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Global low-energy weak solution and large-time behavior for the compressible flow of liquid crystals

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Abstract

In this paper, we consider the weak solution of the simplified Ericksen–Leslie system modeling compressible nematic liquid crystal flows in $\mathbb{R}^3$. When the initial data are of small energy and initial density is positive and essentially bounded, we prove the existence of a global weak solution in $\mathbb{R}^3$. The large-time behavior of a global weak solution is also established.
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1. Introduction

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials [2,6,22]:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) = \mu \Delta u + \lambda \nabla \mathrm{div} u - \nabla d \Delta d, \\ \partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \end{cases} \tag{1.1}$$

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for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$. Here ρ, $u = (u^1, u^2, u^3)^t$ and P denote the density, the velocity, and the pressure respectively. $d = (d^1, d^2, d^3)^t$ is the unit-vector ($|d| = 1$) on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ representing the macroscopic molecular orientation of the liquid crystal materials. μ and λ are positive viscosity constants, and div and Δ are the usual spatial divergence and Laplace operators.

The above system (1.1) is a simplified version of the Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals. Roughly speaking, the system (1.1) is a coupling between the non-homogeneous Navier–Stokes equations and the transported flow harmonic maps. Due to the physical importance and mathematical challenges, the study on nematic liquid crystals has attracted many physicists and mathematicians. The mathematical analysis of the incompressible liquid crystal flows was initiated by Lin and Liu in [23,24]. For any bounded smooth domain in $\mathbb{R}^2$, Lin, Lin and Wang [25] have proved the global existence of Leray–Hopf type weak solutions to system (1.1) which are smooth everywhere except on finitely many time slices (see [9] for the whole space). The uniqueness of weak solutions in two dimension was studied by [26,35]. Hong and Xin [10] studied the global existence for general Ericksen–Leslie system in dimension two. In [34], Wang proved the global existence of strong solutions in whole space under some small conditions. Recently, Lin et al. [14,27] obtained the global existence of weak solutions the nematic liquid crystal flow in dimension three under geometric angle condition and constructed the examples of finite time singularity for any generic initial data.

When the fluid is allowed to be compressible, the Ericksen–Leslie system becomes more complicate. To our knowledge, there seems very few analytic works available yet. The local-in-time strong solutions to the initial value or initial boundary value problem of system (1.1) with non-negative initial density were studied in [3,12]. Based on [16,17], the blow up criterion of strong solutions were obtained in [12,13]. The global existence and uniqueness of strong solution in critical space were studied in [11]. Motivated by [15], when the initial data was sufficiently smooth and suitably small in some energy-norm, the global well-posedness of classical solutions were proved in [21]. Especially global weak solutions were established in [18,19,28] under some small condition or geometric angle condition (see [20]).

Our aim in this paper is to establish the global existence of low-energy weak solutions of system (1.1), if the following initial value:

$$(\rho(\cdot, 0), u(\cdot, 0), d(\cdot, 0)) = (\rho_0, u_0, d_0), \tag{1.2}$$

satisfies that ρ_0 is bounded above and below away from zero, $|d_0| = 1$, $u_0, \nabla d_0 \in L^p(\mathbb{R}^3)$ for some $p > 6$ satisfying (1.10) and $(\rho_0, u_0, \nabla d_0)$ is small in $L^2(\mathbb{R}^3)$. Thus the total initial energy is small, but no other smallness or regularity conditions are imposed.

When the direction field d does not appear, (1.1) reduces to the compressible Navier–Stokes equations. The global classical solutions were first obtained by Matsumura–Nishida [30,31] for initial data close to a non-vacuum equilibrium in $H^3(\mathbb{R}^3)$. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from the vacuum and the gradient of the density remains bounded uniformly in time. Later, Hoff [7,8] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [29] (see also Feireisl et al. [4]), where he obtains global existence of weak solutions-defined as solutions with finite energy. Suen and Hoff [33] adopted Hoff's techniques to obtain global existence of low-energy weak solutions for the magnetohydrodynamics. In this paper, we shall study the Cauchy problem (1.1)–(1.2) for liquid crystals and establish the global existence and large time behavior of low-energy weak solutions. However, compared with the compressible Navier–Stokes equations, some new difficulties arise

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due to the additional presence of the liquid crystal directional field. Especially, the super critical nonlinearity $|\nabla d|^2 d$ in the transported heat flow of harmonic map equation $(1.1)_3$ and the strong coupling nonlinearity $\nabla d \Delta d$ in the momentum equations $(1.1)_2$ will cause serious difficulties in the proofs of the time-independent global energy estimates.

To state the main results in a precise way, we first introduce some notations and conventions which will be used throughout the paper. For a given unit vector $n \in \mathbb{S}^2$ and a positive integer m, we denote

$$H_n^m(\mathbb{R}^3;\mathbb{S}^2) := \{d : d - n \in H^m(\mathbb{R}^3), |d| = 1 \text{ a.e. in } \mathbb{R}^3\}.$$

We use the usual notation for Hölder seminorms: for $v : \mathbb{R}^3 \longrightarrow \mathbb{R}^m$ and $\alpha \in (0, 1]$,

$$\langle v\rangle^{\alpha} = \sup_{\substack{x_1,x_2\in\mathbb{R}^3\\ x_1\neq x_2}} \frac{|v(x_2)-v(x_1)|}{|x_2-x_1|^{\alpha}};$$

and for $v : Q \subseteq \mathbb{R}^3 \times [0,\infty) \longrightarrow \mathbb{R}^3$ and $\alpha_1, \alpha_2 \in (0, 1]$,

$$\langle v\rangle_Q^{\alpha_1,\alpha_2} = \sup_{\substack{(x_1,t_1),(x_2,t_2)\in Q\\ (x_1,t_1)\neq(x_2,t_2)}} \frac{|v(x_2,t_2)-v(x_1,t_1)|}{|x_2-x_1|^{\alpha_1}+|t_2-t_1|^{\alpha_2}}.$$

If X is a Banach space we will abbreviate X^3 by X for convenience. Finally if $I \subset [0,\infty)$ is an interval, $C^1(I;X)$ will be the elements $v \in C(I;X)$ such that the distribution derivative $v_t \in \mathcal{D}'(\text{int } I;\mathbb{R}^3)$ is realized as an element of $C(I;X)$.

As it was pointed out in [7], the effective viscous flux plays an important role in the mathematical theory of compressible fluid dynamics. More precisely, let F and ω be the effective flux and vorticity defined by

$$F \triangleq (\mu+\lambda)\text{div}u - (P(\rho) - P(\tilde\rho)) \quad \text{and} \quad \omega \triangleq \nabla \times u, \tag{1.3}$$

where $\tilde\rho$ is a positive reference density. It is not hard to check that

$$\nabla d\Delta d = \text{div}(\nabla d \odot \nabla d) - \nabla\frac{|\nabla d|^2}{2}, \tag{1.4}$$

where $\nabla d \odot \nabla d$ denotes the symmetric 3×3 matrix: $(\nabla d \odot \nabla d)_{ij} = \nabla_i d \cdot \nabla_j d$, $1 \le i, j \le 3$. Throughout this paper, we denote $v \cdot w$ as the inner product in $\mathbb{R}^3$ for $v, w \in \mathbb{R}^3$. So, it follows from $(1.1)_2$ that

$$\Delta F = \text{div}(\rho\dot u + \text{div}(\nabla d\odot\nabla d) - \nabla\frac{|\nabla d|^2}{2}), \quad \mu\Delta\omega = \nabla\times(\rho\dot u + \text{div}(\nabla d\odot\nabla d)), \tag{1.5}$$

where " $\dot{}$ " denotes the material derivative, i.e.,

$$\dot f := \partial_t f + u\cdot\nabla f.$$

Now we give a precise formulation of our results. First concerning the pressure P, we focus our interest on the case of isentropic flows and assume that

$$P(\rho) = a\rho^{\gamma} \quad \text{with } a > 0, \gamma \geq 1. \tag{1.6}$$

Next we fix a positive reference density $\tilde{\rho}$ and then choose positive bounding densities $\underline{\rho}$ and $\bar{\rho}$ satisfying

$$\underline{\rho} < \tilde{\rho} < \bar{\rho}, \tag{1.7}$$

and finally we define a positive number δ by

$$\delta = \min\{\tilde{\rho} - \underline{\rho}, \bar{\rho} - \tilde{\rho}, \frac{1}{2}(\bar{\rho} - \underline{\rho})\}. \tag{1.8}$$

Notice that δ need not be "small" in the usual sense. Concerning the viscosity coefficients μ and λ we assume that

$$0 \leq \lambda < \frac{3+\sqrt{21}}{6}\mu. \tag{1.9}$$

It follows that

$$\frac{1}{4}\mu(p-2) - \frac{[\frac{1}{4}\lambda(p-2)]^2}{\frac{1}{3}\mu + \lambda} > 0 \tag{1.10}$$

for $p = 6$ and consequently for some $p > 6$, which we now fix.

Concerning the initial data (ρ_0, u_0, d_0), we assume there is a positive number N, which may be arbitrary large, and a positive number $b < \delta$ such that

$$\|u_0\|_{L^p} + \|\nabla d_0\|_{L^p} \leq N \tag{1.11}$$

and

$$\underline{\rho} + b < \operatorname{ess\,inf} \rho_0 \leq \operatorname{ess\,sup} \rho_0 < \bar{\rho} - b. \tag{1.12}$$

We assume also that

$$d_0 \in H_n^1(\mathbb{R}^3; \mathbb{S}^2) \tag{1.13}$$

and write

$$C_0 \triangleq \int_{\mathbb{R}^3} (\frac{1}{2}\rho_0|u_0|^2 + G(\rho_0) + \frac{1}{2}|\nabla d_0|^2)dx, \tag{1.14}$$

where $G(\rho)$ is the potential energy density defined by

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$$G(\rho)=\rho\int_{\tilde\rho}^{\rho}\frac{P(s)-P(\tilde\rho)}{s^2}ds. \tag{1.15}$$

It is clear that there exist two positive constant c_1, c_2 only depending on $\underline{\rho}$, $\bar\rho$, and $\tilde\rho$ such that

$$c_1(\underline{\rho},\bar\rho,\tilde\rho)(\rho-\tilde\rho)^2\le G(\rho)\le c_2(\underline{\rho},\bar\rho,\tilde\rho)(\rho-\tilde\rho)^2. \tag{1.16}$$

The weak solutions of (1.1)–(1.2) are defined in a usual way.

Definition 1.1. A pair of functions (ρ, u, d) is said to be a weak solution of (1.1)–(1.2) provided that $(\rho-\tilde\rho, \rho u)\in C([0,\infty); H^{-1}(\mathbb{R}^3))$, $d-n\in C([0,\infty); L^2(\mathbb{R}^3))$, $u, \nabla d\in L^\infty([0,\infty); L^2(\mathbb{R}^3))$, $\nabla u\in L^2((0,\infty); L^2(\mathbb{R}^3))$, and $|d(\cdot,t)|=1$ a.e. in $\mathbb{R}^3$ for $t\ge 0$. Moreover, the following identities hold for $t_2>t_1\ge 0$ and C^1 test functions ψ having uniformly bounded support in x for $t\in[t_1,t_2]$:

$$\int_{\mathbb{R}^3}\rho(x,t)\psi(x,t)dx\Big|_{t_1}^{t_2}=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\rho\psi_t+\rho u\cdot\nabla\psi)dxdt,$$

$$\begin{aligned}&\int_{\mathbb{R}^3}\rho(x,t)u^j(x,t)\psi(x,t)dx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\mu\nabla u^j\cdot\nabla\psi+\lambda(\mathrm{div}u)\psi_{x_j})dxdt\\&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\rho u^j\psi_t+\rho u^ju\cdot\nabla\psi+P(\rho)\psi_{x_j}-\tfrac12|\nabla d|^2\psi_{x_j}+d_{x_j}\nabla d\cdot\nabla\psi)dxdt,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3}(d^j-n^j)(x,t)\psi(x,t)dx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla d^j\nabla\psi dxdt\\&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}((d^j-n^j)\psi_t-u\cdot\nabla d^j\psi+|\nabla d|^2d^j\psi)dxdt,\end{aligned}$$

with $t_2\ge t_1\ge 0$ and $j=1,2,3$.

Our main results are formulated as the following theorem.

