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Invariant regions for systems of lattice reaction–diffusion equations

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Abstract

In this paper, we study systems of lattice differential equations of reaction–diffusion type. First, we establish some basic properties such as the local existence and global uniqueness of bounded solutions. Then we proceed to our main goal, which is the study of invariant regions. Our main result can be interpreted as an analogue of the weak maximum principle for systems of lattice differential equations. It is inspired by existing results for parabolic differential equations, but its proof is different and relies on the Euler approximations of solutions to lattice differential equations. As a corollary, we obtain a global existence theorem for nonlinear systems of lattice reaction–diffusion equations. The results are illustrated on examples from population dynamics.

Keywords: Lattice differential equation; Reaction–diffusion equation; Invariant region; Maximum principle; Existence and uniqueness

1. Introduction

The most studied example of a lattice differential equation has the form

$$\frac{\partial u}{\partial t}(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \geq 0, \tag{1.1}$$

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where $u : \mathbb{Z} \times [0, \infty) \to \mathbb{R}$ is the unknown function. This equation is obtained from the classical one-dimensional reaction–diffusion equation

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t),x,t), \quad x \in \mathbb{R}, \quad t \in [0,\infty), \tag{1.2}$$

by discretizing the space variable. For some applications in biology, chemistry, kinematics or population dynamics, the semidiscrete equation seems to be more appropriate than the classical reaction–diffusion equation (see, e.g., [2,11,14,21,22]).

For various choices of the reaction function f, numerous authors have studied the properties of solutions to Eq. (1.1), such as the asymptotic behavior [4,35,36], existence of traveling wave solutions [8,11,24,40,41] or pattern formation [6–8]. On the other hand, the recent papers [30, 31] have focused on well-posedness results and maximum principles for Eq. (1.1) with a general reaction function f. Let us mention that the maximum principles are important for the study of traveling wave solutions (cf. [24,39]).

Systems of two or more lattice differential equations were also considered by numerous authors. The motivation for the study of such systems often comes from population dynamics – see, e.g., [5,16–19,23] and the references there. Again, most papers focus on equations of reaction–diffusion type with specific choices of the reaction function. A fairly general class of linear lattice systems with continuous, discrete or mixed time was studied in [28].

The present paper is devoted to general systems of nonlinear lattice differential equations of the form

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= A(x,t)u(x+1,t) + B(x,t)u(x,t) + C(x,t)u(x-1,t)\\ &\quad + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \geq 0,\end{aligned} \tag{1.3}$$

where u takes values in $\mathbb{R}^m$ and A, B, C are matrix-valued functions. Obviously, Eq. (1.1) represents a special case of Eq. (1.3) with $m = 1$, $A(x,t) = C(x,t) = k$ and $B(x,t) = -2k$.

In Section 2, we present some basic results on the existence and uniqueness of solutions to nonlinear systems of lattice equations. We focus on initial-values problems with bounded initial conditions. Such problems generally have infinitely many solutions (see, e.g., [29, Section 3]); to get uniqueness, we restrict ourselves to the class of bounded solutions. As explained in [14], the space of bounded sequences is a quite natural choice for the study of diffusion-type lattice differential equations.

The core of the paper is in Section 3, where we study invariant regions for systems of the form (1.3). The invariance results can be interpreted as a generalization of the weak maximum principle: In the scalar case (1.1), the weak maximum principle says that under suitable assumptions on the reaction function f, the values of the solution always remain in the interval determined by the infimum and supremum of the initial values. Thus, the interval is an invariant region for the given equation. In the higher-dimensional setting, the interval is replaced by a closed convex set S, and the problem is to find sufficient conditions guaranteeing that S is an invariant region, i.e., that solutions with initial values in S never leave this set. The key assumption is that the vector field f points inward S or is tangent to the boundary at all boundary points of S. This condition is well known from the invariance results for classical parabolic equations; see [1,9,25,34,38]. The proofs of these classical results are fairly straightforward for bounded spatial domains, while the treatment of unbounded domains is more difficult.

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For example, Weinberger [38] uses the fact that solutions to initial-value problems on unbounded spatial domains can be obtained as limits of solutions to initial-boundary-value problems on bounded domains. As far as we are aware, no such result is available for lattice reaction–diffusion systems. Chueh et al. [9] simply add the additional hypothesis that for all $t > 0$, $u(x,t) \in S$ whenever x is sufficiently large. Redheffer and Walter [25] provide a more general invariance theorem for solutions satisfying a certain growth condition as $|x| \to \infty$; however, their method does not seem to be applicable to lattice equations. The method used by València [34] is elegant and can be used in the context of lattice equations, but it seems to work only in the case when S is a hyperrectangle.

The proof of our invariance result for lattice equations is different from the existing proofs for parabolic equations: We start by deriving an invariance result for the Euler approximations to Eq. (1.3), and only later pass to the continuous-time limit. Therefore, the results of Section 3 also contribute to the theory of partial difference equations.

In Section 4, we illustrate our invariance result on examples from population dynamics, including a predator-prey model of Lotka–Volterra type, and a model of two competing species.

Let us remark that in Sections 3 and 4, we consider only those equations of the form (1.3) that satisfy $A + B + C = 0$. Already in the scalar case (i.e., when $m = 1$), this condition is necessary for the validity of the weak maximum principle; see [29, Section 4]. For applications, this restriction is not a serious one: Consider a system of lattice reaction–diffusion equations of the form

$$\frac{\partial u}{\partial t}(x,t) = D_1(x,t)u^{\Delta\nabla}(x,t) + D_2(x,t)u^{\Delta}(t) + D_3(x,t)u^{\nabla}(x,t) + f(u(x,t),x,t), \tag{1.4}$$

where $u^{\Delta\nabla}(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t)$ denotes the second-order central difference of u with respect to x, $u^{\Delta}(x,t) = u(x+1,t) - u(x,t)$ is the forward difference and $u^{\nabla}(x,t) = u(x,t) - u(x-1,t)$ is the backward difference. The system (1.4) is the semidiscrete counterpart of the parabolic system

$$\frac{\partial u}{\partial t}(x,t) = D(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + E(x,t)\frac{\partial u}{\partial x}(x,t) + f(u(x,t),x,t),$$

which has been studied in [1,9,25,34,38]; the second-order derivative on the right-hand side is replaced by the second-order central difference, and the first-order derivative is replaced either with the forward or the backward difference.

By expanding the differences in (1.4), we see that the system is equivalent to (1.3) with

$$\begin{aligned} A(x,t) &= D_1(x,t) + D_2(x,t), \\ B(x,t) &= -2D_1(x,t) - D_2(x,t) + D_3(x,t), \\ C(x,t) &= D_1(x,t) - D_3(x,t), \end{aligned}$$

which means that $A + B + C = 0$. Conversely, each equation of the form (1.3) satisfying $A + B + C = 0$ can be rewritten in the form (1.4). The corresponding coefficients D_1, D_2, D_3 are not uniquely determined; for example, it is possible to choose

$$D_1(x,t) = 0, \quad D_2(x,t) = A(x,t), \quad D_3(x,t) = -C(x,t).$$

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2. Well-posedness results

Throughout the rest of this paper, we use the symbol $\ell^\infty(\mathbb{Z})$ to denote the vector space of all bounded real sequences $\{u_x\}_{x\in\mathbb{Z}}$. This space is equipped with the supremum norm

$$\|u\| = \sup_{x\in\mathbb{Z}} |u_x|, \quad u \in \ell^\infty(\mathbb{Z}).$$

The symbol $\ell^\infty(\mathbb{Z})^m$ denotes the product space whose elements have the form $u = (u^1, \dots, u^m)$ with $u^1, \dots, u^m \in \ell^\infty(\mathbb{Z})$. This space is equipped with the supremum norm

$$\|u\| = \max\{\|u^1\|, \dots, \|u^m\|\}, \quad u \in \ell^\infty(\mathbb{Z})^m,$$

and it is a Banach space. For an arbitrary $u \in \ell^\infty(\mathbb{Z})^m$ and $x \in \mathbb{Z}$, we use the symbol u_x to denote the vector $(u_x^1, \dots, u_x^m) \in \mathbb{R}^m$.

In this section, we generalize the results from [30] and obtain some basic well-posedness results for the initial-value problem

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(x,t) = \sum_{j=-k}^{k} A_j(x,t)u(x+j,t) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \in [0,T], \\ &u(x,0) = u_x^0, \quad x \in \mathbb{Z}, \end{aligned}\right\} \tag{2.1}$$

where $u^0 = \{u_x^0\}_{x\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})^m$, $u : \mathbb{Z} \times [0,T] \to \mathbb{R}^m$, $A_{-k}, \dots, A_k : \mathbb{Z} \times [0,T] \to \mathbb{R}^{m\times m}$, and $f : \mathbb{R}^m \times \mathbb{Z} \times [0,T] \to \mathbb{R}^m$. This system generalizes (1.3), which corresponds to the special case $k = 1$. Systems with $k > 1$ are useful, for example, in the study of stochastic processes; see [15].

We impose the following conditions on the functions $A_{-k}, \dots, A_k$, and f:

- (A_1) *$A_{-k}, \dots, A_k$ are bounded.*
- (A_2) *For each $j \in \{-k, \dots, k\}$, $\varepsilon > 0$ and $t \in [0,T]$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [0,T]$, then $\|A_j(x,t) - A_j(x,s)\| < \varepsilon$ for all $x \in \mathbb{Z}$.*
- (F_1) *f is bounded on each set $B \times \mathbb{Z} \times [0,T]$, where $B \subset \mathbb{R}^m$ is bounded.*
- (F_2) *f is Lipschitz-continuous in the first variable on each set $B \times \mathbb{Z} \times [0,T]$, where $B \subset \mathbb{R}^m$ is bounded.*
- (F_3) *For each bounded set $B \subset \mathbb{R}^m$ and each choice of $\varepsilon > 0$ and $t \in [0,T]$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [0,T]$, then $\|f(u,x,t) - f(u,x,s)\| < \varepsilon$ for all $u \in B$, $x \in \mathbb{Z}$.*

The proof of the next theorem is similar to the proof of [30, Theorem 2.1]; we include it here for completeness.

