in $W^{1,p}_{\text{per}}(0,b)$ and $u_n \to u$ in $C(T)$. Hence, if in (8) we let $h = u_n - u$ and pass to the limit as $n \to \infty$, then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

$$\Rightarrow \quad u_n \to u \quad \text{in } W^{1,p}_{\text{per}}(0,b) \quad (\text{see Proposition 2.2}).$$

This proves that ψ_+ satisfies the *C-condition*. Similarly for ψ_-. $\square$

Proposition 3.2. If *hypotheses* H *hold*, **then** φ *satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\bigl|\varphi(u_n)\bigr| \leqslant M_2 \quad \text{for some } M_2 > 0, \text{ all } n \geqslant 1 \tag{16}$$

and

$$\bigl(1 + \|u_n\|\bigr)\varphi'(u_n) \to 0 \quad \text{in } W^{1,p}_{\text{per}}(0,b)^* \text{ as } n \to \infty. \tag{17}$$

From (17) we have

$$\left|\bigl\langle A(u_n), h\bigr\rangle - \int_0^b f(t,u_n)h\,dt\right| \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{18}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$ with $\varepsilon_n \to 0^+$.

In (18) we choose $h = u_n$ and obtain

$$-\bigl\|u_n'\bigr\|_p^p + \int_0^b f(t,u_n)u_n\,dt \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1. \tag{19}$$

On the other hand from (16), we have

$$\bigl\|u_n'\bigr\|_p^p - \int_0^b pF(t,u_n)\,dt \leqslant pM_2 \quad \text{for all } n \geqslant 1. \tag{20}$$

Adding (19) and (20), we obtain

$$\int_0^b \bigl[f(t,u_n)u_n - pF(t,u_n)\bigr]\,,dt \leqslant M_3 \quad \text{for some } M_3 > 0, \text{ all } n \geqslant 1. \tag{21}$$

Claim. *$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded.*

We argue indirectly. So, suppose that $\|u_n\| \to \infty$ and set $y_n = \frac{u_n}{\|u_n\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \quad \text{and} \quad y_n \to y \quad \text{in } C(T) \text{ as } n \to \infty. \tag{22}$$

From (18) we have

$$\left| \langle A(y_n), h \rangle - \int_0^b \frac{f(t,u_n)}{\|u_n\|^{p-1}} h\,dt \right| \leqslant \frac{\varepsilon_n \|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \quad \text{for all } n \geqslant 1. \tag{23}$$

It is clear from hypothesis H(i) that $\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\}_{n\geqslant 1} \subseteq L^1(T)$ is uniformly integrable. Hence, if we set $h = y_n - y$ and pass to the limit as $n \to \infty$, then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y \rangle = 0,$$
$$\Rightarrow \quad y_n \to y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b), \text{ hence } \|y\| = 1 \quad (\text{see Proposition 2.2}). \tag{24}$$

Since $\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\}_{n\geqslant 1} \subseteq L^1(T)$ is uniformly integrable, by the Dunford–Pettis theorem, we may assume that

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\theta} \quad \text{in } L^1(T) \tag{25}$$

with $\widehat{\theta} = \xi|y|^{p-2}y$, $\widehat{\lambda}_m \leqslant \xi(t) \leqslant \widehat{\lambda}_{m+1}$ a.e. on T (see the proof of Proposition 3.1). Passing to the limit as $n \to \infty$ in (23) and using (24) and (25), we obtain

$$\langle A(y), h \rangle = \int_0^b \xi|y|^{p-2}yh\,dt \quad \text{for all } h \in W^{1,p}_{\mathrm{per}}(0,b),$$
$$\Rightarrow \quad A(y) = \xi|y|^{p-2}y,$$
$$\Rightarrow \quad -\left(\left|y'(t)\right|^{p-2}y'(t)\right)' = \xi(t)\left|y(t)\right|^{p-2}y(t) \quad \text{a.e. on } T,$$
$$y(0) = y(b), \qquad y'(0) = y'(b). \tag{26}$$

We know that $\widehat{\lambda}_m \leqslant \xi(t) \leqslant \widehat{\lambda}_{m+1}$ a.e. on T. First suppose that $\xi \neq \widehat{\lambda}_m$ and $\xi \neq \widehat{\lambda}_{m+1}$. Then from (26) and Proposition 2.3 it follows that $y = 0$, which contradicts (24). So, we assume that $\xi(t) = \widehat{\lambda}_m$ a.e. on T or $\xi(t) = \widehat{\lambda}_{m+1}$ a.e. on T. Then being an eigenfunction for (4), we have $y(t) \neq 0$ for a.a. $t \in T$ (see Binding and Rynne [6]). Therefore $|u_n(t)| \to \infty$ for a.a. $t \in T$ and this by virtue of hypothesis H(ii) implies

$$f\left(t,u_n(t)\right)u_n(t) - pF\left(t,u_n(t)\right) \to +\infty \quad \text{for a.a. } t \in T,$$
$$\Rightarrow \quad \int_0^b \left[f\left(t,u_n(t)\right)u_n(t) - pF\left(t,u_n(t)\right)\right]dt \to +\infty \quad \left(\text{by Fatou's lemma}\right). \tag{27}$$

Comparing (21) and (27), we reach a contradiction. This proves the Claim.

By virtue of the Claim, we may assume that $u_n \xrightarrow{w} u$ in $W^{1,p}_{\text{per}}(0,b)$ and $u_n \to u$ in $C(T)$. Using $h = u_n - u$ in (18) and passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

$$\Rightarrow \quad u_n \to u \quad \text{in } W^{1,p}_{\text{per}}(0,b) \quad (\text{see Proposition 2.2}).$$

This proves the proposition. □

Proposition 3.3. **If** *hypotheses* H *hold,* **then** $u = 0$ *is a local minimizer of* $\psi_\pm$ *and of* φ.

Proof. We do the proof for ψ_+, the proofs for ψ_- and φ being similar. By virtue of hypothesis H(iii), given $\widehat{\varepsilon} > 0$, we can find $\widehat{\delta} = \widehat{\delta}(\widehat{\varepsilon}) > 0$ s.t.

$$F(x,t) \leqslant \frac{1}{p}\big(\theta(t) + \widehat{\varepsilon}\big)|x|^p \quad \text{for a.a. } t \in T, \text{ all } |x| \leqslant \widehat{\delta}. \tag{28}$$

Let $u \in \widehat{C}^1(T)$ with $\|u\|_{C^1(T)} \leqslant \widehat{\delta}$. Then

$$\begin{aligned}
\psi_+(u) &= \frac{1}{p}\|u'\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_0^b G_+(t,u)\,dt \\
&\geqslant \frac{1}{p}\|u'\|_p^p - \int_0^b F\big(t,u^+\big)\,dt \quad \big(\text{see (5)}\big) \\
&\geqslant \frac{1}{p}\|u'\|_p^p - \frac{1}{p}\int_0^b \theta|u|^p\,dt - \frac{\widehat{\varepsilon}}{p}\|u\|^p \quad \big(\text{see (28)}\big) \\
&\geqslant \frac{\xi_0 - \widehat{\varepsilon}}{p}\|u\|^p \quad (\text{see Lemma 2.5}).
\end{aligned} \tag{29}$$