Theorem 1.1. *Assume that the system parameters in (1.1) satisfy the conditions (1.6)–(1.10) and let positive numbers N and $b<\delta$ be given. Then there are positive constants ε, C, and θ depending on the parameters and assumptions in (1.6)–(1.10), on N, and on a positive lower bound for b, such that, if initial data (ρ_0, u_0, d_0) are given satisfying (1.11)–(1.13) with*

$$C_0<\varepsilon, \tag{1.17}$$

then there is a weak solution (ρ, u, d) to (1.1)–(1.2) in the sense of the Definition 1.1. Moreover, the solution satisfies the following:

$$\rho-\tilde\rho,\ \rho u\in C([0,\infty); H^{-1}(\mathbb{R}^3)), \tag{1.18}$$

$$d-n\in C([0,\infty); L^2(\mathbb{R}^3)), \tag{1.19}$$

$$\nabla u \in L^2([0,\infty);\mathbb{R}^3), \tag{1.20}$$

$$u(\cdot,t), \nabla d(\cdot,t) \in H^1(\mathbb{R}^3), \quad t>0, \tag{1.21}$$

$$F(\cdot,t), \omega(\cdot,t) \in H^1(\mathbb{R}^3), \quad t>0, \tag{1.22}$$

$$\langle u\rangle^{\frac12,\frac18}_{\mathbb{R}^3\times[\tau,\infty)}, \langle \nabla d\rangle^{\frac12,\frac18}_{\mathbb{R}^3\times[\tau,\infty)} \le C(\tau)C_0^\theta, \tag{1.23}$$

where $C(\tau)$ may depend additionally on a positive lower bound for τ,

$$\underline{\rho} \le \rho(x,t) \le \bar{\rho} \text{ a.e. on } \mathbb{R}^3 \times [0,\infty), \tag{1.24}$$

and

$$\begin{aligned}
&\sup_{t>0} \int_{\mathbb{R}^3} [|\rho-\tilde{\rho}|^2 + |u|^2 + \tfrac12|\nabla d|^2 + \sigma(|\nabla u|^2 + |\nabla^2 d|^2) + \sigma^5(F^2 + |\nabla\omega|^2)]dx \\
&+ \int_0^\infty \int_{\mathbb{R}^3} [|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2 + \sigma(|\dot{u}|^2 + |\nabla d_t|^2 + |\nabla \omega|^2) \\
&+\sigma^5(|\nabla \dot{u}|^2 + |\nabla^2 d_t|^2)]dxds \\
\le\ & CC_0^\theta,
\end{aligned} \tag{1.25}$$

where $\sigma(t) = \min\{1,t\}$. Moreover we also have the following large-time behavior:

$$\lim_{t\to\infty} (\|\rho-\tilde{\rho}\|_{L^l(\mathbb{R}^3)} + \|u\|_{W^{1,r}(\mathbb{R}^3)} + \|\nabla d\|_{W^{1,r}(\mathbb{R}^3)}) = 0, \tag{1.26}$$

holds for $l \in (2,\infty)$, $r \in (2,6)$.

The rest of this paper is devoted to prove Theorem 1.1. In Section 2, we collect some useful inequalities and basic results. In Section 3, we derive the time-independent energy estimates of the solution. The key pointwise upper and lower bound of the density are established in Section 4. Section 5 is devoted to prove the global existence of strong solutions with initial data satisfying the low-energy condition (1.17). Finally, a global weak solution are established through weak convergence of the smooth solutions in Section 5.

2. Preliminaries

In this section, we state some auxiliary lemmas, which will be frequently used in the sequel. We start with the well-known Gagliardo–Nirenberg inequality (see [1,36]).

Lemma 2.1. *First, given $r \in [2,6]$ there is a constant $C(r)$ such that for $f \in H^1(\mathbb{R}^3)$,*

$$\|f\|_{L^r(\mathbb{R}^3)} \le C(r)\|f\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|\nabla f\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r}. \tag{2.1}$$

Next, for any $r \in (3,\infty)$ and $q > 1$, there is a constant $C(r,q)$ such that for $f \in L^q(\mathbb{R}^3) \cap W^{1,r}(\mathbb{R}^3)$,

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le C(r,q)\|f\|_{L^q(\mathbb{R}^3)}^{q(r-3)/(3r+q(r-3))} \|\nabla f\|_{L^r(\mathbb{R}^3)}^{3r/(3r+q(r-3))}, \tag{2.2}$$

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and

$$\langle f\rangle^{\alpha}_{\mathbb{R}^3} \le C(r)\|\nabla f\|_{L^r(\mathbb{R}^3)}, \tag{2.3}$$

where $\alpha = 1 - \frac{3}{r}$.

The next lemma is due to Hoff [7], which will be used to prove the uniform (in time) bound of density.

Lemma 2.2. *If* Γ *is the fundamental solution for the Laplace operator in* $\mathbb{R}^3$*, then given* $p_1 \in [1,3)$ *and* $p_2 \in (3,\infty]$*, there is a constant* $C = C(n, p_1, p_2)$ *such that*

$$\|\Gamma_{x_j} * f\|_{L^\infty(\mathbb{R}^3)} \le C(n, p_1, p_2)[\|f\|_{L^{p_1}(\mathbb{R}^3)} + \|f\|_{L^{p_2}(\mathbb{R}^3)}]. \tag{2.4}$$

Finally, we need the local existence and a blowup criterion of strong solution for system (1.1)–(1.2). In particular, the following results can be proved rigorously by the standard method of Huang–Wang–Wen [12].

Proposition 2.1. *For some* $\ell \in (3,6]$*, assume that the initial data* (ρ_0, u_0, d_0) *satisfies*

$$\rho_0 - \tilde\rho \in W^{1,\ell}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \text{ and } \inf_{x\in\mathbb{R}^3}(\rho_0(x)) > 0, \tag{2.5}$$

$$u_0 \in H^2(\mathbb{R}^3),\ d_0 \in H^3_n(\mathbb{R}^3;\mathbb{S}^2). \tag{2.6}$$

Then there exists a positive time T_0*, such that the Cauchy problem* (1.1)–(1.2) *has a unique smooth solution* (ρ, u, d) *on* $\mathbb{R}^3 \times [0, T_0]$ *with* $\inf\limits_{(x,t)\in\mathbb{R}^3\times[0,T_0]}(\rho_0(x)) > 0$ *and satisfying*

$$\rho - \tilde\rho \in C([0,T_0]; W^{1,\ell} \cap H^1) \cap C^1([0,T_0]; L^2 \cap L^\ell), \tag{2.7}$$

$$u \in C([0,T_0]; H^2) \cap C^1([0,T_0]; L^2), \tag{2.8}$$

$$d \in C([0,T_0]; H^3_n(\mathbb{R}^3;\mathbb{S}^2)) \cap C^1([0,T_0]; H^1_n(\mathbb{R}^3;\mathbb{S}^2)). \tag{2.9}$$

Moreover, if μ, λ *satisfy* (1.9) *and* $0 < T_0 < +\infty$ *is the maximum time of existence, then*

$$\lim_{T\to T_0}(\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} + \|\nabla d\|_{L^3(0,T;L^\infty(\mathbb{R}^3))}) = \infty. \tag{2.10}$$

In view of Lemma 2.1 and the classical estimates of elliptic system, we have

Lemma 2.3. *Let* (ρ, u, d) *be as in Proposition 2.1, then there exists a generic positive constant* C*, depending only on* μ *and* λ *and* p*, such that for* $p \in (1,\infty)$

$$\|\nabla F\|_{L^p(\mathbb{R}^3)} + \|\nabla\omega\|_{L^p(\mathbb{R}^3)} \le C(\|\rho\dot u\|_{L^p(\mathbb{R}^3)} + \|\nabla d\Delta d\|_{L^p(\mathbb{R}^3)}), \tag{2.11}$$

$$\|\nabla u\|_{L^p(\mathbb{R}^3)} \le C(\|F\|_{L^p(\mathbb{R}^3)} + \|\omega\|_{L^p(\mathbb{R}^3)} + \|P(\rho) - P(\tilde\rho)\|_{L^p(\mathbb{R}^3)}), \tag{2.12}$$

where F *and* ω *are defined in* (1.3).

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Proof. An application of the L^p-estimate of elliptic system to (1.5) gives (2.11). On the other hand, since $-\Delta u = -\nabla \mathrm{div} u + \nabla \times \omega$, it holds that

$$\nabla u = -\nabla(-\Delta)^{-1}\nabla \mathrm{div} u + \nabla(-\Delta)^{-1}\nabla \times \omega,$$

which, combined with the Marcinkiewicz multiplier theorem (see [32] p. 96), we arrive at

$$\begin{aligned}\|\nabla u\|_{L^p(\mathbb{R}^3)} &\le C(\|\mathrm{div} u\|_{L^p(\mathbb{R}^3)}) + \|\omega\|_{L^p(\mathbb{R}^3)} \\ &\le C(\|F\|_{L^p(\mathbb{R}^3)} + \|\omega\|_{L^p(\mathbb{R}^3)} + \|P(\rho) - P(\tilde{\rho})\|_{L^p(\mathbb{R}^3)}).\end{aligned}$$

Thus the proof the lemma is completed. □

3. A priori estimates

This section is devoted to establish a number of a priori bounds for local-in-time strong solution, corresponding roughly to (1.25). Those are rather long and technical. We have therefore omitted those which are identical to or nearly identical to arguments given elsewhere in [7] of whose details we regard as routine. On the other hand, we have endeavored to describe the flow of the arguments in such a way that the diligent reader can reconstruct the details without undue difficulty.

Let $T > 0$ be fixed and assume that (ρ, u, d) is a strong solution of (1.1)–(1.2). We define a functional $A(t)$ for a given such solution that

$$\begin{aligned}A(t) = \sup_{0\le s\le t} &\int_{\mathbb{R}^3} [\sigma(|\nabla u|^2 + |\nabla^2 d|^2) + \sigma^5(|\dot{u}|^2 + |\nabla\omega|^2 + |\nabla d_t|^2)]dx \\ &+ \int_0^t \int_{\mathbb{R}^3} [\sigma(|\dot{u}|^2 + |\nabla d_t|^2 + |\nabla\omega|^2) + \sigma^5(|\nabla\dot{u}|^2 + |\nabla^2 d_t|^2)]dxds,\end{aligned} \tag{3.1}$$

where $\sigma(t) = \min\{1, t\}$, and we obtain the following a priori bound for $A(t)$ under the assumptions that the initial energy C_0 in (1.14) is small enough and that the density remains bounded above and below away from zero.

Proposition 3.1. *Assume that the system parameters in (1.1) satisfy the conditions in (1.6)–(1.9) and let positive numbers N and $b < \delta$ be given. Assume (ρ, u, d) is a solution of (1.1) on $\mathbb{R}^3 \times [0, T]$ in the sense of Proposition 2.1 with initial data $(\rho_0 - \tilde{\rho}, u_0) \in H^3(\mathbb{R}^3)$ and $d_0 \in H_n^4(\mathbb{R}^3; \mathbb{S}^2)$ satisfying (1.11)–(1.14), then there are positive constants ε, M, and θ depending on the parameters and assumptions in (1.6)–(1.9), N, and a positive lower bound for b, such that if $C_0 < \varepsilon$ and*

$$\underline{\rho} \le \rho(x, t) \le \bar{\rho} \text{ on } \mathbb{R}^3 \times [0, T],$$

then

$$A(T) \le MC_0^\theta.$$

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The proof will be given in a sequence of lemmas in which we estimate a number of auxiliary functionals. To describe these we first recall the definition of p in (1.10), which is an open condition, and which therefore allows us to choose $q \in [6, \min\{p, 12\})$ which also satisfies (1.10). Then for given (ρ, u, d) we define

$$\begin{aligned}
\bar{A}(t) = & \sup_{1\le s\le t} \int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla^2 d|^2 + |\dot{u}|^2 + |\nabla\omega|^2 + |\nabla d_t|^2)dx \\
& + \int_1^t \int_{\mathbb{R}^3}(|\dot{u}|^2 + |\nabla d_t|^2 + |\nabla\omega|^2 + |\nabla\dot{u}|^2 + |\nabla^2 d_t|^2)dxds, \\
B_q(t) = & \sup_{0\le s\le t} \int_{\mathbb{R}^3}(|u|^q + |\nabla d|^q)dx + \int_0^t \int_{\mathbb{R}^3}(|u|^{q-2}|\nabla u|^2 + |\nabla d|^{q-2}|\nabla^2 d|^2)dxds \\
& + \int_0^t \int_{\mathbb{R}^3}(|u|^{q-4}|\nabla(|u|^2)|^2 + |\nabla d|^{q-4}|\nabla(|\nabla d|^2)|^2)dxds, \\
D(t) = & \sup_{0< s\le t} \int_{\mathbb{R}^3} \sigma^5(|\nabla d|^2|\nabla^2 d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla u|^2|\nabla d|^2)dx, \\
\bar{D}(t) = & \sup_{1\le s\le t} \int_{\mathbb{R}^3} (|\nabla d|^2|\nabla^2 d|^2 + |u|^2|\nabla^2 d|^2 + |\nabla u|^2|\nabla d|^2)dx, \\
E(t) = & \int_0^t \int_{\mathbb{R}^3}[\sigma^{\frac{3}{2}}(|\nabla u|^3 + |\nabla^2 d|^3) + \sigma^5(|\nabla u|^4 + |\nabla^2 d|^4)]dxds \\
& + \Big|\sum_{1\le k_i, j_m\le 3} \int_0^t \int_{\mathbb{R}^3} \sigma u^{j_1}_{x_{k_1}} u^{j_2}_{x_{k_2}} u^{j_3}_{x_{k_3}} dxds\Big|,
\end{aligned}$$

and

$$\bar{E}(t) = \int_1^t \int_{\mathbb{R}^3} (|\nabla u|^3 + |\nabla d|^2|\nabla^2 d|^2 + |\nabla u|^4 + |\nabla^2 d|^4)dxds.$$

It will be seen that the assumed regularity (2.6)–(2.9) suffices to justify the estimates that follow. We begin with the following L^2 energy estimate.