Theorem 2.1. *Assume that $A_{-k}, \dots, A_k : \mathbb{Z} \times [0,T] \to \mathbb{R}^{m\times m}$ satisfy (A_1), (A_2), and $f : \mathbb{R}^m \times \mathbb{Z} \times [0,T] \to \mathbb{R}^m$ satisfies (F_1)–(F_3). Then for each $u^0 \in \ell^\infty(\mathbb{Z})^m$, the initial-value problem (2.1) has a bounded local solution defined on $\mathbb{Z} \times [0,\delta]$, where $\delta > 0$. The solution is obtained by letting $u(x,t) = U(t)_x$, where $U : [0,\delta] \to \ell^\infty(\mathbb{Z})^m$ is a solution of the abstract differential equation*

$$U'(t) = \Phi(U(t), t), \quad U(0) = u^0,$$

with $\Phi : \ell^\infty(\mathbb{Z})^m \times [0, T] \to \ell^\infty(\mathbb{Z})^m$ *being given by*

$$\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{A_k(x, t)u_{x+k} + \cdots + A_{-k}(x, t)u_{x-k} + f(u_x, x, t)\}_{x\in\mathbb{Z}}. \tag{2.2}$$

Proof. Conditions (A_1) and (F_1) guarantee that Φ indeed takes values in $\ell^\infty(\mathbb{Z})^m$. Choose an arbitrary $\rho > 0$, and denote $\mathcal{B} = \{u \in \ell^\infty(\mathbb{Z})^m; \|u - u^0\| \le \rho\}$. For each $i \in \{1, \dots, m\}$, let $B_i = \left[\inf_{x\in\mathbb{Z}}(u^0_x)_i - \rho, \sup_{x\in\mathbb{Z}}(u^0_x)_i + \rho\right] \subset \mathbb{R}$, and denote $B = B_1 \times \cdots \times B_m$. Note that if $u, v \in \mathcal{B}$, then $u_x, v_x \in B$ for all $x \in \mathbb{Z}$. Let L, M be the Lipschitz constant and the bound for the function f on $B \times \mathbb{Z} \times [0, T]$, and $M_{-k}, \dots, M_k$ the bounds for the functions $A_{-k}, \dots, A_k$ on $\mathbb{Z} \times [0, T]$.

Observe that Φ is bounded on $\mathcal{B} \times [0, T]$: For each $u \in \mathcal{B}$, we have $u_x \in B$ for all $x \in \mathbb{Z}$, and consequently

$$\begin{aligned}\|\Phi(u, t)\| &\le M_k \cdot \|\{u_{x+k}\}_{x\in\mathbb{Z}}\| + \cdots + M_{-k} \cdot \|\{u_{x-k}\}_{x\in\mathbb{Z}}\| + \|\{f(u_x, x, t)\}_{x\in\mathbb{Z}}\| \\ &\le (M_{-k} + \cdots + M_k)\|u\| + M \le (M_{-k} + \cdots + M_k)(\|u^0\| + \rho) + M.\end{aligned}$$

Next, we show that Φ is Lipschitz-continuous in the first variable on $\mathcal{B} \times [0, T]$:

$$\begin{aligned}\|\Phi(u, t) - \Phi(v, t)\| &\le M_k \cdot \|\{u_{x+k} - v_{x+k}\}_{x\in\mathbb{Z}}\| + \cdots + M_{-k} \cdot \|\{u_{x-k} - v_{x-k}\}_{x\in\mathbb{Z}}\| \\ &+ \|\{f(u_x, x, t) - f(v_x, x, t)\}_{x\in\mathbb{Z}}\| \le (M_{-k} + \cdots + M_k)\|u - v\| + L\|u - v\|.\end{aligned}$$

Finally, we claim that Φ is continuous on $\mathcal{B} \times [0, T]$. To see this, consider an arbitrary $\varepsilon > 0$ and a fixed pair $(u, t) \in \mathcal{B} \times [0, T]$. Let $\delta_{\min} > 0$ be the minimum of all numbers δ obtained from conditions (A_2) and (F_3). Then for all $(v, s) \in \mathcal{B} \times [0, T]$ with $\|u - v\| < \varepsilon$ and $s \in (t - \delta_{\min}, t + \delta_{\min}) \cap [0, T]$, we have

$$\begin{aligned}&\|\Phi(u, t) - \Phi(v, s)\| \le \|\Phi(u, t) - \Phi(v, t)\| + \|\Phi(v, t) - \Phi(v, s)\| \\ &\le (M_{-k} + \cdots + M_k + L)\|u - v\| + \|\{f(v_x, x, t) - f(v_x, x, s)\}_{x\in\mathbb{Z}}\| \\ &+ \|\{(A_k(x, t) - A_k(x, s))u_{x+k} + \cdots + (A_{-k}(x, t) - A_{-k}(x, s))u_{x-k}\}_{x\in\mathbb{Z}}\| \\ &\le (M_{-k} + \cdots + M_k + L)\|u - v\| + \varepsilon + (2k + 1)\varepsilon\|u\|,\end{aligned}$$

which proves that Φ is continuous at the point (u, t).

By the Picard–Lindelöf theorem, the initial-value problem

$$U'(t) = \Phi(U(t), t), \quad U(0) = u^0,$$

has a local solution defined on $[0, \delta]$, where $\delta > 0$. Letting $u(x, t) = U(t)_x$, $x \in \mathbb{Z}$, we see that u is a solution of the initial-value problem (2.1). □

The next result is a slight generalization of [30, Theorem 2.2], which corresponds to the special case when $m = 1$. The proof for a general $m \in \mathbb{N}$ can be carried out in the same way and we omit it.

Theorem 2.2. *Assume that* $\varphi : \ell^\infty(\mathbb{Z})^m \times \mathbb{Z} \times [0,T] \to \mathbb{R}^m$ *satisfies the following conditions:*

1. φ *is bounded on each set* $\mathcal{B} \times \mathbb{Z} \times [0,T]$, *where* $\mathcal{B} \subset \ell^\infty(\mathbb{Z})^m$ *is bounded.*
2. φ *is Lipschitz-continuous in the first variable on each set* $\mathcal{B} \times \mathbb{Z} \times [0,T]$, *where* $\mathcal{B} \subset \ell^\infty(\mathbb{Z})^m$ *is bounded.*

Then for each $u^0 \in \ell^\infty(\mathbb{Z})^m$, *the initial-value problem*

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(x,t) = \varphi(\{u(x,t)\}_{x\in\mathbb{Z}}, x, t), \quad x \in \mathbb{Z}, \quad t \in [0,T], \\ &u(x,0) = u_x^0, \quad x \in \mathbb{Z}, \end{aligned}\right\} \tag{2.3}$$

has at most one bounded solution $u : \mathbb{Z} \times [0,T] \to \mathbb{R}^m$.

As a corollary of the previous result, we obtain the uniqueness of bounded solutions to the initial-value problem (2.1).

Corollary 2.3. *Assume that* $A_{-k}, \dots, A_k : \mathbb{Z} \times [0,T] \to \mathbb{R}^{m\times m}$ *satisfy* (A_1), *and* $f : \mathbb{R}^m \times \mathbb{Z} \times [0,T] \to \mathbb{R}^m$ *satisfies* (F_1), (F_2). *Then for each* $u^0 \in \ell^\infty(\mathbb{Z})^m$, *the initial-value problem* (2.1) *has at most one bounded solution* $u : \mathbb{Z} \times [0,T] \to \mathbb{R}^m$.

Proof. Note that (2.1) is a special case of (2.3) with the function $\varphi : \ell^\infty(\mathbb{Z})^m \times \mathbb{Z} \times [0,T] \to \mathbb{R}^m$ being given by

$$\varphi(\{u_x\}_{x\in\mathbb{Z}}, x, t) = A_k(x,t)u_{x+k} + \cdots + A_{-k}(x,t)u_{x-k} + f(u_x, x, t).$$

Hence, it is enough to verify that the two assumptions of Theorem 2.2 are satisfied.

Given an arbitrary bounded set $\mathcal{B} \subset \ell^\infty(\mathbb{Z})^m$, there exists a bounded set $B \subset \mathbb{R}^m$ such that $u \in \mathcal{B}$ implies $u_x \in B$, $x \in \mathbb{Z}$. Hence, the first condition in Theorem 2.2 is an immediate consequence of (A_1) and (F_1). To verify the second condition, let L be the Lipschitz constant for the function f on $B \times \mathbb{Z} \times [0,T]$, and $M_{-k}, \dots, M_k$ the bounds for the functions $A_{-k}, \dots, A_k$ on $\mathbb{Z} \times [0,T]$. Then, for each pair of sequences $u, v \in \mathcal{B} \subset \ell^\infty(\mathbb{Z})^m$, we have

$$\begin{aligned} |\varphi(u,x,t) - \varphi(v,x,t)| &\le (M_{-k} + \cdots + M_k) \cdot \|u - v\| + \|f(u_x,x,t) - f(v_x,x,t)\| \\ &\le (M_{-k} + \cdots + M_k + L) \cdot \|u - v\|, \end{aligned}$$

which means that φ is Lipschitz-continuous in the first variable on $\mathcal{B} \times \mathbb{Z} \times [0,T]$. □

We conclude this section with two continuous dependence results concerning ordinary differential equations in Banach spaces. Thanks to Theorem 2.1, these results are also applicable in the study of Eq. (2.1). The first result is a special case of [30, Theorem 3.2]; it provides sufficient conditions ensuring that the solution of a given ordinary differential equation is the limit of the Euler approximations.

Theorem 2.4. *Let* X *be a Banach space and* $\mathcal{B} \subseteq X$. *Suppose that* $\Phi : \mathcal{B} \times [0,T] \to X$ *is continuous and Lipschitz-continuous with respect to the first variable. Assume that* $u^0 \in \mathcal{B}$ *and* $U : [0,T] \to \mathcal{B}$ *satisfies*

$$U'(t) = \Phi(U(t), t), \quad t \in [0, T], \quad U(0) = u^0.$$

For each $n \in \mathbb{N}$, let $h = T/n$, and assume that $U_n : \{0, h, 2h, \dots, (n-1)h, nh\} \to \mathcal{B}$ satisfies

$$\frac{U_n(t+h) - U_n(t)}{h} = \Phi(U_n(t), t), \quad t \in \{0, h, 2h, \dots, (n-1)h\}, \quad U_n(0) = u^0.$$

Moreover, let $U_n^* : [0, T] \to \mathcal{B}$ be the piecewise constant extension of U_n given by

$$U_n^*(t) = \begin{cases} U_n(0) & \text{if } t = 0, \\ U_n(kh) & \text{if } t \in ((k-1)h, kh] \text{ for some } k \in \{1, \dots, n\}. \end{cases}$$

Then the sequence $\{U_n^*\}_{n=1}^\infty$ is uniformly convergent to U on $[0, T]$.