Choosing $\widehat{\varepsilon} \in (0, \xi_0)$ from (29) we infer that

$$\psi_+(u) \geqslant 0 \quad \text{for all } u \in \widehat{C}^1(T) \text{ with } \|u\|_{C^1(T)} \leqslant \widehat{\delta},$$

$$\Rightarrow \quad u = 0 \text{ is a local } \widehat{C}^1(T)\text{-minimizer of } \psi_+,$$

$$\Rightarrow \quad u = 0 \text{ is a local } W^{1,p}_{\text{per}}(0,b)\text{-minimizer of } \psi_+$$

$$\big(\text{see Proposition 3.3 of Kyritsi and Papageorgiou [19]}\big).$$

Similarly for the functionals ψ_- and φ. □

We may assume that $u = 0$ is an isolated critical point of ψ_-. Indeed, otherwise we can find $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\text{per}}(0,b) \setminus \{0\}$ such that $u_n \to 0$ in $W^{1,p}_{\text{per}}(0,b)$ and

$$\begin{aligned}
&\psi_+'(u_n) = 0 \quad \text{for all } n \geqslant 1,\\
&\quad \Rightarrow \quad A(u_n) + \varepsilon |u_n|^{p-2} u_n = N_{g_+}(u_n) \quad \text{for all } n \geqslant 1,\\
&\quad \text{where } N_{g_+}(u)(\cdot) = g_+\big(\cdot, u(\cdot)\big) \text{ for all } u \in W^{1,p}_{\mathrm{per}}(0,b).
\end{aligned} \tag{30}$$

Acting on (30) with $-u_n^- \in W^{1,p}_{\mathrm{per}}(0,b)$, we obtain $u_n \geqslant 0$ for all $n \geqslant 1$ and so (30) becomes

$$\begin{aligned}
&A(u_n) = N_f(u_n) \quad \text{for all } n \geqslant 1,\\
&\quad \text{where } N_f(u)(\cdot) = f\big(\cdot, u(\cdot)\big) \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b),\\
&\quad \Rightarrow \quad u_n \in C^1(T) \text{ is a solution of (1) for all } n \geqslant 1.
\end{aligned}$$

Hence we have produced a whole sequence of distinct nontrivial (and in fact positive) solutions of (1) and so we are done.

Reasoning as in Aizicovici, Papageorgiou and Staicu [2] (see the proof of Proposition 29), we can find $\rho_+ \in (0,1)$ small s.t.

$$\psi_+(0) = 0 < \inf\big[\psi_+(u)\colon \|u\| = \rho_+\big] = \widehat{\eta}_+. \tag{31}$$

In a similar way, we show that we can find $\rho_- \in (0,1)$ small s.t.

$$\psi_-(0) = 0 < \inf\big[\psi_-(u)\colon \|u\| = \rho_-\big] = \widehat{\eta}_-. \tag{32}$$

Now we are ready to produce two constant sign solutions for problem (1).

Proposition 3.4. **If** *hypotheses* H *hold*, **then** *problem* (1) *has at least two constant sign solutions*

$$u_0 \in \operatorname{int} \widehat{C}_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} \widehat{C}_+.$$

Proof. Let $\xi \in \mathbb{R}$, $\xi > 0$. Then

$$\psi_+(\xi) = -\int_0^b F(t,\xi)\,dt \quad \big(\text{see (5)}\big).$$

From hypothesis H(ii) it follows that

$$\widehat{\lambda}_m \leqslant \liminf_{|\xi|\to\infty} \frac{pF(t,\xi)}{|\xi|^p} \leqslant \limsup_{|\xi|\to\infty} \frac{pF(t,\xi)}{|\xi|^p} \leqslant \widehat{\lambda}_{m+1} \quad \text{uniformly for a.a. } t \in T$$

(see, for example, Aizicovici, Papageorgiou and Staicu [2, Remark 26]). Therefore

$$\psi_+(\xi) \to -\infty \quad \text{as } \xi \to +\infty. \tag{33}$$

From (31), (33) and Proposition 3.1, it follows that we can apply Theorem 2.1 (the mountain pass theorem), and obtain $u_0 \in W^{1,p}_{\mathrm{per}}(0,b)$ s.t.

$$\psi_+(0) = 0 < \widehat{\eta}_+ \leqslant \psi_+(u_0), \tag{34}$$

$$\psi'_+(u_0) = 0. \tag{35}$$

From (34) we have $u_0 \neq 0$. From (35) we have

$$A(u_0) + \varepsilon |u_0|^{p-2} u_0 = N_{g_+}(u_0). \tag{36}$$

Acting on (36) with $-u_0^- \in W^{1,p}_{\text{per}}(0,b)$ and using (5), we show that $u_0 \geqslant 0$. So (36) becomes

$$\begin{aligned} &A(u_0) = N_f(u_0), \\ &\quad \Rightarrow \quad -\left(\left|u_0'(t)\right|^{p-2} u_0'(t)\right)' = f\left(t, u_0(t)\right) \quad \text{a.e. on } T, \\ &u_0(0) = u_0(b), \qquad u_0'(0) = u_0'(b), \\ &\quad \Rightarrow \quad u_0 \in \widehat{C}_+ \setminus \{0\} \text{ solves problem (1).} \end{aligned} \tag{37}$$

Let $r = \|u_0\|_\infty$. Then by virtue of hypothesis H(iv), we can find $\xi_r > 0$ s.t.

$$\begin{aligned} &f\left(t, u_0(t)\right) + \xi_r u_0(t)^{p-1} \geqslant 0 \quad \text{a.e. on } T \\ &\quad \Rightarrow \quad \left(\left|u_0'(t)\right|^{p-2} u_0'(t)\right)' \leqslant \xi_r u_0(t)^{p-1} \quad \text{a.e. on } T \ \left(\text{see (37)}\right), \\ &\quad \Rightarrow \quad u_0 \in \operatorname{int} \widehat{C}_+ \quad \left(\text{see Vazquez [27]}\right). \end{aligned}$$

Similarly, working this time with ψ_- and using (32), we obtain a second constant sign solution $v_0 \in -\operatorname{int} \widehat{C}_+$. □

Next using Morse theory, we will produce a third nontrivial solution for problem (1). We start by calculating the critical groups at infinity of φ.

Proposition 3.5. **If** *hypotheses* H *hold*, **then** $C_{m+1}(\varphi, \infty) \neq 0$ ($m \geqslant 1$ *as in hypothesis* H(ii)).

Proof. Let $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ and consider the C^1-functional $\chi : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\chi(u) = \frac{1}{p} \|u'\|_p^p - \frac{\mu}{p} \|u\|_p^p \quad \text{for all } u \in W^{1,p}_{\text{per}}(0,b).$$

We consider the homotopy

$$h(\tau, u) = (1-\tau)\varphi(u) + \tau \chi(u) \quad \text{for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

Clearly we may assume that K_φ is finite (otherwise we already have infinitely many distinct nontrivial solutions of (1) and so we are done). Note that $h(0,\cdot) = \varphi$ satisfies the *C-condition* (see Proposition 3.2) and $h(1,\cdot) = \chi$ also satisfies the *C-condition* since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$.

Claim. *There exist* $\beta \in \mathbb{R}$ *and* $\delta > 0$ *s.t.*

$$h(\tau, u) \leqslant \beta \quad \Rightarrow \quad \left(1 + \|u\|\right) \left\|h_u'(\tau, u)\right\|_* \geqslant \delta \quad \textit{for all } \tau \in [0,1].$$

We argue by contradiction. So, suppose that the Claim is not true. Since h is bounded, we can find $\{\tau_n\}_{n\geqslant 1}\subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1}\subseteq W^{1,p}_{\mathrm{per}}(0,b)$ s.t.

$$\tau_n\to\tau,\qquad \|u_n\|\to\infty,\qquad h(\tau_n,u_n)\to-\infty\quad\text{and}\quad \bigl(1+\|u_n\|\bigr)h'_u(\tau_n,u_n)\to 0. \tag{38}$$

By virtue of the last convergence in (38), we have

$$\left|\bigl\langle A(u_n),h\bigr\rangle-(1-\tau_n)\int_0^b f(t,u_n)h\,dt-\tau_n\mu\int_0^b |u_n|^{p-2}u_nh\,dt\right|\leqslant\frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{39}$$

for all $h\in W^{1,p}_{\mathrm{per}}(0,b)$ with $\varepsilon_n\to 0^+$.