Lemma 3.1. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then*

$$\begin{aligned}
& \sup_{0\le t\le T} \int_{\mathbb{R}^3}(|\rho - \tilde{\rho}|^2 + |u|^2 + |\nabla d|^2)dx \\
& + \int_0^T \int_{\mathbb{R}^3}(|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2)dxdt \le MC_0.
\end{aligned} \tag{3.2}$$

Proof. Multiplying $(1.1)_2$ by u and integrating over $\mathbb{R}^3$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}\rho|u^2|dx + \int_{\mathbb{R}^3} \nabla P(\rho)udx + \int_{\mathbb{R}^3} (\mu|\nabla u|^2 + \lambda|\mathrm{div}u|^2)dx = -\int_{\mathbb{R}^3} u\cdot(\nabla d\Delta d)dx. \tag{3.3}$$

By the mass equation $(1.1)_1$ and the definition of $G(\rho)$ in (1.15), we have

$$G(\rho)_t + \mathrm{div}(G(\rho)u) + (P(\rho) - \tilde{P})\mathrm{div}u = 0.$$

Integrating and adding the result to (3.3) we obtain

$$\frac{d}{dt}\int\limits_{\mathbb{R}^3}\frac{1}{2}\rho|u^2|+G(\rho)dx+\int\limits_{\mathbb{R}^3}(\mu|\nabla u|^2+\lambda|\mathrm{div}u|^2)dx=-\int\limits_{\mathbb{R}^3}u\cdot(\nabla d\Delta d)dx. \tag{3.4}$$

Multiplying $(1.1)_3$ by $\Delta d+|\nabla d|^2 d$ and integrating over $\mathbb{R}^3$, using integration by parts and the fact that $|d|=1$ we obtain

$$\frac{d}{dt}\int\limits_{\mathbb{R}^3}\frac{1}{2}|\nabla d|^2dx+\int\limits_{\mathbb{R}^3}|\Delta d+|\nabla d|^2d|^2dx=\int\limits_{\mathbb{R}^3}u\cdot(\nabla d\Delta d)dx. \tag{3.5}$$

Adding (3.4) to (3.5) and integrating over $[0,t]$, yields (3.2) by (1.16). Thus the proof of lemma is completed. □

Lemma 3.2. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for* $0<t\le 1\wedge T$

$$\begin{aligned}&\sup_{0<s\le t}\sigma\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla^2 d|^2)dx+\int_0^t\int_{\mathbb{R}^3}\sigma(|\dot{u}|^2+|\nabla d_t|^2)dxds\\&\le M[C_0+C_0^{\frac{q-4}{q-2}}B_q^{\frac{2}{q-2}}+C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}}+E],\end{aligned} \tag{3.6}$$

and if $T>1$ *and* $1\le t\le T$, *then*

$$\begin{aligned}&\sup_{1\le s\le t}\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla^2 d|^2)dx+\int_1^t\int_{\mathbb{R}^3}(|\dot{u}|^2+|\nabla d_t|^2)dxds\\&\le M[C_0+C_0^{\frac{3}{2}}\bar{A}^{\frac{1}{2}})+\bar{E}]+A(1),\end{aligned} \tag{3.7}$$

where $1\wedge T=\min\{1,T\}$.

Proof. For $0\le t\le 1\wedge T$, multiplying the equation $(1.1)_2$ by $\sigma\dot{u}$ and integrating over $\mathbb{R}^3\times[0,t]$, we have

$$\begin{aligned}&\sup_{0<s\le t}\sigma\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_0^t\int_{\mathbb{R}^3}\sigma|\dot{u}|^2dxds\\&\le M\{C_0+\left|\int_0^t\int_{\mathbb{R}^3}\sigma[\dot{u}(\mathrm{div}(\nabla d\odot\nabla d)-\tfrac{1}{2}\nabla|\nabla d|^2)dxds\right|+E\}.\end{aligned} \tag{3.8}$$

Differentiating $(1.1)_3$ with respect to x, we have

$$\nabla d_t-\nabla\Delta d=\nabla(|\nabla d|^2d-u\cdot\nabla d). \tag{3.9}$$

Multiplying the above equation by $\sigma\nabla d_t$ and integrating over $\mathbb{R}^3\times[0,t]$, we have

$$\begin{aligned}&\tfrac{1}{2}\sigma\int_{\mathbb{R}^3}|\Delta d|^2dx+\int_0^t\int_{\mathbb{R}^3}\sigma|\nabla d_t|^2dxds\\&=\tfrac{1}{2}\int_0^t\int_{\mathbb{R}^3}\sigma'|\Delta d|^2dxds+\int_0^t\int_{\mathbb{R}^3}\sigma\nabla d_t\nabla(|\nabla d|^2d-u\cdot\nabla d)dxds.\end{aligned}$$

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Adding this to (3.8) and combining with Cauchy's inequality we then get

$$\begin{aligned}
&\sup_{0<s\le t}\sigma\int_{\mathbb{R}^3}(|\nabla u|^2+|\Delta d|^2)dx+\int_0^t\int_{\mathbb{R}^3}\sigma(|\dot{u}|^2+|\nabla d_t|^2)dxds\\
\le\ & M\{C_0+E+\int_0^t\int_{\mathbb{R}^3}\sigma'|\Delta d|^2dxds \qquad (3.10)\\
&+\int_0^t\int_{\mathbb{R}^3}[\sigma(|\nabla^2 d|^2|\nabla d|^2+|\nabla u|^2|\nabla d|^2+|\nabla^2 d|^2|u|^2)]dxds\}.
\end{aligned}$$

The right terms can be estimated as follows:

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^3}\sigma'|\Delta d|^2dxds &\le M\int_0^t\int_{\mathbb{R}^3}|\Delta d+|\nabla d|^2d|^2+|\nabla d|^4dxds\\
&\le M[C_0+(\int_0^t\int_{\mathbb{R}^3}|\nabla d|^2dxds)^{\frac{q-4}{q-2}}(\int_0^t\int_{\mathbb{R}^3}|\nabla d|^qdxds)^{\frac{2}{q-2}}]\\
&\le M(C_0+C_0^{\frac{q-4}{q-2}}B_q^{\frac{2}{q-2}}),
\end{aligned}$$

where we used the fact $|d|=1$. By Young's inequality, we have

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^3}\sigma|\nabla u|^2|\nabla d|^2dxds &\le M(\int_0^t\int_{\mathbb{R}^3}\sigma^{\frac{3}{2}}|\nabla u|^3dxds+\int_0^t\int_{\mathbb{R}^3}|\nabla d|^6dxds)\\
&\le M[E+(\int_0^t\int_{\mathbb{R}^3}|\nabla d|^2dxds)^{\frac{q-6}{q-2}}(\int_0^t\int_{\mathbb{R}^3}|\nabla d|^qdxds)^{\frac{4}{q-2}}]\\
&\le M(E+C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}}).
\end{aligned}$$

The other two terms in the integral on the right side of (3.10) are bounded in a similar way, and (3.6) follows.

For $1\le t\le T$, as in (3.10), we have

$$\begin{aligned}
&\sup_{1\le s\le t}\int_{\mathbb{R}^3}(|\nabla u|^2+|\Delta d|^2)dx+\int_1^t\int_{\mathbb{R}^3}(|\dot{u}|^2+|\nabla d_t|^2)dxds\\
\le\ & M\{C_0+\bar{E}+\int_1^t\int_{\mathbb{R}^3}[(|\nabla^2 d|^2|\nabla d|^2+|\nabla u|^2|\nabla d|^2+|\nabla^2 d|^2|u|^2)]dxds\}+A(1).
\end{aligned} \tag{3.11}$$

Using the fact that

$$|\nabla d|^2=-d\Delta d\ (\text{since } |d|=1), \tag{3.12}$$

the right terms can be bounded as follows:

$$\begin{aligned}
\int_1^t\int_{\mathbb{R}^3}|\nabla u|^2|\nabla d|^2dxds &\le M\int_1^t\int_{\mathbb{R}^3}|\nabla u|^3+|\nabla d|^6dxds\\
&\le M\int_1^t\int_{\mathbb{R}^3}|\nabla u|^3+|\nabla d|^2|\nabla^2 d|^2dxds\\
&\le M\bar{E},\\
\int_1^t\int_{\mathbb{R}^3}|\nabla^2 d|^2|u|^2dxds &\le \int_1^t\int_{\mathbb{R}^3}|\nabla^2 d|^4+|u|^4dxds\\
&\le \bar{E}+\int_1^t(\int_{\mathbb{R}^3}|u|^2dx)^{\frac{1}{2}}(\int_{\mathbb{R}^3}|\nabla u|^2dx)^{\frac{3}{2}}ds\\
&\le M(\bar{E}+C_0^{\frac{3}{2}}\bar{A}^{\frac{1}{2}}).
\end{aligned}$$

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Taking the above results into (3.11), then (3.7) follows. Thus the proof of lemma is completed. □

Next we derive preliminary bounds for $\dot{u}$ and ∇d_t in $L^\infty([0,T];L^2(\mathbb{R}^3))$.

Lemma 3.3. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for $0<t\le 1\wedge T$,*

$$\begin{aligned}
&\sup_{0<s\le t}\sigma^5\int_{\mathbb{R}^3}(|\dot{u}|^2+|\nabla d_t|^2)dx+\int_0^t\int_{\mathbb{R}^3}\sigma^5(|\nabla\dot{u}|^2+|\nabla^2 d_t|^2)dxds\\
&\le M[C_0+E+C_0^{\frac{q-4}{q-2}}B_q^{\frac{2}{q-2}}+C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}}+C_0^{\frac{q-4}{q-2}}B_q^{\frac{2}{q-2}}(E+C_0^{\frac{q-4}{q-2}}B_q^{\frac{2}{q-2}})\\
&\quad+(C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}})^{\frac{1}{3}}A],
\end{aligned}\tag{3.13}$$

and if $T>1$ and $1\le t\le T$, then

$$\begin{aligned}
&\sup_{1\le s\le t}\int_{\mathbb{R}^3}(|\dot{u}|^2+|\nabla d_t|^2)dx+\int_1^t\int_{\mathbb{R}^3}(|\nabla\dot{u}|^2+|\nabla^2 d_t|^2)dxds\\
&\le M\{C_0+C_0\bar{A}\bar{E}+C_0^{\frac{2(q-3)}{3(q-2)}}\bar{B}_q^{\frac{2}{3(q-2)}}\bar{A}+\bar{E}\}+A(1).
\end{aligned}\tag{3.14}$$

Proof. By the definition of material derivative, we can rewrite (1.2) as follows,

$$\rho\dot{u}+\nabla(P(\rho))=\mu\Delta u+\lambda\nabla\mathrm{div}u-\nabla d\Delta d.\tag{3.15}$$

Differentiation (3.15) with respect to t and using (1.1), we have

$$\begin{aligned}
&\rho\dot{u}_t+\rho u\cdot\nabla\dot{u}+\nabla(P(\rho)_t)+(\nabla d\Delta d)_t\\
&=\mu\Delta\dot{u}+\lambda\nabla\mathrm{div}\dot{u}-[\mu\Delta(u\cdot\nabla u)+\lambda\nabla\mathrm{div}(u\cdot\nabla u)]\\
&\quad+\mathrm{div}[(\mu\Delta u+\lambda\nabla\mathrm{div}u)\otimes u-\nabla P(\rho)\otimes u-(\nabla d\Delta d)\otimes u].
\end{aligned}\tag{3.16}$$