The second result, which is a consequence of [27, Theorem 4.7], is concerned with continuous dependence of solutions on the right-hand side of a differential equation.

Theorem 2.5. *Let X be a Banach space, $\mathcal{C} \subseteq X$ and $u^0 \in \mathcal{C}$. Consider functions $\Phi : \mathcal{C} \times [0, T] \to X$ and $\Phi_n : \mathcal{C} \times [0, T] \to X$, $n \in \mathbb{N}$, which are continuous, bounded by the same constant, Lipschitz-continuous in the first variable with the same Lipschitz constant, and such that $\Phi_n \to \Phi$ on $\mathcal{C} \times [0, T]$. Assume that $U : [0, T] \to \mathcal{C}$ satisfies*

$$U'(t) = \Phi(U(t), t), \quad t \in [0, T], \quad U(0) = u^0.$$

Finally, suppose there exists a $\rho > 0$ such that the open ρ-neighborhood of U in X is contained in $\mathcal{C}$. Then there is an $n_0 \in \mathbb{N}$ and a sequence of functions $U_n : [0, T] \to \mathcal{C}$, $n \ge n_0$, such that

$$U_n'(t) = \Phi_n(U_n(t), t), \quad t \in [0, T], \quad U_n(0) = u^0,$$

and $\{U_n\}_{n=n_0}^\infty$ is uniformly convergent to U on $[0, T]$.

3. Invariance results

Throughout this section, we consider compact convex sets S described as intersections of sublevel sets of certain functions $G_1, \dots, G_k$. More precisely, we introduce the following condition:

(S) Assume that $k \in \mathbb{N}$, $U_1, \dots, U_k \subseteq \mathbb{R}^m$ are open sets, and for each $i \in \{1, \dots, k\}$, $G_i : U_i \to \mathbb{R}$ is a $\mathcal{C}^1$ function. Suppose also that the closed sets

$$S_i = \{u \in U_i;\ G_i(u) \le 0\}, \quad i \in \{1, \dots, k\},$$

are convex, their intersection

$$S = S_1 \cap \dots \cap S_k = \{u \in U_1 \cap \dots \cap U_k;\ G_1(u) \le 0, \dots, G_k(u) \le 0\}$$

is bounded and has nonempty interior, and that $\nabla G_i(u) \ne 0$ for each $i \in \{1, \dots, k\}$, $u \in \partial S_i$.

If condition (S) is satisfied, then S is a compact convex set with nonempty interior. Note that $\nabla G_i(u)$ is the outward normal to S_i at $u \in \partial S_i$, the set

$$\{z \in \mathbb{R}^m; \nabla G_i(u) \cdot z = \nabla G_i(u) \cdot u\} \tag{3.1}$$

is the unique supporting hyperplane (and also the tangent hyperplane) of S_i at u, and S_i is contained in the supporting half-space

$$\{z \in \mathbb{R}^m; \nabla G_i(u) \cdot z \le \nabla G_i(u) \cdot u\}.$$

Remark 3.1. The description of the convex set S in terms of the functions $G_1, \dots, G_k$ is taken over from the paper [9]. In practice, the set S is often chosen in one of the following two ways:

1. S is the interior of a closed hypersurface described by the equation $G(u) = 0$, where $u \in U \subset \mathbb{R}^m$, and $G : U \to \mathbb{R}$ is a $\mathcal{C}^1$ function with nonzero gradient on ∂S. In this case, we have $k = 1$ and $S = \{u \in U;\ G(u) \le 0\}$. A simple illustration of this case will be given in Example 4.1.
2. S is the m-dimensional hyperrectangle $S = [a_1, b_1] \times \cdots \times [a_m, b_m]$. In this situation, we let $k = 2m$, $U_1 = \cdots = U_{2m} = \mathbb{R}^m$, and

$$S_i = \{u \in \mathbb{R}^m;\ G_i(u) = a_i - u_i \le 0\}, \qquad i \in \{1, \dots, m\},$$
$$S_{m+i} = \{u \in \mathbb{R}^m;\ G_{m+i}(u) = u_i - b_i \le 0\}, \quad i \in \{1, \dots, m\}.$$

An illustration of this situation will be provided in Example 4.2.

Remark 3.2. Since a nonempty closed convex set is the intersection of all its supporting half-spaces (see, e.g., [26, Corollary 1.3.5]), we have

$$S_i = \bigcap_{u \in \partial S_i} \{z \in \mathbb{R}^m; \nabla G_i(u) \cdot z \le \nabla G_i(u) \cdot u\},$$

and consequently

$$S = \bigcap_{i=1}^{k} \bigcap_{u \in \partial S_i} \{z \in \mathbb{R}^m; \nabla G_i(u) \cdot z \le \nabla G_i(u) \cdot u\}.$$

However, for our purposes, it is more convenient to express S in the form

$$S = \bigcap_{i=1}^{k} \bigcap_{u \in \partial S_i \cap S} \{z \in \mathbb{R}^m; \nabla G_i(u) \cdot z \le \nabla G_i(u) \cdot u\}. \tag{3.2}$$

To see why the last relation holds, we use the fact (see, e.g., [10, pp. 27–28]) that S, which is a compact convex set with nonempty interior, can be expressed as the intersection of only those of its supporting half-spaces that correspond to extreme supporting hyperplanes. (A supporting hyperplane is called extreme if its normal vector a cannot be written in the form $a = \alpha a^1 + \beta a^2$, where $\alpha, \beta > 0$, $\alpha + \beta = 1$, and a^1, a^2 are normal vectors of two distinct supporting hyperplanes at the given point.) Now, if $u \in \partial S$ is such that $u \in \partial S_i$ for a unique index i, then (3.1) is the unique supporting half-plane of S at u, and thus it is extreme. Otherwise, we have $u \in \partial S_i$ for several indices $i \in I$, where $I \subset \{1, \dots, k\}$ and $|I| > 1$. There might be more than one supporting hyperplane of S at u, but the normal cone of S at u (i.e., the set of all outer normal vectors of S at u together with the zero vector) is the sum of the normal cones of the sets S_i, $i \in I$, at u (see [26, Theorem 2.2.1]). Since the latter cones are half-lines in the direction of $\nabla G_i(u)$, $i \in I$, it follows that all extreme supporting hyperplanes of S at u have the form (3.1) with $i \in I$, and this establishes the formula (3.2).

We now consider initial-value problems for lattice reaction–diffusion equations having the form

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(x,t) = A(x,t)u(x+1,t) + B(x,t)u(x,t) + C(x,t)u(x-1,t)\\ &\qquad\qquad + f(u(x,t),t), \quad x \in \mathbb{Z}, \quad t \in [0,T],\\ &u(x,0) = u_x^0, \quad x \in \mathbb{Z}, \end{aligned}\right\} \tag{3.3}$$

where $u^0 = \{u_x^0\}_{x\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})^m$, $u : \mathbb{Z} \times [0,T] \to \mathbb{R}^m$, A, B, $C : \mathbb{Z} \times [0,T] \to \mathbb{R}^{m\times m}$, and $f : \mathbb{R}^m \times [0,T] \to \mathbb{R}^m$. This initial-value problem is a special case of (2.1) with $k = 1$ and

$$A_{-1}(x,t) = A(x,t), \quad A_0(x,t) = B(x,t), \quad A_1(x,t) = C(x,t).$$

Whenever we refer to conditions (A_1), (A_2) from Section 2, we always assume that A_{-1}, A_0, A_1 are defined in this way.

To avoid technical difficulties, we restrict ourselves to the case when f does not explicitly depend on x. In this setting, the conditions (F_1)–(F_3) can be simplified as follows:

(D_1) *f is Lipschitz-continuous in the first variable on each set $B \times [0,T]$, where $B \subset \mathbb{R}^m$ is bounded.*

(D_2) *f is continuous in the second variable.*

Obviously, (D_1) implies (F_2). If $(v,s) \in \mathbb{R}^m \times [0,T]$ and V is an arbitrary bounded neighborhood of v, then the estimate

$$\|f(u,t) - f(v,s)\| \le \|f(u,t) - f(v,t)\| + \|f(v,t) - f(v,s)\|$$

together with the Lipschitz-continuity of f in the first variable on $V \times [0,T]$ and continuity in the second variable at (v,s) imply the continuity of f (as a function of two variables) at (v,s). Thus, the conditions (D_1) and (D_2) imply that f is continuous. For each bounded set $B \subset \mathbb{R}^m$, f is uniformly continuous and bounded on the compact set $\overline{B} \times [0,T]$, which means that the conditions (F_1) and (F_3) are satisfied.

Our goal is to obtain sufficient conditions guaranteeing that S is an invariant region for bounded solutions of Eq. (3.3), i.e., that each bounded solution of Eq. (3.3) with $u_x^0 \in S$, $x \in \mathbb{Z}$, satisfies $u(x,t) \in S$ for each $t \in [0,T]$, $x \in \mathbb{Z}$. We introduce the following conditions:

(C_1) *For each* $i \in \{1, \dots, k\}$ *and* $u \in \partial S_i \cap S$, *we have* $\nabla G_i(u) \cdot f(u,t) \le 0$ *for all* $t \in [0, T]$.
(C_2) *For each* $i \in \{1, \dots, k\}$, $u \in \partial S_i \cap S$, $x \in \mathbb{Z}$ *and* $t \in [0, T]$, *there exist numbers* $a \ge 0$, $b \le 0$, $c \ge 0$ *such that*

$$\nabla G_i(u)^\top A(x,t) = a\nabla G_i(u)^\top, \quad \nabla G_i(u)^\top B(x,t) = b\nabla G_i(u)^\top,$$
$$\nabla G_i(u)^\top C(x,t) = c\nabla G_i(u)^\top.$$

(C_3) *For each* $x \in \mathbb{Z}$ *and* $t \in [0, T]$, *we have* $A(x,t) + B(x,t) + C(x,t) = 0$.