Let $y_n=\frac{u_n}{\|u_n\|}$, $n\geqslant 1$. Then $\|y_n\|=1$ for all $n\geqslant 1$ and so we may assume that

$$y_n\xrightarrow{w}y\quad\text{in } W^{1,p}_{\mathrm{per}}(0,b)\quad\text{and}\quad y_n\to y\quad\text{in } C(T). \tag{40}$$

From (39) we have

$$\begin{aligned}&\left|\bigl\langle A(y_n),h\bigr\rangle-(1-\tau_n)\int_0^b\frac{f(t,u_n)}{\|u_n\|^{p-1}}h\,dt-\tau_n\mu\int_0^b|y_n|^{p-2}y_nh\,dt\right|\\&\quad\leqslant\frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}}\quad\text{for all } n\geqslant 1.\end{aligned} \tag{41}$$

Recall (see (25)) that

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\xrightarrow{w}\widehat{\theta}=\xi|y|^{p-2}y\quad\text{in } L^1(T)\text{ with }\widehat{\lambda}_m\leqslant\xi(t)\leqslant\widehat{\lambda}_{m+1}\text{ a.e. on } T. \tag{42}$$

In (41), we choose $h=y_n-y$ and pass to the limit as $n\to\infty$. Using (40), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\bigl\langle A(y_n),y_n-y\bigr\rangle=0,\\&\Rightarrow\quad y_n\to y\quad\text{in } W^{1,p}_{\mathrm{per}}(0,b)\text{ and so }\|y\|=1.\end{aligned} \tag{43}$$

Hence, if in (41) we pass to the limit as $n\to\infty$ and use (42) and (43), then

$$\begin{aligned}&\bigl\langle A(y),h\bigr\rangle=(1-\tau)\int_0^b\xi|y|^{p-2}yh\,dt+\tau\mu\int_0^b|y|^{p-2}yh\,dt\quad\text{for all } h\in W^{1,p}_{\mathrm{per}}(0,b),\\&\Rightarrow\quad A(y)=\xi_\tau|y|^{p-2}y\quad\text{with }\xi_\tau=(1-\tau)\xi+\tau\mu,\\&\Rightarrow\quad -\bigl(\bigl|y'(t)\bigr|^{p-2}y'(t)\bigr)'=\xi_\tau(t)\bigl|y(t)\bigr|^{p-2}y(t)\quad\text{a.e. on } T,\\&y(0)=y(b),\qquad y'(0)=y'(b).\end{aligned} \tag{44}$$

Note that $\widehat{\lambda}_m \leqslant \xi_\tau(t) \leqslant \widehat{\lambda}_{m+1}$ a.e. on T.
If $\tau \in (0,1]$, then

$$\xi_\tau \neq \widehat{\lambda}_m \quad \text{and} \quad \xi_\tau \neq \widehat{\lambda}_{m+1},$$
$$\Rightarrow \quad y = 0 \quad \big(\text{see (44) and Proposition 2.3}\big), \text{ which contradicts (43).}$$

So, suppose $\tau = 0$. Then $\xi_0 = \xi$ and we proceed as in the proof of Proposition 3.2 to reach a contradiction, using hypothesis H(ii) and the third convergence in (38). This proves the Claim.

Then from Lemma 8 of O'Regan, Papageorgiou and Smyrlis [22] (see also Chang [8, p. 334] and Perera and Schechter [25]) we have

$$C_k(\varphi, \infty) = C_k(\chi, \infty) \quad \text{for all } k \geqslant 0. \tag{45}$$

Since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, $u = 0$ is the only critical point of χ. Hence

$$C_k(\chi, \infty) = C_k(\chi, 0) \quad \text{for all } k \geqslant 0. \tag{46}$$

Let $r > 0$ and set $E_0 = \{u \in W^{1,p}_{\text{per}}(0,b)\colon \|u'\|_p^p < \mu\|u\|_p^p,\ \|u\| = r\}$ and $D = \{u \in W^{1,p}_{\text{per}}(0,b)\colon \|u'\|_p^p \geqslant \mu\|u\|_p^p\}$. Evidently $E_0 \cap D = \emptyset$. Also $\partial B_r = \{u \in W^{1,p}_{\text{per}}(0,b)\colon \|u\| = r\}$ is a Banach C^1-manifold, hence locally contractible. Since E_0 is an open subset of ∂B_r, E_0 is locally contractible. Similarly $W^{1,p}_{\text{per}}(0,b) \setminus D$ is locally contractible. Note that since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, we have $i(E_0) = m+1$, where i denotes the index introduced by Fadell and Rabinowitz [13]. Similarly $i(W^{1,p}_{\text{per}}(0,b) \setminus D) = m+1$. Invoking Theorem 3.6 of Cingolani and Degiovanni [9], we know that there exists $C \subseteq W^{1,p}_{\text{per}}(0,b)$ compact s.t. the pair $(E_0 \cup C, E_0)$ and D homologically link in dimension $m+1$ and so $C_{m+1}(\chi, 0) \neq 0$ (see Chang [7, p. 89]). From (45) and (46) we conclude that $C_{m+1}(\varphi, \infty) \neq 0$. $\square$

Next we compute the critical groups at infinity of $\psi_\pm$.

Proposition 3.6. **If** *hypotheses* H *hold*, **then** $C_k(\psi_+, \infty) = C_k(\psi_-, \infty) = 0$ *for all* $k \geqslant 0$.

Proof. We do the proof for ψ_+, the proof for ψ_- being similar.

Let $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ and consider the C^1-functional $\sigma_+ : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\|u'\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \frac{\mu+\varepsilon}{p}\|u^+\|_p^p$$

for all $u \in W^{1,p}_{\text{per}}(0,b)$, with $\varepsilon \in (0, \widehat{\lambda}_2)$.

We consider the homotopy $h_+ : [0,1] \times W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$h_+(\tau, u) = (1-\tau)\psi_+(u) + \tau\sigma_+(u) \quad \text{for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

As before, without any loss of generality, we assume that K_{ψ_+} is finite.

Claim. *There exist* $\beta \in \mathbb{R}$ *and* $\delta > 0$ *s.t.*

$$h_+(\tau, u) \leqslant \beta \quad \Rightarrow \quad \big(1 + \|u\|\big)\big\|(h_+)'_u(\tau, u)\big\|_* \geqslant \delta \quad \textit{for all } \tau \in [0,1].$$

As before, we argue by contradiction. So, suppose we can find $\{\tau_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ s.t.

$$\tau_n \to \tau \in [0,1], \qquad \|u_n\| \to \infty, \qquad h_+(\tau_n, u_n) \to -\infty \quad \text{and} \quad \big(1+\|u_n\|\big)(h_+)'_u(\tau_n,u_n) \to 0. \tag{47}$$

From the last convergence in (47), we have

$$\left| \langle A(u_n), h\rangle + \varepsilon \int_0^b |u_n|^{p-2} u_n h\,dt - (1-\tau_n)\int_0^b g_+(t,u_n) h\,dt \right. \\ \left. - \tau_n(\mu+\varepsilon)\int_0^b \big(u_n^+\big)^p h\,dt \right| \leqslant \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \tag{48}$$

for all $h \in W^{1,p}_{\mathrm{per}}(0,b)$ with $\varepsilon_n \to 0^+$.