Multiplying (3.16) by $\sigma^5\dot{u}$, and integrating over $\mathbb{R}^3\times[0,t]$, we obtain that for $0\le t\le 1\wedge T$,

$$\begin{aligned}
&\sup_{0<s\le t}\sigma^5\int_{\mathbb{R}^3}|\dot{u}|^2dx+\int_0^t\int_{\mathbb{R}^3}\sigma^5|\nabla\dot{u}|^2dxds\\
&\le M[C_0+E+C_0^{\frac{q-4}{q-2}}B_q^{\frac{2}{q-2}}+C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}}\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\sigma^5|\nabla d|^2(|\nabla d_t|^2+|u|^2|\nabla^2 d|^2)dxds].
\end{aligned}\tag{3.17}$$

Next we differentiate (3.9) with respect to t, multiply by $\sigma^5\nabla d_t$ and integrate over $\mathbb{R}^3\times[0,t]$ to obtain

$$\begin{aligned}
&\frac{1}{2}\sigma^5\int_{\mathbb{R}^3}|\nabla d_t|^2dx+\int_0^t\int_{\mathbb{R}^3}\sigma^5|\nabla^2 d_t|^2dxds\\
&=\frac{5}{2}\int_0^t\int_{\mathbb{R}^3}\sigma^4\sigma'|\nabla d_t|^2dxds+\int_0^t\int_{\mathbb{R}^3}\sigma^5\nabla(|\nabla d|^2 d-u\cdot\nabla d)_t\nabla d_t dxds.
\end{aligned}$$

Adding this to (3.17), integrating by parts, using Cauchy's inequality and Lemma 3.2, we then have

$$\begin{aligned}
&\sup_{0<s\le t} \sigma^5 \int_{\mathbb{R}^3} |\dot{u}|^2 + |\nabla d_t|^2 dx + \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla \dot{u}|^2 + |\nabla^2 d_t|^2) dxds \\
\le\ & M[C_0 + E + C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}} + C_0^{\frac{q-6}{q-2}} B_q^{\frac{4}{q-2}} \\
&+ \int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2 |u|^2 (|\nabla u|^2 + |\nabla^2 d|^2) + \sigma^5 |\nabla d|^4 |d_t|^2 dxds \\
&+ \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^2 |\dot{u}|^2 + |\nabla d|^2 |\nabla d_t|^2 + |\nabla d_t|^2 |u|^2) dxds].
\end{aligned} \tag{3.18}$$

By (2.2), the terms on right side can be bounded by

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2 |u|^2 |\nabla u|^2 dxds \\
\le\ & \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^8 + |u|^8 + |\nabla u|^4) dxds \\
\le\ & E + \sup_{0\le s\le t} \|(u, \nabla d)\|_{L^4(\mathbb{R}^3)}^4 \int_0^t \sigma^5 \|(u, \nabla d)\|_{L^\infty(\mathbb{R}^3)}^4 ds \\
\le\ & E + M[C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}][E + C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}],
\end{aligned}$$

where $\|(f, g)\|_X$ denotes $\|f\|_X + \|g\|_X$. By $(1.1)_3$ and (3.12), we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^4 |d_t|^2 dxds \\
\le\ & M \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^4 |\nabla^2 d|^2 + |\nabla d|^4 |u|^2 |\nabla d|^2 + |\nabla d|^8) dxds \\
\le\ & M \int_0^t \int_{\mathbb{R}^3} \sigma^5 (|\nabla d|^4 |\nabla^2 d|^2 + |\nabla^2 d|^2 |u|^2 |\nabla d|^2) dxds \\
\le\ & ME + M[C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}][E + C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}].
\end{aligned}$$

The last term on the right side in (3.18) can be bounded by

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\nabla d|^2 |\dot{u}|^2 dxds \\
\le\ & \left(\int_0^t \int_{\mathbb{R}^3} |\nabla d|^6 dxds\right)^{\frac{1}{3}} \left(\int_0^t \int_{\mathbb{R}^3} \sigma^{\frac{15}{2}} |\dot{u}|^3 dxds\right)^{\frac{2}{3}} \\
\le\ & (C_0^{\frac{q-6}{q-2}} B_q^{\frac{4}{q-2}})^{\frac{1}{3}} \left(\int_0^t \sigma^{\frac{15}{2}} \|\dot{u}\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|\nabla \dot{u}\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} ds\right)^{\frac{2}{3}} \\
\le\ & (C_0^{\frac{q-6}{q-2}} B_q^{\frac{4}{q-2}})^{\frac{1}{3}} \left(\int_0^t \sigma^{15} \|\dot{u}\|_{L^2(\mathbb{R}^3)}^6 dxds\right)^{\frac{1}{6}} \left(\int_0^t \sigma^5 \|\nabla \dot{u}\|_{L^2(\mathbb{R}^3)}^2 dxds\right)^{\frac{1}{2}} \\
\le\ & (C_0^{\frac{q-6}{q-2}} B_q^{\frac{4}{q-2}})^{\frac{1}{3}} A.
\end{aligned}$$

The other integrals on the right side of (3.18) are bounded in a similar way, and (3.13) follows.

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For $1 \le t \le T$, as in (3.18), we have

$$\begin{aligned}
&\sup_{1\le s\le t} \int_{\mathbb{R}^3} |\dot{u}|^2 + |\nabla d_t|^2 dx + \int_1^t \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |\nabla^2 d_t|^2) dxds \\
\le\ & M[\int_1^t \int_{\mathbb{R}^3} |\nabla d|^2 |u|^2 (|\nabla u|^2 + |\nabla^2 d|^2) + |\nabla d|^4 |d_t|^2 dxds \\
&+ \int_1^t \int_{\mathbb{R}^3} (|\nabla d|^2 |\dot{u}|^2 + |\nabla d|^2 |\nabla d_t|^2 + |\nabla d_t|^2 |u|^2) dxds \\
&+ C_0 + \bar{E}] + A(1).
\end{aligned} \tag{3.19}$$

The terms on the right can be bounded by

$$\begin{aligned}
&\int_1^t \int_{\mathbb{R}^3} |\nabla d|^2 |u|^2 |\nabla u|^2 dxds \\
\le\ & \int_1^t \int_{\mathbb{R}^3} (|\nabla d|^8 + |u|^8 + |\nabla u|^4) dxds \\
\le\ & \bar{E} + \sup_{1\le s\le t} \|(u, \nabla d)\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\mathbb{R}^3)} \int_1^t \|(u, \nabla d)\|^{\frac{14}{3}}_{L^\infty(\mathbb{R}^3)} ds \\
\le\ & \bar{E} + \sup_{1\le s\le t} [\|(u, \nabla d)\|^{\frac{4}{3}}_{L^2(\mathbb{R}^3)} \|\nabla (u, \nabla d)\|^2_{L^2(\mathbb{R}^3)}] \\
&\times \int_1^t \|(u, \nabla d)\|^{\frac{2}{3}}_{L^2(\mathbb{R}^3)} \|\nabla (u, \nabla d)\|^4_{L^4(\mathbb{R}^3)} ds \\
\le\ & \bar{E} + MC_0 \bar{A} \bar{E}.
\end{aligned}$$

By $(1.1)_3$ and (3.12), we have

$$\begin{aligned}
&\int_1^t \int_{\mathbb{R}^3} |\nabla d|^4 |d_t|^2 dxds \\
\le\ & M \int_1^t \int_{\mathbb{R}^3} (|\nabla d|^4 |\nabla^2 d|^2 + |\nabla d|^4 |u|^2 |\nabla d|^2 + |\nabla d|^8) dxds \\
\le\ & M \int_1^t \int_{\mathbb{R}^3} (|\nabla d|^4 |\nabla^2 d|^2 + |\nabla^2 d|^2 |u|^2 |\nabla d|^2) dxds \\
\le\ & M\bar{E} + MC_0 \bar{A} \bar{E}.
\end{aligned}$$

The last term on the right side in (3.19) can be bounded by

$$\begin{aligned}
&\int_1^t \int_{\mathbb{R}^3} |\nabla d|^2 |\dot{u}|^2 dxds \\
\le\ & \int_1^t (\int_{\mathbb{R}^3} |\nabla d|^3 dx)^{\frac{2}{3}} (\int_{\mathbb{R}^3} |\dot{u}|^6 dx)^{\frac{1}{3}} ds \\
\le\ & \sup_{1\le s\le t} \int_{\mathbb{R}^3} |\nabla d|^3 dx)^{\frac{2}{3}} \int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dxds \\
\le\ & C_0^{\frac{2(q-3)}{3(q-2)}} \bar{B}_q^{\frac{2}{3(q-2)}} \bar{A}.
\end{aligned}$$

The other integrals on the right side of (3.19) are bounded in a similar way, and (3.14) follows. Thus the proof of lemma is completed. □

Next we derive a number of auxiliary estimates needed to close the bounds in the previous two lemmas. We begin with a bound for the vorticity ω.

Lemma 3.4. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for* $0 < t \le 1 \wedge T$,

$$\begin{aligned} &\sup_{0<s\le t} \int_{\mathbb{R}^3} \sigma^5(|\nabla F|^2 + |\nabla \omega|^2)dx + \int_0^t \int_{\mathbb{R}^3} \sigma(|\nabla F|^2 + |\nabla \omega|^2)dxds \\ &\le M(D + E^{\frac{2}{3}} C_0^{\frac{q-6}{3q-6}} B_q^{\frac{4}{3q-4}} + \sup_{0<s\le t} \int_{\mathbb{R}^3} \sigma^5|\dot{u}|^2dx + \int_0^t \int_{\mathbb{R}^3} \sigma|\dot{u}|^2dxds), \end{aligned} \tag{3.20}$$

and if $T > 1$ *and* $1 \le t \le T$, *then*

$$\begin{aligned} &\sup_{1\le s\le t} \int_{\mathbb{R}^3} (|\nabla F|^2 + |\nabla \omega|^2)dx + \int_1^t \int_{\mathbb{R}^3} (|\nabla F|^2 + |\nabla \omega|^2)dxds \\ &\le M(\bar{D} + \bar{E} + \sup_{1\le s\le t} \int_{\mathbb{R}^3} |\dot{u}|^2dx + \int_1^t \int_{\mathbb{R}^3} |\dot{u}|^2dxds). \end{aligned} \tag{3.21}$$

Proof. By (2.11) and the definition of D, E, we can easily get (3.20) and (3.21). The proof of lemma is completed. □

Next we derive an estimate for the functional B_q.