Remark 3.3. The fact that the condition (C_3) is necessary even in the scalar case was already noticed in the introduction. Let us provide some additional comments concerning the first two conditions:

- Recall that if $u \in \partial S_i \cap S$, then $\nabla G_i(u)$ is the outward normal to S_i at u. Thus, condition (C_1) says that the vector field f at u points inward S_i or is tangent to ∂S_i for all values of t. This condition, which was mentioned in the introduction of this paper, is the standard condition in the study of invariant regions for both ordinary and partial differential equations (see, e.g., [1,9,25,33,34,38]).
- Condition (C_2) says that $\nabla G_i(u)$ is a left eigenvector of the matrices $A(x,t)$, $B(x,t)$, $C(x,t)$ for each $x \in \mathbb{Z}$ and $t \in [0, T]$; equivalently, it is the eigenvector of $A(x,t)^\top$, $B(x,t)^\top$, $C(x,t)^\top$. Moreover, it is required that the corresponding eigenvalues a, c are nonnegative, while b is nonpositive (note that the eigenvalues might depend on x and t). Condition of a similar type can be found in [9,12,13,37], and it is also implicitly present in [34]. Let us mention two typical situations when (C_2) is satisfied:
  1. The matrices A, B, C are scalar multiples of the identity matrix, where the scalars corresponding to A, C are nonnegative and the scalar corresponding to B is nonpositive (the scalars might depend on x and t). This happens, e.g., for weakly coupled systems of lattice reaction–diffusion equations, where all equations have the same diffusion coefficient; see Example 4.1. Since each vector in $\mathbb{R}^m$ is a left eigenvector to a scalar multiple of the identity matrix, the condition (C_2) is satisfied for an arbitrary set S.
  2. The matrices A, B, C are diagonal, the diagonal elements of A, C are nonnegative, and the diagonal elements of B are nonpositive. This happens, e.g., for weakly coupled systems of lattice reaction–diffusion equations where different equations might have different diffusion coefficients; see Example 4.2. Since the eigenvectors of a diagonal matrix are precisely the vectors of the canonical basis in $\mathbb{R}^m$ (and their multiples), condition (C_2) is satisfied if S is the m-dimensional hyperrectangle described in Remark 3.1.

  Clearly, if A, B, C are not scalar multiples of the identity matrix, then condition (C_2) imposes a serious restriction on the shape of S – it says that the boundary of S has to be such that the normal vectors ∇G_i are left eigenvectors of A, B, C. In general, a condition of this type cannot be avoided. The necessity of an analogous condition for systems of parabolic differential equations was proved in [9, Theorem 4.2]. For example, if we have a decoupled system of two linear diffusion equations with different diffusion coefficients, it can easily happen that a solution leaves a compact convex set that has a non-rectangular shape; a convincing pictorial argument can be found in [12, Section 3.4]. The situation when A, B, C are not scalar multiples of the identity matrix will be illustrated in Example 4.3.

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We begin our investigation of invariant regions by considering the linear case $f \equiv 0$; in this situation, condition (C_1) is trivially satisfied.

Lemma 3.4. *Suppose that the conditions* (S), (C_2), (C_3) *are satisfied and there exists a* $\beta > 0$ *such that if* $x \in \mathbb{Z}$, $t \in [0, T]$ *and* λ *is an eigenvalue of* $B(x, t)$, *then* $|\lambda| \le \beta$. *If* $h \in (0, 1/\beta]$ *and* $\{u_x\}_{x \in \mathbb{Z}}$ *is a sequence such that* $u_x \in S$ *for each* $x \in \mathbb{Z}$, *then*

$$hA(x,t)u_{x+1} + (I + hB(x,t))u_x + hC(x,t)u_{x-1}$$

is an element of S *for all* $t \in [0, T]$, $x \in \mathbb{Z}$.

Proof. Assume that $h \in (0, 1/\beta]$ and $\{u_x\}_{x \in \mathbb{Z}}$ is a sequence such that $u_x \in S$ for each $x \in \mathbb{Z}$. Consider a fixed pair $t \in [0, T]$, $x \in \mathbb{Z}$, and denote

$$\tilde{u} = hA(x,t)u_{x+1} + (I + hB(x,t))u_x + hC(x,t)u_{x-1}.$$

We will show that $\tilde{u} \in S$. Taking into account (cf. Remark 3.2) that

$$S = \bigcap_{i=1}^{k} \bigcap_{y \in \partial S_i \cap S} \{z \in \mathbb{R}^m; \nabla G_i(y) \cdot z \le \nabla G_i(y) \cdot y\},$$

we need to prove that if $i \in \{1, \dots, k\}$ and $y \in \partial S_i \cap S$, then $\nabla G_i(y) \cdot \tilde{u} \le \nabla G_i(y) \cdot y$.

We know that u_{x+1}, u_x and u_{x-1} are elements of S_i, and therefore

$$\nabla G_i(y) \cdot u_{x+1} \le \nabla G_i(y) \cdot y, \quad \nabla G_i(y) \cdot u_x \le \nabla G_i(y) \cdot y, \quad \nabla G_i(y) \cdot u_{x-1} \le \nabla G_i(y) \cdot y.$$

Using condition (C_2), we get

$$\begin{aligned}\nabla G_i(y) \cdot \tilde{u} &= \nabla G_i(y)^\top \tilde{u} = \nabla G_i(y)^\top (hau_{x+1} + (1 + hb)u_x + hcu_{x-1}) \\ &= \nabla G_i(y) \cdot (hau_{x+1} + (1 + hb)u_x + hcu_{x-1}),\end{aligned}$$

where $a, c \ge 0$ and $b \le 0$. Note that the three identities in condition (C_2) together with condition (C_3) imply that $a + b + c = 0$. Moreover, the relation $\nabla G_i(u)^\top B(x,t) = b\nabla G_i(u)^\top$ from condition (C_2) implies that the number b is an eigenvalue of $B(x,t)$, and therefore $1 + hb = 1 - h|b| \ge 1 - \frac{1}{\beta}\beta = 0$. Consequently,

$$\nabla G_i(y) \cdot \tilde{u} \le ha\nabla G_i(y) \cdot y + (1 + hb)\nabla G_i(y) \cdot y + hc\nabla G_i(y) \cdot y = \nabla G_i(y) \cdot y,$$

which completes the proof. □

The previous lemma will be needed in the proof of our main result; however, we can immediately derive the following interesting corollary, which shows that S is an invariant region for the Euler approximations of the linear system whenever the time step is sufficiently small.

Corollary 3.5. *Suppose that the conditions* (S), (C_2), (C_3) *are satisfied and there exists a* $\beta > 0$ *such that if* $x \in \mathbb{Z}$, $t \in [0, T]$ *and* λ *is an eigenvalue of* $B(x, t)$, *then* $|\lambda| \le \beta$. *Let* $n \in \mathbb{N}$, $h = T/n$, *and consider the partial difference equation*

$$\frac{u(x,t+h)-u(x,t)}{h} = A(x,t)u(x+1,t)+B(x,t)u(x,t)+C(x,t)u(x-1,t), \quad x \in \mathbb{Z}, \tag{3.4}$$

where $t \in \{0, h, 2h, \ldots, (n-1)h\}$. *If* $h \in (0, 1/\beta]$, *then* S *is an invariant region for Eq.* (3.4).

Proof. The statement is an immediate consequence of Lemma 3.4, because Eq. (3.4) is equivalent to the relation

$$u(x,t+h) = hA(x,t)u(x+1,t)+(I+hB(x,t))u(x,t)+hC(x,t)u(x-1,t), \quad x \in \mathbb{Z}. \qquad \square$$

Remark 3.6. Obviously, the statement of the previous corollary remains valid if we consider Euler approximations with nonconstant step size, which does not exceed $1/\beta$. Note that if B is bounded, then its eigenvalues are also bounded. Thus, our result generalizes [29, Theorem 4.7] in the case of a discrete time scale.

We now turn our attention to nonlinear systems and temporarily replace the condition (C_1) by the following stronger version:

(C_1') *For each* $i \in \{1, \ldots, k\}$ *and* $u \in \partial S_i \cap S$, *we have* $\nabla G_i(u) \cdot f(u, t) < 0$ *for all* $t \in [0, T]$.

This stronger assumption will enable us to show that S is an invariant region for the Euler approximations of Eq. (3.3) provided that the step size is sufficiently small. We need the following auxiliary lemma.

Lemma 3.7. *Assume that conditions* (S), (D_1), (D_2), (C_1') *are satisfied. Then there exist numbers* $\delta > 0$, $\varepsilon > 0$ *with the following property: If* $i \in \{1, \ldots, k\}$, $z, w \in \mathbb{R}^m$, $d(z, \partial S_i \cap S) \le \delta$ *and* $\|w - z\| \le \delta$, *then* $z, w \in U_i$ *and* $\nabla G_i(w) \cdot f(z, t) < -\varepsilon$ *for all* $t \in [0, T]$.

Proof. Choose an arbitrary $i \in \{1, \ldots, k\}$. Recall that U_i is open and $S \subset U_i$. Thus, there exists a $\rho_i > 0$ such that the closed ρ_i-neighborhood of S, i.e., the set $S_{\rho_i} = \{y \in \mathbb{R}^m;\ d(y, S) \le \rho_i\}$, is contained in U_i. (If $U_i \ne \mathbb{R}^m$, note that $d(\partial S, \partial U_i) > 0$, because $u \mapsto d(u, \partial U_i)$ is a continuous positive function on the compact set ∂S.)

Since f and ∇G_i are continuous and $(\partial S_i \cap S) \times [0, T]$ is compact, (C_1') implies

$$d_i = \max_{y \in \partial S_i \cap S,\, t \in [0,T]} \nabla G_i(y) \cdot f(y, t) < 0.$$

The function

$$g(w, z, t) = \nabla G_i(w) \cdot f(z, t)$$

is continuous on the compact set $S_{\rho_i} \times S_{\rho_i} \times [0, T]$, and therefore uniformly continuous. Thus, there exists $\eta_i > 0$ such that

$$|g(z_1, w_1, t) - g(z_2, w_2, t)| < \frac{|d_i|}{2}$$

whenever $\|z_1 - z_2\| \le \eta_i$, $\|w_1 - w_2\| \le \eta_i$.