In (48) we choose $h = -u_n^- \in W^{1,p}_{\mathrm{per}}(0,b)$ and

$$\big\|\big(u_n^-\big)'\big\|_p^p + \varepsilon\big\|u_n^-\big\|_p^p \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1, \\ \Rightarrow \quad u_n^- \to 0 \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \text{ as } n \to \infty. \tag{49}$$

From (47) (second convergence) and (49) it follows that $\|u_n^+\| \to \infty$. We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \quad \text{and} \quad y_n \to y \quad \text{in } C(T). \tag{50}$$

From (48) and (49), we have

$$\left| \langle A(y_n), h\rangle + \varepsilon \int_0^b y_n^{p-1} h\,dt - (1-\tau_n)\int_0^b \frac{g_+(t,u_n^+)}{\|u_n^+\|^{p-1}} h\,dt \right. \\ \left. - \tau_n(\mu+\varepsilon)\int_0^b y_n^{p-1} h\,dt \right| \leqslant \varepsilon_n' \|h\| \tag{51}$$

with $\varepsilon_n' \to 0$.

In (51) we choose $h = y_n - y$. Passing to the limit as $n \to \infty$ and using (50) we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ \Rightarrow \quad y_n \to y \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \text{ and so } \|y\| = 1,\ y \geqslant 0. \tag{52}$$

Recall that

$$\frac{g_+(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\theta}_+ = (\xi+\varepsilon) y^{p-1} \quad \text{in } L^1(T) \quad \text{and} \quad \widehat{\lambda}_m \leqslant \xi(t) \leqslant \widehat{\lambda}_{m+1} \quad \text{for a.a. } t \in T. \tag{53}$$

Therefore, if in (51) we pass to the limit as $n \to \infty$ and use (52) and (53), then

$$\begin{aligned}
&\langle A(y), h\rangle = \int_0^b \xi_\tau y^{p-1} h\,dt \quad \text{for all } h \in W^{1,p}_{\mathrm{per}}(0,b) \text{ with } \xi_\tau = (1-\tau)\xi + \tau\mu, \\
&\quad \Rightarrow \quad A(y) = \xi_\tau y^{p-1}, \\
&\quad \Rightarrow \quad -\left(\left|y'(t)\right|^{p-2} y'(t)\right)' = \xi_\tau(t) y(t)^{p-1} \quad \text{a.e. on } T, \\
&y(0) = y(b), \qquad y'(0) = y'(b).
\end{aligned} \tag{54}$$

We know that $\widehat{\lambda}_m \leqslant \xi_\tau(t) \leqslant \widehat{\lambda}_{m+1}$ a.e. on T. If $\tau \in (0,1]$, then $\xi_\tau \neq \widehat{\lambda}_m$, $\xi_\tau \neq \widehat{\lambda}_{m+1}$ and so by virtue of (54) and Proposition 2.3, we have $y = 0$, which contradicts (52). The same is true if $\tau = 0$ and $\xi_0 \neq \widehat{\lambda}_m$, $\xi_0 \neq \widehat{\lambda}_{m+1}$. Finally, if $\tau = 0$ and $\xi_0 = \widehat{\lambda}_m$, or $\xi_0 = \widehat{\lambda}_{m+1}$ a.e. on T, then from (54) and since $m \geqslant 1$, $y(\cdot)$ must be nodal again a contradiction (see (52)). This proves the claim.

The claim permits the use of Lemma 8 of [22] and we have

$$C_k(\psi_+, \infty) = C_k(\sigma_+, \infty) \quad \text{for all } k \geqslant 0. \tag{55}$$

Since $\mu \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, $u = 0$ is the only critical point of σ_+ and so

$$C_k(\sigma_+, \infty) = C_k(\sigma_+, 0) \quad \text{for all } k \geqslant 0. \tag{56}$$

Let $\eta \in L^\infty(0,b)$, $\eta \geqslant 0$, $\eta \neq 0$ and consider the homotopy $\widehat{h}_+ : [0,1] \times W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\widehat{h}_+(\tau, u) = \sigma_+(u) - \tau\eta u \quad \text{for all } (\tau, u) \in [0,1] \times W^{1,p}_{\mathrm{per}}(0,b).$$

We claim that

$$(\widehat{h}_+)'(\tau, u) \neq 0 \quad \text{for all } \tau \in [0,1],\ u \neq 0. \tag{57}$$

Suppose that (57) is not true. We can find $\tau \in [0,1]$ and $u \neq 0$ s.t.

$$\begin{aligned}
&(\widehat{h}_+)'(\tau, u) = 0, \\
&\quad \Rightarrow \quad A(u) + \varepsilon|u|^{p-2}u = (\mu + \varepsilon)\left(u^+\right)^{p-1} + \tau\eta.
\end{aligned} \tag{58}$$

On (58) we act with $-u^- \in W^{1,p}_{\mathrm{per}}(0,b)$ and obtain $\|(u^-)'\|_p^p + \varepsilon\|u^-\|_p^p = 0$, i.e., $u \geqslant 0$. So, (58) becomes

$$A(u) = \mu u^{p-1} + \tau\eta, \quad u \geqslant 0,\ u \neq 0. \tag{59}$$

First suppose that $\tau = 0$. Then

$$\begin{aligned}
&A(u) = \mu u^{p-1} \quad \big(\text{see (59)}\big), \\
&\quad \Rightarrow \quad -\left(\left|u'(t)\right|^{p-2} u'(t)\right)' = \mu u(t)^{p-1} \quad \text{a.e. on } T, \\
&u(0) = u(b), \qquad u'(0) = u'(b), \\
&\quad \Rightarrow \quad u \text{ must be nodal (recall } m \geqslant 1\text{), which contradicts (59).}
\end{aligned}$$

So, we assume that $\tau \in (0,1]$. Then

$$\begin{aligned} &A(u) = \mu u^{p-1} + \tau\eta, \\ &\quad\Rightarrow\quad -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \mu u(t)^{p-1} + \tau\eta(t) \quad \text{a.e. on } T, \\ &u(0) = u(b), \qquad u'(0) = u'(b). \end{aligned} \tag{60}$$

We have $u \in C_+ \setminus \{0\}$ and $(|u'(t)|^{p-2}u'(t))' \leqslant 0$ a.e. on T. It follows that $u \in \operatorname{int}\widehat{C}_+$ (see Vazquez [27]).

Let $y \in \widehat{C}_+$ and consider

$$R(y,u)(t) = \left|y'(t)\right|^p - \left|u'(t)\right|^{p-2}u'(t)\left(\frac{y^p}{u^{p-1}}\right)'(t).$$

From the generalized Picone identity of Allegretto and Huang [3], we have

$$\begin{aligned} 0 &\leqslant \int_0^b R(y,u)(t)\,dt \\ &= \|y'\|_p^p - \int_0^b -\left(\left|u'\right|^{p-2}u'\right)'\frac{y^p}{u^{p-1}}\,dt \quad \text{(by integration by parts)} \\ &= \|y'\|_p^p - \int_0^b \left(\mu y^p + \tau\eta\right)dt \quad \left(\text{see (60)}\right) \\ &\leqslant \|y'\|_p^p - \mu\|y\|_p^p \quad (\text{recall } \eta \geqslant 0). \end{aligned}$$

We choose $y = \widehat{u}_0 \in \operatorname{int}\widehat{C}_+$. Then

$$0 \leqslant -\mu\widehat{u}_0^p b < 0, \quad \text{a contradiction.}$$

This proves that (57) holds. Then the homotopy invariance property of critical groups (see Chang [8, p. 334]) implies that

$$C_k(\sigma_+,0) = C_k(\widehat{\sigma}_+,0) \quad \text{for all } k \geqslant 0, \tag{61}$$

where $\widehat{\sigma}_+(u) = \sigma_+(u) - \eta u$ for all $u \in W^{1,p}_{\mathrm{per}}(0,b)$. From the previous argument, we know that $\widehat{\sigma}_+$ has no critical points. Then

$$\begin{aligned} &C_k(\widehat{\sigma}_+,0) = 0 \quad \text{for all } k \geqslant 0, \\ &\quad\Rightarrow\quad C_k(\psi_+,\infty) \quad \text{for all } k \geqslant 0 \quad \left(\text{see (61), (56) and (55)}\right). \end{aligned}$$

Similarly, we show that $C_k(\psi_-,\infty) = 0$ for all $k \geqslant 0$. $\square$

Having this proposition, we can have a precise computation of the critical groups of φ at $u_0 \in \operatorname{int}\widehat{C}_+$ and $v_0 \in -\operatorname{int}\widehat{C}_+$. Recall that u_0, v_0 are the two constant sign solutions of (1) obtained in Proposition 3.4.