Lemma 3.5. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for any* $0 < t \le T$,

$$B_q \le M(C_0^{\frac{p-q}{p-2}} N^{\frac{q-2}{p-2}} + C_0^{\frac{q+3}{3(q-2)}} B_q^{\frac{3q-11}{3(q-2)}} + C_0^{\frac{q-3}{3(q-2)}} B_q^{\frac{3q-5}{3(q-2)}}). \tag{3.22}$$

Proof. We multiply $(1.1)_2$ by $|u|^{q-2}u$ and integrate over $\mathbb{R}^3 \times (0, t)$ to obtain that

$$\begin{aligned} &q^{-1} \int_{\mathbb{R}^3} \rho|u|^q dx|_0^t + \int_0^t \int_{\mathbb{R}^3} \mu|u|^{q-2}|\nabla u|^2dxds \\ &+ \int_0^t \int_{\mathbb{R}^3} [\tfrac{1}{4}\mu(q-2)|u|^{q-4}|\nabla(|u|^2)|^2 + \lambda|u|^{q-2}(\mathrm{div} u)^2]dxds \\ &= \int_0^t \int_{\mathbb{R}^3} [(P(\rho) - \tilde{P}(\rho))\mathrm{div}(|u|^{q-2}u) - |u|^{q-2}u \cdot (\nabla d \Delta d)]dxds \\ &- \int_0^t \int_{\mathbb{R}^3} \tfrac{1}{2}\lambda(q-2)|u|^{q-4}(\mathrm{div} u)u \cdot \nabla(|u|^2)dxds. \end{aligned} \tag{3.23}$$

For any $\eta > 0$,

$$\begin{aligned} &|-\int_0^t \int_{\mathbb{R}^3} \tfrac{1}{2}\lambda(q-2)|u|^{q-4}(\mathrm{div} u)u \cdot \nabla(|u|^2)dxds| \\ &\le \tfrac{1}{2}\lambda(q-2) \int_0^t \int_{\mathbb{R}^3} |u|^{\frac{q-2}{2}}|(\mathrm{div} u)||u|^{\frac{q-4}{2}}|\nabla(|u|^2)|dxds| \\ &\le \tfrac{1}{4}\lambda(q-2)[\eta \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|(\mathrm{div} u)|^2dxds + \eta^{-1} \int_0^t \int_{\mathbb{R}^3} |u|^{q-4}|\nabla(|u|^2)|^2dxds], \end{aligned}$$

so if we choose

$$\frac{1}{4}\lambda(q-2)\eta = \beta\mu + \lambda$$

for a positive β to be determined, then the term in question will be bounded by

$$3\beta\mu \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 dxds + \lambda \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\mathrm{div} u|^2 dxds$$
$$+\frac{[\frac{1}{4}\lambda(q-1)]^2}{\beta\mu+\lambda} \int_0^t \int_{\mathbb{R}^3} |u|^{q-4}|\nabla(|u|^2)|^2 dxds.$$

Substituting this into (3.23), we then get

$$\begin{aligned}&q^{-1} \int_{\mathbb{R}^3} \rho|u|^q dx|_0^t + \mu(1-3\beta) \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 dxds\\&+[\tfrac{1}{4}\mu(q-2) + \frac{[\frac{1}{4}\lambda(q-1)]^2}{\beta\mu+\lambda}] \int_0^t \int_{\mathbb{R}^3} |u|^{q-4}|\nabla(|u|^2)|^2 dxds\\\le\ & |\int_0^t \int_{\mathbb{R}^3} (P(\rho) - \tilde{P}(\rho))\mathrm{div}(|u|^{q-2}u) dxds|\\&+|\int_0^t \int_{\mathbb{R}^3} |u|^{q-2}u(\nabla d\Delta d) dxds|.\end{aligned}$$

Recall that $q \in [6, p)$, thus (1.10) holds with p replaced by q, and this is the condition that brackets on the left here is positive when $\beta = \frac{1}{3}$. It follows this term is positive for some $\beta \in (0, \frac{1}{3})$, which we now fix. It then follows that

$$\begin{aligned}&q^{-1} \int_{\mathbb{R}^3} \rho|u|^q dx\Big|_0^t + \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 dxds + \int_0^t \int_{\mathbb{R}^3} |u|^{q-4}|\nabla(|u|^2)|^2 dxds\\\le\ & M[|\int_0^t \int_{\mathbb{R}^3} (P(\rho) - \tilde{P}(\rho))\mathrm{div}(|u|^{q-2}u) dxds| + |\int_0^t \int_{\mathbb{R}^3} |u|^{q-2}u(\nabla d\Delta d) dxds|].\end{aligned} \tag{3.24}$$

We multiply (3.9) by $|\nabla d|^{q-2}\nabla d$ and integrate over $\mathbb{R}^3 \times (0, t)$ to obtain that

$$\begin{aligned}&q^{-1} \int_{\mathbb{R}^3} |\nabla d|^q dx\Big|_0^t + \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-2}|\nabla^2 d|^2 dxds\\&+\int_0^t \int_{\mathbb{R}^3} \tfrac{1}{4}(q-2)|\nabla d|^{q-4}|\nabla(|\nabla d|^2)|^2 dxds\\=\ & \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-2}\nabla d\nabla(|\nabla d|^2 d - u\cdot\nabla d) dxds.\end{aligned} \tag{3.25}$$

Adding (3.25) to (3.24) and applying the Cauchy's inequality in an elementary way we then obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} |u|^q + |\nabla d|^q dx + \int_0^t \int_{\mathbb{R}^3} |u|^{q-2}|\nabla u|^2 dxds + \int_0^t \int_{\mathbb{R}^3} |u|^{q-4}|\nabla(|u|^2)|^2 dxds\\&+\int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-2}|\nabla^2 d|^2 dxds + \int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-4}|\nabla(|\nabla d|^2)|^2 dxds\\\le\ & M[\int_{\mathbb{R}^3} |u_0|^q + |\nabla d_0|^q dx + |\int_0^t \int_{\mathbb{R}^3} (P(\rho) - \tilde{P}(\rho))\mathrm{div}(|u|^{q-2}u) dxds|\\&+|\int_0^t \int_{\mathbb{R}^3} |u|^{q-2}u(\nabla d\Delta d) dxds| + |\int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q-2}\nabla d\nabla(|\nabla d|^2 d - u\cdot\nabla d) dxds|]\\=\ & \sum_{i=1}^4 I_i.\end{aligned} \tag{3.26}$$

Since $q \in [6, \min\{p, 12\})$, then by Hölder's inequality and Sobolev's inequality, we have

$$I_1 \le (\int_{\mathbb{R}^3} |u_0|^2 + |\nabla d_0|^2 dx)^{\frac{p-q}{p-2}} (\int_{\mathbb{R}^3} |u_0|^p + |\nabla d_0|^p dx)^{\frac{q-2}{p-2}} \le M C_0^{\frac{p-q}{p-2}} N^{\frac{q-2}{p-2}},$$

and

$$\begin{aligned}
I_2 &\le [\int_0^t \int_{\mathbb{R}^3} |u|^{2q-4} dxds]^{\frac{1}{2}} [\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxds]^{\frac{1}{2}} \\
&\le C_0^{\frac{1}{2}} [\int_0^t (\int_{\mathbb{R}^3} |u|^{3q} dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |u|^{\frac{3}{2}(q-4)} dx)^{\frac{2}{3}} ds]^{\frac{1}{2}} \\
&\le C_0^{\frac{1}{2}} [\int_0^t (\int_{\mathbb{R}^3} |u|^{q-2} |\nabla u|^2 dx)(\int_{\mathbb{R}^3} |u|^{\frac{3}{2}(q-4)} dx)^{\frac{2}{3}} ds]^{\frac{1}{2}} \\
&\le C_0^{\frac{1}{2}} B_q^{\frac{1}{2}} \sup_{0<s\le t} [\int_{\mathbb{R}^3} |u(s)|^2 dx]^{\frac{12-q}{6(q-2)}} \int_{\mathbb{R}^3} |u(s)|^q dx]^{\frac{3q-16}{6(q-2)}} \\
&\le M C_0^{\frac{q+3}{3(q-2)}} B_q^{\frac{3q-11}{3(q-2)}}.
\end{aligned}$$

Using (1.4) and integrating by parts, we obtain

$$\begin{aligned}
I_3 &\le [\int_0^t \int_{\mathbb{R}^3} (|u|^{q-2} |\nabla u|^2 + |u|^{q-4} |\nabla(|u|^2)|^2) dxds]^{\frac{1}{2}} [\int_0^t \int_{\mathbb{R}^3} |\nabla d|^4 |u|^{q-2} dxds]^{\frac{1}{2}} \\
&\le M B_q^{\frac{1}{2}} [(\int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q+2} dxds)^{\frac{1}{2}} + (\int_0^t \int_{\mathbb{R}^3} |u|^{q+2} dxds)^{\frac{1}{2}}] \\
&\le M B_q^{\frac{1}{2}} [(\int_0^t (\int_{\mathbb{R}^3} |\nabla d|^{3q} dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |\nabla d|^3 dx)^{\frac{2}{3}} ds)^{\frac{1}{2}} \\
&\quad + (\int_0^t (\int_{\mathbb{R}^3} |u|^{3q} dx)^{\frac{1}{3}} (\int_{\mathbb{R}^3} |u|^3 dx)^{\frac{2}{3}} ds)^{\frac{1}{2}}] \\
&\le M C_0^{\frac{q-3}{3(q-2)}} B_q^{\frac{3q-5}{3(q-2)}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_4 &\le M[\int_0^t \int_{\mathbb{R}^3} |\nabla^2 d| |\nabla d|^{q-2} (|\nabla d|^2 + |u| |\nabla d|^2) dxds] \\
&\le M[\int_0^t \int_{\mathbb{R}^3} (|\nabla d|^{q-2} |\nabla^2 d|^2 dxds]^{\frac{1}{2}} [\int_0^t \int_{\mathbb{R}^3} |\nabla d|^{q+2} + |\nabla d|^q |u|^2 dxds]^{\frac{1}{2}} \\
&\le M C_0^{\frac{q-3}{3(q-2)}} B_q^{\frac{3q-5}{3(q-2)}}.
\end{aligned}$$

Substituting these results into (3.26) gives (3.22). Thus the proof of lemma is completed. □

Next we derive a bound for the functional D and $\bar{D}$.

Lemma 3.6. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then for* $0 < t \le 1 \wedge T$,

$$D \le M[C_0^{\frac{q-4}{2q-4}} B_q^{\frac{1}{q-2}} A + A^2 + A^5], \tag{3.27}$$

and if $T > 1$ *and* $1 \le t \le T$*, then*

$$\bar{D} \le M[C_0^{\frac{1}{4}} \bar{A}^{\frac{7}{4}} + \bar{A}^2 + \bar{A}^5]. \tag{3.28}$$

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Proof. We give the proof of (3.27), that of (3.28) being similar. By Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sigma^5 |\nabla^2 d|^2 |\nabla d|^2 dx \\
\le\ & \sigma^4 \|\nabla d\|^2_{L^\infty(\mathbb{R}^3)} [\sigma \|\nabla^2 d\|^2_{L^2(\mathbb{R}^3)}] \\
\le\ & MA[\sigma^4 \|\nabla d\|^2_{L^4(\mathbb{R}^3)} + \sigma^4 \|\nabla^2 d\|^2_{L^4(\mathbb{R}^3)}] \\
\le\ & MA[C_0^{\frac{q-4}{2q-4}} B_q^{\frac{1}{q-2}} + \|\sigma^{\frac12} \nabla^2 d\|^{\frac12}_{L^2(\mathbb{R}^3)} \|\sigma^{\frac52} \nabla^3 d\|^{\frac32}_{L^2(\mathbb{R}^3)}] \\
\le\ & MA[C_0^{\frac{q-4}{2q-4}} B_q^{\frac{1}{q-2}} + A^{\frac14} \|\sigma^{\frac52} \nabla^3 d\|^{\frac32}_{L^2(\mathbb{R}^3)}].
\end{aligned}$$

From (3.9) and (3.12), we have

$$\begin{aligned}
\|\sigma^{\frac52} \nabla^3 d\|^2_{L^2(\mathbb{R}^3)} &\le M \int_{\mathbb{R}^3} \sigma^5 (|\nabla d_t|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla u|^2 |\nabla d|^2 + |\nabla^2 d|^2 |\nabla d|^2) dx \\
&\le M(A+D),
\end{aligned}$$

so that

$$\int_{\mathbb{R}^3} \sigma^5 |\nabla^2 d|^2 |\nabla d|^2 dx \le MA[C_0^{\frac{q-4}{2q-4}} B_q^{\frac{1}{q-2}} + A^{\frac14} (A+D)^{\frac34}].$$

The other terms included in D are estimated in exactly the same way, and (3.27) follows. The proof of lemma is completed. □

The following lemma contains the required bound for the pressure term in (2.11), which has been proved in Hoff [7] (Lemma 3.3).

Lemma 3.7. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then it holds*

$$\int_0^t \int_{\mathbb{R}^3} \sigma^5 |\rho - \tilde{\rho}|^4 dx ds \le M[C_0 + \int_0^t \int_{\mathbb{R}^3} \sigma^5 |F|^4 dx ds]. \tag{3.29}$$

We can now obtain the required estimates for the functional E and $\bar{E}$.

Lemma 3.8. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then there are polynomials φ_1 and φ_2 whose degrees and coefficients depend on the same M quantities as M in the statement of Proposition 3.1 such that: for $0 < t \le 1 \wedge T$, we have*

$$E \le M[\varphi_1(C_0) + \varphi_2(A + B_q)], \tag{3.30}$$

and if $T > 1$ and $1 \le t \le T$, then

$$\bar{E} \le M[\varphi_1(C_0 + A(1) + B_q(1)) + \varphi_2(\bar{A} + B_q)]. \tag{3.31}$$

The polynomial φ_1 contains no constant term and the monomials in φ_2 all have degrees strictly greater than 1.

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Proof. Since the term $|\sum\limits_{1\le k_i,j_m\le 3}\int_0^t\int_{\mathbb{R}^3}\sigma u^{j_1}_{x_{k_1}}u^{j_2}_{x_{k_2}}u^{j_3}_{x_{k_3}}dxds|$ has been bounded exactly in Hoff [7]. So here we just bound the other terms for simplicity.