Choose a positive number $\delta_i \le \min(\eta_i/2, \rho_i/2)$ and let $\varepsilon_i = -d_i/2$. Consider an arbitrary pair $z, w \in \mathbb{R}^m$ satisfying $d(z, \partial S_i \cap S) \le \delta_i$ and $\|w - z\| \le \delta_i$. Note that $z, w \in S_{\rho_i} \subset U_i$. Let $p(z)$ be the closest point to z on $\partial S_i \cap S$. Then $\|z - p(z)\| \le \delta_i \le \eta_i/2$, $\|w - p(z)\| \le \|w - z\| + \|z - p(z)\| \le 2\delta_i \le \eta_i$, and consequently

$$\begin{aligned}\nabla G_i(w) \cdot f(z, t) &= g(w, z, t) = g(p(z), p(z), t) + g(w, z, t) - g(p(z), p(z), t) \\ &< d_i + \frac{|d_i|}{2} = \frac{d_i}{2} = -\varepsilon_i.\end{aligned}$$

Thus, the assertion of the theorem holds with $\delta = \min(\delta_1, \dots, \delta_k)$ and $\varepsilon = \min(\varepsilon_1, \dots, \varepsilon_k)$. □

We now proceed to the promised result concerning the invariance of S under the Euler approximations of Eq. (3.3). Parts of the proof are inspired by the proof of [33, Lemma 5.1], which deals with the invariance of convex sets under the Euler approximations to ordinary differential equations.

Theorem 3.8. *Suppose that conditions* (S), (A_1), (A_2), (D_1), (D_2), (C_1'), (C_2), (C_3) *are satisfied. Let $n \in \mathbb{N}$, $h = T/n$, and consider the partial difference equation*

$$\begin{aligned}\frac{u(x, t+h) - u(x, t)}{h} &= A(x, t)u(x+1, t) + B(x, t)u(x, t) + C(x, t)u(x-1, t) \\ &\quad + f(u(x, t), t), \quad x \in \mathbb{Z}, \end{aligned} \tag{3.5}$$

where $t \in \{0, h, 2h, \dots, (n-1)h\}$. If h is sufficiently small, then S is an invariant region for Eq. (3.5).

Proof. Since the function B is bounded, there exists a $\beta > 0$ with the following property: If λ is an eigenvalue of $B(x, t)$ for some $x \in \mathbb{Z}$, $t \in [0, T]$, then $|\lambda| \le \beta$. Thus, the assumptions of Lemma 3.4 are satisfied.

Let δ, ε be the numbers from Lemma 3.7. The lemma implies that if S_δ is the closed δ-neighborhood of S, then $S_\delta \subset U_1 \cap \dots \cap U_k$. Denote

$$M_1 = \sup_{y \in S, t \in [0, T]} \|f(y, t)\|, \tag{3.6}$$

$$M_2 = \max_{i \in \{1, \dots, k\}} \left(\max_{y \in S_\delta} \|\nabla G_i(y)\| \right), \tag{3.7}$$

$$M_3 = \sup_{t \in [0, T]} (\|A(x, t)\| + \|B(x, t)\| + \|C(x, t)\|), \tag{3.8}$$

$$M_4 = \max_{y \in S} \|y\|. \tag{3.9}$$

Let L be the Lipschitz constant for f on $S \times [0, T]$ and denote

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$$h_0 = \min\left(\frac{1}{\beta}, \frac{\delta}{M_1}, \frac{\varepsilon}{M_2 M_3 M_4 L}\right).$$

Assume that $h \in (0, h_0]$. We now show that if $t \in \{0, h, 2h, \dots, (n-1)h\}$ and $u(x,t) \in S$ for all $x \in \mathbb{Z}$, then $u(x, t+h) \in S$ for all $x \in \mathbb{Z}$. Fix an arbitrary $i \in \{1, \dots, k\}$; we will prove that $u(x, t+h) \in S_i$, i.e., that $G_i(u(x, t+h)) \le 0$. Denote

$$\tilde{u} = hA(x,t)u(x+1,t) + (I + hB(x,t))u(x,t) + hC(x,t)u(x-1,t)$$

and observe that

$$u(x, t+h) = \tilde{u} + hf(u(x,t),t), \quad x \in \mathbb{Z}.$$

By Lemma 3.4, we have $\tilde{u} \in S$. We now distinguish two cases: either $d(\tilde{u}, \partial S) \ge \delta$, or $d(\tilde{u}, \partial S) < \delta$.

If $d(\tilde{u}, \partial S) \ge \delta$, the fact that $u(x, t+h) \in S$ follows immediately from the estimate $\|u(x,t+h) - \tilde{u}\| = h\|f(u(x,t),t)\| \le hM_1 \le h_0 M_1 \le \delta$.

It remains to consider the case when $d(\tilde{u}, \partial S) < \delta$. Let ℓ be the line segment connecting the points $\tilde{u}$ and $\tilde{u} + hf(u(x,t),t)$. Since $\|u(x,t+h) - \tilde{u}\| \le \delta$ and $\tilde{u} \in S$, the distance between an arbitrary point of ℓ and the set S does not exceed δ, and therefore $\ell \subset S_\delta \subset U_i$. Using the mean-value theorem, we obtain

$$\begin{aligned} G_i(u(x,t+h)) &= G_i(\tilde{u} + hf(u(x,t),t)) = G_i(\tilde{u}) + G_i(\tilde{u} + hf(u(x,t),t)) - G_i(\tilde{u}) \\ &= G_i(\tilde{u}) + h\nabla G_i(w) \cdot f(u(x,t),t), \end{aligned}$$

where $w \in \ell$. Note that $G_i(\tilde{u}) \le 0$ (because $\tilde{u} \in S$) and $\nabla G_i(w) \cdot f(\tilde{u}, t) < -\varepsilon$ (this follows from Lemma 3.7, because $\|w - \tilde{u}\| \le \delta$). Consequently,

$$\begin{aligned} G_i(u(x,t+h)) &\le h\nabla G_i(w) \cdot f(u(x,t),t) \\ &= h\nabla G_i(w) \cdot f(\tilde{u},t) + h\nabla G_i(w) \cdot (f(u(x,t),t) - f(\tilde{u},t)) \\ &< -h\varepsilon + h\|\nabla G_i(w)\| L \|u(x,t) - \tilde{u}\| \\ &\le -h\varepsilon + h^2 M_2 L \|A(x,t)u(x+1,t) + B(x,t)u(x,t) + C(x,t)u(x-1,t)\| \\ &\le -h\varepsilon + h^2 M_2 L M_4 (\|A(x,t)\| + \|B(x,t)\| + \|C(x,t)\|) \\ &\le -h\varepsilon + h^2 M_2 L M_4 M_3 = -h(\varepsilon - hLM_2M_3M_4). \end{aligned}$$

By the definition of h_0, the last term is nonpositive, and therefore $G_i(u(x,t+h)) < 0$. □

Remark 3.9. Note that the previous theorem no longer holds if (C_1') is replaced by the weaker version (C_1). Indeed, suppose that at a certain point $y \in \partial S_i \cap S$, the vector field f is nonzero and tangent to the boundary of S_i. Furthermore, assume that y is the only point of S on the half-line determined by y and the vector f (i.e., the normal section of ∂S_i at y in the direction of f is "curved"). If we start with the initial condition $u(x,0) = y$ for all $x \in \mathbb{Z}$, then the solution of Eq. (3.5) will immediately leave the set S no matter how small step size h we choose.

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We now use our previous result concerning the Euler approximations to show that S is an invariant region for the lattice differential equation (3.3).

Theorem 3.10. *Assume that conditions* (S), (A_1), (A_2), (D_1), (D_2), (C_1'), (C_2), (C_3) *are satisfied. If* $u : \mathbb{Z} \times [0, T] \to \mathbb{R}^m$ *is a bounded solution of Eq.* (3.3) *with* $u^0 \in \ell^\infty(\mathbb{Z})^m$ *and* $u_x^0 \in S$ *for each* $x \in \mathbb{Z}$, *then* $u(x, t) \in S$ *for all* $t \in [0, T]$, $x \in \mathbb{Z}$.

Proof. According to Theorems 2.1 and 2.3, the solution u necessarily has the form $u(x, t) = U(t)_x$, where $U : [0, T] \to \ell^\infty(\mathbb{Z})^m$ is the unique solution of the abstract differential equation

$$U'(t) = \Phi(U(t), t), \quad U(0) = u^0$$

with $\Phi : \ell^\infty(\mathbb{Z})^m \times [0, T] \to \ell^\infty(\mathbb{Z})^m$ being given by

$$\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{A(x, t)u_{x+1} + B(x, t)u_x + C(x, t)u_{x-1} + f(u_x, t)\}_{x\in\mathbb{Z}}. \tag{3.10}$$

For each $n \in \mathbb{N}$, let $\mathbb{T}_n = \{0, h, 2h, \dots, (n-1)h, h\}$, where $h = 1/n$. Also, let $U_n : \mathbb{T}_n \to \ell^\infty(\mathbb{Z})^m$ be the solution of the abstract difference equation

$$\frac{U_n(t+h) - U_n(t)}{h} = \Phi(U_n(t), t), \quad x \in \mathbb{Z}, \quad t \in \{0, h, 2h, \dots, (n-1)h\}, \quad U_n(0) = u^0.$$

Obviously, if we denote $u_n(x, t) = (U_n(t))_x$, then u_n is the solution of the partial difference equation

$$\begin{aligned}\frac{u_n(x, t+h) - u_n(x, t)}{h} &= A(x, t)u_n(x+1, t) + B(x, t)u_n(x, t)\\ &\quad + C(x, t)u_n(x-1, t) + f(u_n(x, t), t)\end{aligned}$$

satisfying $u_n(x, 0) = u_x^0$ for all $x \in \mathbb{Z}$.