Proposition 3.7. **If** *hypotheses* H *hold and* $u_0 \in \operatorname{int} \widehat{C}_+$ *and* $v_0 \in -\operatorname{int} \widehat{C}_+$ *are the two constant sign solutions of* (1) *obtained in* Proposition 3.4, **then** $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all* $k \geqslant 0$.

Proof. We do the proof for u_0, the proof for v_0 being similar.

First note that we may assume that $\{0, u_0\}$ are the only critical points of ψ_+ (otherwise, we already have one more solution $y_0 \in \operatorname{int} \widehat{C}_+$ of (1) distinct from $\{0, u_0, v_0\}$; note that $K_{\psi_+} \subseteq \widehat{C}_+$).

Let $\eta < 0 < \xi < \widehat{\eta}_+$ (see (31)) and consider the following triple of sets

$$\psi_+^{\eta} \subseteq \psi_+^{\xi} \subseteq W^{1,p}_{\text{per}}(0, b).$$

For this triple, we consider the long exact sequence of homology groups

$$\cdots \to H_k\big(W^{1,p}_{\text{per}}(0, b), \psi_+^{\eta}\big) \xrightarrow{i_*} H_k\big(W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi}\big) \xrightarrow{\partial_*} H_{k-1}\big(\psi_+^{\xi}, \psi_+^{\eta}\big) \to \cdots. \tag{62}$$

By i_* we denote the group homomorphism induced by the inclusion $(W^{1,p}_{\text{per}}(0, b), \psi_+^{\eta}) \xrightarrow{i} (W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi})$ and ∂_* is the boundary homomorphism. From the rank theorem, we have

$$\begin{aligned} \mathit{rank}\, H_k\big(W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi}\big) &= \mathit{rank}(\mathit{ker}\, \partial_*) + \mathit{rank}(\mathit{im} \partial_*) \quad \big(\text{see (62)}\big) \\ &= \mathit{rank}(\mathit{im} i_*) + \mathit{rank}(\mathit{im} \partial_*) \quad \big(\text{from the exactness of (62)}\big). \end{aligned} \tag{63}$$

Recalling that $\{0, u_0\}$ are the only critical points of ψ_+ and since

$$\eta < 0 = \psi_+(0) < \widehat{\eta}_+ \leqslant \psi_+(u_0),$$

we have

$$\begin{aligned} &H_k\big(W^{1,p}_{\text{per}}(0, b), \psi_+^{\eta}\big) = C_k(\psi_+, \infty) = 0 \quad \text{for all } k \geqslant 0 \quad (\text{see Proposition 3.6}), \\ &\Rightarrow \quad \mathit{im} i_* = \{0\}. \end{aligned} \tag{64}$$

Also $H_{k-1}(\psi_+^{\xi}, \psi_+^{\eta}) = C_{k-1}(\psi_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z}$ for all $k \geqslant 0$ (see Proposition 3.3). Therefore

$$\mathit{rank}(\mathit{im} \partial_*) \leqslant 1. \tag{65}$$

Finally since $0 < \xi < \widehat{\eta}_+ \leqslant \psi_+(u_0)$, we have

$$H_k\big(W^{1,p}_{\text{per}}(0, b), \psi_+^{\xi}\big) = C_k(\psi_+, u_0) \quad \text{for all } k \geqslant 0. \tag{66}$$

So, if in (63), we use (64), (65), (66), then

$$\mathit{rank}\, C_1(\psi_+, u_0) \leqslant 1. \tag{67}$$

From the proof of Proposition 3.4, we know that u_0 is a critical point of ψ_+ of mountain pass type. Hence $C_1(\psi_+, u_0) \neq 0$ (see Chang [7, p. 89]). Combining this with (67) we infer that

$$C_k(\psi_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{68}$$

Consider the homotopy $\bar{h}_+ : [0,1] \times W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\bar{h}_+(\tau,u) = (1-\tau)\varphi(u) + \tau\psi_+(u) \quad \text{for all } (\tau,u) \in [0,1] \times W^{1,p}_{\mathrm{per}}(0,b).$$

Claim. *We may assume that we can find $\rho \in (0,1)$ small s.t. u_0 is the only critical point for all $\tau \in [0,1]$ of $\bar{h}_+(\tau,\cdot)$ in $\bar{B}_\rho(u_0) = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u-u_0\| = \rho\}$.*

Suppose we can find $\{\tau_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{\bar{u}_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ s.t.

$$\tau_n \to \tau \in [0,1], \bar{u}_n \to u_0 \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \quad \text{and} \quad (\bar{h}_+)'(\tau_n,\bar{u}_n) = 0 \quad \text{for all } n \geqslant 1. \tag{69}$$

We have

$$\begin{aligned}
&A(\bar{u}_n) + \tau_n|\bar{u}_n|^{p-2}\bar{u}_n = (1-\tau_n)N_{g_+}(\bar{u}_n) + \tau_n N_f(\bar{u}_n) \quad \text{for all } n \geqslant 1,\\
&\quad \Rightarrow \quad -\big(\big|\bar{u}_n'(t)\big|^{p-2}\bar{u}_n'(t)\big)' = f\big(t,\bar{u}_n^+(t)\big) + (1-\tau_n)f\big(t,-\bar{u}_n^-(t)\big) + \tau_n\big(u_n^-\big)^{p-1} \quad \text{a.e. on } T,\\
&\bar{u}_n(0) = \bar{u}_n(b), \qquad \bar{u}_n'(0) = \bar{u}_n'(b).
\end{aligned} \tag{70}$$

From (70), arguing as in the proof of Proposition 3.3 of Kyritsi and Papageorgiou [19], we establish that $\{\bar{u}_n\}_{n\geqslant 1} \subseteq C^1(T)$ is relatively compact. Therefore we have

$$\bar{u}_n \to u_0 \quad \text{in } C^1(T) \quad \big(\text{see (69)}\big). \tag{71}$$

Recall that $u_0 \in \operatorname{int}\widehat{C}_+$. So, we can find $n_0 \geqslant 1$ s.t.

$$\begin{aligned}
&\bar{u}_n \in \operatorname{int}\widehat{C}_+ \quad \text{for all } n \geqslant n_0 \quad \big(\text{see (71)}\big),\\
&\quad \Rightarrow \quad -\big(\big|\bar{u}_n'(t)\big|^{p-2}\bar{u}_n'(t)\big)' = f\big(t,\bar{u}_n(t)\big) \quad \text{a.e. on } T,\\
&\bar{u}_n(0) = \bar{u}_n(b), \qquad \bar{u}_n'(0) = \bar{u}_n'(b),\\
&\quad \Rightarrow \quad \{\bar{u}_n\}_{n\geqslant 1} \subseteq \operatorname{int}\widehat{C}_+ \text{ are nontrivial solutions of (1) and so we are done.}
\end{aligned}$$

This proves the Claim.