First for $0<t\le 1\wedge T$, from (2.1), (3.9) and (3.12) we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\sigma^{\frac{3}{2}}|\nabla^2 d|^3dxds\\
\le\ & M\Big(\int_0^t\int_{\mathbb{R}^3}\sigma|\nabla^2 d|dxds\Big)^{\frac{3}{4}}\Big(\int_0^t\int_{\mathbb{R}^3}\sigma|\nabla^3 d|^2dxds\Big)^{\frac{3}{4}}\\
\le\ & MA^{\frac{3}{4}}\Big(\int_0^t\int_{\mathbb{R}^3}\sigma(|\nabla d_t|^2+|\nabla d|^2|\nabla^2 d|^2+|u|^2|\nabla^2 d|^2+|\nabla d|^2|\nabla u|^2)dxds\Big)^{\frac{3}{4}}\\
\le\ & MA^{\frac{3}{4}}\Big(A+\int_0^t\Big(\int_{\mathbb{R}^3}|\nabla d|^6+|u|^6+\sigma^{\frac{3}{2}}(|\nabla^2 d|^3+|\nabla u|^3)dxds\Big)^{\frac{3}{4}}\\
\le\ & MA\Big(A+C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}}+E\Big)^{\frac{3}{4}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\sigma^5|\nabla^2 d|^4dxds\\
\le\ & M\Big(\int_0^t\sigma^5\Big(\int_{\mathbb{R}^3}|\nabla^2 d|dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\nabla^3 d|^2dx\Big)^{\frac{3}{2}}ds\\
\le\ & M\Big(\int_0^t\Big(\sigma\int_{\mathbb{R}^3}|\nabla^2 d|dx\Big)^{\frac{1}{2}}\Big(\sigma\int_{\mathbb{R}^3}|\nabla^3 d|^2dx\Big)^{\frac{3}{4}}\Big(\sigma^5\int_{\mathbb{R}^3}|\nabla^3 d|^2dx\Big)^{\frac{3}{4}}ds\\
\le\ & MA\Big(A+C_0^{\frac{q-6}{q-2}}B_q^{\frac{4}{q-2}}+E\Big)^{\frac{3}{4}}(A+D)^{\frac{3}{4}}.
\end{aligned}$$

From Lemma 2.3, Lemma 3.1, Lemma 3.4 and the definition of F,ω, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\sigma^5|\nabla u|^4dxds\\
\le\ & M\Big[\int_0^t\int_{\mathbb{R}^3}\sigma^5(|\rho-\tilde\rho|^4+|F|^4+|\omega|^4)dxds\Big]\\
\le\ & M\Big[C_0+\Big(\sup_{0\le s\le t}\int_{\mathbb{R}^3}\sigma(|F|^2+|\omega|^2)dx\int_{\mathbb{R}^3}\sigma^5(|\nabla F|^2+|\nabla\omega|^2)dx\Big)^{\frac{1}{2}}\\
&\times\Big(\int_0^t\int_{\mathbb{R}^3}\sigma(|\nabla F|^2+|\nabla\omega|^2)dx\Big]\\
\le\ & M\Big[C_0+(C_0+A)^{\frac{1}{2}}(A+D)^{\frac{1}{2}}\Big(A+E^{\frac{2}{3}}C_0^{\frac{q-6}{3q-6}}B_q^{\frac{4}{3q-4}}\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\sigma^{\frac{3}{2}}|\nabla u|^3dxds\\
\le\ & M\Big[\int_0^t\int_{\mathbb{R}^3}\sigma^{\frac{3}{2}}(|\rho-\tilde\rho|^3+|F|^3+|\omega|^3)dxds\Big]\\
\le\ & M\Big[C_0+\int_0^t\sigma^{\frac{3}{2}}\Big(\int_{\mathbb{R}^3}(|F|^2+|\omega|^2)dx\Big)^{\frac{3}{4}}\Big(\int_{\mathbb{R}^3}(|\nabla F|^2+|\nabla\omega|^2)dx\Big)^{\frac{3}{4}}ds\\
\le\ & M\Big[C_0+(C_0+A)^{\frac{3}{4}}\Big(A+E^{\frac{2}{3}}C_0^{\frac{q-6}{3q-6}}B_q^{\frac{4}{3q-4}}\Big)^{\frac{3}{4}}\Big].
\end{aligned}$$

Thus combining the above results and Lemma 3.6, we yield (3.30).

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Now for $1 \le t \le T$, if we take $q = 4$ in (3.25) and integrate by parts to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla d|^4 dx + \int_1^t \int_{\mathbb{R}^3} |\nabla d|^2 |\nabla^2 d|^2 dxds + \int_1^t \int_{\mathbb{R}^3} |\nabla(|\nabla d|^2)|^2 dxds \\
\le\ & M[\int_{\mathbb{R}^3} |\nabla d(\cdot,1)|^4 dx + |\int_1^t \int_{\mathbb{R}^3} \Delta d |\nabla d|^4 d dxds| + \int_1^t \int_{\mathbb{R}^3} |\nabla d|^3 |\nabla^2 d||u| dxds] \\
\le\ & M[C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}(1) + \int_1^t \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2 d|^2 |\nabla d|^2 dxds \\
& + (\int_0^t \int_{\mathbb{R}^3} |\nabla d|^{\frac{18}{5}} |\nabla^2 d|^{\frac{6}{5}} dxds)^{\frac{5}{6}} (\int_0^t \int_{\mathbb{R}^3} |u|^6 dxds)^{\frac{1}{6}} \\
\le\ & M[C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}(1) + \sup_{1\le s\le t} \|\nabla d\|^2_{L^\infty(\mathbb{R}^3)} \int_1^t \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2 d|^2 dxds \\
& + (\int_0^t \int_{\mathbb{R}^3} |\nabla d|^2 |\nabla^2 d|^2 dxds)^{\frac{5}{6}} (\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxds)^{\frac{1}{2}} \\
\le\ & M[C_0^{\frac{q-4}{q-2}} B_q^{\frac{2}{q-2}}(1) + C_0(C_0^{\frac{q-4}{2q-4}} B_q^{\frac{1}{q-2}} + \bar{A}^{\frac{1}{4}}(\bar{A} + \bar{D})^{\frac{3}{4}}) + C_0^{\frac{1}{2}} \bar{E}^{\frac{5}{6}}].
\end{aligned}$$

Multiplying (3.9) by $\nabla \Delta d$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla^3 d|^2 dx \le\ & \int_{\mathbb{R}^3} |\nabla d_t||\nabla^3 d| dx + 2\int_{\mathbb{R}^3} |\nabla d||\nabla^2 d||\nabla^3 d| dx \\
& + \int_{\mathbb{R}^3} \nabla u \cdot \nabla d \nabla \Delta d + u \cdot \nabla\nabla d \nabla \Delta d dx \\
=\ & \int_{\mathbb{R}^3} |\nabla d_t||\nabla^3 d| dx + 2\int_{\mathbb{R}^3} |\nabla d||\nabla^2 d||\nabla^3 d| dx \\
& + \int_{\mathbb{R}^3} \nabla u \cdot \nabla d \nabla \Delta d - \nabla u \cdot \nabla\nabla d \Delta d + \frac{1}{2}(\mathrm{div} u)|\Delta d|^2 dx.
\end{aligned}$$

By Cauchy's inequality, we have

$$\int_{\mathbb{R}^3} |\nabla^3 d|^2 dx \le \int_{\mathbb{R}^3} |\nabla d_t|^2 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla d|^2 |\nabla u|^2 + |\nabla u||\nabla^2 d|^2 dx.$$

Thus we have

$$\begin{aligned}
&\int_1^t \int_{\mathbb{R}^3} |\nabla^2 d|^4 dxds \\
\le\ & M(\int_1^t (\int_{\mathbb{R}^3} |\nabla^2 d| dx)^{\frac{1}{2}} (\int_{\mathbb{R}^3} |\nabla^3 d|^2 dx)^{\frac{3}{2}} ds \\
\le\ & M\bar{A} \int_1^t (\int_{\mathbb{R}^3} |\nabla d_t|^2 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla d|^2 |\nabla u|^2 + |\nabla u||\nabla^2 d|^2 dx) \\
& \times (\int_{\mathbb{R}^3} |\nabla d_t|^2 + |\nabla d|^2 |\nabla^2 d|^2 + |\nabla d|^2 |\nabla u|^2 + |u|^2 |\nabla^2 d|^2 dx)^{\frac{1}{2}} ds \\
\le\ & M\bar{A}(\bar{A} + \bar{D})^{\frac{1}{2}} (\int_1^t (\int_{\mathbb{R}^3} |\nabla d_t|^2 + |\nabla d|^2 |\nabla^2 d|^2 dxds \\
& + \int_1^t (\int_{\mathbb{R}^3} |\nabla d|^2 |\nabla u|^2 + |\nabla u||\nabla^2 d|^2 dxds) \\
\le\ & M\bar{A}(\bar{A} + \bar{D})^{\frac{1}{2}} (A + C_0^{\frac{q-4}{q-2}} B_q(1)^{\frac{2}{q-2}} \\
& + C_0(C_0^{\frac{q-4}{2q-4}} B_q^{\frac{1}{q-2}} + \bar{A}^{\frac{1}{4}}(\bar{A} + \bar{D})^{\frac{3}{4}}) + C_0^{\frac{1}{2}} \bar{E}^{\frac{5}{6}} + C_0^{\frac{1}{2}} \bar{E}^{\frac{1}{2}}).
\end{aligned}$$

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Bounds for the term $\int_1^t \int_{\mathbb{R}^3} (|\nabla u|^3 + |\nabla u|^4) dxds$ are obtained in a similar way, which in fact is much more simple. Then applying Lemma 3.6, we can bound $\bar{E}$ which gives (3.31). The proof of lemma is completed. □

Combining the results of Lemmas 3.1–3.8, we have the following bound for $A + B_q$.

Lemma 3.9. *Assume that the hypotheses and notations of Proposition 3.1 are in force. Then there are polynomials φ_1 and φ_2 as described in Lemma 3.8 such that for $0 < t \le 1 \wedge T$,*

$$A + B_q \le M[\varphi_1(C_0) + \varphi_2(A + B_q)], \tag{3.32}$$

and if $T > 1$ and $1 \le t \le T$, then

$$\bar{A} + B_q \le M[\varphi_1(C_0 + A(1) + B_q(1)) + \varphi_2(\bar{A} + B_q)]. \tag{3.33}$$

Proof of Proposition 3.1. Proposition now follows immediately from the bounds (3.32) and (3.33) and the fact that the functions A, $\bar{A}$, B_q are continuous in time. □

4. Pointwise bounds for the density

In this section we derive pointwise bounds for the density ρ, which are independent both of time and of initial smoothness. This will then close the estimates of Proposition 2.1 to give an uncontingent estimate for the functional A defined in (3.1).

We begin with two auxiliary lemmas. The first lemma is a maximum-principle arguments applied to integral curves of the velocity field, which has been proved in Hoff [7].

Lemma 4.1. *Let (ρ, u, d) be as in Proposition 3.1 and suppose that $0 < c_1 \le \rho \le c_2$ on $\mathbb{R}^3 \times [0, T]$. Fix $t_0 \ge 0$ and define the particle trajectories $x : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$\begin{cases} \dot{x}(t, y) = u(x(t, y), t), \\ x(t_0, y) = y. \end{cases}$$

Then there is a constant C depending only on c_1 and c_2 such that if $g \in L^1(\mathbb{R}^3)$ is nonnegative and $t \in [0, T]$, then each of the integrals $\int_{\mathbb{R}^3} g(x(t, y))dy$ and $\int_{\mathbb{R}^3} g(x)dx$ is bounded by C times the other.

Next we derive a result relating the Hölder-continuity of $u(\cdot, t)$ to various norms appearing in the definition (2.1) of the functional A.