By Theorem 3.8, there exists an $n_0 \in \mathbb{N}$ such that $u_n(x, t) \in S$ for all $n \geq n_0$, $x \in \mathbb{Z}$, $t \in \mathbb{T}_n$. Thus, if we denote $\mathcal{B} = \{u \in \ell^\infty(\mathbb{Z})^m; u_x \in S \text{ for all } x \in \mathbb{Z}\}$, then $U_n(t) \in \mathcal{B}$ for all $n \geq n_0$, $t \in \mathbb{T}_n$. It follows that the piecewise constant extension U_n^* described in Theorem 2.4 also takes values in $\mathcal{B}$. According to this theorem, the sequence $\{U_n^*\}_{n=1}^\infty$ is uniformly convergent to U on $[0, T]$. Since $\mathcal{B}$ is closed (because S is closed), it follows that U takes values in $\mathcal{B}$, i.e., $u(x, t) \in S$ for all $t \in [0, T]$, $x \in \mathbb{Z}$. □

Our final goal is to prove that the previous theorem remains valid if (C_1') is replaced by the weaker condition (C_1). The idea is to find a perturbation of the reaction function f such that the new reaction function will satisfy the stronger condition (C_1'). This idea was used by Chueh et al. [9] in the context of parabolic equations. Unfortunately, the authors did not provide any details on the construction of such a perturbation. This purpose of the next lemma is to fill this gap.

Lemma 3.11. *If condition* (S) *holds, there exists a function* $N : \mathbb{R}^m \to \mathbb{R}^m$ *that is Lipschitz-continuous on each bounded subset of* $\mathbb{R}^m$, $\|N(u)\| \leq 1$ *for all* $u \in \mathbb{R}^m$, *and* $\nabla G_i(u) \cdot N(u) < 0$ *for each* $i \in \{1, \dots, k\}$ *and* $u \in \partial S_i \cap S$.

Proof. Choose an arbitrary s_0 in the interior of S and let

$$N(u) = \delta(u) \frac{s_0 - u}{\operatorname{diam} S + 1}, \quad u \in \mathbb{R}^m,$$

where diam S denotes the diameter of S and $\delta : \mathbb{R}^m \to \mathbb{R}$ is given by

$$\delta(u) = (1 - d(u, S))^+.$$

It is well known that the distance from a point to a nonempty set is Lipschitz-continuous. Hence, δ is Lipschitz-continuous (because it is a composition of Lipschitz-continuous functions) and takes values in $[0, 1]$. On each bounded subset of $\mathbb{R}^m$, N is the product of two bounded Lipschitz-continuous functions, and therefore Lipschitz-continuous.

If $d(u, S) > 1$, we have $N(u) = 0$. Otherwise, let $p(u)$ be the closest point to u in S. Then $\|p(u) - u\| \le 1$, $\|p(u) - s_0\| \le \operatorname{diam} S$, and therefore

$$\|N(u)\| \le \frac{\|s_0 - u\|}{\operatorname{diam} S + 1} \le \frac{\|s_0 - p(u)\| + \|p(u) - u\|}{\operatorname{diam} S + 1} \le 1.$$

Finally, if $u \in \partial S_i \cap S$, then $\nabla G_i(u)$ is the outward normal of S_i at u. Since the vector $s_0 - u$ points from u to the interior of S_i, we have $\nabla G_i(u) \cdot (s_0 - u) < 0$, and consequently $\nabla G_i(u) \cdot N(u) < 0$. □

Theorem 3.12. *Assume that conditions* (S), (A_1), (A_2), (D_1), (D_2), (C_1)–(C_3) *are satisfied. If* $u : \mathbb{Z} \times [0, T] \to \mathbb{R}^m$ *is a bounded solution of Eq.* (3.3) *with* $u^0 \in \ell^\infty(\mathbb{Z})^m$ *and* $u_x^0 \in S$ *for each* $x \in \mathbb{Z}$, *then* $u(x, t) \in S$ *for all* $t \in [0, T]$, $x \in \mathbb{Z}$.

Proof. According to Theorems 2.1 and 2.3, the solution u necessarily has the form $u(x, t) = U(t)_x$, where $U : [0, T] \to \ell^\infty(\mathbb{Z})^m$ is a solution of the abstract differential equation

$$U'(t) = \Phi(U(t), t), \quad U(0) = u^0,$$

with $\Phi : \ell^\infty(\mathbb{Z})^m \times [0, T] \to \ell^\infty(\mathbb{Z})^m$ being given by (3.10). Since u is bounded, there exists a bounded set $\mathcal{C} \subset \ell^\infty(\mathbb{Z})^m$ that contains the solution U together with its 1-neighborhood. As in the proof of Theorem 2.1, one can show that the restriction of the function Φ to $\mathcal{C} \times [0, T]$ is continuous, bounded, and Lipschitz-continuous in the first variable.

Consider the operator $\Phi_\varepsilon : \ell^\infty(\mathbb{Z})^m \times [0, T] \to \ell^\infty(\mathbb{Z})^m$ given by

$$\Phi_\varepsilon(\{u_x\}_{x \in \mathbb{Z}}, t) = \Phi(\{u_x\}_{x \in \mathbb{Z}}, t) + \varepsilon\{N(u_x)\}_{x \in \mathbb{Z}},$$

where N is the function from Lemma 3.11. It follows from the properties of Φ and N that for each $\varepsilon \in [0, 1]$, the restrictions of the functions Φ_ε to $\mathcal{C} \times [0, T]$ are continuous, bounded by a constant independent of ε, and Lipschitz-continuous in the first variable with a Lipschitz constant independent of ε. Note also that $\lim_{\varepsilon \to 0+} \Phi_\varepsilon(\{u_x\}_{x \in \mathbb{Z}}, t) = \Phi(\{u_x\}_{x \in \mathbb{Z}}, t)$. According to Theorem 2.5, there exists an $n_0 \in \mathbb{N}$ and a sequence of functions $U_n : [0, T] \to \mathcal{C}$, $n \ge n_0$, such that

$$U_n'(t) = \Phi_{1/n}(U_n(t), t), \quad t \in [0, T], \quad U_n(0) = u^0.$$

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Moreover, $\{U_n\}_{n=n_0}^{\infty}$ is uniformly convergent to U on $[0, T]$. If we denote $u_n(x,t) = (U_n(t))_x$, then $u_n : \mathbb{Z} \times [0, T] \to \mathbb{R}^m$ is a bounded solution of the equation

$$\frac{\partial u_n}{\partial t}(x,t) = A(x,t)u_n(x+1,t) + B(x,t)u_n(x,t) + C(x,t)u_n(x-1,t) + f_n(u_n(x,t),t), \quad x \in \mathbb{Z}, \quad t \in [0, T],$$

where

$$f_n(u,t) = f(u,t) + \frac{1}{n}N(u), \quad u \in \mathbb{R}^m, \quad t \in [0, T].$$

For each $i \in \{1, \dots, k\}$ and $u \in \partial S_i \cap S$, it follows from condition (C_1) and the properties of N that

$$\nabla G_i(u) \cdot f_n(u,t) = \nabla G_i(u) \cdot f(u,t) + \frac{1}{n}\nabla G_i(u) \cdot N(u) < 0, \quad t \in [0, T].$$

Thus, f_n satisfies condition (C_1'). Since f fulfills the conditions (D_1) and (D_2), it is clear that f_n has the same properties.

According to Theorem 3.10, we have $u_n(x,t) \in S$ for all $x \in \mathbb{Z}$, $t \in [0, T]$. Thus, if we denote $\mathcal{B} = \{u \in \ell^\infty(\mathbb{Z})^m;\, u_x \in S \text{ for all } x \in \mathbb{Z}\}$, then $U_n(t) \in \mathcal{B}$ for all $n \ge n_0$, $t \in [0, T]$. Since $\mathcal{B}$ is closed (because S is closed), it follows that U takes values in $\mathcal{B}$, i.e., $u(x,t) \in S$ for all $t \in [0, T]$, $x \in \mathbb{Z}$. □

Remark 3.13. Let us mention two special cases of Theorem 3.12, which generalize earlier results:

- If $f \equiv 0$, Eq. (3.3) becomes a linear system of lattice diffusion-type equations. If A, B, C satisfy conditions (A_1), (A_2), (C_2), (C_3), Theorem 3.12 implies that each bounded solution of Eq. (3.3) with initial values in the compact convex set S remains in this set for all $t > 0$. This result generalizes [29, Theorem 4.7] in the case when $\mathbb{T} = \mathbb{R}$.
- If $m = 1$, Eq. (3.3) becomes a scalar reaction–diffusion equation. Assume that $S = [\alpha, \beta]$, i.e.,

$$S_1 = \{u \in \mathbb{R};\ G_1(u) = \alpha - u \le 0\},$$
$$S_2 = \{u \in \mathbb{R};\ G_2(u) = u - \beta \le 0\}$$

(cf. Remark 3.1). Then $\nabla S_1(u) = -1$ and $\nabla S_2(u) = 1$ for all $u \in \mathbb{R}$. Since $\partial S_1 \cap S = \{\alpha\}$ and $\partial S_2 \cap S = \{\beta\}$, condition (C_1) is satisfied if $f(\alpha,t) \le 0$ and $f(\beta,t) \ge 0$ for all $t \in [0, T]$. Obviously, condition (C_2) is satisfied if A, C are nonnegative and B is nonpositive. Provided that the remaining conditions (C_3), (A_1), (A_2), (D_1) and (D_2) are satisfied, we get the invariance of the interval $[\alpha, \beta]$; this result extends [30, Theorem 4.4] in the case when $\mathbb{T} = \mathbb{R}$ (in [30], it was assumed that A, B, C are constant).

A simple corollary of our invariance result is the following global existence theorem; its proof is essentially the same as in [30, Theorem 4.6], but we include it for completeness.

Theorem 3.14. *Assume that conditions* (S), (A_1), (A_2), (D_1), (D_2), (C_1)–(C_3) *are satisfied. If* $u^0 \in \ell^\infty(\mathbb{Z})^m$ *is such that* $u_x \in S$ *for all* $x \in \mathbb{Z}$, *then the initial-value problem* (3.3) *has a unique bounded solution* $u : \mathbb{Z} \times [0, T] \to \mathbb{R}^m$.

Proof. The uniqueness is an immediate consequence of Theorem 2.3. According to Theorem 2.1, it is enough to prove that the initial-value problem

$$U'(t) = \Phi(U(t), t), \quad U(0) = u^0, \tag{3.11}$$

where $\Phi : \ell^\infty(\mathbb{Z})^m \times [0, T] \to \ell^\infty(\mathbb{Z})^m$ is given by Eq. (3.10), has a solution on the whole interval $[0, T]$.