Then the Claim and the homotopy invariance property of the critical groups (see Chang [8, p. 334]), we have

$$\begin{aligned}
&C_k(\varphi,u_0) = C_k(\psi_+,u_0) \quad \text{for all } k \geqslant 0,\\
&\quad \Rightarrow \quad C_k(\varphi,u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0 \quad \big(\text{see (68)}\big).
\end{aligned}$$

In the similar fashion, using this time ψ_-, we show that $C_k(\varphi,v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geqslant 0$. $\square$

Now we can state the multiplicity theorem for problem (1) under double resonance conditions.

Theorem 3.8. **If** *hypotheses* H *hold*, **then** *problem* (1) *has at least three nontrivial solutions*

$$u_0 \in \operatorname{int}\widehat{C}_+, \qquad v_0 \in -\operatorname{int}\widehat{C}_+ \quad \textit{and} \quad y_0 \in C^1(T).$$

Proof. From Proposition 3.4, we already have two nontrivial constant sing solutions of (1)

$$u_0 \in \operatorname{int} \widehat{C}_+ \quad \text{and} \quad v_0 \in -\operatorname{int} \widehat{C}_+.$$

From Proposition 3.7, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{72}$$

Also, by virtue of Proposition 3.3, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{73}$$

Recall that $C_{m+1}(\varphi, \infty) \neq 0$ (see Proposition 3.5). This implies that there exists $y_0 \in K_\varphi$ s.t.

$$C_{m+1}(\varphi, y_0) \neq 0, \quad m \geqslant 2. \tag{74}$$

Comparing (74) with (72) and (73), we infer that $y_0 \notin \{0, u_0, v_0\}$. Also $y_0 \in C^1(T)$ and solves problem (1). □

4. Existence theorem

In the previous section we proved a multiplicity theorem (three nontrivial solutions) for problem (1), by avoiding double resonance in the first spectral interval $[\widehat{\lambda}_0 = 0, \widehat{\lambda}_1]$. It is natural to ask what can be said if double resonance occurs in the first spectral interval $[0, \widehat{\lambda}_1]$. In this case, we show that we can still have an existence theorem.

The new hypotheses on the reaction $f(t,x)$ are:

H′ $f : T \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function s.t. for a.a. $t \in T$, $f(t,0) = 0$ and

- **(i)** $|f(t,x)| \leqslant a(t)(1 + |x|^{p-1})$ for a.a. $t \in T$, all $x \in \mathbb{R}$, with $a \in L^1(T)_+$;
- **(ii)** $0 \leqslant \liminf_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \limsup_{|x|\to\infty} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \widehat{\lambda}_1$ uniformly for a.a. $t \in T$ and $\lim_{|x|\to\infty}[f(t,x)x - pF(t,x)] = +\infty$ uniformly for a.a. $t \in T$;
- **(iii)** there exist an integer $l \geqslant 1$ and functions $\xi, \widehat{\xi} \in L^\infty(T)_+$ s.t.

$$\widehat{\lambda}_l \leqslant \xi(t) \leqslant \widehat{\xi}(t) \leqslant \widehat{\lambda}_{l+1} \quad \text{for a.a. } t \in T,\ \widehat{\lambda}_l \neq \xi,\ \widehat{\lambda}_{l+1} \neq \widehat{\xi}$$
$$\text{and} \quad \xi(t) \leqslant \liminf_{x\to 0} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \limsup_{x\to 0} \frac{f(t,x)}{|x|^{p-2}x} \leqslant \widehat{\xi}(t)$$

uniformly for a.a. $t \in T$.

Remark. Hypothesis H′(ii) implies that we have double resonance in the spectral interval $[0, \widehat{\lambda}_1]$. Hypothesis H′(iii) implies that at zero we have nonuniform nonresonance with respect to the first two eigenvalues $\widehat{\lambda}_0 = 0$ and $\widehat{\lambda}_1 > 0$. Again we emphasize that no differentiability hypothesis is assumed on $f(t,\cdot)$.

Example. The following function $f(x)$ satisfies hypotheses H′ (for the sake of simplicity we drop the t-dependence):

$$f(x) = \begin{cases} \xi|x|^{p-2}x & \text{if } |x| \leqslant 1, \\ \widehat{\lambda}_1|x|^{p-2}x + (\xi - \widehat{\lambda}_1)|x|^{q-2}x & \text{if } |x| > 1, \end{cases}$$

with $\xi \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$, $l \geqslant 1$ and $1 < q < p < \infty$.

Let $\mu \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$ and as before we consider the C^1-functional $\chi : W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\chi(u) = \frac{1}{p}\|u'\|_p^p - \frac{\mu}{p}\|u\|_p^p \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b).$$

We will show that the first two critical groups of χ at the origin are trivial (i.e., $C_0(\chi,0) = C_1(\chi,0) = 0$). To this end, we will need the following lemma.

Lemma 4.1. **If** $\mu \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$ *with* $l \geqslant 1$ *and* $V_\mu = \{u \in W^{1,p}_{\mathrm{per}}(0,b)\colon \|u'\|_p^p < \mu\|u\|_p^p\}$, **then** V_μ *is path-connected.*

Proof. Note that $\pm\widehat{u}_0 \in V_\mu$. To prove the lemma it suffices to connect an arbitrary $\widetilde{u} \in V_\mu$ with $\widehat{u}_0$ by a continuous path which stays in the open set V_μ. So, let $\widetilde{V}$ be the path component of V_μ containing $\widetilde{u}$. We define

$$\widetilde{m} = \inf\left[\frac{\|u'\|_p^p}{\|u\|_p^p}\colon u \in \widetilde{V},\ u \neq 0\right] = \inf\left[\|u\|_p^p\colon u \in M \cap \widetilde{V}\right],$$

where $M = W^{1,p}_{\mathrm{per}}(0,b) \cap \partial B_1^{L^p}$. By virtue of the Ekeland variational principle and Lemma 3.5($\underline{\text{iii}}$) of Cuesta, de Figueiredo and Gossez [10], we can find $\{u_n\}_{n\geqslant 1} \subseteq M \cap \widetilde{V}$ s.t.

$$\|u_n'\|_p^p \to \widetilde{m} \tag{75}$$

and

$$\left|\langle A(u_n), h\rangle\right| \leqslant \varepsilon_n\|h\| \quad \text{for all } h \in T_{u_n}M \text{ with } \varepsilon_n \to 0^+. \tag{76}$$

Here by $T_{u_n}M$ we denote the tangent space at u_n of the Banach C^1-manifold M. We know that $T_{u_n}M = \{h \in W^{1,p}_{\mathrm{per}}(0,b)\colon \int_0^b |u_n|^{p-2}u_n h\,dt = 0\}$. For $y \in W^{1,p}_{\mathrm{per}}(0,b)$ we set

$$h = y - \left(\int_0^b |u_n|^{p-2}u_n y\,dt\right)u_n \in T_{u_n}M \quad \text{for all } n \geqslant 1.$$

In (76) we use this h and obtain

$$\left|\langle A(u_n), y\rangle - \int_0^b |u_n|^{p-2}u_n y\,dt\,\|u_n'\|_p^p\right| \leqslant \varepsilon_n c\|y\| \quad \text{for some } c > 0,\ \text{all } n \geqslant 1. \tag{77}$$

From (75) and since $\|u_n\|_p = 1$ for all $n \geqslant 1$, we see that $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}_{\mathrm{per}}(0,b)$ is bounded. So, we may assume that $u_n \xrightarrow{w} \widetilde{u}$ in $W^{1,p}_{\mathrm{per}}(0,b)$. In (77) we choose $y = u_n - \widetilde{u}$. Then

$$\lim_{n\to\infty}\langle A(u_n), u_n - \widetilde{u}\rangle = 0,$$

$$\Rightarrow \quad u_n \to \widetilde{u} \quad \text{in } W^{1,p}_{\mathrm{per}}(0,b) \quad (\text{see Proposition 2.2}).$$