Lemma 4.2. *Let (ρ, u, d) be as in Proposition 3.1. Then for $\alpha \in (0, \frac{1}{2}]$ and $t \in (0, T]$, we have*

$$\begin{aligned} \langle u(\cdot, t)\rangle^{\alpha} \le\ & M[\|\nabla u(\cdot, t)\|_{L^2(\mathbb{R}^3)}^{\frac{1-2\alpha}{2}} \|\nabla \omega(\cdot, t)\|_{L^2(\mathbb{R}^3)}^{\frac{1+2\alpha}{2}} + (C_0 + \|\nabla u(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1-2\alpha}{4}} \\ & \times (\|\dot{u}(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \|(\nabla d \Delta d)(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1+2\alpha}{4}} + C_0^{\frac{1-\alpha}{3}}]. \end{aligned} \tag{4.1}$$

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Proof. Let $\alpha \in (0, \frac{1}{2}]$ and define $r \in (3, 6]$ by $r = \frac{3}{1-\alpha}$. Then by (2.3) and (2.12), we have

$$\langle u(\cdot,t)\rangle^{\alpha} \leq M[\|F(\cdot,t)\|_{L^r(\mathbb{R}^3)} + \|\omega(\cdot,t)\|_{L^r(\mathbb{R}^3)} + \|(\rho-\tilde{\rho})(\cdot,t)\|_{L^r(\mathbb{R}^3)}]. \tag{4.2}$$

By (2.1), we obtain

$$\begin{aligned}\|\omega(\cdot,t)\|_{L^r(\mathbb{R}^3)} \leq\ & M\|\omega(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{\frac{6-r}{2r}}\|\nabla\omega(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{\frac{3r-6}{2r}} \\ \leq\ & M\|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{\frac{1-2\alpha}{2}}\|\nabla\omega(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{\frac{1+2\alpha}{2}},\end{aligned}$$

and

$$\begin{aligned}\|F(\cdot,t)\|_{L^r(\mathbb{R}^3)} \leq\ & M\|F(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{\frac{6-r}{2r}}\|\nabla F(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{\frac{3r-6}{2r}} \\ \leq\ & M(\|(\rho-\tilde{\rho})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1-2\alpha}{4}} \\ & \times(\|\dot{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + \|(\nabla d\Delta d)(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1+2\alpha}{4}}] \\ \leq\ & M(\|C_0 + \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1-2\alpha}{4}} \\ & \times(\|\dot{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + \|(\nabla d\Delta d)(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2)^{\frac{1+2\alpha}{4}}].\end{aligned}$$

Putting the above results into (4.2) yields (4.1). Thus the proof the lemmas is completed. □

Now we derive the upper and lower pointwise bounds for the density.

Proposition 4.1. *Assume that the system parameters in (1.1) satisfy the conditions (1.6)–(1.9) and let positive numbers N and $b \leq \delta$ be given. Assume (ρ, u, d) is a solution of (1.1) on $\mathbb{R}^3 \times [0, T]$ in the sense of Proposition 2.1 with initial data (ρ_0, u_0, d_0) satisfying (1.11)–(1.14), (2.5) and (2.6). Then there are positive constants ε, M, and θ depending on the parameters and assumptions in (1.6)–(1.9), N, and a positive lower bound for b, such that, if $C_0 < \varepsilon$ and $\rho(x,t) > 0$ on $\mathbb{R}^3 \times [0, T]$, then in fact*

$$\underline{\rho} \leq \rho \leq \bar{\rho} \quad on \quad \mathbb{R}^3 \times [0, T], \tag{4.3}$$

and

$$A(T) \leq MC_0^{\theta}. \tag{4.4}$$

Proof. First we choose positive numbers κ and κ' satisfying

$$\underline{\rho} < \kappa < \underline{\rho} + b < \bar{\rho} - b < \kappa' < \bar{\rho}.$$

Recall that ρ_0 takes values in $[\underline{\rho} + b, \bar{\rho} - b]$, so that $\rho \in [\underline{\rho}, \bar{\rho}]$ on $\mathbb{R}^3 \times [0, \tau]$ for some positive τ by the time regularity (2.7). It then follows from Proposition 3.1 that $A(\tau) \leq MC_0^{\theta}$, where M is now fixed. We shall that if C_0 is further restricted, then in fact that $\kappa < \rho < \kappa'$ on all of

$\mathbb{R}^3 \times [0, T]$, and therefore that $A(T) \le MC_0^\theta$ as well. We shall prove the required upper bound, the proof of the lower bound being similar.

For $y \in \mathbb{R}^3$ and define the corresponding particle path $x(t)$ by

$$\begin{cases} \dot{x}(t, y) = u(x(t, y), t), \\ x(t_0, y) = y. \end{cases}$$

Suppose that there is a time $t_1 \le \tau$ such that $\rho(x(t_1), t_1) = \kappa'$. We may take t_1 minimal and then choose $t_0 < t_1$ maximal such that $\rho(x(t_0), t_0) = \bar{\rho} - b$. Thus $\rho(x(t), t) \in [\bar{\rho} - b, \kappa']$ for $t \in [t_0, t_1]$. We divide into two cases:

Case 1: $t_0 < t_1 \le T \wedge 1$

We have from the definition (1.3) of F and the mass equation that

$$(\mu + \lambda)\frac{d}{dt}[\log \rho(x(t), t) - \log \tilde{\rho}] + P(\rho(x(t), t)) - P(\tilde{\rho}) = -F(x(t), t).$$

Integrating from t_0 to t_1 and abbreviating $\rho(x(t), t)$ by $\rho(t)$, etc., we then obtain

$$(\mu + \lambda) \log \rho(s)|_{t_0}^{t_1} + \int_{t_0}^{t_1} [P(s) - P(\tilde{\rho})]ds = -\int_{t_0}^{t_1} F(s)ds. \tag{4.5}$$

We shall show that

$$\Big|\int_{t_0}^{t_1} F(s)ds\Big| \le \tilde{M}C_0^\theta \tag{4.6}$$

for a constant $\tilde{M}$ which depends on the same quantities as the M from Proposition 3.1 (which has been fixed). If so, then from (4.5), we have

$$(\mu + \lambda)[\log \kappa' - \log(\bar{\rho} - b)] \le -\int_{t_0}^{t_1} [P(s) - P(\tilde{\rho})]ds + \tilde{M}C_0^\theta \le \tilde{M}C_0^\theta, \tag{4.7}$$

where the last inequality holds because $\rho(t)$ takes values in $[\bar{\rho} - b, \kappa'] \subset [\tilde{\rho}, \bar{\rho}]$, and P is increasing on $[\tilde{\rho}, \bar{\rho}]$. But (4.7) cannot hold if C_0 is small depending on $\tilde{M}, \kappa'$, and $\bar{\rho} - b$. Stipulating the smallness condition, we therefore conclude that there is no time t_1 such that $\rho(t_1) = \rho(x(t_1), t_1) = \kappa'$. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $\rho < \kappa'$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed. The proof that $\rho > \kappa$ is similar.

To prove (4.6) we let Γ be the fundamental solution of the Laplace operator in $\mathbb{R}^3$ and apply (1.5) to rewrite

$$\begin{aligned} \int_{t_0}^{t_1} F(s)ds = &\int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\nabla_x \Gamma(x(s) - y))\rho\dot{u}(y, s)dyds \\ &+ \int_{t_0}^{t_1} \int_{\mathbb{R}^3} (\nabla_x \Gamma(x(s) - y))(\nabla d \Delta d)(y, s)dyds. \end{aligned} \tag{4.8}$$

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We note the first integral on the right side of (4.8) is bounded exactly as in Hoff [7] (Lemma 4.2)

$$
\begin{aligned}
&\Big|\int_{t_0}^{t_1}\int_{\mathbb{R}^3}(\nabla_x\Gamma(x(s)-y))\rho\dot{u}(y,s)dyds\Big|\\
&\le \|\nabla\Gamma*(\rho u)(\cdot,t_1)\|_{L^\infty(\mathbb{R}^3)}+\|\nabla\Gamma*(\rho u)(\cdot,t_2)\|_{L^\infty(\mathbb{R}^3)}\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\Gamma_{x_jx_k}(x(s)-y)[u^k(x(s),s)-u^k(y,s)](\rho u^j)(y,s)dyds\\
&\le \tilde{M}C_0^\theta+\tilde{M}C_0^\theta\int_0^1\langle u(\cdot,s)\rangle^\alpha ds\\
&\le \tilde{M}C_0^\theta+\tilde{M}C_0^\theta\int_0^1\langle u(\cdot,s)\rangle^\alpha ds\\
&\le \tilde{M}C_0^\theta+\tilde{M}C_0^\theta\Big(\int_0^1 s^{-\frac{3+6\alpha}{4}}ds\Big)^{\frac12}\int_0^1\big(C_0+\|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}ds\big)^{\frac{1-2\alpha}{4}}\\
&\quad\times\Big(\int_0^1 s^{\frac32}(\|\dot{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\|(\nabla d\Delta d)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)})ds\Big)^{\frac{1+2\alpha}{4}}\\
&\le \tilde{M}C_0^\theta,
\end{aligned}
$$

if $\alpha<\frac{1}{6}$. Note that (3.22) holds for $q=6$, thus if $2<r<\frac{3q}{q+3}$, by (2.4), the second integral on the right side of (4.8) can be bounded in as

$$
\begin{aligned}
&\Big|\int_{t_0}^{t_1}\int_{\mathbb{R}^3}(\nabla_x\Gamma(x(s)-y))(\nabla d\Delta d)(y,s)dyds\Big|\\
&\le \tilde{M}\int_0^1\|(\nabla d\Delta d)(s)\|_{L^2(\mathbb{R}^3)}+\|(\nabla d\Delta d)(s)\|_{L^r(\mathbb{R}^3)}ds\\
&\le \tilde{M}\int_0^1\|(|\nabla d|^4|\Delta d|^2)(s)\|^{\frac12}_{L^2(\mathbb{R}^3)}\||\Delta d|^2(s)\|^{\frac12}_{L^2(\mathbb{R}^3)}ds\\
&\quad+\int_0^1\|(\Delta d)(s)\|_{L^3(\mathbb{R}^3)}\|(\nabla d)(s)\|_{L^{\frac{3r}{3-r}}(\mathbb{R}^3)}ds\\
&\le \tilde{M}C_0^\theta.
\end{aligned}
$$

Thus the proof of (4.6) is completed.

Case 2: $1\le t_0<t_1$

Again by the mass equation and the definition (1.3) of F,

$$\frac{d}{dt}(\rho(t)-\tilde{\rho})+(\mu+\lambda)^{-1}\rho(t)(P(t)-\tilde{P})=(\mu+\lambda)^{-1}\rho(t)F(t).$$

Multiplying by $(\rho(t)-\tilde{\rho})$ we get

$$\frac{1}{2}(\rho(t)-\tilde{\rho})^2+(\mu+\lambda)^{-1}f(t)\rho(t)(\rho(t)-\tilde{\rho})^2=-(\mu+\lambda)^{-1}\rho(t)(\rho(t)-\tilde{\rho})F(t), \tag{4.9}$$

where

$$f(t)=(P(t)-\tilde{P})(\rho(t)-\tilde{\rho})^{-1}.$$

Since $f(t)\ge 0$ on $[t_0,t_1]$, thus integrating (4.9) over $[t_0,t_1]$, we arrive at

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$$|\kappa' - \tilde{\rho}|^2 - |\bar{\rho} - b - \tilde{\rho}|^2 \leq \tilde{M} \int_{t_0}^{t_1} \|F(\cdot, s)\|^2_{L^\infty(\mathbb{R}^3)} ds. \tag{4.10}$$

So that if we show that

$$\int_{t_0}^{t_1} \|F(\cdot, s)\|^2_{L^\infty} ds \leq \tilde{M} C_0^\theta. \tag{4.11}$$

Then as in Case 1, (4.10) cannot hold if C_0 is sufficiently small. Since $y \in \mathbb{R}^3$ was arbitrary, it follows that $\rho < \kappa'$ on $\mathbb{R}^3 \times [0, \tau]$, as claimed.