Let $\mathcal{T}$ be the set of all $\tau \in [0, T]$ such that Eq. (3.11) has a solution on $[0, \tau]$, and denote $t_1 = \sup \mathcal{T}$. We know from Theorem 2.1 that $t_1 > 0$; let us prove that $t_1 \in \mathcal{T}$. It follows from the definition of t_1 that Eq. (3.11) has a solution U defined on $[0, t_1)$. According to Theorem 3.12, U takes values in the bounded set $\mathcal{S} = \{u \in \ell^\infty(\mathbb{Z})^m;\ u_x \in S \text{ for all } x \in \mathbb{Z}\}$. As in the proof of Theorem 2.1, one can show that Φ is continuous on its domain and bounded on $\mathcal{S} \times [0, T]$ by a constant M. Since U is a solution of (3.11), we have

$$U(t) = U(0) + \int_0^t \Phi(U(s), s)\,\mathrm{d}s \tag{3.12}$$

for each $t \in [0, t_1)$. Hence, $\|U(s_1) - U(s_2)\| \le M|s_1 - s_2|$ for all $s_1, s_2 \in [0, t_1)$, which means that the Cauchy condition for the existence of the limit $U(t_1-) = \lim_{s \to t_1-} U(s)$ is satisfied. If we extend U to $[0, t_1]$ by letting $U(t_1) = U(t_1-)$, we see that (3.12) holds also for $t = t_1$. Since the mapping $s \mapsto \Phi(U(s), s)$ is continuous on $[0, t_1]$, it follows that U is a solution of Eq. (3.11) on $[0, t_1]$, i.e., $t_1 \in \mathcal{T}$.

If $t_1 < T$, Theorem 2.1 implies that U can be extended from $[0, t_1]$ to a larger interval, which contradicts the definition of t_1. Thus, we necessarily have $t_1 = T$. $\square$

4. Examples

Let us illustrate the previous results on three examples. Following the discussion in Remark 3.3, we consider three typical cases: 1) A, B, C are scalar multiples of the identity matrix. 2) A, B, C are diagonal matrices. 3) A, B, C are nondiagonal matrices.

Throughout this section, we use the symbol $u^{\Delta\nabla}$ to denote the second-order central difference of u with respect to x, i.e., $u^{\Delta\nabla}(x, t) = u(x + 1, t) - 2u(x, t) + u(x - 1, t)$; cf. Section 1.

Example 4.1. Consider the pair of weakly coupled lattice reaction–diffusion equations

$$\begin{aligned}
\frac{\partial u_1}{\partial t}(x, t) &= k u_1^{\Delta\nabla}(x, t) + a u_1(x, t) - b u_1(x, t) u_2(x, t), \\
\frac{\partial u_2}{\partial t}(x, t) &= k u_2^{\Delta\nabla}(x, t) - c u_2(x, t) + d u_1(x, t) u_2(x, t),
\end{aligned}$$

where $k \ge 0$ is the diffusion coefficient and $a, b, c, d > 0$ are parameters. This system can be interpreted as a predator-prey model of Lotka–Volterra type with diffusion; $u_1(x, t)$ is the number of prey and $u_2(x, t)$ the number of predators at point $x \in \mathbb{Z}$ and time $t \in [0, T]$.

The given system is a special case of Eq. (3.3) with $m = 2$,

$$A(x,t) = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B(x,t) = -2k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C(x,t) = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$f_1(u,t) = au_1 - bu_1u_2, \quad f_2(u,t) = -cu_2 + du_1u_2.$$

It is obvious that conditions (A_1), (A_2), (D_1), (D_2) are satisfied. Let $U = (0,\infty) \times (0,\infty)$. In the classical Lotka–Volterra model, it is well known that the quantity

$$du_1 - c\ln u_1 + bu_2 - a\ln u_2$$

remains constant along each solution $u = (u_1, u_2)$ contained in U (see, e.g., [20, Section 11.2]). More precisely, each solution with positive initial values is a closed orbit in U given by the equation

$$du_1 - c\ln u_1 + bu_2 - a\ln u_2 = K \tag{4.1}$$

for a certain $K \in \mathbb{R}$. Let S be the planar region enclosed by the curve (4.1), i.e., $S = \{u \in U;\ G(u) \le 0\}$, where $G(u) = du_1 - c\ln u_1 + bu_2 - a\ln u_2 - K$.

Our previous considerations imply that $\nabla G(u) \cdot f(u,t) = 0$ for each $u \in \partial S$, and this fact is confirmed by simple calculation:

$$\nabla G(u) \cdot f(u,t) = \begin{pmatrix} d - c/u_1 \\ b - a/u_2 \end{pmatrix} \cdot \begin{pmatrix} au_1 - bu_1u_2 \\ -cu_2 + du_1u_2 \end{pmatrix} = 0$$

Thus, condition (C_1) is satisfied. Condition (C_2) also holds because A, B, C are scalar multiples of the identity matrix (cf. Remark 3.3):

$$\begin{aligned} \nabla G(u)^\top A(x,t) &= k\nabla G(u)^\top, \\ \nabla G(u)^\top B(x,t) &= -2k\nabla G(u)^\top, \\ \nabla G(u)^\top C(x,t) &= k\nabla G(u)^\top. \end{aligned}$$

Obviously, condition (C_3) holds as well. Note that G is a convex function of two variables on U, because its Hessian matrix

$$\begin{pmatrix} c/u_1^2 & 0 \\ 0 & a/u_2^2 \end{pmatrix}$$

is positive definite (cf. [26, Theorem 1.5.13]). Consequently, the set $S \subset U$ is convex, because it is a sublevel set of the convex function G. Hence, condition (S) holds and Theorem 3.12 implies that S is an invariant region for the given system of equations.

Example 4.2. Consider the pair of weakly coupled lattice reaction–diffusion equations

$$\left.\begin{aligned}\frac{\partial u_1}{\partial t}(x,t)&=d_1u_1^{\Delta\nabla}(x,t)+u_1(x,t)\big(a_1-b_1u_1(x,t)-c_1u_2(x,t)\big),\\ \frac{\partial u_2}{\partial t}(x,t)&=d_2u_2^{\Delta\nabla}(x,t)+u_2(x,t)\big(a_2-b_2u_2(x,t)-c_2u_1(x,t)\big),\end{aligned}\right\}\tag{4.2}$$

where $d_1, d_2 \geq 0$ are diffusion coefficients and $a_1, a_2, b_1, b_2 > 0$, $c_1, c_2 \geq 0$ are parameters. This system can be interpreted as a model of Lotka–Volterra type with two competing populations; for each $i \in \{1, 2\}$, $u_i(x, t)$ is the number of individuals from the i-th population at point $x \in \mathbb{Z}$ and time $t \in [0, T]$. In the absence of one population, the other population obeys the logistic law. This competition model has been studied in numerous papers, see [16–18] and the references there.

The given system of equations is a special case of (3.3) with $m = 2$,

$$A(x,t)=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},\quad B(x,t)=-2\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},\quad C(x,t)=\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},$$
$$f_1(u,t)=u_1(a_1-b_1u_1-c_1u_2),\qquad f_2(u,t)=u_2(a_2-b_2u_2-c_2u_1).$$

It is obvious that conditions (A_1), (A_2), (D_1), (D_2) are satisfied.

Let us try to find rectangles of the form $S = [\alpha, \beta] \times [\gamma, \delta]$, $0 \leq \alpha < \beta$, $0 \leq \gamma < \delta$, that are invariant regions for the given system. Such a rectangle S is the intersection of the four closed half-planes

$$S_1=\{u\in\mathbb{R}^2;\ G_1(u)=\alpha-u_1\leq 0\},$$
$$S_2=\{u\in\mathbb{R}^2;\ G_2(u)=\gamma-u_2\leq 0\},$$
$$S_3=\{u\in\mathbb{R}^2;\ G_3(u)=u_1-\beta\leq 0\},$$
$$S_4=\{u\in\mathbb{R}^2;\ G_4(u)=u_2-\delta\leq 0\}.$$

We have $U_1 = U_2 = U_3 = U_4 = \mathbb{R}^2$ and for each $u \in \mathbb{R}^2$, we get

$$\nabla G_1(u)=\begin{pmatrix}-1\\ 0\end{pmatrix},\quad \nabla G_2(u)=\begin{pmatrix}0\\ -1\end{pmatrix},\quad \nabla G_3(u)=\begin{pmatrix}1\\ 0\end{pmatrix},\quad \nabla G_4(u)=\begin{pmatrix}0\\ 1\end{pmatrix},$$

which means that condition (S) is satisfied. Condition (C_2) holds because A, B, C are diagonal and the vectors $\nabla G_i(u)$ are multiples of the vectors from the canonical basis in $\mathbb{R}^2$ (cf. Remark 3.3):

$$\begin{aligned}
\nabla G_i(u)^\top A(x,t)&=d_1\nabla G_i(u)^\top &&\text{for } i\in\{1,3\},\\
\nabla G_i(u)^\top A(x,t)&=d_2\nabla G_i(u)^\top &&\text{for } i\in\{2,4\},\\
\nabla G_i(u)^\top B(x,t)&=-2d_1\nabla G_i(u)^\top &&\text{for } i\in\{1,3\},\\
\nabla G_i(u)^\top B(x,t)&=-2d_2\nabla G_i(u)^\top &&\text{for } i\in\{2,4\},\\
\nabla G_i(u)^\top C(x,t)&=d_1\nabla G_i(u)^\top &&\text{for } i\in\{1,3\},\\
\nabla G_i(u)^\top C(x,t)&=d_2\nabla G_i(u)^\top &&\text{for } i\in\{2,4\}.
\end{aligned}$$

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Obviously, condition (C_3) holds as well, and it remains to check condition (C_1). For each $i \in \{1, 2, 3, 4\}$ and $u \in \partial S_i \cap S$, we find necessary and sufficient conditions guaranteeing that $\nabla G_i(u) \cdot f(u,t) \le 0$:

- If $i = 1$ and $u \in \partial S_1 \cap S$, then $u_1 = \alpha$ and $u_2 \in [\gamma, \delta]$. Therefore, we need

$$\nabla G_1(u) \cdot f(u,t) = -f_1(u,t) = -\alpha(a_1 - b_1\alpha - c_1u_2) \le 0 \quad \text{for all } u_2 \in [\gamma, \delta].$$

  Since $\alpha \ge 0$, this is equivalent to

$$\alpha = 0 \quad \text{or} \quad a_1 - b_1\alpha - c_1\delta \ge 0. \tag{4.3}$$