Hence $\widetilde{u} \in \overline{M \cap \widetilde{V}}$ and $\widetilde{m} = \|\widetilde{u}'\|_p^p$. If $\widetilde{u}$ is boundary point of $\overline{M \cap \widetilde{V}}$, since $M \cap \widetilde{V}$ is a path component of $M \cap V_\mu$ which is open in M, then by virtue of Lemma 3.5(iii) of Cuesta, de Figueiredo and Gossez [10], we have $\widetilde{u} \notin M \cap V_\mu$. But $\|\widetilde{u}'\|_p^p = \widetilde{m} < \mu$ and so $\widetilde{u} \in M \cap V_\mu$, a contradiction. This proves that $\widetilde{u}$ cannot be a boundary point of $\overline{M \cap \widetilde{V}}$ and so $\widetilde{u} \in M \cap \widetilde{V}$, which means that $\widetilde{u}$ is a critical point of the functional $\sigma(u) = \|u'\|_p^p$ on the Banach C^1-manifold M. Therefore, to prove the lemma, it suffices to connect $\widehat{u}_0$ and this particular $\widetilde{u}$ with a continuous path staying in V_μ.

First we assume that $\widetilde{u} \leqslant 0$. We know that the only L^p-normalized constant sign eigenfunctions of the negative periodic scalar p-Laplacian are $\pm\widehat{u}_0$. Since $\|\widetilde{u}'\|_p^p = \widetilde{m}$, by the Lagrange multiplier rule $\widetilde{u}$ is an eigenfunction in M and so $\widetilde{u} = -\widehat{u}_0$. Then Proposition 2.4 guarantees that there is a continuous curve in V_μ connecting $\widehat{u}_0$ and $-\widehat{u}_0 = \widetilde{u}$.

Therefore, we way assume that $\widetilde{u}^+ \neq 0$. We set

$$\widetilde{u}_\tau = \frac{\widetilde{u}^+ - (1-\tau)\widetilde{u}^-}{\|\widetilde{u}^+ - (1-\tau)\widetilde{u}^-\|_p} \in M \quad \text{for all } \tau \in [0,1].$$

As we said, since $\|\widetilde{u}'\|_p^p = \widetilde{m}$, by virtue of the Lagrange multiplier rule, we have

$$\langle A(\widetilde{u}), h\rangle = \widetilde{m} \int_0^b |\widetilde{u}|^{p-2}\widetilde{u} h \, dt \quad \text{for all } h \in W^{1,p}_{\mathrm{per}}(0,b). \tag{78}$$

In (78) we choose $h = \widetilde{u}^+ \in W^{1,p}_{\mathrm{per}}(0,b)$ and obtain

$$\|(\widetilde{u}^+)'\|_p^p = \widetilde{m}\|\widetilde{u}^+\|_p^p. \tag{79}$$

Next in (78) we choose $h = -\widetilde{u}^- \in W^{1,p}_{\mathrm{per}}(0,b)$ and have

$$\|(\widetilde{u}^-)'\|_p^p = \widetilde{m}\|\widetilde{u}^-\|_p^p. \tag{80}$$

Note that $\widetilde{u}^+$ and $\widetilde{u}^-$ have disjoint supports. So, (79) and (80) imply

$$\begin{aligned} &\|\widetilde{u}_\tau'\|_p^p = \widetilde{m} \quad \text{for all } \tau \in [0,1], \\ \Rightarrow\quad &\widetilde{u}_1 = \frac{\widetilde{u}^+}{\|\widetilde{u}^+\|_p} \text{ is a minimizer of } u \to \sigma(u) = \|u'\|_p^p \text{ on } M, \\ \Rightarrow\quad &\widetilde{u}_1 = \widehat{u}_0 \quad (\text{since } \widetilde{u}_1 \geqslant 0). \end{aligned}$$

On the other hand

$$\widetilde{u}_0 = \frac{\widetilde{u}^+ - \widetilde{u}^-}{\|\widetilde{u}^+ - \widetilde{u}^-\|_p} = \frac{\widetilde{u}}{\|\widetilde{u}\|_p} = \widetilde{u} \quad (\text{since } \widetilde{u} \in M).$$

There $\tau \to \widetilde{u}_\tau$ is a continuous curve connecting $\widetilde{u}$ and $\widehat{u}_0$ which stays in V_μ since $\|\widetilde{u}_\tau'\| = \widetilde{m} < \mu$ for all $\tau \in [0,1]$. $\square$

This lemma leads to the following observation concerning the first two critical groups of $\chi(u) = \frac{1}{p}\|u'\|_p^p - \frac{\mu}{p}\|u\|_p^p$ for all $u \in W^{1,p}_{\mathrm{per}}(0,b)$ with $\mu \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$.

Proposition 4.2. **If** *hypotheses* H *hold*, **then** $C_0(\chi,0) = C_1(\chi,0) = 0$.

Proof. Since $\mu \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$, $u = 0$ is the only critical point of χ. So, by definition

$$C_k(\chi, 0) = H_k\big(\chi^0, \chi^0 \setminus \{0\}\big)$$

(we can take $U = W^{1,p}_{\text{per}}(0, b)$ in the definition of the critical groups; see Section 2).

We have $V_\mu \subseteq \chi^0 \setminus \{0\}$. Let $* \in V_\mu$. Then from the reduced exact homology sequence (see Granas and Dugundji [17, p. 388]), we have

$$\cdots \to H_k\big(\psi^0, *\big) \xrightarrow{i_*} H_k\big(\psi^0, \psi^0 \setminus \{0\}\big) \xrightarrow{\partial_*} H_{k-1}\big(\psi^0 \setminus \{0\}, *\big) \to \cdots, \tag{81}$$

where i_* is the group homomorphism induced by the inclusion map $(\chi^0, *) \xrightarrow{i} (\chi^0, \chi^0 \setminus \{0\})$ and ∂_* is the boundary homomorphism. Since χ is p-homogeneous, χ^0 is radially contractible and so

$$C_k\big(\chi^0, *\big) = 0 \quad \text{for all } k \geqslant 0 \quad \big(\text{see Granas and Dugundji [17, p. 389]}\big). \tag{82}$$

From the exactness of the long sequence (81) we have

$$0 = im i_* = ker\, \partial_* \quad \big(\text{see (82)}\big).$$

This means that $H_k(\chi^0, \chi^0 \setminus \{0\}) = C_k(\chi, 0)$ is isomorphic to a subgroup of $H_{k-1}(\chi^0 \setminus \{0\}, *)$. We have $V_\mu = \dot{\chi}^0 = \{u \in W^{1,p}_{\text{per}}(0, b)\colon\ \chi(u) < 0\}$. Then using the second deformation theorem (see, for example, Papageorgiou and Kyritsi [23, p. 349] and the result of Granas and Dugundji [17, p. 407]), we have

$$\chi^0 \setminus \{0\} \text{ is homotopy equivalent to } \chi^{-\varepsilon} \quad (\varepsilon > 0 \text{ small}) \tag{83}$$

and

$$\dot{\chi}^0 = V_\mu \text{ is homotopy equivalent to } \chi^{-\varepsilon} \quad (\varepsilon > 0 \text{ small}). \tag{84}$$

From (83) and (84) it follows that

$$\begin{aligned} &\chi^0 \setminus \{0\} \text{ is homotopy equivalent to } V_\mu = \dot{\chi}^0, \\ \Rightarrow \quad & H_k\big(\chi^0 \setminus \{0\}, *\big) \simeq H_k(V_\mu, *) \quad \text{for all } k \geqslant 0. \end{aligned} \tag{85}$$

From Lemma 4.1 we know that V_μ is path connected. Hence

$$\begin{aligned} &H_0(V_\mu, *) = 0 \quad \big(\text{see, for example, Maunder [20, p. 109]}\big), \\ \Rightarrow \quad & H_0\big(\chi^0 \setminus \{0\}, *\big) = 0 \quad \big(\text{see (85)}\big). \end{aligned} \tag{86}$$

Recall that $C_k(\chi, 0)$ is isomorphic to a subgroup of $H_{k-1}(\chi^0 \setminus \{0\}, *)$ for $k \geqslant 0$. Hence $C_0(\chi, 0) = C_1(\chi, 0) = 0$ (recall that for $k < 0$, all singular homology groups are by definition trivial). □

Now we are ready for the existence theorem.