To prove (4.11) we apply (1.5) and (2.4) to get

$$\begin{aligned}
\int_{t_0}^{t_1} \|F(\cdot, s)\|^2_{L^\infty(\mathbb{R}^3)} ds \leq{} & \int_{t_0}^{t_1} \|\dot{u}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} ds + \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} ds \\
& + \int_{t_0}^{t_1} \|\dot{u}(\cdot, s)\|^2_{L^4(\mathbb{R}^3)} ds + \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|^2_{L^4(\mathbb{R}^3)} ds \\
\leq{} & \tilde{M} C_0^\theta + \int_{t_0}^{t_1} \|\dot{u}(\cdot, s)\|^2_{L^4(\mathbb{R}^3)} ds + \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|^2_{L^4(\mathbb{R}^3)} ds.
\end{aligned}$$

The terms integral on the right side above can be bounded as

$$\begin{aligned}
\int_{t_0}^{t_1} \|\dot{u}(\cdot, s)\|^2_{L^4(\mathbb{R}^3)} ds \leq{} & \int_{t_0}^{t_1} \|\dot{u}(\cdot, s)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla \dot{u}(\cdot, s)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} ds \\
\leq{} & (\int_{t_0}^{t_1} \|\dot{u}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} ds)^{\frac{1}{4}} (\int_{t_0}^{t_1} \|\nabla \dot{u}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} ds)^{\frac{3}{4}} \\
\leq{} & \tilde{M} C_0^\theta
\end{aligned}$$

and

$$\begin{aligned}
& \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|^2_{L^4(\mathbb{R}^3)} ds \\
\leq{} & \int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla (\nabla d \Delta d)(\cdot, s)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^3)} ds \\
\leq{} & (\int_{t_0}^{t_1} \|(\nabla d \Delta d)(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} ds)^{\frac{1}{4}} (\int_{t_0}^{t_1} \|\nabla (\nabla d \Delta d)(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} ds)^{\frac{3}{4}} \\
\leq{} & \tilde{M} C_0^\theta,
\end{aligned}$$

where the last inequality follows from Proposition 3.1. Thus (4.11) is proved. The proof of proposition is completed. □

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by constructing weak solutions as limits of smooth solutions. So, we first prove the global-in-time existence of strong solutions with strong initial data which is strictly away vacuum and is only of small energy.

Proposition 5.1. *Assume that* (ρ_0, u_0, d_0) *satisfy* (2.5) *and* (2.6). *Then for any* $0 < T < \infty$, *there exists a unique strong solution* (ρ, u, d) *of* (1.1)–(1.14) *on* $\mathbb{R}^3 \times [0, T]$ *satisfying* (2.7)–(2.9) *with* T_0 *being replaced by* T, *provided the initial energy* C_0 *satisfies the smallness condition* (1.17) *with* $\varepsilon > 0$ *being the same one as in* *Proposition 3.1* *and* *Proposition 4.3*.

Proof. The standard local existence result (Proposition 2.1) shows that the Cauchy problem (1.1)–(1.2) admits a unique local smooth solution (ρ, u, d) on $\mathbb{R}^3 \times [0, T_0]$. In view of Lemma 3.1 and Proposition 4.1, we have

$$\begin{aligned} &A(T_0) + B_q(T_0) + E(T_0) + \sup_{0 \le t \le T_0} \int_{\mathbb{R}^3} (|\rho - \tilde{\rho}|^2 + |u|^2 + |\nabla d|^2) dx \\ &+ \int_0^{T_0} \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) dx dt \le M C_0, \end{aligned}$$

and

$$\underline{\rho} \le \rho \le \bar{\rho} \quad \text{on} \quad \mathbb{R}^3 \times [0, T_0]. \tag{5.1}$$

For any small enough positive T_*, we have

$$\begin{aligned} \int_{T_*}^{T_0} \|\nabla d(x,t)\|^3_{L^\infty(\mathbb{R}^3)} dx dt &\le C \int_{T_*}^{T_0} \|\nabla d(x,t)\|_{L^6(\mathbb{R}^3)} \|\nabla^2 d(x,t)\|^2_{L^4(\mathbb{R}^3)} dx dt \\ &\le C(T_*, T_0) B_6(T_0) E(T_0). \end{aligned} \tag{5.2}$$

Then the standard arguments based on the local existence results together with the a priori bounds (5.1)–(5.2), we deduce that (ρ, u, d) is in fact the unique smooth solution of (1.1)–(1.14) on $\mathbb{R}^3 \times [0, T]$ for any $0 < T < \infty$. □

With the help of Proposition 5.1, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. For any map $d_0 \in H_n^1(\mathbb{R}^3; \mathbb{S}^2)$, there exists $d_0^m \in H_n^4(\mathbb{R}^3; \mathbb{S}^2)$ such that

$$\lim_{m \to \infty} \|d_0^m - d_0\|_{H^1(\mathbb{R}^3)} = 0.$$

Let

$$\rho_0^m = J_{\frac{1}{m}} * \rho_0, \quad u_0^m = J_{\frac{1}{m}} * u_0,$$

where $J_{\frac{1}{m}} = J_{\frac{1}{m}}(x)$ is the standard mollifier. Then $\rho_0^m - \tilde{\rho} \in W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, and $\inf_{x \in \mathbb{R}^3}(\rho_0^m(x)) > 0$, $u_0^m \in H^2(\mathbb{R}^3)$, $d_0^m \in H_n^3(\mathbb{R}^3; \mathbb{S}^2)$, and the initial norm for $(\rho_0^m, u_0^m, \nabla d_0^m)$ (i.e., the right side of (1.14) with $(\rho_0, u_0, \nabla d_0)$ replaced by $(\rho_0^m, u_0^m, \nabla d_0^m)$) is bounded by C_0. The above proposition can be applied to obtain a global smooth solution (ρ^m, u^m, d^m) of (1.1)–(1.14) satisfying (3.2), (4.3) and (4.4) for all $t > 0$ uniformly in m.

In view of (2.3) and (2.11), we see from Sobolev embedding theorem that

$$\begin{aligned}\langle u^m(\cdot,t)\rangle^{\frac{1}{2}} &\le C\|\nabla u^m(\cdot,t)\|_{L^6(\mathbb{R}^3)}\\ &\le C(\|F^m(\cdot,t)\|_{L^6(\mathbb{R}^3)}+\|\omega^m(\cdot,t)\|_{L^6(\mathbb{R}^3)}+\|P^m(\cdot,t)-\tilde{P}\|_{L^6(\mathbb{R}^3)})\\ &\le C(\tau)(1+\|\dot{u}^m(\cdot,t)\|_{L^2(\mathbb{R}^3)}+\|\nabla d^m(\cdot,t)\Delta d^m(\cdot,t)\|_{L^2(\mathbb{R}^3)})\\ &\le C(\tau),\end{aligned} \tag{5.3}$$

for $t \ge \tau > 0$. Here F^m, ω^m and P^m are the functions F, ω and P with (ρ, u, d) being replaced by (ρ^m, u^m, d^m).

In addition to (5.3), we also have

$$|u^m(x,t) - \frac{1}{|B_{R(x)}|}\int\limits_{B_{R(x)}} u^m(y,t)dy| \le C(\tau)R^{\frac{1}{2}},$$

and hence, for $0 < \tau \le t_1 \le t_2$, we have

$$\begin{aligned}&|u^m(x,t_2)-u^m(x,t_1)|\\ \le\ & \tfrac{1}{|B_{R(x)}|}\textstyle\int_{B_{R(x)}}|u^m(y,t_2)-u^m(y,t_1)|dy + C(\tau)R^{\frac{1}{2}}\\ \le\ & CR^{-\frac{3}{2}}|t_2-t_1|^{\frac{1}{2}}(\textstyle\int_{t_1}^{t_2}\int_{B_{R(x)}}|u_t^m(y,t)|^2dydt)^{\frac{1}{2}}+C(\tau)R^{\frac{1}{2}}\\ \le\ & CR^{-\frac{3}{2}}|t_2-t_1|^{\frac{1}{2}}(\textstyle\int_{t_1}^{t_2}\int_{B_{R(x)}}(|\dot{u}^m(y,t)|^2+|u^m|^2|\nabla u^m|^2)dydt)^{\frac{1}{2}}+C(\tau)R^{\frac{1}{2}}\\ \le\ & C(\tau)[R^{-\frac{3}{2}}|t_2-t_1|^{\frac{1}{2}}+R^{\frac{1}{2}}].\end{aligned} \tag{5.4}$$

Taking $R = |t_2 - t_1|^{\frac{1}{4}}$ in (5.4), we get

$$|u^m(x,t_2) - u^m(x,t_1)| \le C(\tau)|t_2 - t_1|^{\frac{1}{8}}, \quad 0 < \tau \le t_1 \le t_2 < \infty. \tag{5.5}$$

The same estimates in (5.3) and (5.5) also hold for d and ∇d. Thus, we have proved that $\{u^m\}$, $\{d^m\}$ and $\{\nabla d^m\}$ are uniform Hölder continuity away from $t = 0$. As a result, it follows from Ascoli–Arzela theorem that

$$u^m \longrightarrow u, \quad d^m \longrightarrow d \ \text{ uniformly on compact sets in } \mathbb{R}^3 \times (0,\infty). \tag{5.6}$$

Moreover, by argument in [29] (see also [4]), we know that

$$\rho^m \longrightarrow \rho \ \text{ strongly in } L^p(\mathbb{R}^3 \times (0,\infty)),\ \forall p \in [2,\infty). \tag{5.7}$$

Therefore, passing to the limit as $m \longrightarrow \infty$ by (5.6) and (5.7) we obtain the limited (ρ, u, d) which is indeed a weak solution of (1.1)–(1.14) in the sense of Definition 1.1 and satisfies (1.18)–(1.25).

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Next we derive the large-time behavior of (ρ, u, d) in (1.26). This can be done as the ones in [5], however, for completeness we sketch the proof here. We first deduce from the mass equation that

$$(P(\rho) - \tilde{P})_t + u \cdot \nabla(P(\rho) - \tilde{P}) + \gamma P(\rho)\mathrm{div} u = 0.$$

Multiplying the above equation by $4(P(\rho) - \tilde{P})^3$ and integrating it over $\mathbb{R}^3$, we get that

$$\frac{d}{dt}\|P(\rho) - \tilde{P}\|^4_{L^4(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|P(\rho) - \tilde{P}|^4)\mathrm{div} u - 3\gamma P(\rho)(P(\rho) - \tilde{P})^3 \mathrm{div} u dx,$$

which, together with (3.29) shows that

$$\int_1^\infty |\frac{d}{dt}\|P(\rho) - \tilde{P}\|^4_{L^4(\mathbb{R}^3)}|dt \le C(1 + \int_1^\infty \|F\|^4_{L^4(\mathbb{R}^3)} ds) \int_1^\infty \|\nabla u\|^2_{L^2(\mathbb{R}^3)} ds \le C.$$

As a result, we have

$$\|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^3)} \longrightarrow 0 \ \text{ as } \ t \to \infty.$$

This, together with (3.2) and the uniform lower and upper bound of density, shows that

$$\lim_{t\to\infty} \|\rho - \tilde{\rho}\|_{L^l(\mathbb{R}^3)} = 0 \tag{5.8}$$

holds for any $l \in (2, \infty)$.

Following the same argument in [5], we take a sequence

$$\mathbf{u}^m(t, x) := u(t + m, x),$$

for all integer m, and $(x, t) \in \mathbb{R}^3 \times [1, 2]$. Then from (1.25), we have

$$\lim_{m\to\infty} \int_0^1 \|\nabla \mathbf{u}^m\|_{L^2(\mathbb{R}^3)} = 0.$$

From (1.25) again, we have

$$\|\mathbf{u}^m\|_{H^1(\mathbb{R}^3)} \le C \text{ uniformly for } t, m.$$

Thus we arrive at

$$\lim_{m\to\infty} \|\mathbf{u}^m\|_{L^2(\mathbb{R}^3)} = 0 \text{ uniformly for } t.$$

That means

$$\lim_{t\to\infty} \|u(t)\|_{L^2(\mathbb{R}^3)} = 0. \tag{5.9}$$

For $t \geq 1$, from (2.11) and (2.12), we obtain that

$$\begin{aligned} \|\nabla u(t)\|_{L^6(\mathbb{R}^3)} &\leq C(\|F(t)\|_{L^6(\mathbb{R}^3)}) + \|\omega(t)\|_{L^6(\mathbb{R}^3)} + \|(P(\rho) - \tilde{P})(t)\|_{L^6(\mathbb{R}^3)}) \\ &\leq C(1 + \|\nabla F(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla\omega(t)\|_{L^2(\mathbb{R}^3)}) \\ &\leq C(1 + \|\dot{u}(t)\|_{L^2(\mathbb{R}^3)} + \|(\nabla d \Delta d)(t)\|_{L^2(\mathbb{R}^3)}) \\ &\leq C. \end{aligned} \tag{5.10}$$

Combining (1.25), (5.9) and (5.10), we have

$$\lim_{t\to\infty} \|u\|_{W^{1,r}(\mathbb{R}^3)} = 0, \tag{5.11}$$

holds for $r \in$ (2.6).

Similarly, we have

$$\lim_{t\to\infty} \|\nabla d\|_{W^{1,r}(\mathbb{R}^3)} = 0, \tag{5.12}$$

holds for $r \in (2, 6)$. Putting (5.8), (5.11) and (5.12) together gives (1.26). Thus the proof of Theorem 1.1 is completed. □

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