- If $i = 2$ and $u \in \partial S_2 \cap S$, then $u_2 = \gamma$ and $u_1 \in [\alpha, \beta]$. Therefore, we need

$$\nabla G_2(u) \cdot f(u,t) = -f_2(u,t) = -\gamma(a_2 - b_2\gamma - c_2u_1) \le 0 \quad \text{for all } u_1 \in [\alpha, \beta].$$

  Since $\gamma \ge 0$, this is equivalent to

$$\gamma = 0 \quad \text{or} \quad a_2 - b_2\gamma - c_2\beta \ge 0. \tag{4.4}$$

- If $i = 3$ and $u \in \partial S_3 \cap S$, then $u_1 = \beta$ and $u_2 \in [\gamma, \delta]$. Therefore, we need

$$\nabla G_3(u) \cdot f(u,t) = f_1(u,t) = \beta(a_1 - b_1\beta - c_1u_2) \le 0 \quad \text{for all } u_2 \in [\gamma, \delta].$$

  Since $\beta > 0$, this is equivalent to

$$a_1 - b_1\beta - c_1\gamma \le 0. \tag{4.5}$$

- If $i = 4$ and $u \in \partial S_4 \cap S$, then $u_2 = \delta$ and $u_1 \in [\alpha, \beta]$. Therefore, we need

$$\nabla G_4(u) \cdot f(u,t) = f_2(u,t) = \delta(a_2 - b_2\delta - c_2u_1) \le 0 \quad \text{for all } u_1 \in [\alpha, \beta].$$

  Since $\delta > 0$, this is equivalent to

$$a_2 - b_2\delta - c_2\alpha \le 0. \tag{4.6}$$

Theorem 3.12 implies that if conditions (4.3)–(4.6) are satisfied, then the rectangle $S = [\alpha, \beta] \times [\gamma, \delta]$ is an invariant region for our system of equations.

For example, if we let $\alpha = \gamma = 0$, then (4.3) and (4.4) are satisfied, while (4.5) and (4.6) require that $\beta \ge a_1/b_1$ and $\delta \ge a_2/b_2$, respectively. Note that $(0, 0)$, $(a_1/b_1, 0)$ and $(0, a_2/b_2)$ are equilibrium points of the vector field f. We also remark that Theorem 3.14 guarantees the existence of a unique bounded global solution on $[0, T]$ for all initial conditions $u^0 \in \ell^\infty(\mathbb{Z})^2$ with nonnegative components (take $S = [0, \beta] \times [0, \delta]$ with $\beta \ge a_1/b_1$ and $\delta \ge a_2/b_2$ chosen in such a way that $u_x^0 \in S$ for all $x \in \mathbb{Z}$).

Several authors (e.g., [16–18]) have considered the special case of the system (4.2) with $a_1 = a_2 = b_1 = b_2 = 1$, $c_1 = k$, $c_2 = h$, i.e., the system

$$\left.\begin{aligned}\frac{\partial u_1}{\partial t}(x,t) &= d_1 u_1^{\Delta\nabla}(x,t) + u_1(x,t)\big(1 - u_1(x,t) - k u_2(x,t)\big),\\ \frac{\partial u_2}{\partial t}(x,t) &= d_2 u_2^{\Delta\nabla}(x,t) + u_2(x,t)\big(1 - u_2(x,t) - h u_1(x,t)\big).\end{aligned}\right\} \tag{4.7}$$

The case when $h, k > 1$ is referred to as the bistable case, since both equilibria $(1, 0)$ and $(0, 1)$ are stable. Moreover, there is another equilibrium point at

$$\left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right).$$

As a consequence of our previous calculation, we can show the existence of two invariant rectangles adjacent to the two stable equilibria (for a different approach, see the proof of Theorem 1 in [16]).

1. To find an invariant rectangle adjacent to $(0, 1)$, we let $S = [\alpha, \beta] \times [\gamma, \delta]$ with $\alpha = 0$, $0 < \gamma \le 1$, $\delta \ge 1$. Then (4.3) and (4.6) are satisfied, while (4.4) and (4.5) reduce to

   $$1 - \gamma - h\beta \ge 0 \quad \text{and} \quad 1 - \beta - k\gamma \le 0.$$

   Solving for β, we get

   $$1 - k\gamma \le \beta \le \frac{1-\gamma}{h}.$$

   This pair of inequalities can be satisfied if and only if $1 - k\gamma \le \frac{1-\gamma}{h}$, which is easily shown to be equivalent to

   $$\gamma \le \frac{1-h}{1-hk}.$$

2. To find an invariant rectangle adjacent to $(1, 0)$, we let $S = [\alpha, \beta] \times [\gamma, \delta]$ with $\gamma = 0$, $0 < \alpha \le 1$, $\beta \ge 1$. Then (4.4) and (4.5) are satisfied, while (4.3) and (4.6) reduce to

   $$1 - \alpha - k\delta \ge 0 \quad \text{and} \quad 1 - \delta - h\alpha \le 0.$$

   Solving for δ, we get

   $$1 - h\alpha \le \delta \le \frac{1-\alpha}{k}.$$

   This pair of inequalities can be satisfied if and only if $1 - h\alpha \le \frac{1-\alpha}{k}$, which is easily shown to be equivalent to

   $$\alpha \le \frac{1-k}{1-hk}.$$

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Example 4.3. Suppose that $a, b > 0$ and consider the linear system of lattice diffusion equations

$$\left.\begin{aligned} \frac{\partial u_1}{\partial t}(x,t) &= a u_1^{\Delta\nabla}(x,t) + b u_2^{\Delta\nabla}(x,t), \\ \frac{\partial u_2}{\partial t}(x,t) &= b u_1^{\Delta\nabla}(x,t) + a u_2^{\Delta\nabla}(x,t), \end{aligned}\right\} \tag{4.8}$$

or, equivalently,

$$\frac{\partial u}{\partial t}(x,t) = D u^{\Delta\nabla}(x,t), \quad \text{where } D = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}.$$

In contrast to the previous examples, the diffusion matrix D is no longer diagonal. The cross-diffusion terms (together with suitable reaction terms) are used in various population or epidemic models (see, e.g., [3,32] in the context of parabolic equations). For example, u_1 and u_2 might describe two different types of individuals with the same diffusion rate a. Moreover, we assume that the individuals of each type try to keep away from places with high concentration of individuals of the other type; this behavior is sometimes modeled by the cross-diffusion terms with diffusion rates b.

Unfortunately, simple models of this kind have the unpleasant property that nonnegative initial conditions need not lead to nonnegative solutions. For example, consider the initial condition

$$u_1(x,0) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0, \end{cases} \qquad u_2(x,0) = 0 \text{ for all } x \in \mathbb{Z}.$$

Using Eq. (4.8), we get $\frac{\partial u_2}{\partial t}(0,0) = -2b < 0$, which means that $t \mapsto u_2(0,t)$ is negative for small positive values of t. A similar argument shows that no rectangle of the form $S = [\alpha, \beta] \times [\gamma, \delta]$ can be an invariant region for Eq. (4.8). To see this, take the initial condition

$$u_1(x,0) = \begin{cases} \frac{\alpha+\beta}{2} & \text{for } x = 0, \\ \alpha & \text{for } x \neq 0, \end{cases} \qquad u_2(x,0) = \gamma \text{ for all } x \in \mathbb{Z},$$

and observe that $\frac{\partial u_2}{\partial t}(0,0) = b(\alpha - \beta) < 0$.

Nevertheless, if $a \geq b$, then Eq. (4.8) has invariant regions of a different type, namely all rectangles whose sides make a 45° angle with the coordinate axes. This follows from Theorem 3.12, because we have

$$A(x,t) = D, \quad B(x,t) = -2D, \quad C(x,t) = D, \quad f_1(u,t) = f_2(u,t) = 0.$$

The eigenvalues of D are $a+b$ and $a-b$, and the corresponding left eigenvectors are $(1,1)$ and $(-1,1)$. Hence, if S is a rectangle of the above-mentioned type, then condition (C_2) holds; the remaining assumptions of Theorem 3.12 are trivially satisfied.

Remark 4.4. Sometimes it is necessary to deal with lattice systems of the form (3.3) where the reaction function f is not defined on the whole set $\mathbb{R}^m \times [0, T]$. For example, the authors of [5] study a predator-prey model consisting of two weakly coupled lattice reaction–diffusion equations

$$\frac{\partial u_1}{\partial t}(x,t) = u_1^{\Delta\nabla}(x,t) + r u_1(x,t)\big(1 - u_1(x,t) - k u_2(x,t)\big),$$
$$\frac{\partial u_2}{\partial t}(x,t) = d u_2^{\Delta\nabla}(x,t) + s u_2(x,t)\left(1 - \frac{u_2(x,t)}{u_1(x,t)}\right),$$

where $d, r, s, k > 0$ are parameters. Here, the reaction function is undefined on the line $u_1 = 0$.

In general, suppose that f is defined on a set $Y \times [0,T]$, where $Y \subset \mathbb{R}^m$. Assume that f is Lipschitz-continuous in the first variable on each set $B \times [0,T]$, where $B \subset Y$ is bounded, and that f is continuous in the second variable.

If $S \subset Y$ is a compact convex set, the question whether S is an invariant region for bounded solutions of Eq. (3.3) still makes sense. To be able to apply the results from Section 3, we can extend f to $\mathbb{R}^m \times [0,T]$ by letting

$$f(u,t) = f(p(u),t), \quad u \in \mathbb{R}^m, \quad t \in [0,T],$$

where $p(u)$ is the projection of u to S, i.e., the unique point of S that minimizes the distance to u. We claim that this extension of f satisfies conditions (D_1), (D_2). Continuity in the second variable is obvious. To verify that (D_1) holds, let L be the Lipschitz constant for f on $S \times [0,T]$. Then for each pair $u, v \in \mathbb{R}^m$, we have

$$\|f(u,t) - f(v,t)\| = \|f(p(u),t) - f(p(v),t)\| \le L\|p(u) - p(v)\| \le L\|u - v\|,$$

where the last inequality follows from the well-known fact that projection onto closed convex sets is a nonexpansive mapping. This shows that (D_1) is fulfilled and therefore the results from Section 3 are applicable. In particular, Theorems 3.12 and 3.14 still hold: Although their proofs require that f is defined on $\mathbb{R}^m \times [0,T]$, the values of f outside S are unimportant thanks to the invariance of S.

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