Theorem 4.3. If *hypotheses* H′ *hold,* **then** *problem* (1) *has a nontrivial solution* $\widehat{u} \in C^1(T)$.

Proof. Reasoning as in Perera [24] (see Lemma 4.1), we can find $R > 0$ and $\widehat{\varphi} \in C^1(W^{1,p}_{\text{per}}(0,b))$ s.t.

$$\widehat{\varphi}(u) = \begin{cases} \chi(u) & \text{if } \|u\| \leqslant R, \\ \varphi(u) & \text{if } \|u\| \geqslant 2^{\frac{1}{p}} R \end{cases} \tag{87}$$

and $K_{\widehat{\varphi}} \subseteq K_\varphi$. From (87) we see that

$$C_k(\widehat{\varphi}, 0) = C_k(\chi, 0) \quad \text{for all } k \geqslant 0. \tag{88}$$

From Proposition 4.2, we know that

$$\begin{aligned} &C_0(\chi, 0) = C_1(\chi, 0) = 0, \\ &\Rightarrow \quad C_0(\widehat{\varphi}, 0) = C_1(\widehat{\varphi}, 0) = 0 \quad \big(\text{see (88)}\big). \end{aligned} \tag{89}$$

On the other hand, again from (87) we see that

$$C_k(\widehat{\varphi}, \infty) = C_k(\varphi, \infty) \quad \text{for all } k \geqslant 0. \tag{90}$$

As in the proof of Proposition 3.5, we show that

$$\begin{aligned} &C_1(\varphi, \infty) \neq 0, \\ &\Rightarrow \quad C_1(\widehat{\varphi}, \infty) \neq 0 \quad \big(\text{see (90)}\big). \end{aligned}$$

This means that we can find $\widehat{u} \in K_{\widehat{\varphi}}$ s.t.

$$C_1(\widehat{\varphi}, \widehat{u}) \neq 0. \tag{91}$$

Comparing (89) and (91) we infer that $\widehat{u} \neq 0$. Moreover, $\widehat{u} \in K_{\widehat{\varphi}} \subseteq K_\varphi$ and so $\widehat{u} \in C^1(T)$ solves problem (1). □

So our work here shows that existence of nontrivial solutions is guaranteed if double resonance occurs at any spectral interval and multiplicity (producing at least three nontrivial solutions) can happen when we have double resonance at any spectral interval beyond the "principal" one $[\widehat{\lambda}_0 = 0, \widehat{\lambda}_1]$.

Acknowledgment

The authors wish to thank a knowledgeable referee for his/her corrections, remarks and criticisms which helped improve the paper.

References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, Periodic solutions for second order differential inclusions with the scalar p-Laplacian, J. Math. Anal. Appl. 322 (2006) 913–929.
[2] S. Aizicovici, N.S. Papageorgiou, V. Staicu, Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints, Mem. Amer. Math. Soc. 196 (2008) 70.
[3] W. Allegretto, Y. Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal. 32 (1998) 819–830.
[4] H. Berestycki, D. de Figueiredo, Double resonance in semilinear elliptic problems, Comm. Partial Differential Equations 6 (1981) 91–120.
[5] P. Binding, B.P. Rynne, The spectrum of the periodic p-Laplacian, J. Differential Equations 235 (2007) 199–218.
[6] P. Binding, B.P. Rynne, Variational and nonvariational eigenvalues of the p-Laplacian, J. Differential Equations 244 (2008) 24–39.

[7] K.C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Boston, 1993.
[8] K.C. Chang, Methods in Nonlinear Analysis, Springer, Berlin, 2005.
[9] S. Cingolani, M. Degiovanni, Nontrivial solutions for p-Laplace equations with right hand side having p-linear growth at infinity, Comm. Partial Differential Equations 30 (2005) 1191–1205.
[10] M. Cuesta, D. de Figueiredo, J.-P. Gossez, The beginning of the Fučik spectrum for the p-Laplacian, J. Differential Equations 159 (2002) 212–238.
[11] C. Fabry, A. Fonda, Periodic solutions of nonlinear differential equations with douple resonance, Ann Mat. Pura Appl. 157 (1990) 99–116.
[12] C. Fabry, D. Fayyad, Periodic solutions of second order differential equations with a p-Laplacian and asymmetric nonlinearities, Rend. Istit. Mat. Univ. Trieste 24 (1994) 207–227.
[13] E.R. Fadell, P. Rabinowitz, Generalized cohomological index theories for Lie group actions with applications to bifurcation questions for Hamiltonian systems, Invent. Math. 45 (1978) 139–174.
[14] L. Gasiński, N.S. Papageorgiou, Nonlinear Analysis, Chapman & Hall/CRC Press, Boca Raton, FL, 2006.
[15] L. Gasiński, N.S. Papageorgiou, A multiplicity theorem for double resonant problems, Adv. Nonlinear Stud. 10 (2010) 819–836.
[16] L. Gasiński, N.S. Papageorgiou, Dirichlet problems with double resonance and an indefinite potential, Nonlinear Anal. 75 (2012) 4560–4595.
[17] A. Granas, J. Dugundji, Fixed Point Theory, Springer, New York, 2003.
[18] S. Kyritsi, N.S. Papageorgiou, Solutions for doubly resonant nonlinear nonsmooth periodic problems, Proc. Edinb. Math. Soc. 48 (2005) 199–211.
[19] S. Kyritsi, N.S. Papageorgiou, On the multiplicity of solutions for nonlinear periodic problems with the nonlinearity crossing several eigenvalues, Glasg. Math. J. 52 (2010) 271–302.
[20] C.R.E. Maunder, Algebraic Topology, Dover Publications, Mineola, New York, 1996.
[21] P. Omari, F. Zanolin, Nonresonance conditions on the potential for a second order periodic boundary value problem, Proc. Amer. Math. Soc. 117 (1993) 125–135.
[22] D. O'Regan, N.S. Papageorgiou, G. Smyrlis, Neumann problems with double resonance, Topol. Methods Nonlinear Anal. 39 (2012) 151–173.
[23] N.S. Papageorgiou, S. Kyritsi, Handbook of Applied Analysis, Springer, New York, 2009.
[24] K. Perera, Nontrivial critical groups in p-Laplacian problems via the Yang index, Topol. Methods Nonlinear Anal. 21 (2003) 301–310.
[25] K. Perera, M. Schechter, Solution of nonlinear equations having asymptotic limits at zero and infinity, Calc. Var. Partial Differential Equations 12 (2001) 359–369.
[26] J. Su, L. Zhao, Multiple periodic solutions of ordinary differential equations with double resonance, Nonlinear Anal. 700 (2009) 1520–1527.
[27] J. Vazquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.
[28] M. Zhang, The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, J. Lond. Math. Soc. 64 (2001) 125–143.